Design of Reinforced Concrete Halls

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شكر ودعاء

جزيل الشكر لكل من حرص على اقتناء هذا الكتاب

ودعاء الى الله

ان ينتفع بما حواه من علم وان يتغمد المولى برحمته

مؤلفه الاستاذ الدكتور / محمد هلال

سامح هلال

P R E F A C E

The contents of this edition are mainly the same as those of the previous one with the necessary corrections and the addition of one of the biggest structures designed by the author, namely the bulk urea store shown in Fig VIII-9

It has further been found more convenient to replace chapter XI on Foundations by a new chapter, published for the first time, on Folded Plate Structures A simple sysematic method of design, based on the fundamentals of mechanics and theory of plane structures has been shown To illustrate the application of the method, the design and details of three different folded-plate structures worked out by the author are included in this chapter The chapter on Foundations is in the author's textbook on Fundamentals of Reinforced and Prestressed Concrete"

The author hopes that this book remains of benefit to structural engineers, graduate and undergraduate students of the engineering faculties and higher institutes in the design of reinforced concrete structures of common use

March 1978 M Hilal

C O N T E N T S

I - INTRODUCTION

The object of a hall is to cover a limited area that has to be utilised for a certain purpose such as meetings, sports, storage, exhibitions, industry, etc

Reinforced concrete halls and their supporting elements must satisfy the following conditions

a) The disposition, layout, lighting, ventilation, drainage and in general the architectural composition must satisfy the requirements of the owner for an economic, efficient and good-looking structure

b) The structure and its structural members should be so designed and constructed that they are able, with appropriate safety, to withstand all the loads, superimposed loads and other actions (such as differential settlements and temperature changes) liable to occur during construction and in use

The object of the design calculations is to guarantee sufficient safety against the structure being rendered 'unfit' for service

A structure is considered to have become 'unfit' when one or more of its members ceases to perform the function for which it was designed, owing to failure , buckling due to elastic, plastic or dynamic instability, excessive cracking, excessive elastic or plastic deformations etc

c) The structural supporting elements must be so chosen that they give the most economic solution within the available possible means

d) The initial and maintainance cost must be the minimum possible

e) The structure should preferably show clearly the statical system adopted and it is generally recommended not to hide the supporting elements The proportions and dimensions chosen according to a convenient statical system are generally the most convenient and best looking

The structures dealt with in the following chapters are supposed to be of reinforced concrete The general principles given here cannot be applied directly to prestressed concrete structures without the necessary adaptation

The main supporting elements of big span halls can be classified, regarding their statical systems, as follows:

1) Girder Types

Under these systems, we understand girders or trusses giving vertical reactions for vertical loads such as

a) Simple girders Fig I-1

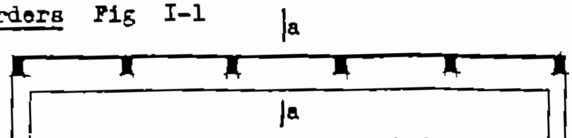


Fig I-1 A simple girder

In this system, the maximum bending moment takes place at one section only (section a-a) and is relatively high The design of the girder is governed by the extreme fiber stress of the same section and hence, uneconomic!

b) Cantilever girders Fig I-2

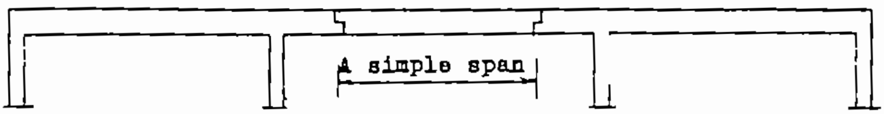


Fig I-2 A cantilever girder

This statically determinate system is generally used in long girders or where differential settlements are expected

c) Continuous girders: Fig I-3

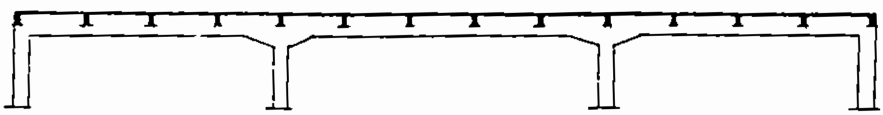


Fig I-3 A continuous girder

Such statically indeterminate systems are generally used in girders shorter than about 45 ms and where differential settlements do not give high internal forces

In reinforced concrete structures, it is generally not possible to construct the knife edge hinges allowing the full rotation or the rollers allowing the displacement and rotation assumed in such ideal statical systems

Girders supported on reinforced concrete columns possess generally some sort of fixation

For girders monolithically cast with the columns, if no exact calculation as a building frame is done, it is allowed to consider them as freely supported on the interior columns and rigidly connected to the exterior columns The fixing moment may approximately be estimated in the following manner Fig I-4

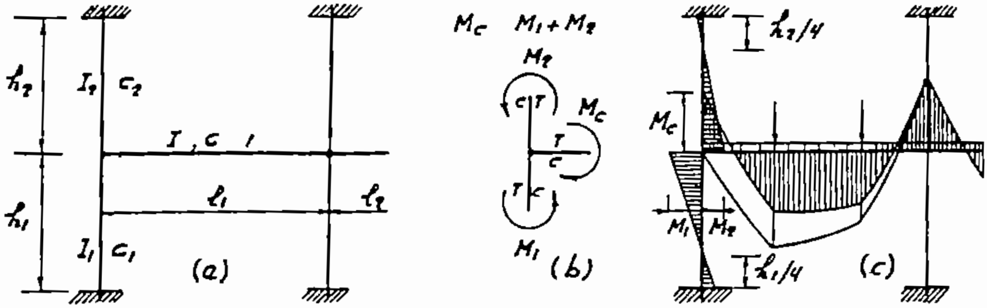


Fig I-4 Moment distribution in a building frame

Assume I_o, I_1 and I_2 to be the moments of inertia of the girder of span l, the lower column of height h_1, and the upper column of height h_2 Their relative rigidity can be given by the factors

$$c_o = \frac{I_o}{l} \quad \frac{l}{I_o} = 1 \qquad \text{reference value}$$

$$c_1 = \frac{I_1}{h_1} \quad \frac{l}{I_o} \qquad \text{relative rigidity of lower column}$$

$$c_2 = \frac{I_2}{h_2} \quad \frac{l}{I_o} \qquad \text{relative rigidity of upper column}$$

Then, the fixing moment M_c at the exterior columns is given by

$$M_c = \bar{M} \quad \frac{c_1 + c_2}{1 + c_1 + c_2}$$

in which $\bar{M}$ is the fixed-end-moment of the loaded span l_1 This moment will be distributed on the columns in proportion to their relative rigidity c Thus

$$M_1 = M_c \quad \frac{c_1}{c_1 + c_2} \qquad \text{and} \qquad M_2 = M_c \quad \frac{c_2}{c_1 + c_2}$$

For girders of top floor $c_2 = 0$ and $M_1 = M_c$

The sense of the bending moments can be adjusted by the system of arrows shown in Fig I-4 b

The exterior columns are to be designed for the induced bending moments shown in Fig I-4 c plus the maximum axial loads, whereas the interior columns may be calculated for the maximum axial loads only

d) Vierendeel girders Fig I-5

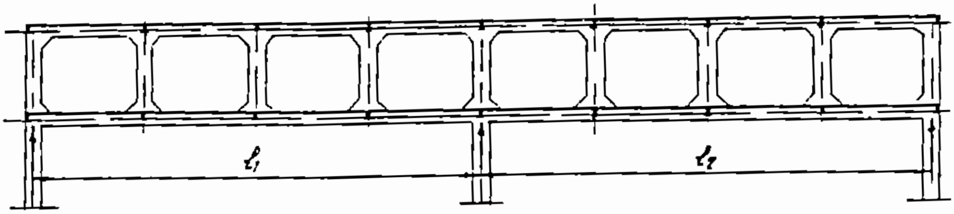


Fig I-5 A Vierendeel girder

Vierendeel girders are composed of a top chord, a bottom chord and verticals This system is internally high grade statically indeterminate while externally, it might be statically determinate as in freely supported simple and cantilever spans or indeterminate as in continuous spans

e) Trusses Fig I-6

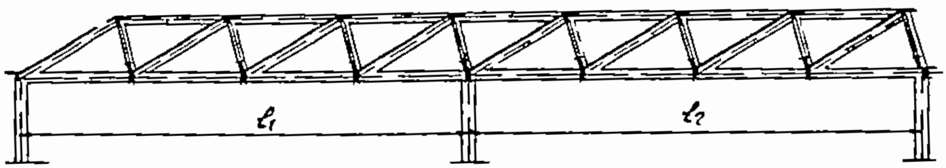


Fig I-6 A Continuous truss

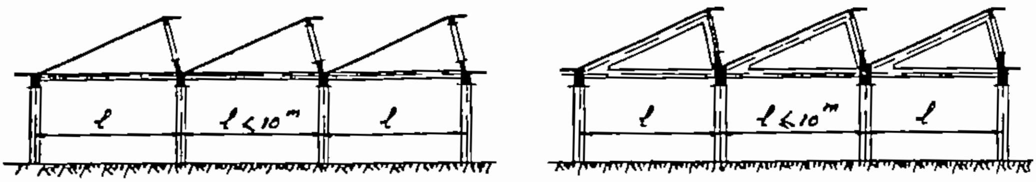
Trusses may, in some cases, give a convenient solution for the main supporting element of the hall They may be simple, externally statically determinate, or continuous, externally statically indeterminate

The joints being monolithically cast and rigidly connected, the induced bending moments may, in some cases, be of considerable values and in order to have trusses free from corner cracks, such moments must be considered in the design

2) Sheds

These are polygonal or curved slabs (or girders) with ties giv- vertical reactions for vertical loads As typical examples, we give the following types

a) Saw-tooth slab and girder types Fig I-7



Slab type Girder type

Fig I-7 Saw-tooth sheds

b) Polygonal sheds with a tie Fig I-8

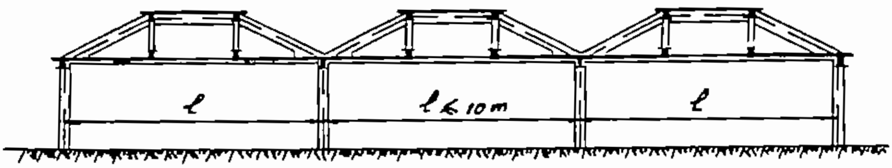


Fig I-8 A Polygonal shed with a tie

In these two types, the elongation of the tie for spans l≤10 ms is generally small and can be neglected Hence, we get vertical reactions for vertical loads

c) Arched slabs (or girders) with a tie Fig I-9

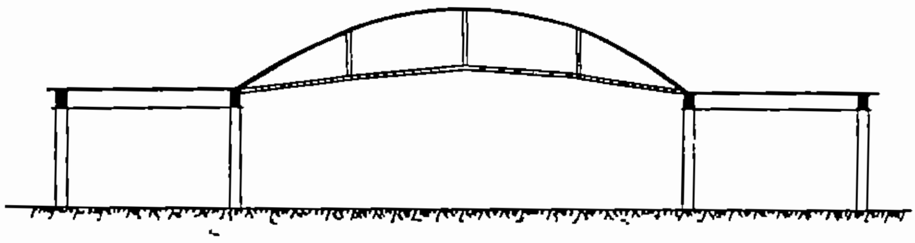


Fig I-9 Arched-slab roof

3) Frames

The main feature of a frame is the continuity and rigid connection of the horizontal, inclined or curved members of the roof with the vertical or inclined supporting members The continuity gives inclined reactions for vertical loads, and the bending moments due to the loads will be distributed on the different members of the frame

Frames may be classified to the following systems

a) Statically determinate frames Fig I-10

One may recognize here cantilever frames (Fig I-10a), simple frames (Fig I-10b) and three hinged frames (Fig I-10c)

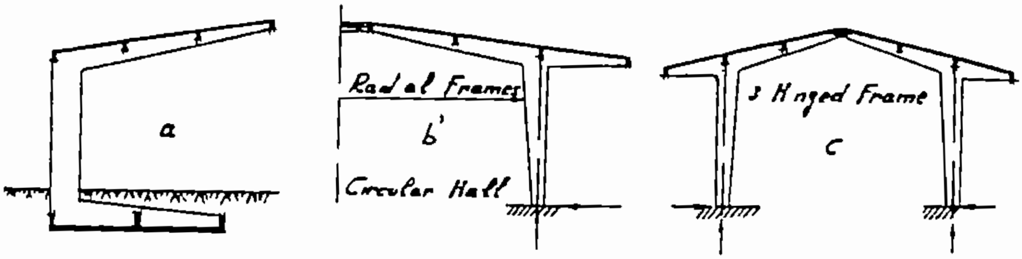


Fig I-10 Statically determinate frames

b) Once statically indeterminate frames with or without ties

(Fig I-11)

