University of Benghazi
Faculty of Education Al marj
Global Libyan Journal

جامعة بنغازي
كلية التربية المرج
المجلة الليبية العالمية

العدد السابع والعشرون – 02/ سبتمبر (2017)

SOME APPLICATIONS OF A LINEAR OPERATOR WITH NEGATIVE COEFFICIENTS

Aisha Ahmed Amer,[1] Nagat Muftah Alabbar,[2] Rabeaa Abd Allah Alshbear,[3] Abeer Mustafa Hasek[4]

[1] Mathmatics Department, Faculty of Science -Al-Khomus, Al-Margib University. [2] Mathmatics Department, Faculty of Education of Benghazi, University of Benghazi. [3] Mathmatics Department, Faculty of Science -Al-Khomus, Al-Margib University. [4] Mathmatics Department, Faculty of Science -Al-Khomus, Al-Margib University.

University of Benghazi
Faculty of Education Al marj
Global Libyan Journal

جامعة بنغازي
كلية التربية المرج
المجلة الليبية العالمية

العدد السابع والعشرون – 02/ سبتمبر (2017)

SOME APPLICATIONS OF A LINEAR OPERATOR WITH NEGATIVE COEFFICIENTS

Abstract:

The main object of this paper is to introduce and study the new subclasses $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ of analytic functions with negative coefficients defined by a linear operator .Coefficient bounds for functions belonging to these subclasses are determined. Further, an application involving fractional calculus we are also given.

بعض التطبيقات لمعامل خطي للمعاملات سالبة

عائشة احمد عامر ، نجاة مفتاح العبار، ربيعة عبدالله الشبير، عبير مصطفى

قسم الرياضيات بكلية التربية بنغازي ، جامعة بنغازي2 قسم الرياضيات بكلية العلوم الخمس- جامعة المرقب1و3و4

ملخص.

الهدف الرئيسي من هذه الورقةالبحثية هو تقديم ودراسة فئات فرعية جديدة لدوال تحليلية ذات معاملات سالبة التي تم تحدديها بمعامل خطي. تم حساب المعاملات التي تنتمي إلى هذه الفئات الفرعية. وعلاوة على ذلك، تم تطبيق هذا المعاملات على المعامل الكسري لحساب التفاضل والتكامل

1 Introduction

The theory of derivative and integral plays an important role in the theory of univalent functions. It is believed that Ruscheweyh (1975) was the first to give a generalized derivative operator in the theory of univalent function. Later, Salagean (1983) gave another generalized derivative operator. In the same paper, he introduced an integral operator. Many properties have been discussed and studied by many researchers for these two operators. For example, Al-Oboudi (2004) introduced a generalized Salagean operator, Al-Shaqsi and Darus (2009) generalized the operator given by Ruscheweyh (1975), while Darus and Al-Shaqsi (2008) studied both derivatives of Ruscheweyh and Salagean. These operators motivate us to create another type of derivative operator.

In this paper is to introduce and study the new subclasses of analytic functions with negative coefficients defined by a linear operator [1,2]

Let $A(n)$ denote the class of all analytic functions in the open unit disc

$\mathrm{U}=\{z\in\mathrm{C}:|z|<1\}$, of the form:

$$f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k,(n\in\mathrm{N}). \tag{1}$$

Denote $T(n)$ the subclass of $A(n)$ consisting of functions of the form:

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k,(a_k\geq 0,n\in\mathrm{N}). \tag{2}$$

For functions $f\in A(n)$ given by (1) and $g\in A(n)$ given by $g(z)=z+\sum_{n=2}^{\infty}b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f*g)(z)=z+\sum_{k=n+1}^{\infty}a_k b_k z^k.$$

If f, g are analytic in U, we say that f is subordinate to g, denoted by $f\prec g$, if there exists a function w analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in\mathrm{U})$, such that $f(z)=g(w(z))$, $(z\in\mathrm{U})$. It is known that $f(z)\prec g(z)$ $(z\in\mathrm{U})\Rightarrow f(0)=g(0)$ *and* $f(\mathrm{U})\subset g(\mathrm{U})$.

