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Orthogonal Sets of Functions and Sturm-Liouville Problem

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ABSTRACT

In this paper, for a regular Sturm-Liouville problem, we prove that eigenfunctions corresponding to different eigenvalues are orthogonal with respect to some weight function along with the assumption that all eigenvalues of Sturm-Liouville problem are real, then we give some examples to illustrate the main results.

Keywords: Sturm-Liouville problems, eigenvalues, eigenfunctions, Orthogonality.

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INTRODUCTION

A differential equation of the following:

$$[r(x)y']' + [q(x) + \lambda p(x)]y = 0 \quad (1)$$

Is Known as Sturm-Liouville equation.

We assume that the functions p , q , r and r' in (1) are continuous in $a \leq x \leq b$ and $p(x) > 0$.

Here λ is a parameter independent of x .

Equation (1) is constant on some interval $a \leq x \leq b$, satisfying boundary conditions at the two end points a and b .

$$a_1y(a) + a_2y'(a) = 0 \quad (2a)$$

$$b_1y(b) + b_2y'(b) = 0 \quad (2b)$$

Suppose that the real constants a_1 , a_2 , b_1 and b_2 in (2) are not equal to zero.

The boundary value problem consisting of (1) and (2) is called a Sturm-Liouville problem. Named for the French mathematicians Charles Sturm (1803-1855) and Joseph Liouville (1809-1882).

Clearly $y = 0$ is always a solution of the Sturm-Liouville problem for any value of the parameter λ . $y = 0$ known as trivial solution of no practical use. The non-zero solutions of the Sturm-Liouville problem given by (1) and (2) are called eigenfunctions of the problem and the values of λ for which such solutions exist, are called eigenvalues of the problem.

Remark: A special case of (1) and (2). Let $p = r = 1$ and $q = 0$ in (1). Also, let $a_1 = b_1 = 1$ and $a_2 = b_2 = 0$, then (1) and (2) reduce to $y'' + \lambda y = 0$ with $y(a) = 0$, $y(b) = 0$. This is simplest form of the Sturm-Liouville problem.

Orthogonality of eigenfunctions:

Theorem (1): Suppose that the functions $p(x)$, $q(x)$, $r(x)$ and $r'(x)$ in the Sturm-Liouville equation (1) are real valued, continuous and $p(x) > 0$ on the interval $a \leq x \leq b$. Let $y_i(x)$ and $y_j(x)$ be eigenfunctions of the Sturm-Liouville problem (given by the above Sturm-Liouville equation and boundary conditions $a_1y(a) + a_2y'(a) = 0$ and $b_1y(b) +$

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$b_2 y'(b) = 0$) that correspond to different eigenvalues λ_i and λ_j respectively. Then y_i , y_j are orthogonal and that interval with respect to the weight function $p(x)$.

Prove that eigenfunctions corresponding to different eigenvalues are orthogonal with respect to some weight function.

Proof: Consider the problems (1), (2a) and (2b):

Where p , q , r and r' are real valued, continuous and $p(x) > 0$ on $a \leq x \leq b$. Let a_1 , a_2 in (2a) be given constants, not both zero and so be b_1 , b_2 in (2b).

Let y_i and y_j be eigenfunctions of the above Sturm-Liouville problem that correspond to different eigenvalues λ_i and λ_j . Then, by division of eigenfunctions, y_i and y_j both satisfy (1).

$$\text{Hence } (ry'_i)' + (q + \lambda_i p)y_i = 0 \quad (3)$$

$$\text{and } (ry'_j)' + (q + \lambda_j p)y_j = 0 \quad (4)$$

Multiplying (3) by y_j and (4) by y_i then subtracting, we get

$$(ry'_i)'y_j - (ry'_j)'y_i + (\lambda_i - \lambda_j)py_i y_j = 0 \text{ or}$$

$$(\lambda_i - \lambda_j)py_i y_j = (ry'_j)'y_i - (ry'_i)'y_j \text{ or}$$

$$(\lambda_i - \lambda_j)py_i y_j = \frac{d}{dx} \{ (ry'_j)y_i - (ry'_i)y_j \} \quad (5)$$