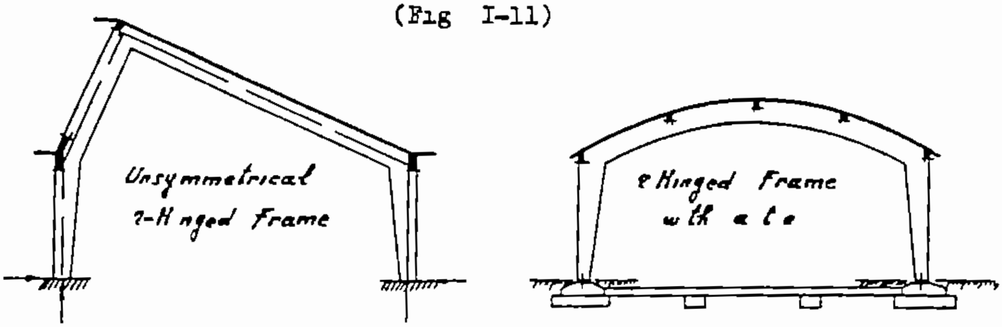


Fig I-11 Once statically indeterminate frames with or without ties

c) Twice statically indeterminate two-hinged frames with ties

(Fig I-12)

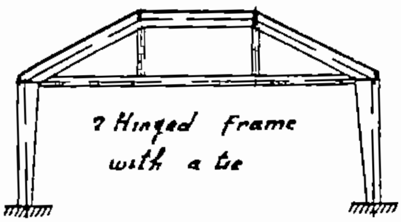


Fig I-12

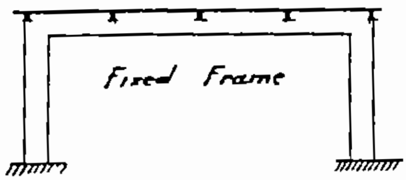


Fig I-13

d) Three times sta-ically indeterminate frames Fig I-13

e) Continuous and multiple frames Fig I-14 a and b

Such frames are generally high grade statically indeterminate and need much time to determine the statically indeterminate values and the internal forces The solution can however be much simplified for symmetrical frames generally used in structures

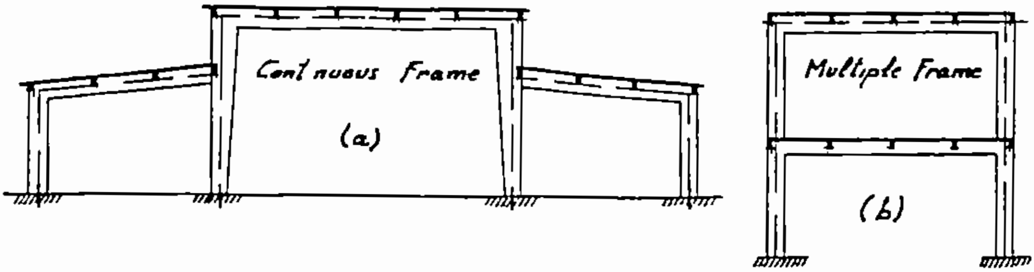


Fig I-14 Continuous and multiple frames

4) Arched Roofs

Structures covering long spans and large areas without intermediate supports using the minimum amount of building materials have always been one of the main aims of structural engineers When a plane roof surfaoe is not necessary to meet the functional requirements of the structpre, an arched roof conveniently formed is normally found to give the least amount of building materials because here, the external loads are mainly resisted by internal compressive forces plus small bending moments and shearing forces Arches may be constructed in different systems as shown in Fig I-15 a to f

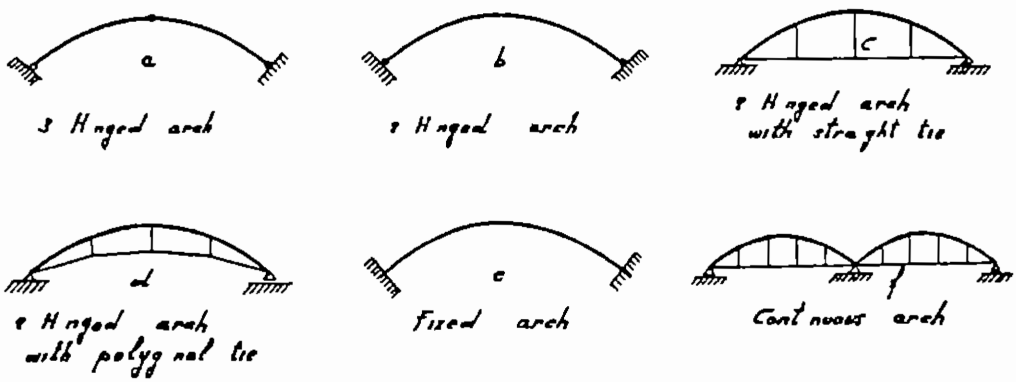


Fig I-15 Different types of arched girders

5) Shell Roofs

As a further development of the arch principle, shell surfaces provide structurally efficient solution to the problem of carrying roof loads over long spans They owe their efficiency to the translation of the applied loads into compressive, tensile and shear stresses in the plane of their surface generallv termed as the membrane stresses We recognize Surfaces of revolution, cylindrical shells, and double curved shells

a) Surfaces of revolution e g domes and cones Fig I-16

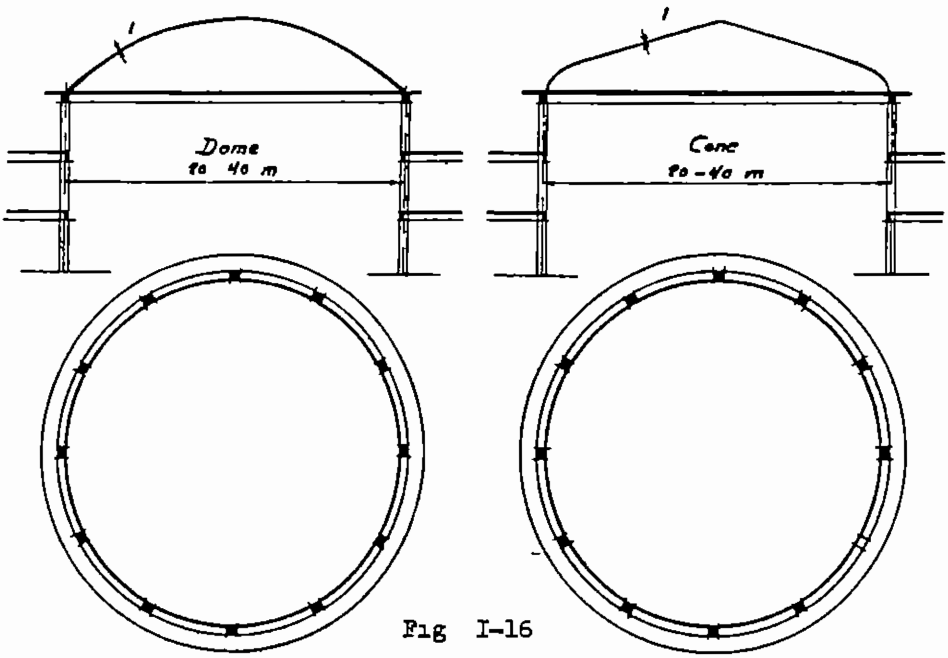


Fig I-16

c) Cylindrical shells

The behavior of a shell is completely different from that of the arch The arch is a plane structure resisting the external loads by plane forces whereas the shell is a space structure supported on the diaphragms and resists the external loads by membrane stresses in its surface

Shells may be long as for example, the barrel vault shown in Fig I-17 and the saw-tooth shell roof shown in Fig I-18 Such shells may

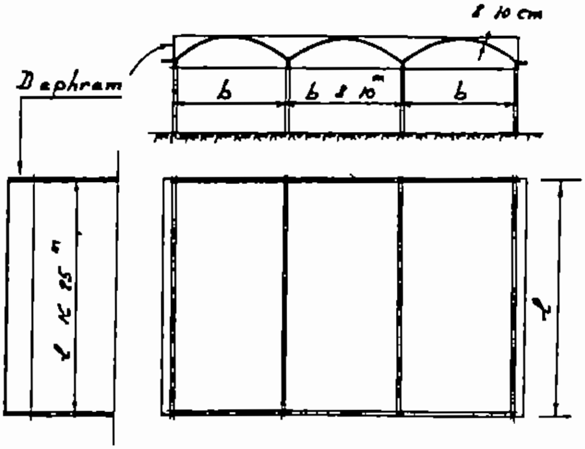


Fig I-17 A long cylindrical barrel-vault shell roof be considered as beams of span l and breadth b

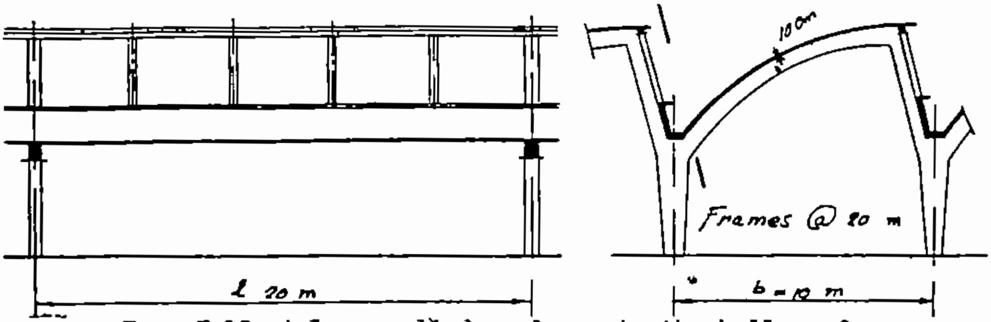
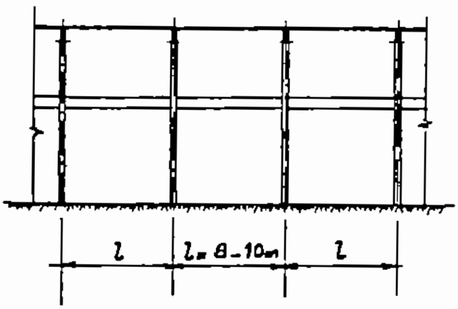


Fig I-18 A Long cylindrical saw-tooth shell roof

They may also be short as for example the shell roof shown in Fig I-19 Such a shell may be treated as a curved membrane supported on the diaphragms



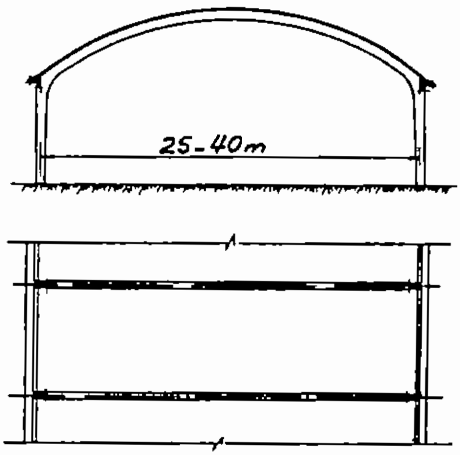


Fig I-19

A short-shell roof

c) Double curved shells

Double curved shells have been recently extensively used in modern structural architecture to cover relatively big areas with the least possible building materials Some of the simple forms are shown in Fig I-20 a to d

