

العدد الرابع عشر - فبراير 2017

## Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients

<sup>1</sup> Dr. Nagat Muftah Alabbar, <sup>2</sup> Dr. Aisha Ahmed Amer.

(<sup>1</sup> Mathematics Department, Faculty of Education of Benghazi, University of Benghazi, <sup>2</sup> Mathematics Department, Faculty of Science -Al-Khomus, Al-Margib University)



العدد الرابع عشر - فبراير 2017

## خصائص المعامل التفاضلي المعمم على فئة فرعية معينة من الدوال التحليلية ذات معاملات سالبة

\* نجاة مفتاح العبار، \*\* عائشة أحمد عامر

( \* قسم الرياضيات بكلية التربية - جامعة بنغازي، \*\* قسم الرياضيات بكلية العلوم - الخمس- جامعة المرقب )

الملخص:

الهدف الرئيسي من هذه الورقة البحثية هو تقديم ودراسة فئة فرعية جديدة من الدوال التحليلية المحددة من قبل المعامل التفاضلي المعمم مع معاملات السالبة في قرص الوحدة . وتشمل النتائج حساب المعاملات ، ونقاط المتطرفة ، النمو و التثوية ومعامل تكاملي في مجال الفئات الفرعية المذكور.

## Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients

### Abstract.

The main object of this paper is to introduce and study the new subclasses  $T^{\alpha,n}(m,q,\lambda,\beta)$  and  $C^{\alpha,n}(m,q,\lambda,\beta)$  of analytic functions defined by generalized derivative operator with negative coefficients in the unit disk. The results presented here include coefficient estimates, extreme points, growth and distortion properties and integral operators for the aforementioned subclasses.

**Keywords:** starlike; convex; distortion theorems; derivative operator.

## 1. Introduction.

Let  $A$  denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the open unit disk  $U = \{z : |z| < 1\}$ . A function  $f \in A$  is called starlike functions of order  $\beta$  and type  $\gamma$ , if it satisfies

$$\left| \frac{\frac{zf'(z) - 1}{f(z)}}{\frac{zf'(z) + 1 - 2\gamma}{f(z)}} \right| < \beta, \quad (2)$$

where  $0 \leq \gamma < 1$ ,  $0 < \beta \leq 1$ . We denote by  $S^*(\gamma, \beta)$  the class of starlike functions of order  $\gamma$  and type  $\beta$ . A function  $f \in A$  is called convex functions of order  $\beta$  and type  $\gamma$ , if it satisfies

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z) + 2 - 2\gamma}{f'(z)}} \right| < \beta, \quad (3)$$

where  $0 \leq \gamma < 1$ ,  $0 < \beta \leq 1$ . We denote by  $K(\gamma, \beta)$  the class of convex functions of order  $\gamma$  and type  $\beta$ .

From (2) and (3), we note that:  $f(z) \in K(\gamma, \beta)$  if, and only if,

$$zf' \in S^*(\gamma, \beta).$$

The classes  $S^*(\gamma, \beta)$  and  $K(\gamma, \beta)$  were considered by Gupta and Jain [9].

Let  $T$  denote the subclass of  $A$  consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0). \quad (4)$$

We denote by  $T^*(\gamma, \beta)$  and  $C(\gamma, \beta)$ , the classes obtained by taking intersections, respectively, of the classes  $S^*(\gamma, \beta)$  and  $K(\gamma, \beta)$  with the class  $T$ . Thus we have

العدد الرابع عشر - فبراير 2017

$$T^*(\gamma, \beta) = S^*(\gamma, \beta) \cap T,$$

and

$$C(\gamma, \beta) = K(\gamma, \beta) \cap T.$$

The classes  $T^*(\gamma, \beta)$  and  $C(\gamma, \beta)$  were studied by [10] Also the classes  $T^*(\gamma, 1) = T^*(\gamma)$  and  $C(\gamma, 1) = C(\gamma)$  were studied by Silverman [8].

For functions  $f \in A$ , given by (1), and  $g$  given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of functions  $f$  and  $g$  is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The theory of derivative play an important role in the theory of univalent functions. It is believed that Ruscheweyh (1975) was the first to give a generalised derivative operator in the theory of univalent function. Later, Salagean (1983) gave another generalised derivative operator. In the same paper, he introduced an integral operator. Many properties have been discussed and studied by many researchers for these two operators. For example, Al-Oboudi (2004) introduced a generalised Salagean operator, Al-Shaqsi and Darus (2009) generalised the operator given by Ruscheweyh (1975), while Darus and Al-Shaqsi (2008) studied both derivatives of Ruscheweyh and Salagean. These operators motivate us to create another type of derivative operator.