which can be verified by performing the indicated differentiation of the expression in brackets on R.H.S of (5). Since $r(x)$ and $r'(x)$ are continuous by assumption and y_i , y_j are solutions of (1), it follows that the expression within brackets on R.H.S of (5). Is continuous on $a \leq x \leq b$. Integrating both sides of (5) over x from a to b , we thus obtain

$$(\lambda_i - \lambda_j) \int_a^b py_i y_j dx = [r(y'_j y_i - y'_i y_j)]_a^b \text{ or}$$

$$(\lambda_i - \lambda_j) \int_a^b py_i y_j dx = r(b) \{ y'_j(b) y_i(b) - y'_i(b) y_j(b) \} \\ - r(a) \{ y'_j(a) y_i(a) - y'_i(a) y_j(a) \} \quad (6)$$

Now we have to consider several cases depending on whether $r(x)$ vanishes or does not vanish at a or b .

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Case (I): Let $r(a) = r(b) = 0$ Then (6) reduces to

$$(\lambda_i - \lambda_j) \int_a^b p y_i y_j dx = 0 \quad (7)$$

Case (II): Let $r(b) = 0$ but $r(a) \neq 0$. Then (6) reduces to

$$(\lambda_i - \lambda_j) \int_a^b p y_i y_j dx = -r(a) \{y_j'(a) y_i(a) - y_i'(a) y_j(a)\} \quad (8)$$

Since y_i and y_j both satisfy (2a) , we have

$$a_1 y_i(a) + a_2 y_i'(a) = 0 \quad (9)$$

and

$$a_1 y_j(a) + a_2 y_j'(a) = 0 \quad (10)$$

Let $a_2 \neq 0$. Multiplying (10) by $y_i(a)$ and (9) by $y_j(a)$ and then subtracting, we get

$$a_2 \{y_j'(a) y_i(a) - y_i'(a) y_j(a)\} = 0$$

$$\text{Since } a_2 \neq 0, \text{ so } y_j'(a) y_i(a) - y_i'(a) y_j(a) = 0 \quad (11)$$

Using (11) , (8) reduces to (7). If $a_2 = 0$, then let $a_1 \neq 0$. Now, multiplying (9) by $y_j'(a)$ and (10) by $y_i'(a)$ and then subtracting, we get

$$a_1 \{y_j'(a) y_i(a) - y_i'(a) y_j(a)\} = 0$$

$$\text{Since } a_1 \neq 0 \text{ so } y_j'(a) y_i(a) - y_i'(a) y_j(a) = 0$$

Hence, as before, (8) reduces to (7).

Case (III): Let $r(a) = 0$ but $r(b) \neq 0$. Then (6) reduces to

$$(\lambda_i - \lambda_j) \int_a^b p y_i y_j dx = r(b) \{y_j'(b) y_i(b) - y_i'(b) y_j(b)\} \quad (12)$$

Since y_i and y_j both satisfy (2b) we have

$$b_1 y_i(b) + b_2 y_i'(b) = 0 \quad (13)$$

$$b_1 y_j(b) + b_2 y_j'(b) = 0 \quad (14) \text{ and}$$

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Let $b_2 \neq 0$. Multiplying (14) by $y_i(b)$ and (13) by $y_j(b)$ and then subtracting, we get

$$b_2 \{y_j'(b)y_i(b) - y_i'(b)y_j(b)\} = 0$$

$$\text{Since } b_2 \neq 0, \text{ so } y_j'(b)y_i(b) - y_i'(b)y_j(b) = 0 \quad (15)$$

Using (15), (12) reduces to (7). If $b_2 = 0$, then let $b_1 \neq 0$. Now, multiplying (13) by $y_j'(b)$ and (14) by $y_i'(b)$ and then subtracting, we get

$$b_1 \{y_j'(b)y_i(b) - y_i'(b)y_j(b)\} = 0$$

$$\text{Since } b_1 \neq 0 \text{ so } y_j'(b)y_i(b) - y_i'(b)y_j(b) = 0$$

Hence as before, (12) reduces to (7).

