

العدد الرابع - ديسمبر 2015

## Hankel determinant for subclass of analytic functions Defined by differential operator

\*S. F. Ramadan and \*\* M. Darus

\*Department of Mathematics, Zawia University, Zawia, Libya. \*\*School of  
Mathematical Sciences, Faculty of Sciences, University Kebangsaan Malaysia



Hankel determinant for subclass of analytic functions Defined by differential operator

## العدد الرابع - ديسمبر 2015

### Abstract:

By making use of the differential operator  $S^m$  given by Sălăgean, a class of analytic functions is introduced. The upper bound for the second Hankel determinant for this class is given.

**Keywords:** Hankel determinant, Differential operator, Upper bound.

### 1. Introduction

Let  $A$  be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

defined in the open unit disk  $U = \{z : |z| < 1\}$ . Let  $P(\alpha)$  denote the class of functions  $p(z)$  of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad (1.2)$$

which are analytic in  $U$  and satisfy  $\operatorname{Re} p(z) > \alpha$ , ( $z \in U$ ) for some  $\alpha$ , ( $0 \leq \alpha < 1$ ). If  $p(z) \in P = P(0)$ , then we say that  $p(z)$  is the Carathéodory function [1].

**Definition 1.1** [11] For a function  $f \in A$  the Sălăgean derivative operator  $S^m : A \rightarrow A$  is defined by

$$S^m f(z) = f(z) * \left( z + \sum_{n=2}^{\infty} n^m a_n z^n \right). \quad (1.3)$$

Also,

$$S^0 f(z) = f(z), \quad S^1 f(z) = z f'(z)$$

**Hankel determinant for subclass of analytic functions Defined by differential operator**

## العدد الرابع - ديسمبر 2015

$$S^m f(z) = S(S^{m-1} f(z)).$$

If  $f$  is given by (1.1), then we see that

$$S^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \quad (m \in \mathbb{N}_0). \quad (1.4)$$

Noonan and Thomas [9] stated the  $q$ -th Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \quad (n, q \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

This determinant was discussed by several authors with  $q = 2$ . For example, we know that the functional  $|H_2(1)| = |a_3 - a_2^2|$  is known as the Fekete-Szegő problem and they consider the further generalized functional  $|a_3 - \mu a_2^2|$  where  $a_1 = 1$  and  $\mu$  is real number [2]. In [10], Noor determined the rate of growth of  $H_q(n)$  as  $n \rightarrow \infty$  for functions  $f$  given by (1.1) with bounded boundary.

Recently, for classes of starlike and convex functions Janteng et.al [7] have shown the following Theorems.

**Theorem 1.2** Let  $f(z) \in S^*$  (starlike function). Then  $|a_2 a_4 - a_3^2| \leq 1$ .

Equality is attained for functions

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \dots$$

And

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + z^7 + \dots$$

**Theorem 1.3** Let  $f(z) \in K$  (convex function). Then  $|a_2 a_4 - a_3^2| \leq \frac{1}{8}$ .

**Hankel determinant for subclass of analytic functions Defined by differential operator**

## العدد الرابع - ديسمبر 2015

Following these results, Hayami and Owa [4], [5] generalized Theorem 1.2 and Theorem 1.3 by finding the upper bounds for the generalized functional  $|a_{p+2} - \mu a_{p+1}^2|$  and  $|a_{p+1} a_{p+3} - \mu a_{p+2}^2|$  for functions  $f$  in the class  $S_p^*(\alpha)$  and  $K_p(\alpha)$ . Indeed the authors [6] also studied the generalized functional  $|a_n a_{n+2} - \mu a_{n+1}^2|$  for functions  $f$  defined in the class  $\text{Re}(f(z)/z) > \alpha$  and  $\text{Re}(f'(z)) > \alpha$  for some  $\alpha (0 \leq \alpha < 1)$ .