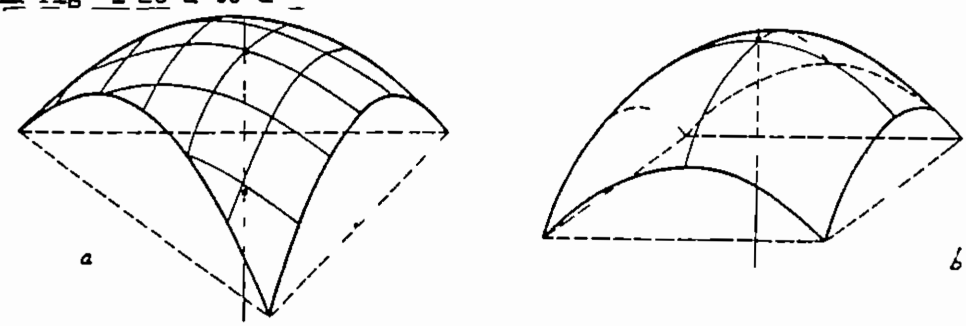


Fig I-20 Double curved shells on triangular and rectangular ground plan

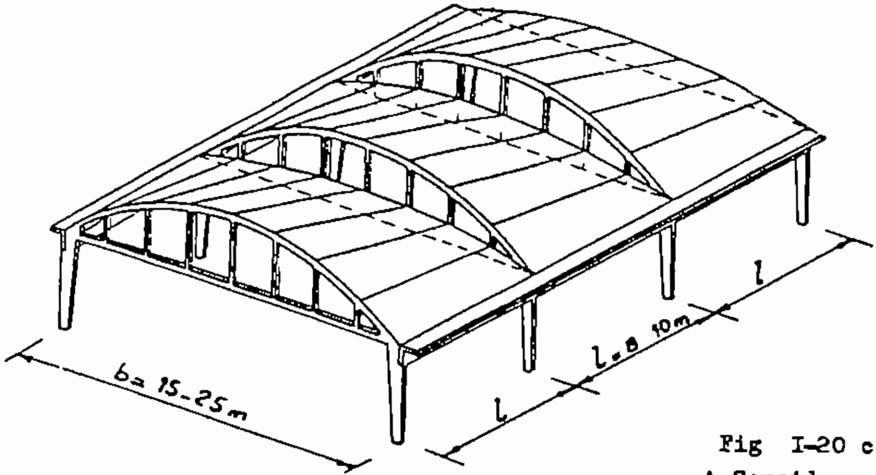


Fig I-20 c
A Conoid roof

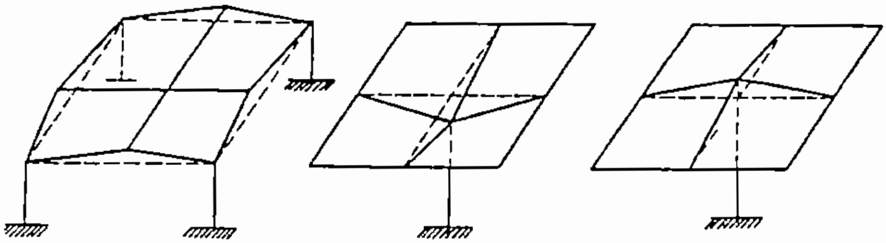


Fig I-20 d Some simple forms of hyperbolic paraboloid roofs

6) <u>Folded-plate roofs</u> Fig I-21

In an attempt to simplify formwork and yet retain many of the ad-

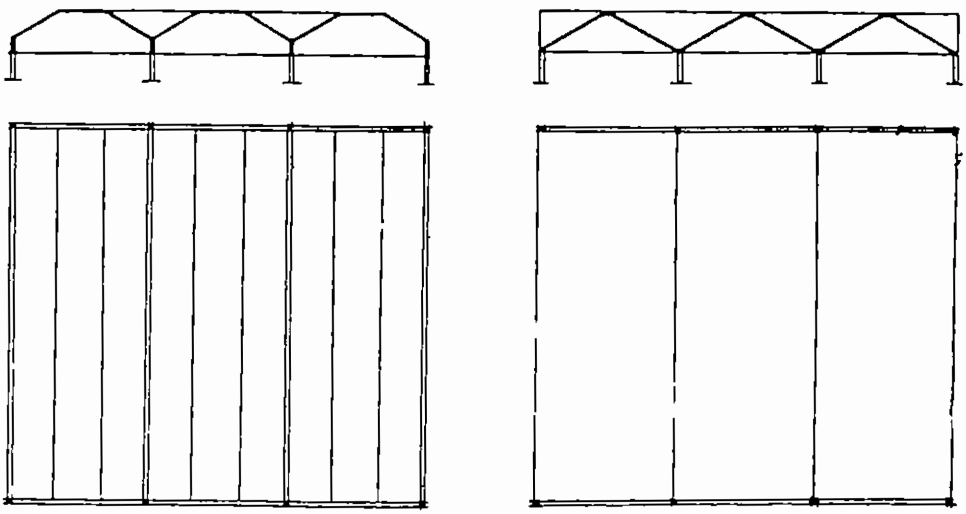


Fig I-21 Folded-plate roofs

vantage characteristics of cylindrical shells, the folded-plate structure is evolved These surfaces have a deep corrugated form similar to that of cylindrical shells, except that plane elements are used, intersecting in fold lines parallel to the span direction as shown in Fig I-21

The following 7 chapters, II to IIIV, include the theory, design and construction of the classic basic systems of the main supporting elements of plane structures namely simple girders, continuous girders, frames, Vierendeel girders, trusses, saw-tooth structures and arched roofs

In chapter IX some constructional details, necessary for the design are treated

In order to reduce the internal forces due to temperature changes, shrinkage displacement of the supports, hinged and eventually free bearings are extensively introduced at the supports of the main supporting elements Chapter X has been devoted for the theory, design and constructional details of such bearings

The analysis of folded-plate structures followed by detailed design examples is presented in chapter XI

It has been found, as stated before, that big spans with the least possible amount of building materials can be achieved by the use of shell structures The exact analysis of such structures is generally very complicated, but due to the remarkable reserve strength of shell construction, which makes it practically impossible for a shell structure to collapse, it has been possible, without detailed mathematical analysis, to use simplified methods of calculation based on the qualitative understanding of the fundamental nature of the behavior of some forms of shells

Some simple basic forms of shell structures, treated in a clear, simple manner are given in chapter XII

II - SIMPLE GIRDERS

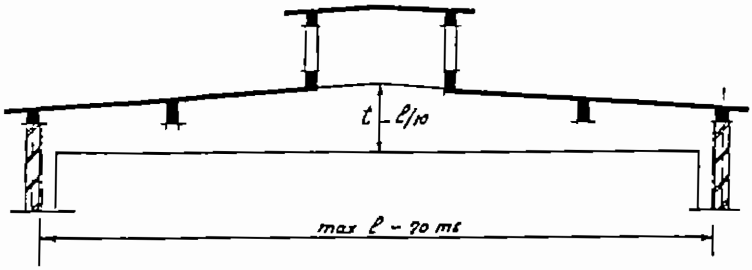


Fig II-1 A simple girder

Simple girders as the main supporting element of a roof of a hall (fig II-1) may be used when differential settlements are liable to take place or in special cases e g when a new roof is to be constructed on existing bearing masonry walls or if the columns used to support the girders are relatively slender etc

If the span of a simple girder is bigger than 10 meters, it is recommended to make one of the supports hinged and the other sliding to allow for the free movement of the girder In this case a clear horizontal joint between the reinforced concrete roof and the walls is to be arranged allowing the necessary rotation and displacement of the roof to take place without creating cracks in the wall

In case of girders supported on slender columns it is recommended to join the columns supporting the girders and lying in masonry walls by horizontal beams arranged at convenient distances (3-5 ms) In this manner, the columns and the outside walls, as one plane, can follow the deformation of the roof without creating vertical cracks between the columns and the walls or horizontal cracks between the roof wall beams and the walls

In masonry or brick walls, it is however a good practice to arrange reinforced concrete connecting members - vertical columns and horizontal beams - every 16 to 20 m^2 Accordingly , if the

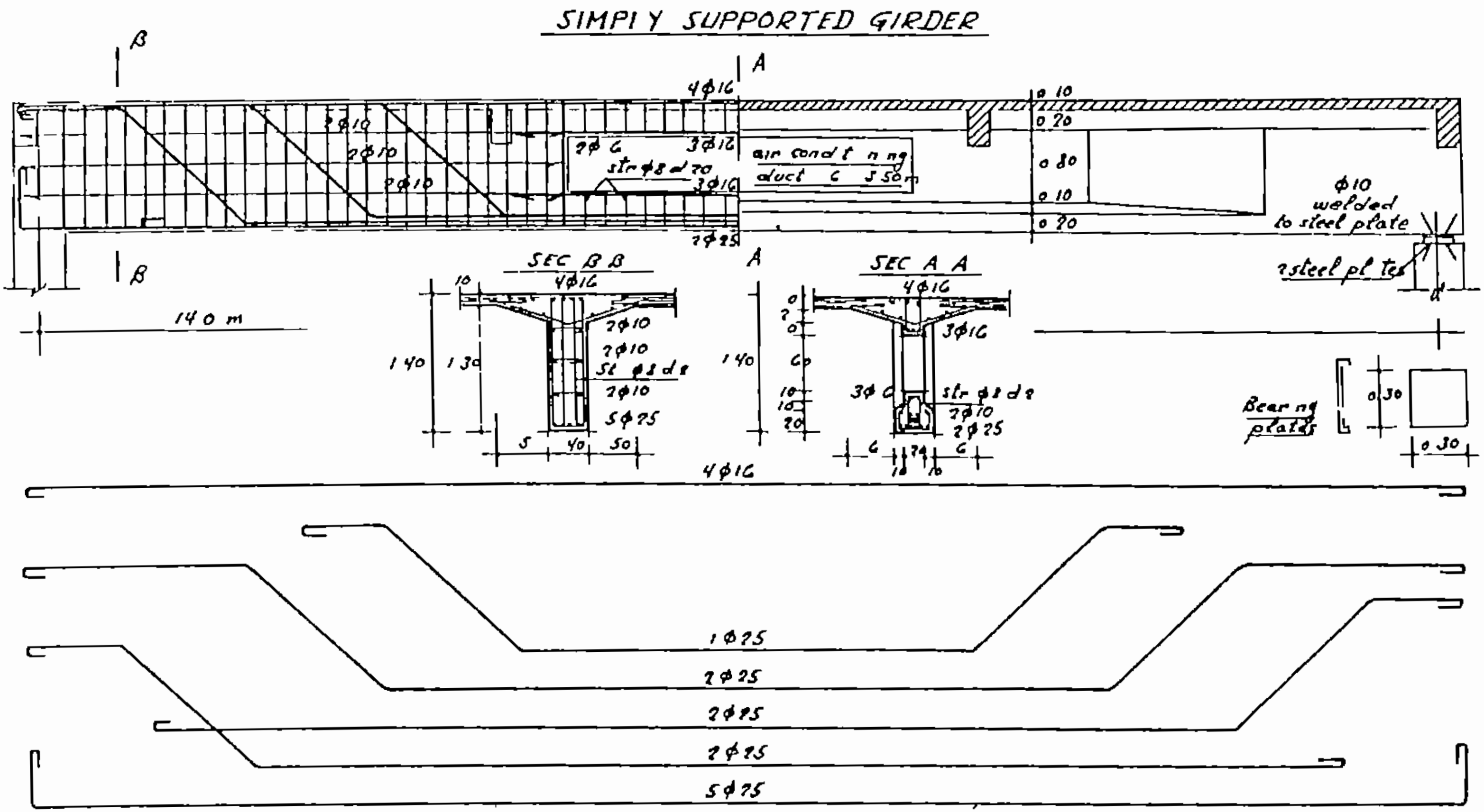


Fig II-2 Details of reinforcements of a simple girder vertical columns in the outside walls of a building are arranged every 6 ms , it is recommended to arrange horizontal connecting beams in the wall every 3-3 5 ms

Figure II-2 shows the details of the roof - main girder of an air-conditioned hall 14 ms span

In order to reduce the own weight of the 140 cms deep main girder, the breadth of 80 cms from the web is shosen 20 cms and an opening 60 x 350 cms for the air-conditioning duct is arranged at the middle of the span

The opening has been chosen in the tension zone, which is statically not acting and, at the middle of the beam where the shear stresses are minmum Its height is to be chosen such that there is sufficient concrete area at the top to resist the compressive stresses in the girder and ample cover for the tension reinforcements at the bottom

The bottom enlarged part of the web is arranged in order to give adequate space for the tension reinforcements of the main girder

In order to resist the shear stresses safely, the breadth of the web has been increased gradually from 20 to 40 cms on the two sides of the girder at zones of high shear stresses For the same reason, it is not recommended to arrange the air-conditioning ducts near the supports at the inner surface of the outside walls

The steel reinforcement shown in figure II-2 gives the classic known arrangement in which the diagonal tension of the beam is mainly resisted by the combined action of the bent bars and the stirrups

In order to avoid the formation of vertical shrinkage cracks along the web, shrinkage reinforcements ϕ 10 mms arranged at distances of about 35 cms along the outside surface of the beam are provided The relatively heavy bars arranged around the air-conditioning opening are recommended

The haunches introduced between the roof slab and the web of the main girder increase the moment of resistance of the girder because in this manner, the effective breadth of the flange is increased

We give in the following some new trends in the design of reinforced concrete girders

1) The use of deformed high grade or cold twisted (e g Tor or Tentor) steel for the main reinforcement is prefered due to bigger bond

and higher resistance If the bond between the steel reinforcement and the concrete is increased the tension cracks are increased in number and decreased in width Due to higher resistance the area of the steel reinforcement is reduced and the girder under consideration is generally more economic In the shown example , the main steel of 12 Φ 25 mm may be replaced by 12 Φ 22 high grade steel or 11 Φ 22 cold twisted steel

2) It has been proved by tests that the use of high grade reinforcements of small diameters distributed on the tension zone of the girder gives a better distribution of the cracks Figure II-3 shows the cross-sections of two beams designed for the same span and loading , and having the same moment of resistance At failure, the number of tension cracks in the beam b reinforced by high grade steel was three times as much and much narrower than those of beam a in spite of the higher stresses in steel