Case (IV): Let $r(a) \neq 0$ and $r(b) \neq 0$. There is no loss of generality by assuming that $a_2 \neq 0$ and $b_2 \neq 0$. Then, proceeding as in cases (II) and (III), relations (11) and (15) can be proved. Then, using (11) and (15), (6) reduces to (7).

Case (V): Let $r(a) = r(b)$. Proceed as in case (IV) to show that (6) reduces to (7).

From the above discussing, we see that in all situations, we get (7).

$$\text{Since } \lambda_i \text{ and } \lambda_j \text{ are different, (7) reduces to } \int_a^b p(x)y_i(x)y_j(x)dx = 0$$

Showing that $y_i(x)$ and $y_j(x)$ are orthogonal with respect to weight function $p(x)$.

Reality of eigenvalues:

Theorem (2): If the Sturm-Liouville problem (1), (2) satisfies the condition stated in theorem (1) and p is positive in the whole interval $a \leq x \leq b$, then all the eigenvalues of the problem are real.

Proof: To prove that all eigenvalues of Sturm-Liouville problem are real.

Consider the problems (1), (2a) and (2b):

Where p, q, r and r' are real valued, continuous and $p(x) > 0$ on $a \leq x \leq b$. Let a_1, a_2 in (2a) be given constants, not both zero, and so be b_1, b_2 in (2b).

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Let $y(x)$ be an eigenfunction corresponding to an eigenvalue $\lambda = \alpha + i\beta$, where α, β are real constants. This eigenfunction $y(x)$ satisfies (1), (2a) and (2b) and may be a complex valued function.

Taking the complex conjugates of all the terms in(1), (2a) and (2b), we get

$$[r(x)\bar{y}']' + [q(x) + \bar{\lambda}p(y)]\bar{y} = 0 \quad (16)$$

$$a_1\bar{y}(a) + a_2\bar{y}'(a) = 0 \quad (17a)$$

$$b_1\bar{y}(b) + b_2\bar{y}'(b) = 0 \quad (17b)$$

The above equations (16) , (17a) and (17b) show that $\bar{y}(x)$ is eigenfunction corresponding the eigenvalue $\bar{\lambda} = \alpha - i\beta$. Multiplying (1) by \bar{y} and (16) by y and subtracting, we get

$$(r'y)\bar{y} - (r\bar{y}')y + (\lambda - \bar{\lambda})py\bar{y} = 0 \quad \text{or} \quad (\lambda - \bar{\lambda})py\bar{y} = (r\bar{y}')y - (r'y)\bar{y}$$

$$\text{Or} \quad (\lambda - \bar{\lambda})py\bar{y} = \frac{d}{dx}\{(r\bar{y}')y - (r'y)\bar{y}\}, \quad (18)$$

Which can be verified by performing the indicated differentiation of the expression in brackets on R.H.S. of (18).

Integrating both sides of (18) w.r.t. "x" from a to b, we thus obtain

$$(\lambda - \bar{\lambda}) \int_a^b py\bar{y} dx = [r(\bar{y}'y - y'\bar{y})]_a^b$$

$$\text{Or} \quad (\lambda - \bar{\lambda}) \int_a^b py\bar{y} dx = r(b)\{\bar{y}'(b)y(b) - y'(b)\bar{y}(b)\} \\ - r(a)\{\bar{y}'(a)y(a) - y'(a)\bar{y}(a)\} \quad (19)$$

Now we have to consider several cases depending on whether $r(x)$ vanishes or does not vanish at a or b.