Let the function $\varphi(a,c;z)$ be given by

$$\varphi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1},\quad (z\in\mathrm{U},c\neq 0,-1,-2,-3,\ldots),$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial).

Corresponding to the function $\varphi(a,c;z)$, Carlson and Shaffer [12] introduced a linear operator $L(a,c)$ by

$$L(a,c)f(z) := \varphi(a,c;z) * f(z) = \sum_{n=0}^{\infty} \frac{(a)_k}{(c)_k} a_k z^{k+1}.$$

Note that:

$L(a,a)$ is the identity operator,

and $L(a,c) = L(a,b)L(b,c) \quad (b,c \neq 0,-1,\ldots).$

The author [1, 2] has recently introduced a new linear operator $D_l^{m,\lambda}(a,b)f(z)$ as the following:

Definition 1.1 *Let*

$$\phi_l^{m,\lambda}(a,b;z) = \sum_{k=0}^{\infty} \left(\frac{1+\lambda k + l}{1+l} \right)^m \frac{(a)_k}{(b)_k} z^{k+1},$$

where $(z \in U, b \neq 0,-1,-2,-3,\ldots), \lambda \geq 0, m \in \mathrm{Z}, l \geq 0,$ and $(x)_k$ is the Pochhammer symbol.

We defines a linear operator $D_l^{m,\lambda}(a,b) : A \to A$ by the following Hadamard product:

$$D_l^{m,\lambda}(a,b)f(z) := \phi_l^{m,\lambda}(a,b;z) * f(z) = \sum_{k=0}^{\infty} \left(\frac{1+\lambda k + l}{1+l} \right)^m \frac{(a)_k}{(b)_k} a_k z^{k+1}. \tag{3}$$

Note that:

$$D_0^{0,\lambda}(a,b)f(z) = L(a,b)f(z),$$

$$(1+l)D_l^{m,\lambda}(a,b)f(z) = (1-\lambda+l)L(a,b)f(z) + \lambda z(L(a,b)f(z))' = D_\lambda(L(a,b)f(z)), \lambda \geq 0,$$

$$D_l^{m,\lambda}(a,b)f(z) = D_\lambda(D_l^{m-1,\lambda}(a,b)f(z)), \text{ where } m \in \mathrm{N} = \{1,2,3,\ldots\}.$$

Special cases of this operator includes:

- $D_0^{m,0}(a,b)f(z) = D_l^{0,\lambda}(a,b)f(z) = L(a,b)f(z).$

- the Ruscheweyh derivative operator [8] in the cases: $D_0^{0,0}(\beta+1,1)f(z)=D^{\beta}f(z);\beta\geq -1.$

- the Salagean derivative operator [10]: $D_0^{m,1}(1,1)f(z).$

- the generalized Salagean derivative operator introduced by Al-Oboudi [9]: $D_0^{m,\lambda}(1,1)f(z).$

- the Catas drivative operator [15]: $D_l^{m,\lambda}(1,1)f(z),$ and finally

- The fractional operator introduced by Owa and Srivastava [6]

$$D_0^{0,0}(2,2-\gamma)f(z)=\Omega^{\gamma}f(z)=\Gamma(2-\gamma)z^{\gamma}D_z^{\gamma}f(z);$$

$D_z^{\gamma}f(z)$ is the fractional derivative of f of order $\gamma;\quad \gamma\neq 2,3,4,\cdots.$

Now, we introduce new subclasses of analytic functions involving our operator $D_l^{m,\lambda}(a,b).$