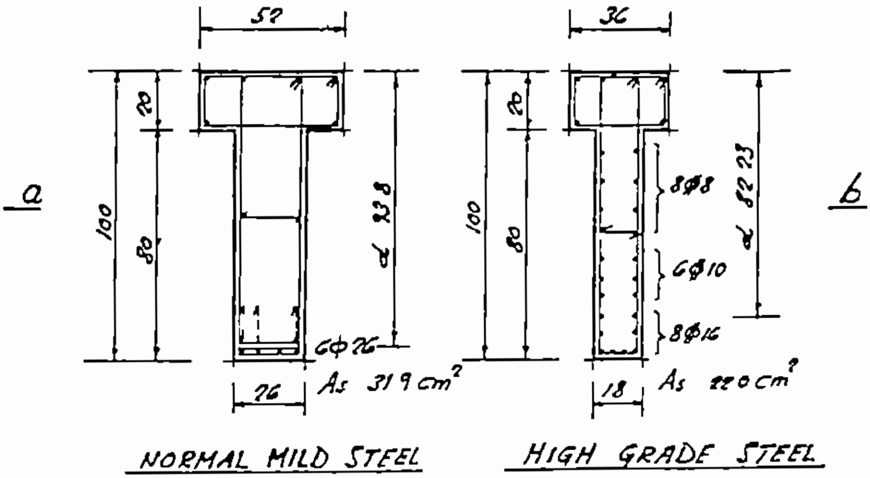


Fig II-3 Sections of the same moment of resistance

3) When tension cracks are developed at the central part of a beam, the bond between the steel and concrete is broken at the position of the cracks Under heavy loads, the width and number of tension cracks is increased and the beam may collapse as the bar is pulled through the concrete To prevent this occurrence , end anchorage is essential , fig II-4 If the anchorage is adequate, such a beam will not collapse even if the bond is broken over the entire length between anchorages This is so because the beam acts as an arch with a tie as shown in figure II-4 in which the shaded uncracked concrete represents the arch and the tension reinforcements the tie In this

case, the force in the tension steel, over the entire unbonded length, is constant and equal to $T = M_{max} / y_{ct}$ In consequence, the total steel elongation in such beams is larger than in those in which bond is preserved, resulting in larger deflections and larger widths of cracks

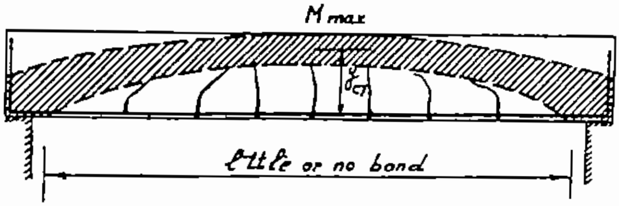


Fig II-4 Tied-arch action in a cracked beam

In this manner, one can easily imagine that the cutting off of the longitudinal tension bars weakens the tie and reduces the bearing capacity of the beam The shearing forces can however be resisted by the inclined compressive forces of the arch

It is con sequently recommended to use for the tension reinforcements deformed bars giving high bond resistance and, to introduce at least one third of such reinforcements to the supports and to anchor them well beyond the center line of the supports

4) The width of the diagonal tension cracks depends on the type of the web reinforcements used, as can be seen from the following test results which show that the use of inclined stirrups arranged at small distances gives the best result (Fig II-5)

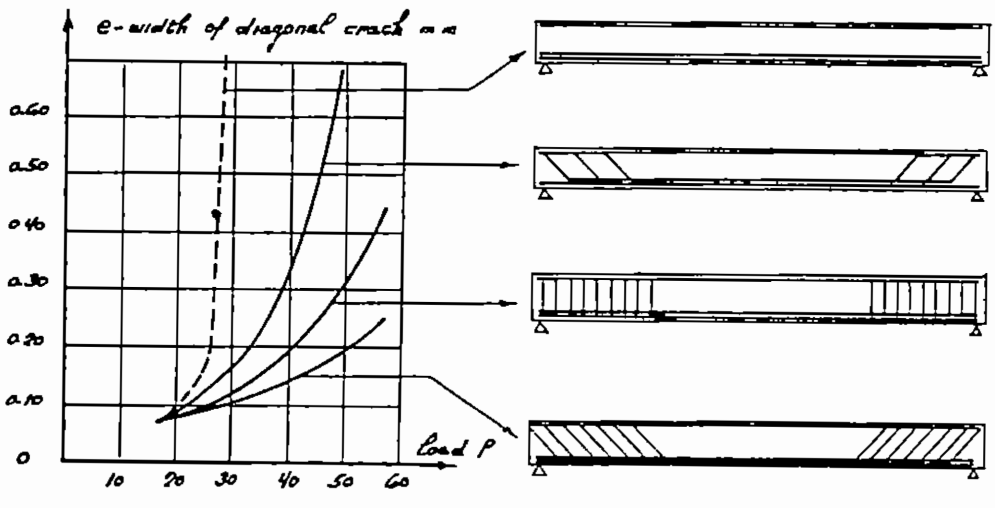


Fig II-5 Effect of type of web reinforcements on width of cracks

5) A reinforced concrete simple beam can however be considered as a complex truss (Fig II-6) with the tension steel reinforcement as the bottom chord, the top concrete – the flange – as the top chord, compression concrete diagonals and tension steel vertical or diagonal stirrups

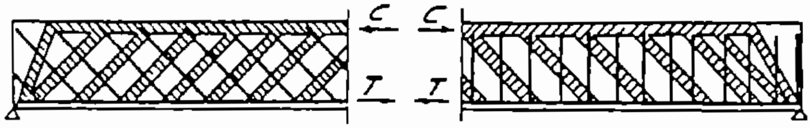


Fig II-6 Action of a simple reinforced concrete beam as a complex truss

6) It has been found that the tension in the longitudinal reinforcement of a cracked simple beam is not equal to zero at the supports, but there exists a value T_0 equal to 1 - 1 6 Q where Q is the shearing force in the bottom chord of a truss with inclined compression diagonals as shown in Fig II-7

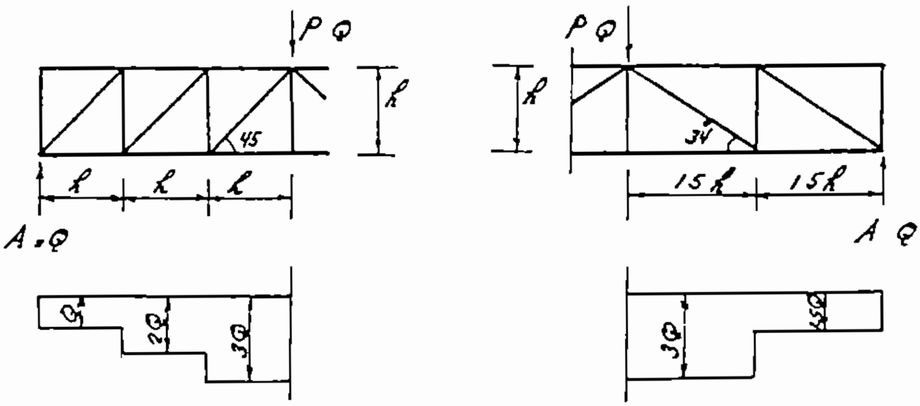


Fig II-7 Tension in bottom chord of trusses with different inclined diagonals

7) The checking of the anchorage conditions at the ends of the longitudinal reinforcing bars should be based on the bending moment (or the tension) diagram, which should be suitably displaced to take account of the need to absorb the horizontal components of the forces in the 'struts' (compression members) of the fictitious lattice system of the complex truss

This displaced diagram (see Fig II-8), which serves as the basis for designing the longitudinal reinforcement, is obtained by shifting the enveloping curve of the bending moments (or the tensions) parallel to the center-line of the member in the most unfavourable direction by by an amount equal to the effective depth d of the section The longitudinal bars should be anchored outside the 'displaced' diagram

The amount of displacement actually required may vary between $\frac{d}{2}$ and d, according to the efficiency of the transverse reinforcement: the value of d is thus certainly on the safe side

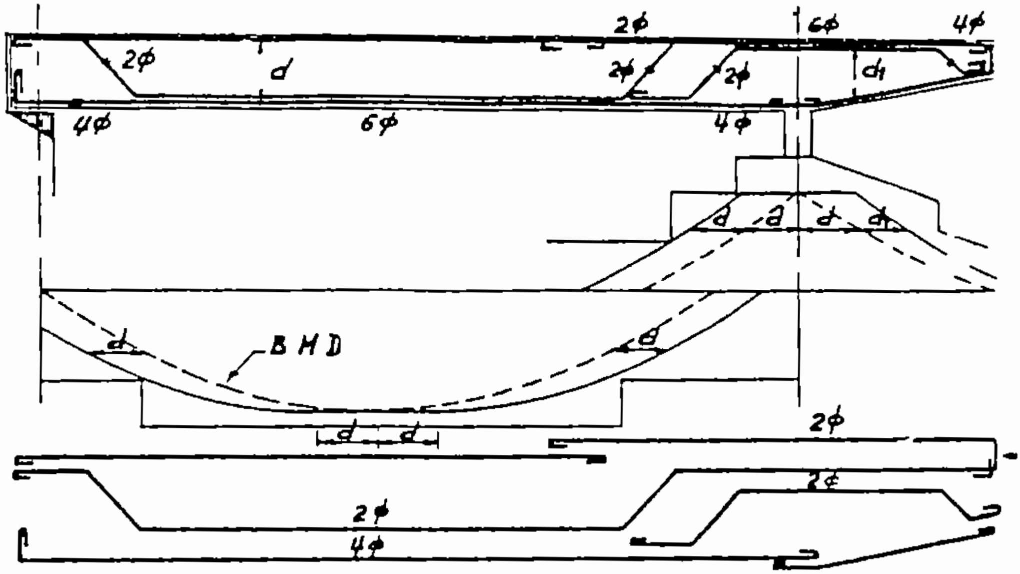


Fig II-8 Displaced bending moment and moment of resistance diagrams

Conclusions

From the previous investigations, we arrive to the following conclusions

1) The use of high grade or cold treated steels as tension reinforcement is preferred especially for bigger diameters

2) The use of longitudinal reinforcements of relatively small diameters 8-10 mm distributed over the two surfaces of the web and at small distances (15 - 20 cms) reduce the width of the cracks and can be considered as resisting a part of the tension in the section relative to the distance of the bars from the neutral axis

3) The maximum possible part of the tension reinforcement is to be introduced to the supports and well anchored there

4) The anchorage conditions at the ends of the longitudinal tension bars should be based on the shifted bending moment diagram i e , the ends of the bars determined according to the classic moment of resistance diagrm must be shifted a distance d in the most unfavourable direction

5) The use of inclined stirrups at small distances (15 - 20 cms) to resist relatively high diagonal tension is most effective

III - CONTINUOUS GIRDERS

Continuous girders of constant and variable moment of inertia give in many cases a convenient solution for big spans If the choice of the spans is free and the magnitude of the loads on the different spans is the same, it is recommended to choose the inner spans somewhat bigger than the outer ones in order to have field moments of nearly the same order (fig III-1)

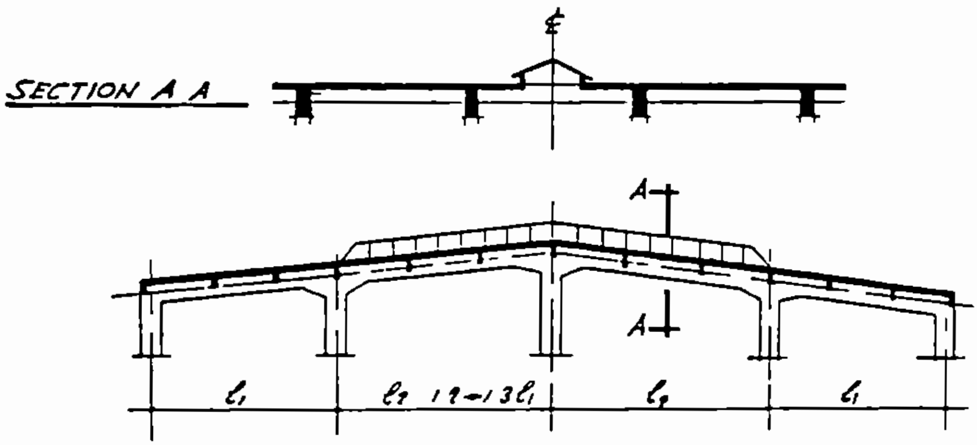


Fig III-1 A continuous girder

The increase of the depth of the girder towards the supports affects the bending moment diagram increasing the connecting moments by 10 - 20% and decreasing the field moments by the coresponding values For spans bigger than 10^{ms} such effects must be taken in consideration as shown in the following example of a crane girder subject to uniform dead loads and concentrated rolling loads (Fig III-2,A)

To reduce the amount of work included in the problem the girder is chosen symmetrical and composed of three spans only

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Refer to "Theory of Elastically Restrained Beams by M Hilal Published 1945.