Case (I): Let $r(a) = r(b) = 0$. Then (6) reduces to

$$(\lambda - \bar{\lambda}) \int_a^b py\bar{y} dx = 0 \quad (20)$$

Case (II): Let $r(b) = 0$ but $r(a) \neq 0$ Then (19) reduces to

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$$(\lambda - \bar{\lambda}) \int_a^b p y \bar{y} dx = -r(a) \{ \bar{y}'(a) y(a) - \dot{y}(a) \bar{y}(a) \} \quad (21)$$

Consider relations (2a) and (17a). Let $a_2 \neq 0$. Multiplying (17a) by $y(a)$ and (2a) by $\bar{y}(a)$ and then subtracting, we obtain

$$a_2 \{ \bar{y}'(a) y(a) - \dot{y}(a) \bar{y}(a) \} = 0$$

$$\text{Since } a_2 \neq 0, \text{ we so } \bar{y}'(a) y(a) - \dot{y}(a) \bar{y}(a) = 0 \quad (22)$$

Using (22), (21) reduces to (20). If $a_2 = 0$, then assume that $a_1 \neq 0$.

Now, multiplying (2a) by $\bar{y}'(a)$ and (17a) by $\dot{y}(a)$ and then subtracting, we obtain

$$a_1 \{ \bar{y}'(a) y(a) - \dot{y}(a) \bar{y}(a) \} = 0$$

$$\text{Since } a_1 \neq 0, \text{ so } \bar{y}'(a) y(a) - \dot{y}(a) \bar{y}(a) = 0$$

Hence as before (21) reduces to (20).

Case (III): Let $r(a) = 0$ but $r(b) \neq 0$ Then (19) reduces to

$$(\lambda - \bar{\lambda}) \int_a^b p y \bar{y} dx = r(b) \{ \bar{y}'(b) y(b) - \dot{y}(b) \bar{y}(b) \} \quad (23)$$

Consider relations (2b) and (17b). Let $b_2 \neq 0$. Multiplying (17b) by $y(b)$ and (2b) by $\bar{y}(b)$ and then subtracting, we get

$$b_2 \{ \bar{y}'(b) y(b) - \dot{y}(b) \bar{y}(b) \} = 0$$

$$\text{Since } b_2 \neq 0, \text{ so } \bar{y}'(b) y(b) - \dot{y}(b) \bar{y}(b) = 0 \quad (24)$$

Using (24), (23) reduces to (20). If $b_2 = 0$, then assume that $b_1 \neq 0$.

Now, multiplying (2b) by $\bar{y}'(b)$ and (17b) by $\dot{y}(b)$ and then subtracting, we get

$$b_1 \{ \bar{y}'(b) y(b) - \dot{y}(b) \bar{y}(b) \} = 0$$

$$\text{Since } b_1 \neq 0, \text{ so } \bar{y}'(b) y(b) - \dot{y}(b) \bar{y}(b) = 0$$

Hence as before (23) reduces to (20)

Case (IV): Let $r(a) \neq 0$ but $r(b) \neq 0$ There is no loss of generality by assuming that $a_2 \neq 0$ and $b_2 \neq 0$. Then proceeding as in case (II) and (III), relations (22) and (24) can be proved, then, using (22) and (24), (19) reduces to (20).

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Case (V): Let $r(a) = r(b)$. Proceed case (IV) to show that (19) reduces to (20).

From the above discussion, we see that all situations we obtain (20).

Now, $\lambda = \alpha + i\beta \Rightarrow \bar{\lambda} = \alpha - i\beta$ and hence

$$\lambda - \bar{\lambda} = (\alpha + i\beta) - (\alpha - i\beta) = 2i\beta$$

Again $y\bar{y} = |y|^2$, where $|y|$ stands for modulus of y . Then, (20) reduces to

$$2i\beta \int_a^b p(x)|y(x)|^2 dx = 0 \quad \text{or} \quad \beta \int_a^b p(x)|y(x)|^2 dx = 0 \quad (25)$$

Since $\int_a^b p(x)|y(x)|^2 dx$ has positive value in the interval $a \leq x \leq b$, (25) reduces to $\beta = 0$ and hence $\lambda = \alpha + i\beta = \alpha$, which is real.