Definition 1.2 *A function* $f\in T(n)$ *is said to be in the subclass* $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c),$ *for* $(0<\beta\leq 1,\quad \gamma\in \mathrm{C}/\{0\})$ *if and only if:*

$$\left|\frac{1}{\gamma}\left(\frac{zv'}{v}-1\right)\right|<\beta, \tag{4}$$

where

$$\frac{zv'}{v}=\frac{z(D_l^{m,\lambda}(a,b))'+\delta z^2(D_l^{m,\lambda}(a,b)f(z))''}{(1-\delta)D_l^{m,\lambda}(a,b)f(z)+\delta z(D_l^{m,\lambda}(a,b)f(z))'}, \tag{5}$$

$(z\in \mathrm{U},\quad 0\leq\delta\leq 1).$

Definition 1.3 *A function* $f\in T(n)$ *is said to be in the subclass* $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ *if and only if*

$$\left|\frac{1}{\gamma}(v'-1)\right|<\beta, \tag{6}$$

$(z\in \mathrm{U},\quad 0\leq\delta\leq 1,\quad 0<\beta\leq 1,\quad \gamma\in \mathrm{C}/\{0\}).$

We note that there are some known subclasses of $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$.

Remark 1.1 *(1) If* $m=0,$ *and* $a=c=0$ *then*

$$TS_{\beta,\gamma}^{0,\delta}(\lambda,l,0,0)=S_n(\beta,\gamma,\delta).$$

(2) If $m=0,$ *and* $a=c=0$ then

$$TR_{\beta,\gamma}^{0,\delta}(\lambda,l,0,0)=R_n(\beta,\gamma,\delta).$$

The classes $S_n(\beta,\gamma,\delta)$ and $R_n(\beta,\gamma,\delta)$ were investigated in [3].

(3)If $m=0,$ $a=c=0$ and $\delta=0,$ $\beta=|b|,$ $\gamma=1$ then

$$TS_{|b|,1}^{0,0}(0,l,\lambda,c)=S_1^*(b),$$

where $(b\in \mathbb{C}/\{0\})$. The class $S_1^*(b)$ was studied in [4].

2 Coefficient bounds

In this section, we obtain necessary and sufficient conditions for a function to be in the subclasses $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ respectively.

Theorem 2.1 *Let the function f be defined by (3). Then f belongs to the subclass* $TS_l^{m,\lambda}(a,b)$ *if and only if*

$$\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|a_k\le\beta|\gamma|, \qquad (7)$$

$(z\in \mathbb{U},\quad 0\le\delta\le1,\quad 0<\beta\le1,\quad \gamma\in \mathbb{C}/\{0\}).$

Proof: Suppose $f\in TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. By making use of (4) we easily obtain

$$Re\left\{\frac{zv'}{v}-1\right\}>-\beta|\gamma|\quad (z\in \mathbb{U}),$$

which, in view of (5), gives :

$$Re\left\{\frac{-\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k-1)\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k-1}}{\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k-1}}\right\}>-\beta|\gamma|. \qquad (8)$$

Setting $z=r \quad (0\le r<1)$ in (8) we observe that the expression in the denominator on the left hand side of (8) is positive for $r=0$ and also for all $r\in(0,1)$. Thus by letting $r\to 1^-$ ،آthrough real values (8) leads us to the desired condition (7) of the theorem.

Conversely, by applying the hypothesis (8) and setting $|z|=1$, we find by using (7)that

$$\left|\frac{zv'}{v}-1\right|=\left|\frac{-\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k-1)\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k}}{z-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k}}\right|$$

$$\le\frac{\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k-1)\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k-1}}$$

$$\le\frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k\right)}{1-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k}$$

$$=\beta|\gamma|.$$

Hence, by the maximum modulus principle, we have $f\in TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$.

Corollary 2.1

Let the function f be defined by (3)and $f\in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$, then

$$a_k\le\frac{\beta|\gamma|}{[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}\right|},\quad (k\ge n+1), \qquad (9)$$

with equality only for functions of the form

$$f_z = z - \frac{\beta|\gamma|}{[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|} z^k, \quad (k \ge n+1).$$

By using the same arguments as in the proof of Theorem 2.1 we can establish the next theorem.