The author in [1] have recently introduced a new generalised derivative operator  $D^{\alpha, n}(m, q, \lambda)f(z)$  as the following:

For the function  $f \in A$  given by (1) we define a new generalised derivative operator

$D^{\alpha, n}(m, q, \lambda)f(z) : A \rightarrow A$  as follows:

$$D^{\alpha, n}(m, q, \lambda)(f)(z) = z + \sum_{k=2}^{\infty} k^{\alpha} \left(1 + \frac{k-1}{1+q} \lambda\right)^m c(n, k) a_k z^k, \quad (5)$$

العدد الرابع عشر - فبراير 2017

where  $n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, m \in \mathbb{Z}, \lambda, q \geq 0$  and  $c(n, k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$ .

If  $m = 0, 1, 2, \dots$ , then

$$\begin{aligned} D^{\alpha, n}(m, q, \lambda)f(z) &= \underbrace{\phi(z) * \dots * \phi(z)}_{(m)\text{-times}} * \left[ \frac{z}{(1-z)^{n+1}} \right] * \sum_{k=1}^{\infty} k^{\alpha} z^k * f(z) \\ &= R^n * D^{\alpha}(m, q, \lambda)f(z), \end{aligned}$$

where  $R^n = z + \sum_{k=2}^{\infty} c(n, k)z^k$ , the Ruscheweyh derivative operator.

If  $m = -1, -2, \dots$ , then

$$\begin{aligned} D^{\alpha, n}(m, q, \lambda)f(z) &= \underbrace{\phi(z) * \dots * \phi(z)}_{(-m)\text{-times}} * \left[ \frac{z}{(1-z)^{n+1}} \right] * \sum_{k=1}^{\infty} k^{\alpha} z^k * f(z) \\ &= R^n * D^{\alpha}(m, q, \lambda)f(z). \end{aligned}$$

Note that:

$$D^{0,0}(0, q, \lambda)f(z) = D^{0,0}(1, 0, 0)f(z) = f(z), \text{ and}$$

$$D^{0,0}(1, q, \lambda)f(z) = zf'(z).$$

By specialising the parameters of  $D^{\alpha, n}(m, q, \lambda)f(z)$ , we get the following derivative and integral operators.

- The derivative operator introduced by Ruscheweyh [2];

$$D^{0, n}(0, q, \lambda) \equiv D^{0, n}(1, 0, 0); (n \in \mathbb{N}_0) \equiv R^n = z + \sum_{k=2}^{\infty} c(n, k)a_k z^k.$$

- The derivative operator introduced by Salagean [3];

$$D^{\alpha, 0}(0, q, \lambda) \equiv D_1^{\alpha, 0}(n, 0, 1); (n \in \mathbb{N}_0) \equiv D^n = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

- The generalised Salagean derivative operator introduced by Oboudi [4];

العدد الرابع عشر - فبراير 2017

$$D^{0,0}(n, 0, \lambda); (n \in \mathbb{N}_0) \equiv D_{\lambda}^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k.$$

- The generalised Ruscheweyh derivative operator introduced by Darus and Al-Shaqsi [5];

$$D^{0,n}(1, 0, \lambda); (n \in \mathbb{N}_0) \equiv R_{\lambda}^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))c(n, k) a_k z^k.$$

- The derivative operator introduced by Catas [6];

$$D^{0,\beta}(m, l, \lambda); (m \in \mathbb{N}_0) \equiv D^m(\lambda, \beta, l) = z + \sum_{k=2}^{\infty} \left( \frac{1 + \lambda(k-1) + l}{1+l} \right)^m c(\beta, k) a_k z^k.$$

- The integral operator introduced by Cho and T. H. Kim [7];

$$D^{1,0}(-n, \lambda, 1) \equiv I_n^{\lambda} = z + \sum_{k=2}^{\infty} k \left( \frac{1 + \lambda}{k + \lambda} \right)^n a_k z^k.$$

Next we define the following new subclasses functions as follows:

**Definition 0.1** Let  $f \in T$  be given by (4). Then  $f$  is said to be in the class  $T^{\alpha,n}(m, q, \lambda, \beta)$  if, and only if,

$$\left| \frac{\frac{z(D^{\alpha,n}(m, q, \lambda)f)'(z) - 1}{D^{\alpha,n}(m, q, \lambda)f(z)} - 1}{\frac{z(D^{\alpha,n}(m, q, \lambda)f)'(z) + 1 - 2\gamma}{D^{\alpha,n}(m, q, \lambda)f(z)}} \right| < \beta,$$

where  $D^{\alpha,n}(m, q, \lambda)f(z)$  is given by (5) and  $n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ ,  $m \in \mathbb{Z}$ ,  $\lambda, q \geq 0$  and  $0 \leq \gamma < 1$ ,  $0 < \beta \leq 1$ . Further, a function  $f \in T$  is said to be in the class  $C^{\alpha,n}(m, q, \lambda, \beta)$  if, and only if,

$$zf' \in T^{\alpha,n}(m, q, \lambda, \beta).$$

We note that, by specializing the parameters  $\alpha, n, m, \lambda, \beta$  we shall obtain the following subclasses which were studied by various authors:

1. For  $\alpha = n = m = 0$ , we have  $T^{0,0}(0, q, \lambda, \beta) = S^*(\gamma, \beta)$ , is the class of starlike function of order  $\gamma$  and type  $\beta$  which was studied by Gupta and Jain [9].

### العدد الرابع عشر - فبراير 2017

- 2 . For  $\alpha = n = m = 0$  and  $\beta = 1$  we obtain the class  $T^{0,0}(0,q,\lambda,1) = T^*(\gamma)$ , which was studied by Silverman [8].
3. For  $\alpha = n = q = 0, m = 1$  we have the class  $C^{0,0}(1,0,\lambda,\beta) = C(\gamma,\beta)$ , which was studied by Gupta and Jain [9].
4. For  $\alpha = n = q = 0, m = 1, \beta = 1$ , we obtain the class  $C^{0,0}(1,0,\lambda,1) = C(\gamma)$ , studied by Silverman [8].

### Main Results.

In this paper we introduce, coefficient Inequalities, growth and distortion Properties, extreme points and the class preserving integral operators of the form:

$$F(z) := \frac{(c+1)}{z^c} \int_0^z t^{c-1} f(t), \quad (6)$$

for the subclasses  $T^{\alpha,n}(m,q,\lambda,\beta)$  and  $C^{\alpha,n}(m,q,\lambda,\beta)$  are considered.

### 2. Coefficient Inequalities

In this segment, we give an important and adequate condition for a function  $f(z)$ , given by (1), to be in subclasses  $T^{\alpha,n}(m,q,\lambda,\beta)$  and  $C^{\alpha,n}(m,q,\lambda,\beta)$ .

**Theorem 2.1.** A function  $f$  belongs to the subclass  $T^{\alpha,n}(m,q,\lambda,\beta)$  if, and only if,

$$\sum_{k=2}^{\infty} \left( ((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k \right) \leq 2\beta(1-\gamma). \quad (7)$$

**Proof.** Let the function  $f$  be in the class  $T^{\alpha,n}(m,q,\lambda,\beta)$ . Then we have

$$\left| \frac{\frac{z(D^{\alpha,n}(m,q,\lambda)f)'(z) - 1}{D^{\alpha,n}(m,q,\lambda)f(z)} - 1}{\frac{z(D^{\alpha,n}(m,q,\lambda)f)'(z) + 1 - 2\gamma}{D^{\alpha,n}(m,q,\lambda)f(z)}} \right| = \left| \frac{\frac{z - \sum_{k=2}^{\infty} (k)(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k}{z - \sum_{k=2}^{\infty} (k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k} - 1}{\frac{z - \sum_{k=2}^{\infty} (k)(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k}{z - \sum_{k=2}^{\infty} (k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+\delta)} a_k z^k} + 1 - 2\gamma} \right| \leq \beta.$$

Since  $|Re(z)| \leq |z|$  for all  $z$ , we have

$$\Re \left\{ \frac{\sum_{k=2}^{\infty} (k-1)(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+\delta)}{k!\Gamma(1+n)} a_k z^k}{-\sum_{k=2}^{\infty} (k+1-2\gamma)(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k + (2-2\gamma)} \right\} \leq \beta.$$

Choosing values of  $z$  on the real axis, so that  $\frac{z(D^{\alpha,n}(m,q,\lambda)f)'(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}$  is real, and letting