In this paper, we consider the Hankel determinant in the case of  $q = 2$  and  $n = 2$ ,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

We seek upper bound for the functional  $|a_2 a_4 - \mu a_3^2|$  where  $\mu$  is real for functions  $f$  belonging to the class which is defined as the following:

**Definition 1.4** The function  $f \in A$  is said to be in the class  $R^m(\alpha)$ ,  $m \in \mathbb{R}$ ,  $0 \leq \alpha < 1$  if it satisfies the inequality

$$\text{Re}\left\{(S^m f(z))'\right\} > \alpha, \quad (z \in U),$$

where  $S^m$  is given by (1.4).

## 2 Preliminaries

To obtain our results, we need some lemmas.

**Lemma 2.1** [3] The power series for  $p$  given in (1.2) converges in  $U$  to a function in  $P$  if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \dots & 2 \end{vmatrix} \quad (n = 1, 2, 3, \dots),$$

## العدد الرابع - ديسمبر 2015

where  $c_{-k} = \overline{c_k}$  are all non-negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^m \rho_k p_0(e^{it_k} z), \quad \rho_k > 0, \quad t_k \text{ real and } t_k \neq t_j \text{ for } k \neq j \text{ where } p_0(z) = \frac{1+z}{1-z};$$

in this case  $D_n > 0$  for  $n < r-1$  and for  $D_n = 0$  for  $n \geq r$ .

This necessary and sufficient condition is due to Carathéodory and Toeplitz, and can be found in [3].

**Lemma 2.2** (see [4]) If  $p \in P(\alpha)$ , then

$$|c_k| \leq 2(1-\alpha), \quad (k = 1, 2, 3, \dots).$$

The result is sharp for

$$p(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + \sum_{k=1}^{\infty} (1-2\alpha)z^k.$$

**Lemma 2.3** (see [4])  $p \in P(\alpha)$ , then

$$\begin{cases} 2(1-\alpha)c_2 = c_1^2 + \{4(1-\alpha)^2 - c_1^2\} \zeta \\ 4(1-\alpha)^2 c_3 = c_1^3 + 2\{4(1-\alpha)^2 - c_1^2\} c_1 \zeta - \{4(1-\alpha)^2 - c_1^2\} c_1 \zeta^2 \\ \quad + 2(1-\alpha)\{4(1-\alpha)^2 - c_1^2\} (1-|\zeta|^2) \eta \end{cases}$$

For some complex numbers  $\zeta$  and  $\eta$  ( $|\zeta| \leq 1, |\eta| \leq 1$ ).

### 3 Main results

**Theorem 3.1** The function  $f$  given by (1.1) is in the class  $R^m(\alpha)$ ,  $m \in \mathbb{N}_0, 0 \leq \alpha < 1$ .

Then

### العدد الرابع - ديسمبر 2015

$$|a_2 a_4 - \mu a_3^2| \leq \begin{cases} \frac{(3^{2m+3} - 2^{3m+4} \mu)^2 (1-\alpha)^2}{144(3^{2m+2} - 2^{3m+3} \mu)} - \frac{4\mu(1-\alpha)^2}{3^{2m+2}}, & (\mu \leq 0), \\ \frac{(3^{2m+3} - 2^{3m+5} \mu)^2 (1-\alpha)^2}{144(3^{2m+2} - 2^{3m+3} \mu)} + \frac{4\mu(1-\alpha)^2}{3^{2m+2}}, & \left(0 \leq \mu \leq \frac{3^{2m+3}}{2^{3m+5}}\right), \\ \frac{4\mu(1-\alpha)^2}{3^{2m+2}}, & \left(\frac{3^{2m+3}}{2^{3m+5}} \leq \mu \leq \frac{3^{2m+3}}{2^{3m+4}}\right), \\ \frac{(2^{3m+4} \mu - 3^{2m+3})^2 (1-\alpha)^2}{144(2^{3m+3} \mu - 3^{2m+2})} + \frac{4\mu(1-\alpha)^2}{3^{2m+2}}, & \left(\mu \geq \frac{3^{2m+3}}{2^{3m+4}}\right). \end{cases}$$

Proof : Since  $f \in R^m(\alpha)$ , it follows from (1.5) that

$$(S^m f(z))' = p(z), \quad (3.1)$$

where is given by (1.2). Comparing the coefficients, we get

$$\begin{aligned} 2^{m+1} a_2 &= c_1 \\ 3^{m+1} a_3 &= c_2 \\ 4^{m+1} a_4 &= c_3. \end{aligned} \quad (3.2)$$