Theorem 2.2 *Let the function f be defined by (3). Then f belongs to the subclass $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ if and only if*

$$\sum_{k=n+1}^{\infty} k[1+\delta(k-1)]\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right| a_k \le \beta|\gamma|, \tag{10}$$

$(z \in \mathrm{U}, \quad 0 \le \delta \le 1, \quad 0 < \beta \le 1, \quad \gamma \in \mathrm{C}/\{0\}).$

Corollary 2.2

Let function f be defined by (3) and $f \in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. Then

$$a_k \le \frac{\beta|\gamma|}{k[1+\delta(k-1)]\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|} \quad (k \ge n+1), \tag{11}$$

with equality only for functions of the form

$$f_z = z - \frac{\beta|\gamma|}{[1+\delta(k-1)]\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|} z^k, \quad (k \ge n+1).$$

3 An Application of Fractional Calculus

From among various definitions of fractional calculus (that is, fractional derivative and fractional integral),we recall here the following definitions which have been used by many authors including, for example(Owa [13], Srivastava and Owa, [14]).

Definition 3.1 *The fractional integral of order η is defined by*

$$D_z^{-\eta} f(z) = \frac{1}{\Gamma(\eta)} \int_0^z \frac{f(t)}{(z-t)^{1-\eta}} dt,$$

where $\eta > 0$ f is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 3.2 *The fractional derivative of order η is defined by*

$$D_z^{\eta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\eta}} dt,$$

where $0 \le \eta < 1$ f is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Theorem 3.3 *Let the function* f *defined by (3) be in the subclass* $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then for* $|z|=r<1$.

$$|D_z^{-\eta} f(z)| \le$$

$$\frac{|z|^{1+\eta}}{\Gamma(2-\eta)} \left\{ 1 + \frac{\beta|\gamma|(k+1)!\Gamma(2+\eta)}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m \frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)} |z|^k \right\}, \tag{12}$$

and

$$|D_z^{-\eta} f(z)| \ge$$

$$\frac{|z|^{1+\eta}}{\Gamma(2-\eta)} \left\{ 1 - \frac{\beta|\gamma|(k+1)!\Gamma(2+\eta)}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m \frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)} |z|^k \right\}. \tag{13}$$

The estimates in (12) and (13) are sharps.

Proof:

From Definition 3.1,we get

$$\Gamma(2+\eta) z^{-\eta} D_z^{-\eta} = z - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\eta)}{\Gamma(k+1+\eta)} a_k z^k$$

$$= z - \sum_{k=n+1}^{\infty} \Phi(k) a_k z^k, \tag{14}$$

where

$$\Phi(k) = \frac{\Gamma(k+1)\Gamma(2+\eta)}{\Gamma(k+1+\eta)}.$$

Since $\Phi(k)$ is a decreasing function of k we have

$$0<\Phi(k)\le\Phi(k+1)=\frac{(k+1)!\Gamma(2+\eta)}{\Gamma(k+2+\eta)},$$

in view of Theorem 2.1, we have

$$[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\sum_{k=n+1}^{\infty}a_k\le$$

$$\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|a_k\le\beta|\gamma|,$$

$$\sum_{k=n+1}^{\infty}a_k\le\frac{\beta|\gamma|}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|}. \tag{15}$$

Using (15) and (10) we have

$$|\Gamma(2+\eta)z^{-\eta}D_z^{-\eta}f(z)|\le|z|+\Phi(k+1)|z|^{k+1}\sum_{k=n+1}^{\infty}a_k\le$$

$$|z|+\frac{(k+1)!\Gamma(2+\eta)}{\Gamma(k+2+\eta)}\frac{\beta|\gamma|}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|}|z|^{k+1},$$

and

$$|\Gamma(2+\eta)z^{-\eta}D_z^{-\eta}f(z)|\ge|z|-\Phi(k+1)|z|^{k+1}\sum_{k=n+1}^{\infty}a_k\ge$$

$$|z|-\frac{(k+1)!\Gamma(2+\eta)}{\Gamma(k+2+\eta)}\frac{\beta|\gamma|}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|}|z|^{k+1},$$