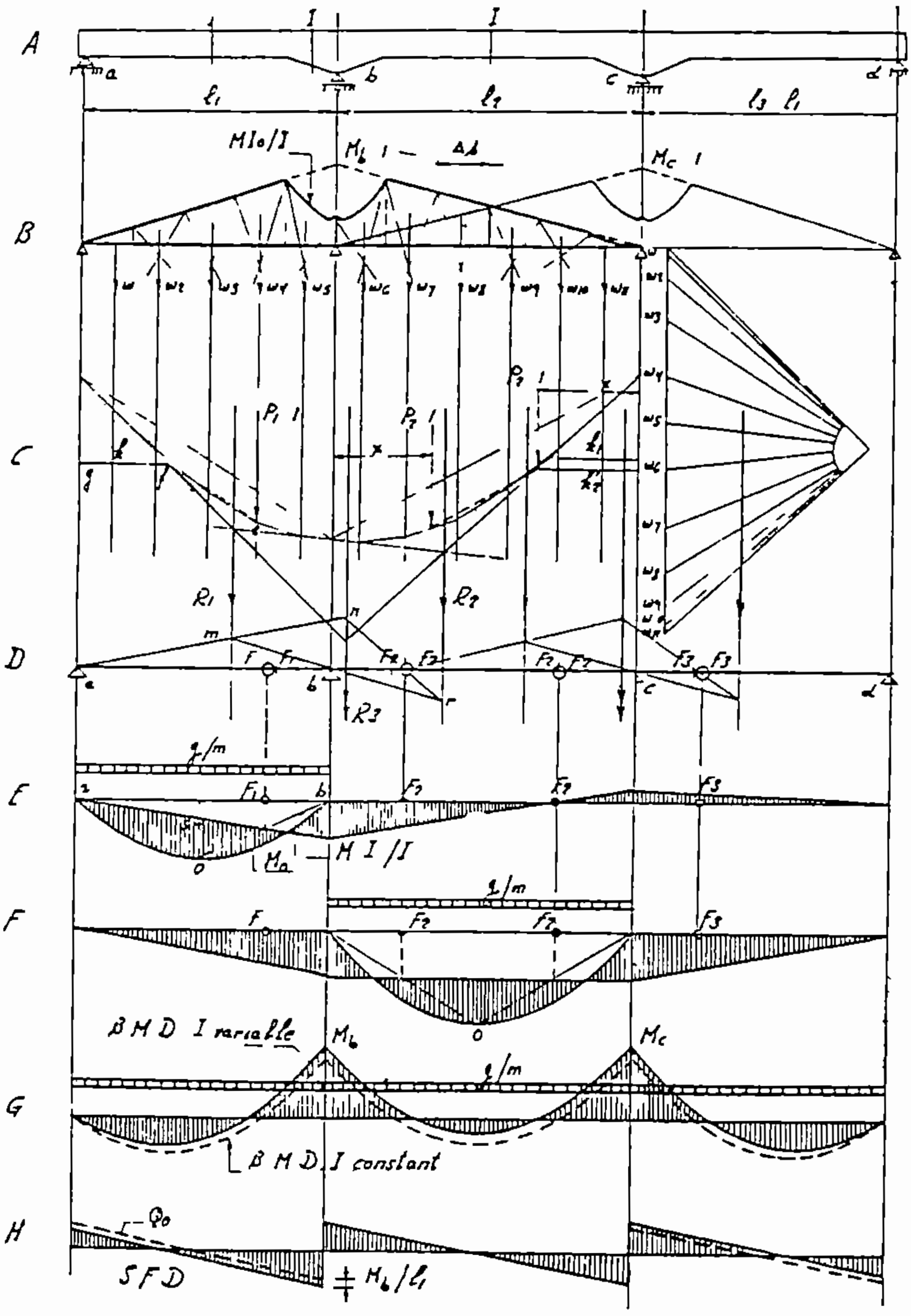


Fig III-2 Fixed points, moments and shears in a continuous girder of variable moment of inertia

We give in the following the main steps required to determine the internal forces using the graphical fixed - points - method

The internal forces due to the permanent uniformly distributed dead loads will be determined by superposition of uniform loads on the different spans The increase of the dead load due to the variation of the depth of the main girder near the supports may be neglected without making an appreciable error

The internal forces due to the rolling loads will be determined by the influence lines It is generally not recommended to use the influence lines for determining the internal forces due to permanent dead loads, the use of the direct method of loading of the different spans gives much better results

1) <u>Determination of the Fixed Points</u>

(Fig III-2 B C and D)

In order to find out the left fixed point F_2 of span l_2, one has to determine the position of the resultant elastic weights R_1 R_2 and R_3 where

R_1 = resultant of elastic weights $w = \Delta s\, MI_o/I$ due to $M_b = 1$ acting on l_1

R_2 = resultant of elastic weights $w = \Delta s\, MI_o/I$ due to $M_b = 1$ acting on l_2

R_3 = resultant of R_1 and R_2

These positions can be determined graphically in the following manner For ab and bc as simple beams draw the bending moment diagrams (B M D) due to $M_b = 1$ then determine the reduced B M D by multiplying the ordinates of the B M D by I_o/I Divide the reduced B M D into convenient number of strips Δs Determine the magnitude of the elastic weights $w = \Delta s\, MI_o/I$ (fig III-2B) and draw their force and link polygons (Fig III-2C)

R_1 & R_2 lie at the point of intersection of the first and last rays of the link polygon of the elastic weights on l_1 and l_2 respectively and R_3 is the resultant of R_1 and R_2

If the beam were of constant moment of inertia then R_1 and R_2 lie at the third points of l_1 and l_2 respectively whereas R_3 lies at the inverted third point

Having determined R_1 R_2 and R_3, the left fixed point F_2 of span l_2 can be determined using the known normal construction of beams of constant moment of inertia (fig III-2D) by drawing a line m n passing through the left fixed point of span l_1 (in our case, point a) , meeting R_1 in m and R_3 in n through m draw line m b and extend it to meet R_2 in r line n r intersects b c at the position of F_2

Repeating the same construction for spans l_2 and l_3 as simple beams due to $M_c = 1$, one can determine the left fixed point F_3 of span l_3

Due to the increase of the moment of inertia of the girder towards the supports one can easily notice that the position of the fixed points F moves towards the center-lines of the spans relative to the fixed points F' of the same girder if it were of constant moment of inertia (Fig III-2D)

2) Bending Moments and Shearing Forces due to Uniform Loads

a) Unsymmetrical exterior span loaded by g t/m'

For such a span of unsymmetical variation of moment of inertia the bending moment diagram can be determined as follows (Fig III-2E)

Draw the M_o - diagram (shown full) and the reduced $M_o I_o / I$ diagram (shown dotted) Determine the center of gravity S of the reduced diagram by dividing it into convenient vertical strips, assuming the area of each strip as an elastic weight acting in its center of gravity, and drawing a force and the corresponding link polygons Through S draw a vertical line to meet the M_o - diagram in O The closing line of the B M D passes through the point of intersection of the crossing line b O and the vertical through F_1

b) Symmetrical intermediate span loaded by g t/m'

This span has a symmetrical variation of the moment of inertia and hence O lies at the middle of the M_o - diagram and the normal construction of the B M D applies (Fig III-2F)

The final B M D due to a dead load g/m acting on all spans (fig III-2G) shows that , due to the increase of I at the supports , the connecting moments are increased and the field moments are decreased compared to beams of constant I

As the depth and reinforcements of the girder along the spans are generally governed by the field moments then a beam of variable I is supposed to be more economic than a beam of constant I

The shearing force in any panel can be determined from the general eouation

$$Q = Q_o + \frac{M_r - M_l}{l}$$

where

Q_o = the shearing force of the simple beam

M_r and M_l = the connecting moments at the right and left supports of the span under consideration

3) Bending Moments due to a Concentrated Unit Load

The internal forces (B M^s , S F^s etc) and reactions of a continuous beam due to rolling loads can be determined by the influence-lines-method

The influence line of a force acting in any section of a beam represents the value of the force in the section due to a simple unit load moving over the beam the ordinates being drown at the position of the load

In order to determine the influence lines of the bending moments of the different sections, one has to draw the B M D due to a concentrated load P = 1 at a series of sections For beams of variable moment of inertia this can be done as follows

For a load P = 1 acting on span l_1 , draw a vertical through the point of application of P to meet the link polygon of the elastic weights w due to M_b = 1 in e (fig III-2C) , then draw the line e f, parallel to the closing line to meet the first ray of the link polygon in f, the horizontal distance f g gives the crossing distance k_1 used for determining the B M D as shown in figure III-3

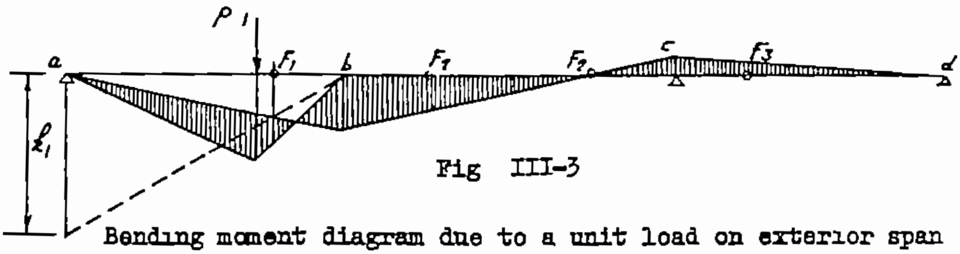


Fig III-3

Bending moment diagram due to a unit load on exterior span

For a load P_2 = 1 acting on span l_2 at a distance x from the left support b , the crossing distances k_1' and k_2' can be determined by assuming P_2 once at x from the left support b and once at the same distance x from the right support c (Fig III-2C) The B M D is drawn by plotting the smaller of the crossing distances k' at the nearer support as shown in figure III-4

It has to be noticed that the values of k are to be measured from figure III-2C to the linear scale of the girder and plotted in figures III-3 and III-4 to the force scale of the B M D

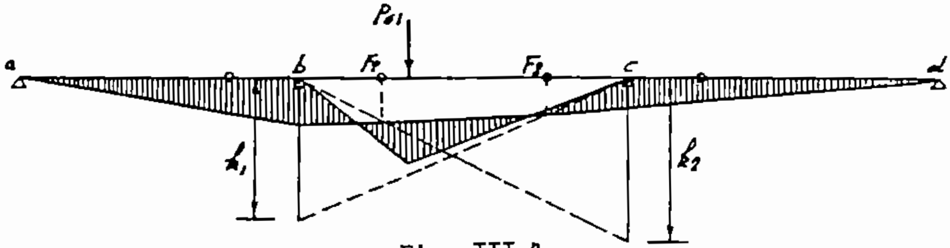


Fig III-4

Bending moment diagram due to a unit load on intermed span

4) Representation of Influence Lines as Elastic Lines

The influence lines of B M ^{s} , S F ^{s} or reactions of a continuous beam (or any statically indeterminate structure) can be represented as elastic lines in the following manner

Choose a main system in which the required force is equal to zero if we apply the same force as an external load acting on the chosen main system, then the resulting elastic line gives the form of the influence line (Fig III-5)

If it is required for example to determine the form of the influence line for the bending moment M_n acting at section n of the continuous beam a b c d , choose a main system in which $M_n = 0$ by introducing a hinge at n This main system will be equivalent to the original system if M_n acts on the beam at n The elastic line of the chosen main system due to M_n gives the form of the influence line of M_n (Fig III-5A) The proof of this statment is as follows

The relative angle of rotation at n of the continuous beam abcd is given according to the law of superposition by the relation

$$\alpha_n = \alpha_{nm} + M_n \alpha_{nn} = 0$$

in which

α_{nm} = angle of rotation of main system at n due to $P_m = 1$

α_{nn} = $M_n = 1$

According to theory of Maxwell, we have

$$\alpha_{nm} = \delta_{mn}$$

in which

δ_{mn} = deflection at point m of main system due to $M_n = 1$

So that

$$M_n = -\alpha_{nm} / \alpha_{nn} = -\delta_{mn} / \alpha_{nn}$$

or

$$M_n = \delta_{mn} \times \text{constant}$$

In the same way, for the influence line of the bending moment at b, M_b, choose a main system in which $M_b = 0$ by introducing a hinge in the beam at b, the elastic line of this main system due to M_b gives the form of the required influence line of M_b (Fig III-5B)