$z \rightarrow 1^-$ , through real axis, we get

$$\sum_{k=2}^{\infty} (k-1)(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{(k)!\Gamma(1+n)} a_k z^k \leq -\beta \left( \sum_{k=2}^{\infty} (k+1-2\gamma)(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+\delta)}{k!\Gamma(1+\delta)} a_k z^k + \beta(2-2\gamma) \right),$$

which implies the assertion (7). Conversely, let the inequality (7) holds true, then

$$\begin{aligned} & \left| z(D^{\alpha,n}(m,q,\lambda)f)'(z) - (D^{\alpha,n}(m,q,\lambda)f(z)) \right| - \beta \\ & \left| z(D^{\alpha,n}(m,q,\lambda)f)'(z) + (1-2\gamma)D^{\alpha,n}(m,q,\lambda)f(z) \right|, \\ & \sum_{k=2}^{\infty} \left( ((k-1) + \beta(k+1-2\gamma))(k)^{\alpha} \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+\delta)}{k!\Gamma(1+\delta)} \right) - \beta(2-2\gamma) \leq 0, \end{aligned}$$

by the assumption. This implies that  $f \in T^{\alpha,n}(m,q,\lambda,\beta)$

**Corollary 2.2.** Let the function  $f$  be in the class  $T^{\alpha,n}(m,q,\lambda,\beta)$ , then

العدد الرابع عشر - فبراير 2017

$$a_k \leq \frac{2\beta(1-\gamma)}{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}}. \quad (8)$$

The result (8) is sharp for the function  $f$  of the form

$$f(z) = z - \frac{2\beta(1-\gamma)}{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}} z^k. \quad (9)$$

By using the same arguments as in the proof of Theorem 2.1, we can establish the next theorem.

**Theorem 2.3.** A function  $f$  belongs to the subclass  $C^{\alpha,n}(m, q, \lambda, \beta)$ , if, and only if,

$$\sum_{k=2}^{\infty} \left( k [(k-1) + \beta(k+1-2\gamma)] (k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k \right) \leq 2\beta(1-\gamma),$$

**Corollary 2.4.** Let the function  $f$  be in the class  $C^{\alpha,n}(m, q, \lambda, \beta)$ . Then

$$a_k \leq \frac{2\beta(1-\gamma)}{k [(k-1) + \beta(k+1-2\gamma)] (k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}},$$

with equality only for functions of the form

$$f(z) = z - \frac{2\beta(1-\gamma)}{k [(k-1) + \beta(k+1-2\gamma)] (k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}} z^k.$$

### 3. Growth and distortion theorems

In this segment, we obtain growth and distortion bounds for the classes  $T^{\alpha,n}(m, q, \lambda, \beta)$  and  $C^{\alpha,n}(m, q, \lambda, \beta)$ .

**Theorem 3.1.** If  $f \in T^{\alpha,n}(m, q, \lambda, \beta)$ , then

$$|f(z)| \geq r - \frac{2\beta(1-\gamma)}{(1 + \beta(3-2\gamma))(2)^\alpha \left(1 + \frac{\lambda}{1+q}\right)^m \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}} r^2 \quad (10)$$

العدد الرابع عشر - فبراير 2017

$$\leq r + \frac{2\beta(1-\gamma)}{(1+\beta(3-2\gamma))(2)^\alpha \left(1+\frac{\lambda}{1+q}\right)^m \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}} r^2, \quad (11)$$

and

$$|f'(z)| \geq 1 - \frac{2\beta(1-\gamma)(2)}{(1+\beta(3-2\gamma))(2)^\alpha \left(1+\frac{\lambda}{1+q}\right)^m \frac{\Gamma(2+n)}{(p+1)!\Gamma(1+n)}} r \quad (12)$$

$$\leq 1 + \frac{2\beta(1-\gamma)(2)}{(1+\beta(3-2\gamma))(2)^\alpha \left(1+\frac{\lambda}{1+q}\right)^m \frac{\Gamma(2+n)}{(p+1)!\Gamma(1+n)}} r, \quad (13)$$

for  $z \in U$ . The estimates for  $|f(z)|$  and  $|f'(z)|$  are sharp.