From (3.2),

$$|a_2 a_4 - \mu a_3^2| = \left| \frac{c_1 c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right|. \quad (3.3)$$

Suppose now that  $c_1 = c$  and  $0 \leq c \leq 2(1-\alpha)$  without loss of generality and applying the triangle inequality, we see that

$$\left| \frac{c_1 c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right| = \frac{1}{4(1-\alpha)^2} \left| \frac{(3^{2m+2} - 2^{3m+3} \mu) c^4}{(72)^{m+1}} + \frac{2(1-\alpha)[4(1-\alpha)^2 - c^2] c \eta}{2^{3m+3}} + \right.$$

**Hankel determinant for subclass of analytic functions Defined by differential operator**

**العدد الرابع - ديسمبر 2015**

$$\frac{2[4(1-\alpha)^2 - c^2]c^2\zeta(3^{2m+2} - 2^{3m+3}\mu)}{(72)^{m+1}} +$$

$$\left| \frac{[4(1-\alpha)^2 - c^2]\zeta^2 \left\{ (3^{2m+2} - 2^{3m+3}\mu)c^2 + 2^{3m+5}\mu(1-\alpha)^2 - 2(3)^{2m+2}(1-\alpha)c \right\}}{(72)^{m+1}} \right|.$$

Triangle inequality gives

$$\left| \frac{c_1c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right| \leq \frac{1}{4(1-\alpha)^2} \frac{|3^{2m+2} - 2^{3m+3}\mu|c^4}{(72)^{m+1}} + \frac{2(1-\alpha)[4(1-\alpha)^2 - c^2]c}{2^{3m+3}} +$$

$$\frac{2[4(1-\alpha)^2 - c^2]c^2|3^{2m+2} - 2^{3m+3}\mu|\rho}{(72)^{m+1}} +$$

$$\frac{[4(1-\alpha)^2 - c^2][|3^{2m+2} - 2^{3m+3}\mu|c^2 + 2^{3m+5}|\mu|(1-\alpha)^2 - 2(3)^{2m+2}(1-\alpha)c]\rho^2}{(72)^{m+1}}$$

$$= \frac{F(\rho)}{4(1-\alpha)^2} \quad (3.4)$$

with  $\rho = |\zeta| \leq 1$  This gives rise to

### العدد الرابع - ديسمبر 2015

$$F'(\rho) = \begin{cases} \frac{2[4(1-\alpha)^2 - c^2]c^2(3^{2m+2} - 2^{3m+3}\mu)}{(72)^{m+1}} + \frac{2[4(1-\alpha)^2 - c^2]\{(3^{2m+2} - 2^{3m+3}\mu)c^2 - 2^{3m+5}\mu(1-\alpha)^2 - 2(3)^{2m+2}(1-\alpha)c\}\rho}{144(3^{2m+2} - 2^{3m+3}\mu)}, & \text{if } \mu \leq 0 \\ \frac{2[4(1-\alpha)^2 - c^2]c^2(3^{2m+2} - 2^{3m+3}\mu)}{(72)^{m+1}} + \frac{2[4(1-\alpha)^2 - c^2]\{(3^{2m+2} - 2^{3m+3}\mu)c^2 + 2^{3m+5}\mu(1-\alpha)^2 - 2(3)^{2m+2}(1-\alpha)c\}\rho}{(72)^{m+1}}, & \text{if } 0 \leq \mu \leq \frac{3^{2m+2}}{2^{3m+3}} \\ \frac{2[4(1-\alpha)^2 - c^2]c^2(2^{3m+3}\mu - 3^{2m+2})}{(72)^{m+1}} + \frac{2[4(1-\alpha)^2 - c^2]\{(2^{3m+3}\mu - 3^{2m+2})c^2 + 2^{3m+5}\mu(1-\alpha)^2 - 2(3)^{2m+2}(1-\alpha)c\}\rho}{(72)^{m+1}}, & \text{if } \mu \geq \frac{3^{2m+2}}{2^{3m+3}} \end{cases}$$