Since λ is arbitrary eigenvalue, it follows that eigenvalues of Sturm-Liouville problem are real

Examples:

Example (1): Solve

$$y'' + \lambda y = 0 \quad 0 \leq x \leq L \quad (26)$$

With boundary conditions

$$y(0) = y(L) = 0 \quad (27)$$

Solution:

We consider three cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$

Case1: $\lambda < 0$

The general solution of Eq. (26) is given by

$$y(x) = c_1 \exp(\sqrt{-\lambda}x) + c_2 \exp(-\sqrt{-\lambda}x)$$

Where $\sqrt{-\lambda}$ is are real numbers. Applying the boundary conditions (27) we obtain

$$c_1 + c_2 = 0$$

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$$c_1 \exp(\sqrt{-\lambda}L) + c_2 \exp(-\sqrt{-\lambda}L) = 0 \quad (28)$$

Since $\lambda \neq 0$, the only solution of the system (28) is $c_1 = c_2 = 0$. Hence for $\lambda < 0$, the only solution of the prob. (26) and (27) is the trivial solution.

Case 2: $\lambda = 0$

In this case the differential equation reduces to

$$y'' = 0$$

Which general solution is the form

$$y(x) = c_1 + c_2 x$$

Applying the boundary conditions (27) we obtain

$$c_1 = 0 \quad \text{and} \quad c_2 L = 0$$

Since $L > 0$, then $c_2 = 0$, and so once again the only solution of the problem for $\lambda = 0$ is the trivial solution, $y(x) \equiv 0$. Finally we have.

Case 3: $\lambda > 0$

In this case the general solution of the differential equation is

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \quad (29)$$

Applying the boundary conditions we find

$$c_1 = 0 \quad \text{and} \quad c_2 \sin\sqrt{\lambda}L = 0$$

Now, either $c_2 = 0$ in which case we obtain the trivial solution (since $c_1 = 0$), or $c_2 \neq 0$ and $\sin\sqrt{\lambda}L = 0$ (30)

Equation (30) will be satisfied if and only if

$$\sqrt{\lambda}L = n\pi \quad (n = 1, 2, 3, \dots)$$

$$\text{Or} \quad \lambda = \frac{n^2 \pi^2}{L^2} \quad (n = 1, 2, 3, \dots)$$

In other words, if λ is a number belonging to the sequence

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$$\frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \dots, \frac{\pi^2 \pi^2}{L^2}, \dots$$

Then the problem dose have nontrivial solutions.

Conclusions:

(1) If $\lambda \leq 0$, the Sturm-Liouville system given by Eqs. (26) and (27) does not have nontrivial solutions.

(2) If $\lambda > 0$, the Sturm-Liouville system (26) and (27) has nontrivial solutions if and only if

$$\lambda = \lambda_n = \frac{n^2 \pi^2}{L^2} \quad (n = 1, 2, 3, \dots)$$

The numbers λ_n are therefore the eigenvalues of this system.

(3) From (29) we find that nontrivial solutions corresponding to λ_n are given by

$$y_n(x) = c_n \sin \frac{n\lambda}{L} x \quad (n = 1, 2, 3, \dots)$$

Where the c_n is arbitrary nonzero real numbers. The functions y_n are therefore the eigenfunctions of this system. The set of all eigenfunctions $\{y_n\}$ is the spectrum of the system.

Example (2):

For the eigenvalu problem given below, obtain the set of orthogonal eigenfunctions in interval $(0, 2c)$:

$$y'' + \lambda y = 0, \quad y(0) = y(2c), \quad y'(0) = y'(2c).$$

Solution:

$$\text{Given } y'' + \lambda y = 0 \quad (31)$$

$$\text{With boundary conditions } y(0) = y(2c) \quad (32)$$

$$\text{And } y'(0) = y'(2c) \quad (33)$$

$$\text{Case 1: Let } \lambda = 0. \text{ then solution (31) is } y(x) = Ax + B \quad (34)$$

$$\text{From (34) } y'(x) = A \quad (35)$$

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From (34), $y'(0) = B$ and $y(2c) = 2cA + B$. So (32) reduces to

$B = 2cA + B$ and hence $A = 0$. Next, from (35), $y'(0) = y'(2c) = A$. So (33) gives $A = A$.