Which prove the inequalities of Theorem 3.3. Finally, we can easily see that the results (12) and (13) are sharp for the function f defined by

$$D_z^{-\eta}f(z)=\frac{z^{1+\eta}}{\Gamma(2-\eta)}\left\{1-\frac{\beta|\gamma|(k+1)!\Gamma(2+\eta)}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)}z^k\right\}.$$

By using the same arguments as in the proof of Theorem 3.3, we can establish the next theorem.

Theorem 3.4 *Let the function* f *defined by (3) be in the class* $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then for* $|z|=r<1$

$$|D_z^{-\eta}f(z)|\le$$

$$\frac{|z|^{1+\eta}}{\Gamma(2-\eta)}\left\{1+\frac{\beta|\gamma|(k+1)!\Gamma(2+\eta)}{(k+1)[1+\delta(k)]\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)}|z|^k\right\}, \tag{16}$$

and

$$|D_z^{-\eta}f(z)|\ge$$

$$\frac{|z|^{1+\eta}}{\Gamma(2-\eta)}\left\{1-\frac{\beta|\gamma|(k+1)!\Gamma(2+\eta)}{(k+1)[1+\delta(k)]\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)}|z|^k\right\}. \tag{17}$$

The estimates in (16) and (17) are sharps.

Theorem 3.5 *Let the function* f *defined by (3) be in the subclass* $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then for* $|z|=r<1$.

$$|D_z^{\eta}f(z)|\le$$

$$\frac{|z|^{1-\eta}}{\Gamma(2-\eta)}\left\{1+\frac{k\beta|\gamma|\Gamma(k+1)\Gamma(2+\eta)}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)}|z|^k\right\}, \tag{18}$$

and

$$|D_z^{\eta}f(z)|\ge$$

$$\frac{|z|^{1-\eta}}{\Gamma(2-\eta)}\left\{1-\frac{k\beta|\gamma|\Gamma(k+1)\Gamma(2+\eta)}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)}|z|^k\right\}. \tag{19}$$

The estimates in (18) and (19) are sharps.

Proof:

From Definition 3.2,we get

$$\Gamma(2-\eta)z^{\eta}D_z^{\eta}=z-\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\eta)}{\Gamma(k+1+\eta)}a_k z^k$$

$$=z-\sum_{k=n+1}^{\infty}k\Psi(k)a_k z^k, \tag{20}$$

where

$$\Psi(k)=\frac{\Gamma(k)\Gamma(2+\eta)}{\Gamma(k+1+\eta)}.$$

Since $\Psi(k)$ is a decreasing function of k we have

$$0<\Psi(k)\le\Psi(k+1)=\frac{\Gamma(k+1)\Gamma(2+\eta)}{\Gamma(k+2+\eta)},$$

in view of Theorem 2.1, we have

$$\left(\frac{1}{k}\right)[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\sum_{k=n+1}^{\infty}ka_k\le$$

$$\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|a_k\le\beta|\gamma|,$$

$$\sum_{k=n+1}^{\infty}ka_k\le\frac{k\beta|\gamma|}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|}. \tag{21}$$

Using(21) we have

$$|\Gamma(2+\eta)z^{\eta}D_z^{\eta}f(z)|\le|z|+\Psi(k+1)|z|^{k+1}\sum_{k=n+1}^{\infty}ka_k\le$$

$$|z|+\frac{k\Gamma(k+1)\Gamma(2+\eta)}{\Gamma(k+2+\eta)}\frac{\beta|\gamma|}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|}|z|^{k+1},$$

and

$$|\Gamma(2+\eta)z^{\eta}D_z^{\eta}f(z)|\ge z-\Psi(k+1)|z|^{k+1}\sum_{k=n+1}^{\infty}ka_k\ge$$