And again for all the cases above,  $F'(\rho) > 0$  for  $\rho > 0$  and  $0 \leq \alpha < 1$ ; implying that  $F$  is an increasing function with  $Max_{\rho \leq 1} F(\rho) = F(1)$ . Now let

$$G(c) = \frac{F(1)}{4(1-\alpha)^2} = \frac{1}{4(1-\alpha)^2} \left\{ \frac{|3^{2m+2} - 2^{3m+3}\mu|c^4}{(72)^{m+1}} + \frac{2(1-\alpha)[4(1-\alpha)^2 - c^2]c}{2^{3m+3}} + \frac{2[4(1-\alpha)^2 - c^2]c^2|3^{2m+2} - 2^{3m+3}\mu|}{(72)^{m+1}} + \frac{[4(1-\alpha)^2 - c^2][|3^{2m+2} - 2^{3m+3}\mu|c^2 + 2^{3m+5}|\mu|(1-\alpha)^2 - 2(3)^{2m+2}(1-\alpha)c]}{(72)^{m+1}} \right\} \quad (3.5)$$

(i)First, let us consider the case  $\mu \leq 0$ ,  $0 \leq \alpha < 1$ . From (3.5), we have

$$G'(c) = \frac{c}{(2)^{3m+2}(3)^{2m+2}(1-\alpha)^2} \times \left\{ -(3^{2m+2} - 2^{3m+3}\mu)c^2 + 3^{2m+3}(1-\alpha)^2 - 2^{3m+4}\mu(1-\alpha)^2 \right\}$$

### العدد الرابع - ديسمبر 2015

Elementary calculation reveals that  $G$  attains its maximum value at

$$c = (1-\alpha) \sqrt{\frac{3^{2m+3} - 2^{3m+4} \mu}{3^{2m+2} - 2^{3m+3} \mu}}.$$

The upper bound for inequality (3.4) corresponds to  $\rho = 1$ ,  $0 \leq \alpha < 1$  and

$$c = (1-\alpha) \sqrt{\frac{3^{2m+3} - 2^{3m+4} \mu}{3^{2m+2} - 2^{3m+3} \mu}}, \text{ in which case}$$

$$\left| \frac{c_1 c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right| \leq \frac{(1-\alpha)^2 (3^{2m+3} - 2^{3m+4} \mu)^2}{144 (3^{2m+2} - 2^{3m+3} \mu)} - \frac{(1-\alpha)^2 4\mu}{3^{2m+2}}$$

(ii) Secondly, we consider the case  $0 \leq \mu \leq \frac{3^{2m+3}}{2^{3m+5}}$ ,  $0 \leq \alpha < 1$ . From (3.5), we have

$$G'(c) = \frac{c}{(2)^{3m+2} (3)^{2m+2} (1-\alpha)^2} \times \left\{ -(3^{2m+2} - 2^{3m+3} \mu) c^2 + 3^{2m+3} (1-\alpha)^2 - 2^{3m+5} \mu (1-\alpha)^2 \right\}$$

where  $G$  attains its maximum value at  $c = (1-\alpha) \sqrt{\frac{3^{2m+3} - 2^{3m+5} \mu}{3^{2m+2} - 2^{3m+3} \mu}}$ . Hence, we obtain

$$\left| \frac{c_1 c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right| \leq \frac{(1-\alpha)^2 (3^{2m+3} - 2^{3m+5} \mu)^2}{144 (3^{2m+2} - 2^{3m+3} \mu)} + \frac{(1-\alpha)^2 4\mu}{3^{2m+2}}$$

(iii) to prove the third result, we consider two cases. First, assume that

$\frac{3^{2m+3}}{2^{3m+5}} \leq \mu \leq \frac{3^{2m+2}}{2^{3m+3}}$ ,  $0 \leq \alpha < 1$ . In this case,  $G$  attains its maximum value at

$$c = (1-\alpha) \sqrt{\frac{2^{3m+5} \mu - 3^{2m+3}}{3^{2m+2} - 2^{3m+3} \mu}}$$