Hence corresponding to the eigenvalue $\lambda = 0$, the eigenfunction is

$y(x) = B$ or $y(x) = 1$ taking $B = 1$.

Case 2: Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (31) becomes $y'' - \mu^2 y = 0$, whose is $y(x) = Ae^{\mu x} + Be^{-\mu x}$ (36)

From (36) $y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ (37)

From (36) $y(0) = A + B$ and $y'(2c) = A\mu e^{2\mu c} + B\mu e^{-2\mu c}$. So (32) gives

$A + B = A\mu e^{2\mu c} + B\mu e^{-2\mu c}$ or $A(1 - e^{2\mu c}) + B(1 - e^{-2\mu c}) = 0$ (38)

From (37), $y'(0) = \mu(A - B)$ and $y'(2c) = \mu(Ae^{2\mu c} - Be^{-2\mu c})$

\therefore (33) gives $\mu(A - B) = \mu(Ae^{2\mu c} - Be^{-2\mu c})$

Or $A(1 - e^{2\mu c}) - B(1 - e^{-2\mu c}) = 0$ (39)

Solving (38) and (39), $A = B = 0$. So (36) reduces to $y(x) = 0$, which is not an eigenfunction. So there is no eigenfunction corresponding to $\lambda = -\mu^2$.

Case 3: Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (31) becomes $y'' + \mu^2 y = 0$ whose solution is $y(x) = A \cos \mu x + B \sin \mu x$ (40)

From (40), $y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ (41)

\therefore (32) gives $A = A \cos 2\mu c + B \sin 2\mu c$

Or $A(1 - \cos 2\mu c) - B \sin 2\mu c = 0$ (42)

From (41), $y'(0) = B\mu$ and $y'(2c) = \mu(-A \sin 2\mu c + B \cos 2\mu c)$

\therefore (33) gives $B\mu = \mu(-A \sin 2\mu c + B \cos 2\mu c)$

Or $A \sin 2\mu c + B(1 - \cos 2\mu c) = 0$ (43)

For non-trivial solution of (42) and (43), we must have

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$$\begin{vmatrix} 1 - \cos 2\mu c & -\sin 2\mu c \\ \sin 2\mu c & 1 - \cos 2\mu c \end{vmatrix} = 0$$

$$\text{Or } (1 - \cos 2\mu c)^2 + \sin^2 2\mu c = 0 \quad \text{or } \cos 2\mu c = 1 = \cos 0$$

$$\therefore 2\mu c = 2n\pi \quad \text{or } \mu = \frac{n\pi}{c}, \quad n = 1, 2, 3, \dots \quad (44)$$

($n = 0$ is omitted here $n = 0 \Rightarrow \mu = 0$, which is contrary to our assumption $\mu \neq 0$.)

With this value of μ , (40) becomes

$$y(x) = A \cos\left(\frac{n\pi x}{c}\right) + B \sin\left(\frac{n\pi x}{c}\right) \quad (45)$$

Taking $A = 1$ and $B = 0$ in (45), the eigenfunctions are given by $y(x) = \cos\left(\frac{n\pi x}{c}\right)$.

Again taking $A = 0$ and $B = 1$ in (45), the eigenfunction are given by $y(x) = \sin\left(\frac{n\pi x}{c}\right)$.

Note that $\int_0^{2c} 1 \cdot \cos\left(\frac{n\pi x}{c}\right) dx = 0$, $\int_0^{2c} 1 \cdot \sin\left(\frac{n\pi x}{c}\right) dx = 0$,

$$\int_0^{2c} \sin\left(\frac{n\pi x}{c}\right) \cos\left(\frac{m\pi x}{c}\right) dx = 0, \text{ for } m \neq n \quad (46)$$

In view of (46) the required eigenfunctions which are orthogonal on $(0, 2c)$ are given by $\left\{1, \cos\left(\frac{n\pi x}{c}\right), \sin\left(\frac{n\pi x}{c}\right)\right\}$, ($n = 1, 2, 3, \dots$)

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