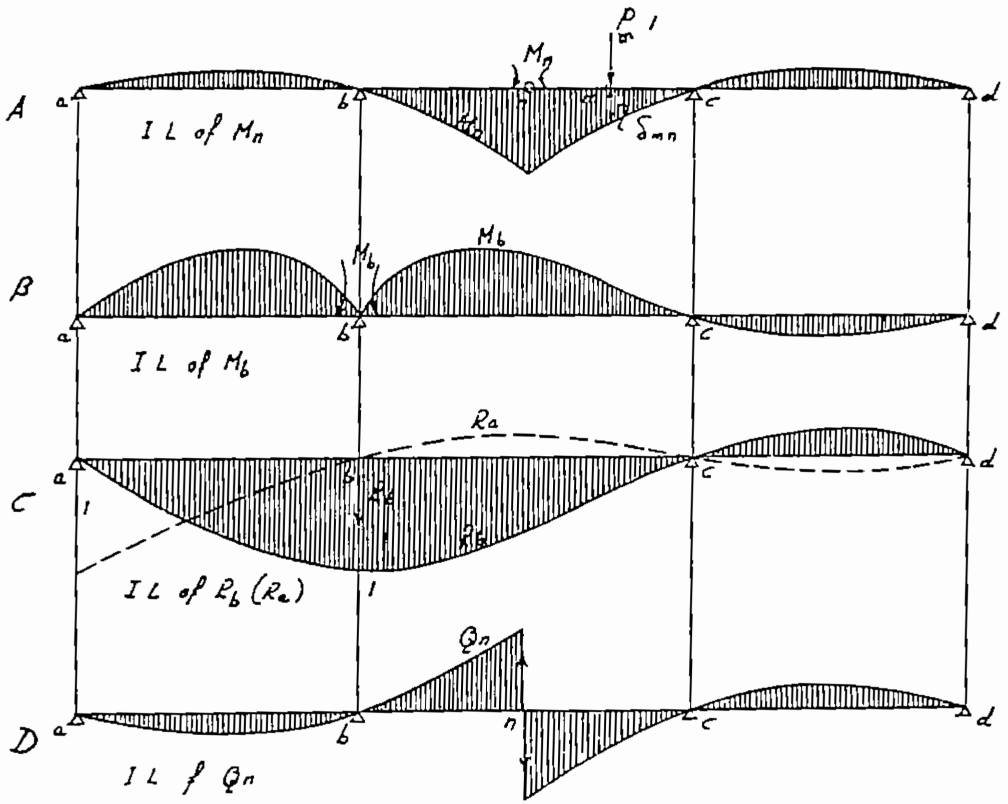


Fig III-5 Representation of influence lines as elastic lines

The same principle can be used for determining the form of the influence line of the reactions, that is, for the influence line of the reaction R_b (or R_a) choose a main system in which R_b (or R_a) = 0, this can be done by removing the support at b (or a), the elastic line of the chosen main system due to R_b (or R_a) as a load gives the form of the influence line of R_b (or R_a) (Fig III-5C)

The form of the influence line of the shearing force at any section can be determined by cutting the continuous beam at the section and applying two equal and opposite forces (Fig III-5D)

This method is generally not used for determining the ordinates of the influence lines but as check for their form

5) Influence Lines for Bending Moments and Shearing Forces

The ordinates of the influence lines of the bending moments are generally determined from the ordinates of the B M D due to P = 1 acting at the different sections, but the influence lines of the shearing forces in all sections of any span can be done in one process as follows (Fig III-6)

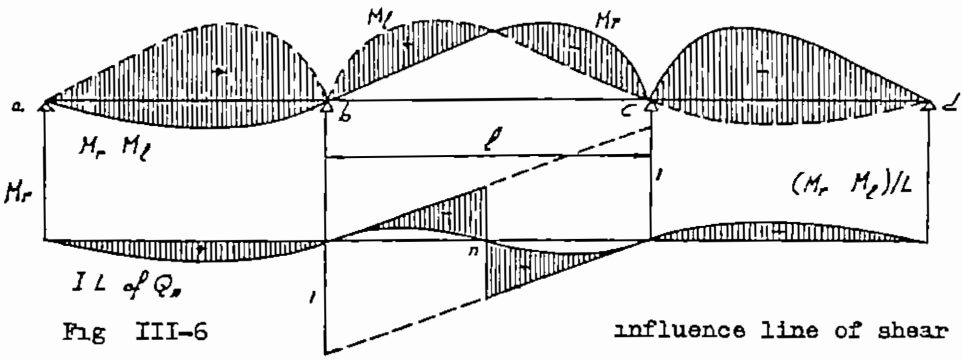


Fig III-6 influence line of shear

The shearing force at any section is given by

$$Q = Q_0 + \frac{M_r - M_l}{l}$$

The influence line of Q_0 for any section in span bc is given by 2 triangles bounded by the two parallel lines with ordinates equal to 1 at b and c It varies from section to section The effect of $(M_r - M_l)/l$ is given by the curve shown in figure III-6, whose ordinates are constant for all sections in bc

6) Absolute Bending Moments and Shearing Forces

The maximum bending moments and shearing forces due to rolling loads will be as shown in figure III-7A & B Adding the B M^S and S F^S due to dead loads given in figure III-2G & H to those due to rolling loads, we get the abslute maximum diagrams given in figure III-7C & D

It has to be noted that for heavy live or rolling loads, negative field moments M_m at the middle of intermediate spans are liable to take place, in which case, the necessary top steel reinforcement must be arranged

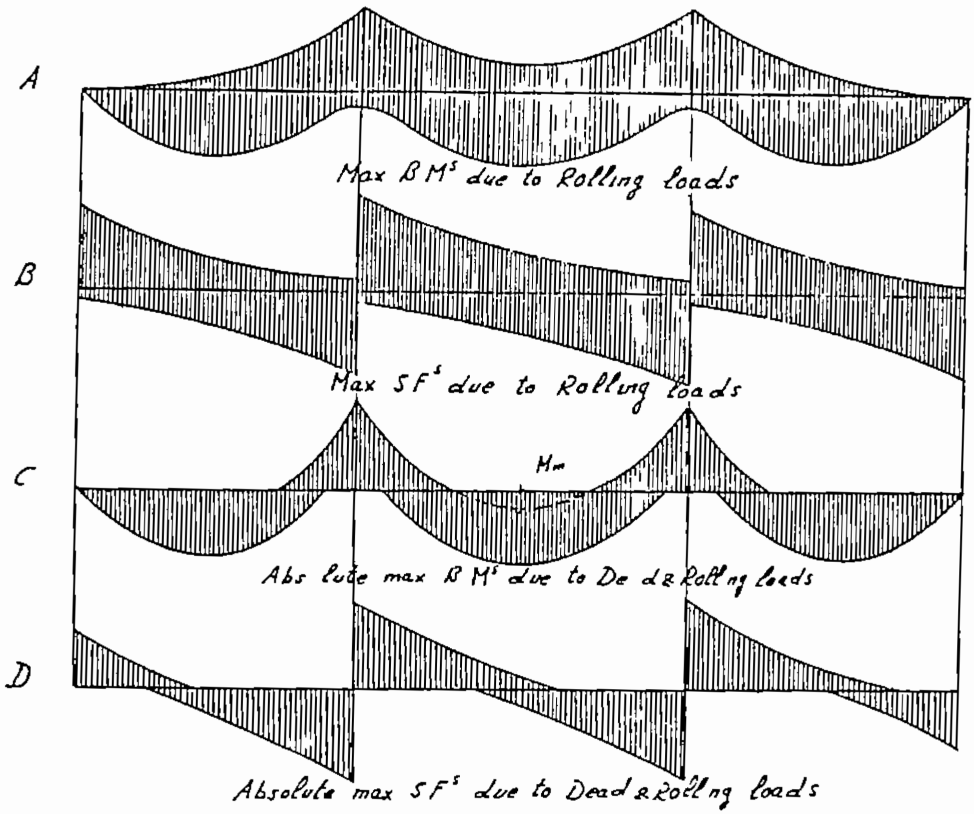


Fig III-7 Absolute maximum bending moments and shearing forces

Recommendations

1) The arrangement of the tension reinforcement is to be done according to the tension line shifted a distance d from the M/y_{CT} - line as shown in figure III-8

2) The tension reinforcement over the supports is not to be concentrated in the rib, a better distribution of the cracks will be attained if a part of the reinforcement is distributed in the flange, if any, as shown in section B-B of figure III-8

3) The maximum shearing force occurs immediately adjacent to the supports but the maximum shear stress to be considered in the design is that at a section d from the face of the support because adjacent to the supports the additional local stresses caused by reactions

counteract crack formation For this reason it is generally recommended in cases where bent bars are arranged to resist diagonal tension , to make the interior bent at a convenient distance from the face of the support so that the top horizontal part of the bent bar can share effectively in resisting the connecting moment (Refer to figure III-8 in which the first bent at the intermediate support is arranged at the top end of a line drawn from the face of the support and making 60° with the horizontal)

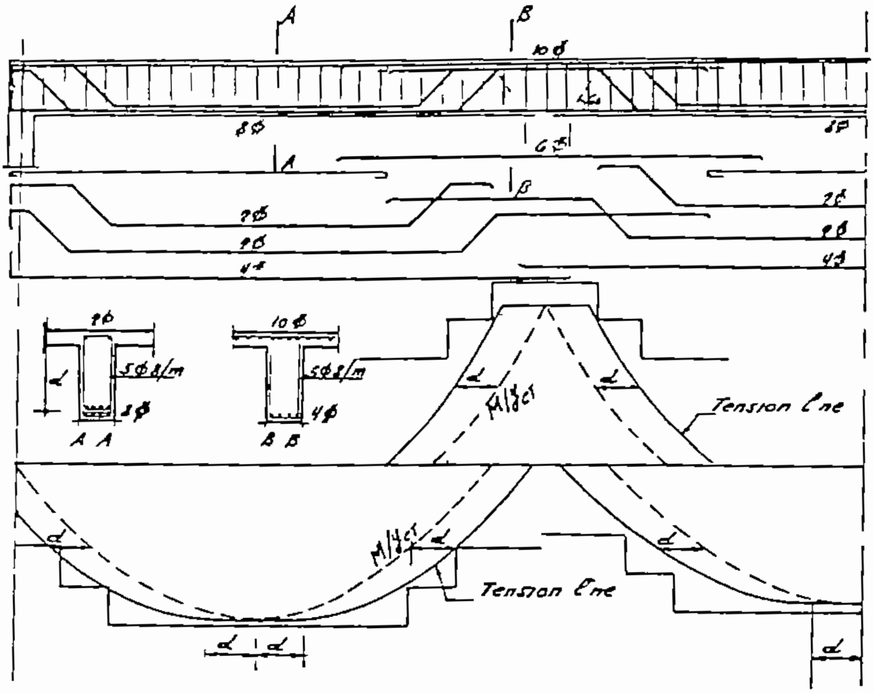


Fig III-8 Details of reinforcements of a continuous girder

4) Some new tests have shown that the best distribution and minimum width of cracks can be attained by using inclined (or vertical) stirrups at zones of high shear stresses as was shown in figure II-6 Accordingly the use of straight bars only for reinforcing continuous (or simple) beams may be the convenient solution Fig III-9

5) The slope of the effective haunch must not be more then 3 1, otherwise , the principal normal compressive stress σ_1 parallel to the outer surface of the haunch is eccessive relative to the

horizontal normal stress σ_c

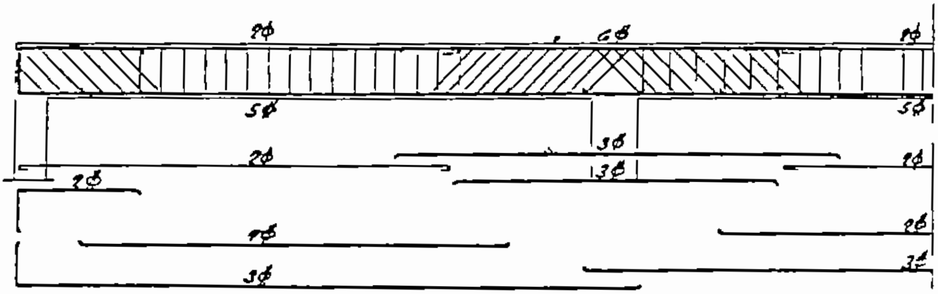


Fig III-9 Use of straight steel bars and inclined stirrups in continuous girders

According to figure III-10, we have

$$(\sigma_1 \, ds \cos\alpha) \cos\alpha = \sigma_c \, ds$$

So that

$$\sigma_1 = \sigma_c / \cos^2\alpha$$

For a slope 1 1 $\cos^2\alpha = \frac{1}{2}$ and $\sigma_1 = 2 \, \sigma_c$ not allowed !