Next consider the case  $\frac{3^{2m+2}}{2^{3m+3}} \leq \mu \leq \frac{3^{2m+3}}{2^{3m+4}}$ ,  $0 \leq \alpha < 1$ . For this case from (3.5), we have

$$G'(c) = \frac{c}{(2)^{3m+3} (3)^{2m+2} (1-\alpha)^2} \times \left\{ -(2^{3m+3} \mu - 3^{2m+2}) c^2 + 2^{3m+4} \mu (1-\alpha)^2 - 3^{2m+3} (1-\alpha)^2 \right\}$$

**Hankel determinant for subclass of analytic functions Defined by differential operator**

### العدد الرابع - ديسمبر 2015

where  $G$  attains its maximum value at  $c = (1-\alpha) \sqrt{\frac{3^{2m+3} - 2^{3m+4} \mu}{2^{3m+3} \mu - 3^{2m+2}}}$ . In both cases, the upper bound is attained as

$$\left| \frac{c_1 c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right| \leq \frac{(1-\alpha)^2 4\mu}{3^{2m+2}}$$

(iv) Finally, consider  $\mu \geq \frac{3^{2m+3}}{2^{3m+4}}$ ,  $0 \leq \alpha < 1$ .

Here,  $G$  attains its maximum value at  $c = (1-\alpha) \sqrt{\frac{2^{3m+4} \mu - 3^{2m+3}}{2^{3m+3} \mu - 3^{2m+2}}}$ . Hence,

$$\left| \frac{c_1 c_3}{2^{3m+3}} - \mu \frac{c_2^2}{3^{2m+2}} \right| \leq \frac{(1-\alpha)^2 (2^{3m+4} \mu - 3^{2m+3})^2}{144(2^{3m+3} \mu - 3^{2m+2})} + \frac{(1-\alpha)^2 4\mu}{3^{2m+2}}$$

This completes the proof of the theorem.

**Remark:** In Theorem 3.1 by taking  $m = 0$ ,  $\alpha = 0$  we obtain the results of Janteng et.al [8].

**Acknowledgement:** This work is supported by the first author.

## References

- [1] P. L. Duren, Univalent functions. *Springer-Verlag, New York, Berlin, Heidelberg, Tokyo*, (1983).
- [2] M. Fekete and G. Szeqö, Eine bemerkung über ungerade schlichte funktionen.  
*J. London Math. Soc.*, 8, (1933), 85-89.
- [3] U. Grenander and G. Szeqö, Toeplitz forms and their application. Berkeley: *Univ. of California Press.*, (1958).
- [4] T. Hayami and S. Owa, Hankel determinant for p-valently starlike and convex functions of order  $\alpha$ . *General Math.* 17 (4), (2009), 29-44.
- [5] T. Hayami and S. owa, Applications of Hankel determinant for p-valently starlike and convex functions of order  $\alpha$  in Extensions of the historical calculus transforms in the geometric function theory. *RIMS Kôkyûroku* 1717, (2010), 8-18.
- [6] T. Hayami and S. owa, Generalized Hankel determinant for certain classes. *Int. J. Math. Anal.* 4 (52) (2010), 2573-2585.
- [7] A. Janteng, S. A. Halim, M. Darus, Hankel detrminant for starlike and convex functions. *Int. J. Math. Anal.* 1 (13) (2007), 619-625.
- [8] A. Janteng, S. A. Halim, M. Darus, Estimate on the second Hankel functional for functions whose derivative has a positive real part . *J. Quality Measurement and Analysis* 4 (1), (2008), 189-195.
- [9] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of areally p-valent functions. *Trans. Amer. Math. Soc.*, 223 (2), (2001), 337-346.
- [10] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation. *Rev. Roum. Math. Pures Et Appl.*, 28 (8), (1983), 731-739.
- [11] Salma Faraj Ramadan , Maslina Darus, New subclass of harmonic functions defined by a generalized differential operator, *Journal of Quality Measurement and Analysis*, JQMA 8 (1), 2012, 97-109.

### العدد الرابع - ديسمبر 2015

- [12] G. S. Sălăgean, Subclasses of univalent functions. *Lecture Notes in Math.1013*  
*Springer, Verlag Berlin* , (1983), 362-372.