**Proof.** Since  $f \in T^{\alpha,n}(m, q, \lambda, \beta)$ , and in view of inequality (7) of Theorem 2.1, we have

$$(1+\beta(3-2\gamma))(2)^\alpha \left(1+\frac{\lambda}{1+q}\right)^m \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)} \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \left( ((k-1)+\beta(k+1-2\gamma))(k)^\alpha \left(1+\frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k \right) \leq 2\beta(1-\gamma),$$

or

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\beta(1-\gamma)}{(1+\beta(3-2\gamma))(2)^\alpha \left(1+\frac{\lambda}{1+q}\right)^m \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}}. \quad (14)$$

Since

$$r - r^2 \sum_{k=2}^{\infty} a_k \leq |f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k, \quad (15)$$

on using (14) and (15), we easily arrive at the desired results of (11) and (10). Furthermore, we observe that

$$1 - (2)r \sum_{k=2}^{\infty} a_k \leq |f'(z)| \leq 1 + (2)r \sum_{k=2}^{\infty} a_k, \quad (16)$$

العدد الرابع عشر - فبراير 2017

On using (14) and (16), we easily arrive at the desired results of (12) and (13).

Finally, we can see that the estimates for  $|f(z)|$  and  $|f'(z)|$  are sharp for the function,

$$f(z) = z^p - \frac{2\beta(p-\gamma)}{(1+(1+2p-2\gamma))(1+\frac{\lambda}{p+q})^\mu} \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}.$$

Similarly, we can prove the following theorem.

**Theorem 3.2.** If  $f \in C^{\alpha,n}(m, q, \lambda, \beta)$ , then

$$\begin{aligned} |f(z)| &\geq r - \frac{2\beta(1-\gamma)}{(2)[1+\beta(3-2\gamma)](2)^\alpha (1+\frac{\lambda}{1+q})^m} \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)} r^2 \\ &\leq r + \frac{2\beta(1-\gamma)}{(2)[1+\beta(2-2\gamma)](2)^\alpha (1+\frac{\lambda}{1+q})^m} \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)} r^2, \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - \frac{4\beta(1-\gamma)}{[1+\beta(3-2\gamma)](2)^\alpha (1+\frac{\lambda}{1+q})^m} \frac{\Gamma(2+n)}{(2)!\Gamma(2+n)} r \\ &\leq 1 + \frac{4\beta(1-\gamma)}{(2)[1+\beta(3-2\gamma)](2\frac{p+1}{p})^\alpha (1+\frac{\lambda}{1+q})^m} \frac{\Gamma(3+n)}{(2)!\Gamma(1+n)} r, \end{aligned}$$

for  $z \in U$ . The estimates for  $|f(z)|$  and  $|f'(z)|$  are sharp.

**4. Extreme Points** Now, we determine extreme points for the subclasses  $T^{\alpha,n}(m, q, \lambda, \beta)$  and  $C^{\alpha,n}(m, q, \lambda, \beta)$ .

**Theorem 4.1.** Let  $f(z) = z$  and,

$$f_k(z) = z - \frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^\alpha (1+\frac{(k-1)\lambda}{1+q})^m} \frac{\Gamma(k+n)}{k!\Gamma(1+n)} z^k.$$

Then  $f$  is in the class  $T_p^{\alpha,n}(m, q, \lambda, \beta)$ , if, and only if, it can be expressed in the form

العدد الرابع عشر - فبراير 2017

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1. \quad (17)$$

**Proof.** Let  $f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z)$

$$f(z) = z - \frac{2\beta(1-\gamma)}{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{(k-1)}{1+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)}} \omega_k z^k.$$

Then, in view of (17), it follows that

$$\sum_{k=2}^{\infty} \frac{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{(k-1)}{1+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)}}{2\beta(1-\gamma)} \times \left\{ \frac{2\beta(1-\gamma)}{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{(k-1)}{1+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)}} \omega_k \right\} = \sum_{k=1}^{\infty} \omega_k = 1 - \omega_1 \leq 1.$$

Thus  $f \in T^{\alpha,n}(m, q, \lambda, \beta)$ .

Conversely, assume that a function  $f$  defined by (4) belongs to class  $T^{\alpha,n}(m, q, \lambda, \beta)$ . Then

$$a_k \leq \frac{2\beta(1-\gamma)}{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{(k-1)}{1+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)}}.$$

We set

$$\omega_k = \frac{((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{(k-1)}{p+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)}}{2\beta(1-\gamma)},$$

and  $\omega_k = 1 - \sum_{k=1}^{\infty} \omega_k$ . Then we have  $f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z)$ . This completes the proof.