" 3 1 $\cos^2\alpha = 9/10$ $\sigma_1 = 1.11 \, \sigma_c$ accepted

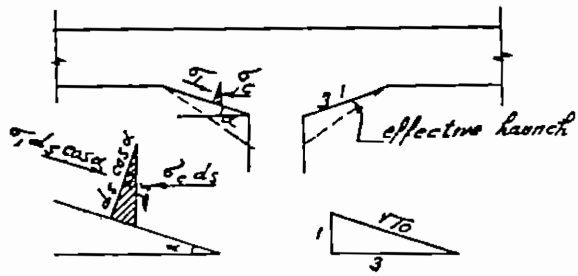


Fig III-10 Principal compressive stresses in haunches

6) The maximum moment that can be resisted by a section is that which causes a stress in tension steel equal to its yield stress so that if in a continuous beam, the moment in any of the critical sections-sections of maximum connecting and field moments - is bigger than the mentioned maximum value, the increase will be resisted by the other critical sections on condition that the equilibrium of the beam is maintained A statically indeterminate beam remains in equilibrium until the stress in steel of all critical sections (max three in number) reaches the yield stress at which moment collapse is liable to

take place

Accordingly, a redistribution of the bending moments determined according to the theory of elasticity is possible any of the critical values may be changed within a range of ± 16% of M_o (M_o = the max B M of the simple beam) on condition that the tension reinforcement is not chosen smaller than half the value required according to the theory of elasticity Fig III-11

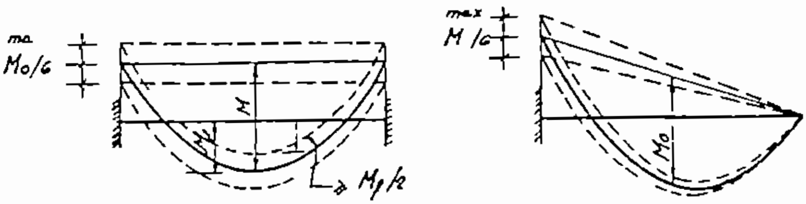


Fig III-11 Limits of redistribution of bending moments

In order to avoid visible cracks in the lower surface of beams (and slabs) and to have convenient depths - in case of beams of small breadth - and reinforcements it is generally advantageous to reduce the connecting moments and to increase the field moments by the corresponding values

Example

A continuous beam of 2 equal spans is subject to a dead load g = 1 t/m and a live load p = 1 t/m Determine the extreme values of the bending moments for the minimum posible design connecting moment The span of the beam is 8 m

Solution (Fig III-12)

Connecting moment for one span only loaded by g = 1 t/m

$_b = gl^2/16 = 1x8^2/16 = 4$ mt

Max connecting moment for the 2 pans loaded by g + p = 2 t/m

max $\bar{M}_b$ = 4 x 4 = 16 mt

Allowed reduction of connecting moment

$\bar{M}_b$ = 0 16 x 16 = 2 56 mt

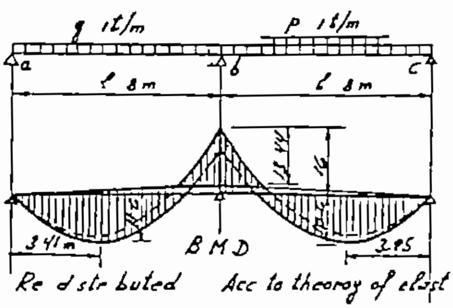


Fig III-12

Design connecting moment = 16 - 2 56 = <u>13 44</u> mt

For max field moment the two spans will be loaded by g and one span only by p , in which case

M_b = 3 x 4 = 12 mt

Outer reaction of loaded span R = 2 x 8/2 - 12/8 = 6 5t

Max field moment lies at point of zero shear which is assumed to be a distance x from the exterior support where

x = R/w = 6 5 /2 = 3 25 m

Max field moment M_f = $wx^2/2$ = $2 \times 3\ 25^2/2$ = 10 6 mt

After re-distribution, the connecting moment for one span loaded is given by

$M_b = 12 - \bar{M}_b$ = 12 - 2 56 = 9 44 mt

Outer reaction of loaded span R = 2 x 8/2 - 9 44/8 = 6 84 t

Max field moment lies at x = 6 82/2 = 3 41 m

Max field moment $M_f = w\,x^2/2 = 2 \times 3\ 41^2/2$ = <u>11 64</u> mt

We give in the following some examples of continuous reinforced concrete girders

The example shown in figure III-13 gives a continuous girder with a relatively short span between long spans In this case the short span will be subject to negative connecting moment over its whole length the reinforcements can be arranged as shown

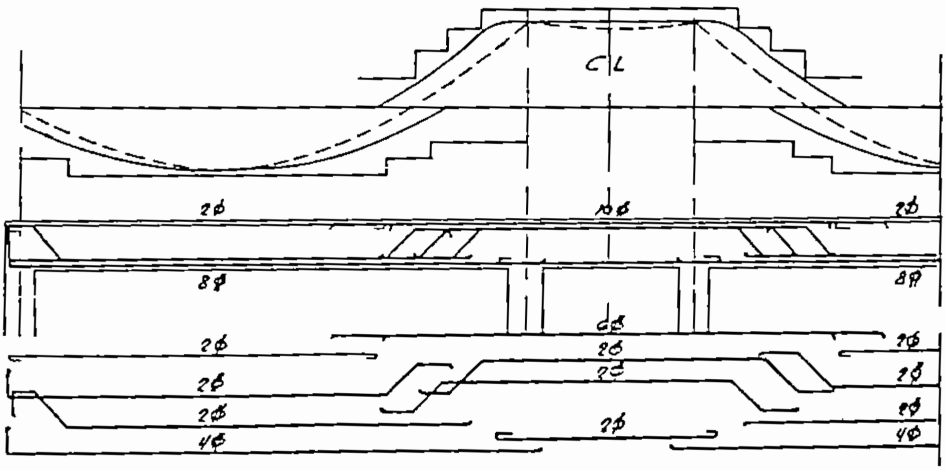


Fig III-13 Details of a continuous girder with a short span between long spans

If the shorter span is an outside one the details can be as shown in figure III-14

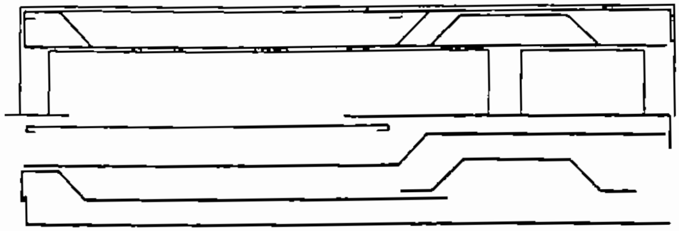


Fig III-14 Continuous girder with an outside short span

Fig III-15 shows a continous girder 15 ms span supporting a 10 m saw tooth roof in factory 135 at Helwan

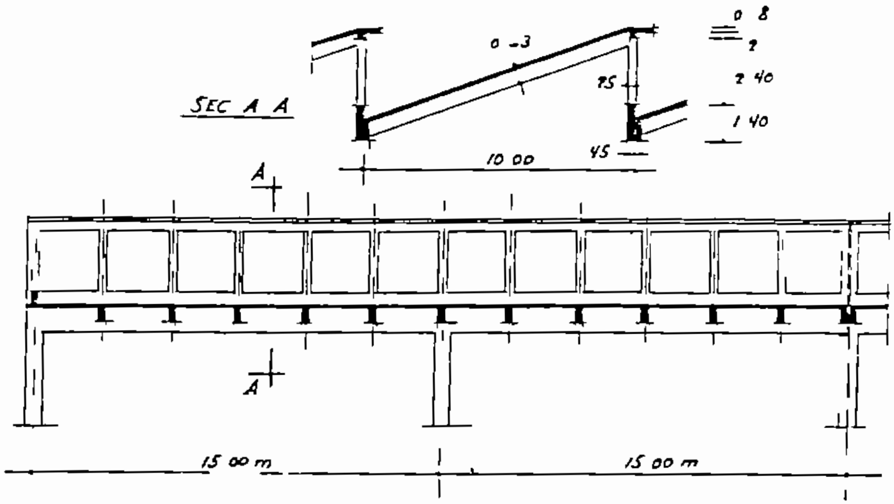


Fig III-15 Cross and longitudinal sections of factory 135 at Helwan

The details of reinforcements of the roof slab and secondary beams as well as the details of the main girder are shown in fig III-16

Due to constructional requirements , the bearing area of some continous girders on the supporting columns is relatively small - e g the bearing at the right support of the girder shown in figure III-15 if the cross reinforcements of the head of the column are not sufficient to resist the tension due to friction and splitting a crack which may endanger the safety of the whole girder is liable to be developed

The required cross reinforcement can be calculated as follows (fig III-17)

Assuming that the frictional force resulting from the possible re - sistance to shrinkage is $H = \mu$

in which

$\mu =$ 0 2 to 1 according to condition and type of bearing plates, and the cross tensile splitting force due to the concentration of A is $T \simeq \frac{A}{3}$

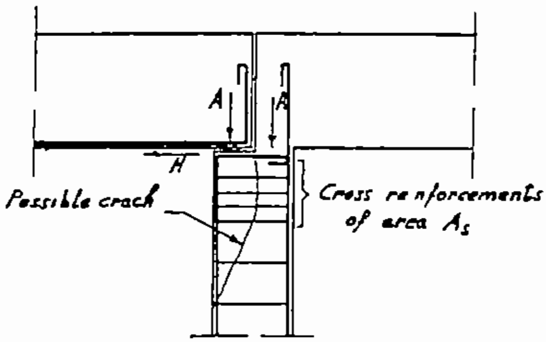


Fig III-17 Horizontal forces and cross reinforcements at the head of a supporting column

Then the total cross tensile force is H + T , and the required cross reinforcement is given by

$$A_s = (H + T) / \sigma_s = (0\ 55 \text{ to } 1\ 3) A/\sigma_s$$

Fig III-18 shows the roof slab and the main girders of an air conditioned wollen textile factory The air-conditioning ducts, 10 meters apart, have a trapezoidal section 1 5 ms clear depth their clear width is 1 00 m at top and 0 80 m at bottom The supporting columns are arranged below the ducts at distances of 24 ms

The walls, top and bottom slabs of the duct have been chosen 20 cms thick, so that it was possible to use the 1 90 ms high trape-zoidal section of the ducts as continuous main girders 24 ms span to support a ribbed roof slab 8 6 ms clear span

The ribbed slab is one way and 30 cms thick It is composed of a solid slab 6 cms thick and ribs arranged every 50 cms The ribs have a trapezoidal section 24 cms deep and 8/10 cms wide In order to distribute the load over the ribs and to assure their combined action two stiffening ribs having the same section as the main ribs are arranged in the longitudinal direction parallel to the ducts In order to have adequate space for resisting the connecting moments two, 40 cms long, solid parts are arranged adjacent to the ducts

The reinforcement of the solid slab is 5Φ 8/m normal to the ribs and 4 φ 6/m parallel to the ribs

The main ribs are reinforced by 2Φ 16 at the bottom in the span and at the top over the supports The stiffening ribs have 2φ 16 at bottom and 2 φ 13 at top

The reinforcement of the main girders is placed in the lower slab at the middle of the spans and in the top slab over the supports. The bent bars are placed in the walls In order to avoid splicing of the reinforcements special long bars of length $<$ 32 ms are used

In continuous girders of equal spans which are o common use in buildings the connecting moments over the supports are generally bigger than the field moments If the slabs are arranged at the upper fibers of the girder then the sections of bigger moments at the supports behave as rectangular and require big depths, whereas the sections of smaller moments at mid-span behave as T-sections and require relatively smaller depth

In case a girder of constant depth is required, it is recommended

a) to redistribute the bending moments by reducing the connecting moments by an amount $< M_o/6$ The reduction may however vary from support to support to give a simplified distribution of the reinforcements

b) to reduce the connecting moments over the supports according to a parabolic curve due to the distribution of the reaction over the width of the support and

c) to design the section of maximum connecting moment for the minimum depth and the maximum allowed compression reinforcement which must be $< \sim 0.4$ the tension reinforcement

IV- FRAMES

A frame is a structure in which the rigid connections between the girders and the supporting columns are utilized so that the internal forces due to the loads are resisted by the combind action of the girders and the columns i e the bending moment M_o is distributed on both of them (Fig IV-1)

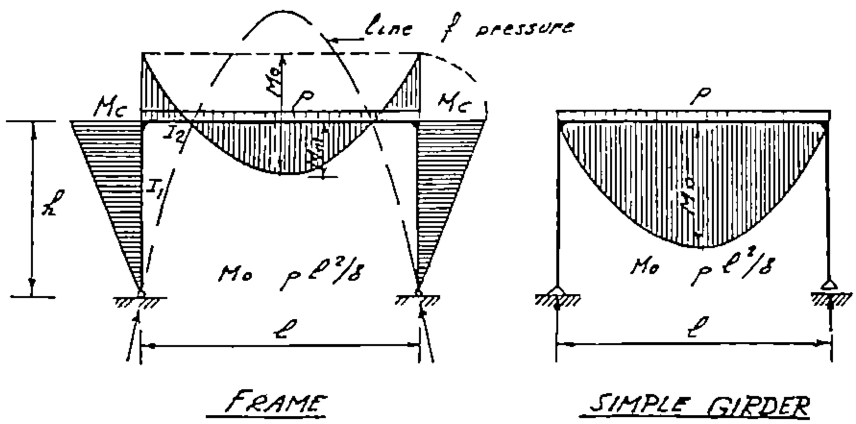


Fig IV-1 Frames and simple girders

In a simple girder we get for vertical loads vertical reactions while in a frame, vertical loads give inclined reactions