$$|z|-\frac{k\Gamma(k+1)\Gamma(2+\eta)}{\Gamma(k+2+\eta)}\frac{\beta|\gamma|}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|}|z|^{k+1},$$

which prove the inequalities of Theorem 3.5. Finally, we can easily see that the results (18)and (19) are sharp for the function f defined by

$$D_z^{\eta}f(z)|=\frac{z^{1-\eta}}{\Gamma(2-\eta)}\left\{1-\frac{k\beta|\gamma|\Gamma(k+1)\Gamma(2+\eta)}{[1+\delta(k)](k+\beta|\gamma|)\left|\left(\frac{\lambda(k)+1+l}{1+l}\right)^m\frac{(a)_k}{(c)_k}\right|\Gamma(k+2+\eta)}z^k\right\}.$$

Many other work on analytic functions related to derivative operator and integral operator can be read in [16] ، [17] and [18] .

References

[1] Aisha Ahmed Amer , Second Hankel Determinant for New Subclass Defined by a Linear Operator, Springer International Publishing Switzerland 2016, Chapter 6 .

[2] Aisha Ahmed Amer and Maslina Darus, A distortion theorem for a certain class of Bazilevic function , *Int. Journal of Math. Analysis*, 6 (2012),12, 591-597.

[3] O.Altintas, O.Ozkan, H. M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Applied Math. Letters.* 13(2000), 63-67, .

[4] A. Y. Lashin, Starlike and convex functions of complex order involving a certain linear operator, *Indian J. Pure and Appl.* 34(7) (2003), 1101-1108, .

[5] J. E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* 23 481-519 (1925).

[6] S. Owa and H. M. Srivastava, Some applications of the generalized Libera operator, *Proc. Japan Acad. Set. A Math. Sei.* 62 (1986), 125-128, .

[7] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* 38 (1972), 746-765.

[8] St. Ruscheweyh , New criteria for univalent functions, *Proc. Amer. Math.Soc.* 49, (1975), 109-115, .

[9] F.M. Al-Oboudi, On univalent functions defined by a generalized Salagean Operator, *Int, J. Math. Math. Sci.* 27 (2004), 1429-1436 .

[10] G. S. Salagean , Subclasses of univalent functions, *Lecture Notes in Math Springer-Verlag.* 1013 (1983), 362-372 .

[11] M. Darus and K. Al-Shaqsi , Differential Differential sandwich theorem with generalized derivative operator, *Int. J . Math. Sci.* 2(2) (2008), 75-78.

[12] B.C. Carlson, D.B. Shaffer, Starlike and prestarlike hypergeometric functions , *SIAM J. Math. Anal.* 15(4) (1984), 737-745 .

[13] H. M. Srivastava and S. Owa, An application of the fractional derivative, *Mathematica Japonica.* 29(3), (1984), 383-389 .

[14] S. Owa, On the distortion theorems, *Kyungpook Math. J*,18, (1978),53-59 .

[15] Catas.A, On a Certain Differential Sandwich Theorem Associated with a New Generalized Derivative Operator. *General Mathematics*, 4,2009,83-95.

[16] Aisha Ahmed Amer, Rabeaa Abd Allah Alshbear & Nagat Muftah Alabbar ,Some Applications Of A Linear Operator To A Certain Subclasses Of Analytic Functions With Negative Coefficients. Journal of Education University Al-Khomus, Al-Margib University 10(2017).441-454. -

[17] Aisha Ahmed Amer and Maslina Darus, Some properties for a linear operator, AIP Conference Proceedings 1522, 908 (2013).

[18] Nagat.M. Mustafa & Maslina Darus, M. 2011. On a subclass of analytic functions with negative coefficient associated to an integral operator involving Hurwitz- Lerch Zeta function, ”Vasile Alecsandri” University of Bacau Faculty of Sciences Scientific Studies and Research Series Mathematics and Informatics. 21(2) : 45 - 56.