العدد الرابع عشر - فبراير 2017

Similarly, we can prove the following result:

**Theorem 4.2.** Let  $f(z) = z$  and,

$$f_k(z) = z - \frac{2\beta(1-\gamma)}{k[(k-1) + \beta(k+1-2\gamma)](k)^\alpha \left(1 + \frac{(k-1)}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{(k)!\Gamma(1+n)}} z^k.$$

Then  $f$  is in the class  $C^{\alpha,n}(m, q, \lambda, \beta)$ , if, and only if, it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1.$$

### 5. Integral Operators.

**Theorem 5.1** If the function  $f(z)$  given by (4) is in the subclass  $T^{\alpha,n}(m, q, \lambda, \beta)$ , where  $0 \leq \gamma < 1$ ,  $0 < \beta \leq 1$ ,  $n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ ,  $m \in \mathbb{Z}$ ,  $\lambda, q \geq 0$  and let  $c$  be a real number such that  $c > -1$ .  $f$  belongs to the class  $T^{\alpha,n}(m, q, \lambda, \beta)$ , then the function  $F$  defined by

$$F(z) := \frac{(c+1)}{z^c} \int_0^z t^{c-1} f(t), \quad (18)$$

also belongs to  $T^{\alpha,n}(m, q, \lambda, \beta)$ .

**Proof.** Let  $f \in T$  Then from representation of  $F$ , it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad (b_k \geq 0).$$

where  $b_k = \left(\frac{c+1}{c+k}\right) a_k$ . Therefore using Theorem 2.1 for the coefficients of  $F$ , we have

$$\sum_{k=2}^{\infty} \left( ((k-1) + \beta(k+1-2\gamma))(k)^\alpha \left(1 + \frac{k-1}{1+q}\lambda\right)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} b_k \right) =$$

العدد الرابع عشر - فبراير 2017

$$\sum_{k=2}^{\infty} \left( ((k-1) + \beta(k+1-2\gamma))(k)^{\alpha} \left(1 + \frac{k-1}{1+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)} \left(\frac{c+1}{c+k}\right) a_k \right) \leq$$

$$\sum_{k=2}^{\infty} \left( ((k-1) + \beta(k+1-2\gamma))(k)^{\alpha} \left(1 + \frac{k-1}{1+q} \lambda\right)^m \frac{\Gamma(k+n)}{k! \Gamma(1+n)} a_k \right) \leq 2\beta(1-\gamma).$$

since  $\left(\frac{c+1}{c+k}\right) < 1$  and  $f \in T^{\alpha,n}(m, q, \lambda, \beta)$ . Hence  $F \in T^{\alpha,n}(m, q, \lambda, \beta)$ .

Many other work on analytic functions functions related to derivative operator and integral operator can be read in [11], [12]

## References.

- [1] Nagat.M.Mustafa, and maslina Darus. (2012) The Fekete-Szegő problem for starlike functions of order  $\beta$  associated with generalised derivative operator. AIP Conf. Proc) 15 (22): 938-944.
- [2] St. Ruscheweyh, (1975), New criteria for univalent functions, *Proc. Amer. Math.Soc.* 49 109-115.

العدد الرابع عشر - فبراير 2017

- [3] Sălăgean, G. S. (1983) Subclasses of univalent functions. Lecture Notes in Mathematics 1013: 362–372.
- [4] F.M. AL-Oboudi, (2004), On univalent functions defined by a generalized Salagean Operator, *int, J. Math. Math. Sci.* **27** 1429-1436.
- [5] M. Darus and K. Al-Shaqsi, (2008), Differential subordination with generalized derivative operator, *Int. J. Comp. Math. Sci.* **2** (275-78).
- [6] Catas, A. ( 2009) On a certain differential sandwich theorem associated with a new generalized derivative operator. *General Mathematics* 17 (4): 83–95.
- [7] N. E. Cho and T. H. Kim, (2003), Multiplier transformations and strongly close -to-convex functions, *Bull. Korean Math. Soc.* **40** (3) 399-410.
- [8] H. Silverman, (1975), Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* 51109-116.
- [9] V. P. Gupta and P. K. Jain, (1976), Certain classes of univalent functions with negative coefficients, *Bull. Austral. Math. Soc.* 14409-416.
- [10] Nagat.M.Mustafa and M. Darus, (2012) Some properties of a subclass of analytic function defined by a generalized Srivastava and Attiya operator. *Facta Universitatis (NIS)* **27** (3): 309-320.
- [11] Aisha.Ahmed. Amer and M. Darus. (2012) Certain properties for analytic functions with negative coefficients defined by a generalised derivative operator, *Journal of Quality Measurement and Analysis, (JQMA)* **8**, 37-44. ISSN 1823-5670.