The magnitude of the bending moments resisted by the columns depends on the relative stiffness $\varkappa$ of the girder with respect to the columns where

$$\varkappa = \frac{h}{l} \quad \frac{I_2}{I_1}$$

For constant I_1 and I_2 and uniform load p, the connecting moment M_c of a rectangular frame is given by

$$M_c = - \frac{pl^2}{4(3 + 2\varkappa)} = - \frac{pl^2}{k_1}$$

The field moment is therefore

$$M_m = \frac{p l^2}{8} - M_c = + \frac{p l^2}{k_2}$$

We give in the following table the values of k_1 and k_2 for different h/l and I_2 / I_1 values

h/l	I_2/I_1	$\varkappa$	k_1	k_2
0 4		0 60	16 8	15 3
0 5	1 5	0 75	18 0	14 4
0 6		0 90	19 2	13 7
0 4		0 80	18 4	14 0
0 5	2 0	1 00	20 0	13 3
0 6		1 20	21 6	12 7
0 4		1 00	20 0	13 3
0 5	2 5	1 25	22 0	12 6
0 6		1 50	24 0	12 0

The table shows that the bigger the value of $\varkappa$ the smaller is the moment resisted by the columns This moment depends also on the variation of the moment of inertia in such a way that the portions of bigger moment of inertia resist bigger moments as shown in fig IV-2 which shows three two-hinged frames of the same span subject to uniform loads Case a shows a frame with a stiff girder and a slender column the bending moment $M_o = pl^2/8$ is mainly resisted by the girder and a small bending moment is resisted by the columns In case b the girder and columns are approximately of the same stiffness the

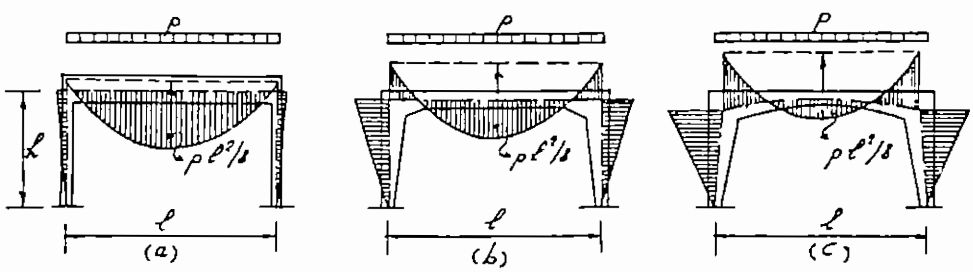


Fig IV-2 Distribution of moments between girder and columns field and corner connecting moments are nearly of the same order Case c shows a frame of relatively big moment of inertia at the corner, causing big connecting moments and small field moments

The minimum bending moments in a frame take place if its axis coincides on the line of pressure of the loads The axis of a reinforced concrete frame may be assumed as the line connecting the centers of gravity of the plain concrete sections the width of flange to be considered in T-sections is $B = 6t_s + b_o$, where t_s = thickness of flange and b_o = breadth of web The line of pressure gives the position of the resultant of the loads and reactions in any section (Refer to fig IV-1)

Accordingly if the form of the frame is not specified, the economic frames are those in which the axis coincides on the line of pressure of the loads i e for a single concentrated load choose a triangular frame , for a series of concentrated loads,, choose a polygonal frame and for a uniform load a parablic frame is most convenient (Fig IV-3)

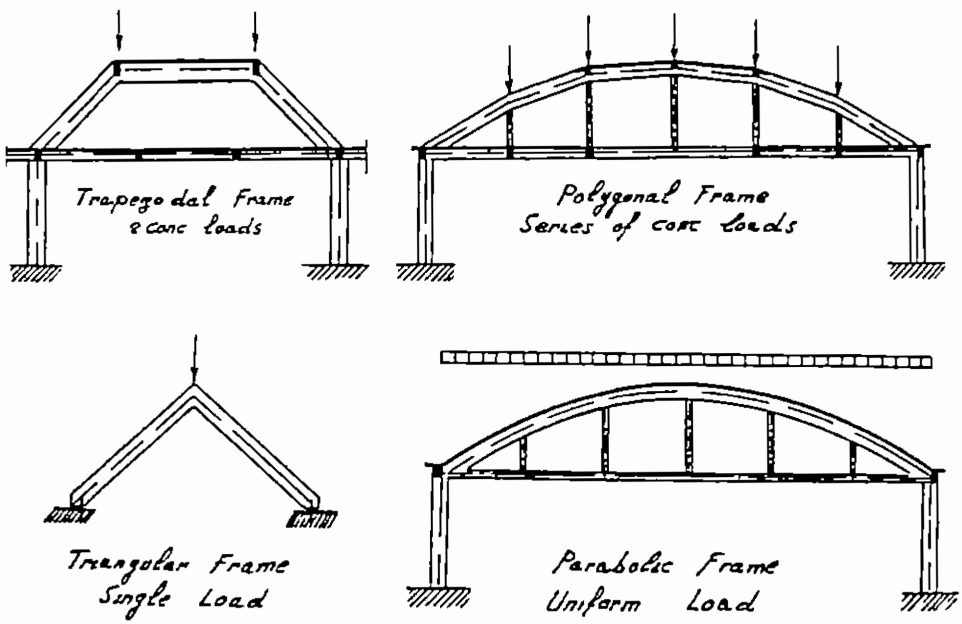


Fig IV-3 Line of pressure and axis of frame

The choice of the form of a frame is generally governed by the external and internal architectural considerations as well as the purpose for which it is used The sratical system depends on the conditions at the supports

Statically determinate three hinged frames are used on weak soils that may be subject to small horizontal or vertical movements of the bearing hinges (Fig IV-4)

Two hinged frames are generally used on medium soils as they are not very sensitive to displacements of the supports

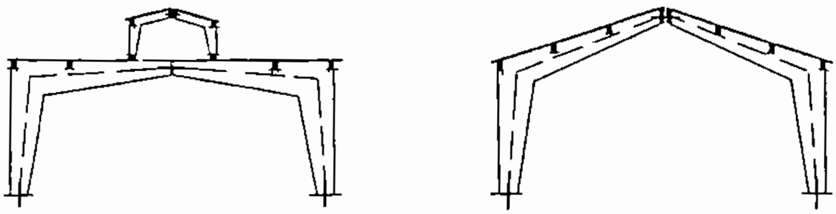


Fig IV-4 Three-hinged frames

Temperature changes and shrinkage cause moderate stresses that can be easily resisted Fig IV-5 shows some of the forms extensively used in reinforced concrete structures

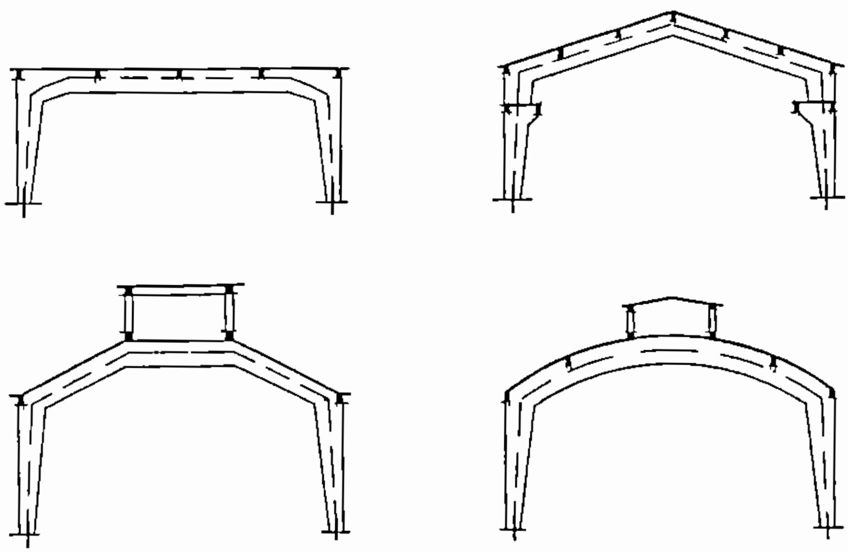


Fig IV-5 Two-hinged frames

On good firm soils , fixed frames may be used In this system, the intrnal stresses due to horizontal or vertical displacement of the supports as those due to temperature changes and shrinkage are relatively high and must be considered

Three Hinged Frames

A three hinged frame is statically determinate The external reactions can be determined from the conditions

$\sum X = 0$, $\sum Y = 0$, $\sum M = 0$ and $M_c = 0$

The example shown in fig IV-6 gives the reactions, the connecting moments at d and e and the line of pressure of a three hinged polygonal frame

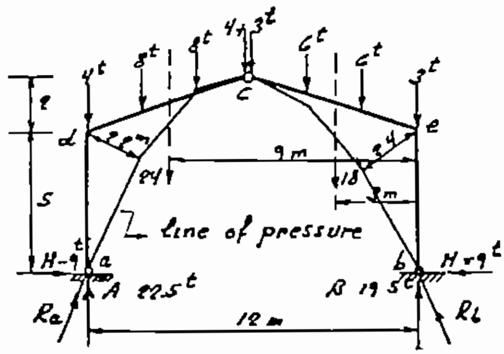
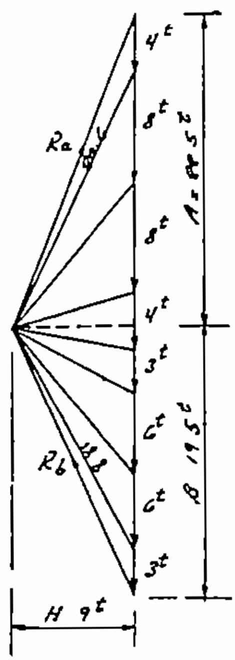


Fig IV-6 Example of a three-hinged frame



$\Sigma Y = 0$ or

$A + B = 3 \times 8 + 3 \times 6 = 42^t$

Moments @ b = 0 i e

$A \times 12 = 24 \times 9 + 18 \times 3$ or

$\underline{A = 22.5}$ ton

Therefore

$\underline{B = 19.5}$ ton

Moments about c = 0 i e

$22.5 \times 6 - H \times 7 - 24 \times 3 = 0$ or

$H = \underline{9.0}$ ton

Therefore $M_d = M_c = 9 \times 5 = \underline{45 \text{ mt}}$

The line of pressure is the link polygon of the loads and reactions with R_a as the first ray at a and R_b as the last ray at b it must pass through the hinge c The internal forces can be easily determined analytically or graphically

Two Hinged Frames

A two hinged frame is once statically indeterminate The statically indeterminate value H can be determined if we choose the statically determinate simple frame with a hinge at a and a roller at b as a main system (Fig IV-7) Due to the effect of the load p the roller

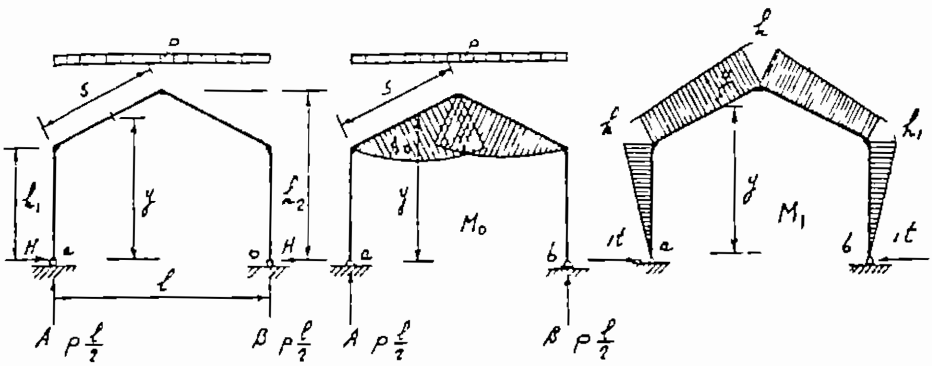


Fig IV-7 Analysis of a two-hinged frame

of the main system at b moves outwards a distance δ_o A horizontal force H acting inwards at b such that the horizontal displacement of point b equals zero gives the required statically indeterminate value So that for the given frame we have

Stat det main system	Simple frame with a roller at b
Condition of elasticity	Horiz displacement $\delta = 0$
Stat indet value	Horiz thrust H
Equation of elasticity	$\delta = 0 = \delta_o + H\,\delta_1$

in which

δ_o = horiz displacement of main system at b due to load temp

δ_1 = H = 1^t

According to theory of virtual work

$$1 \quad \delta_o = \int \frac{M_o\, M_1\, ds}{EI} + \int \frac{N_o\, N_1\, ds}{EA} + \int \frac{Q_o\, Q_1\, ds}{GA} + \alpha\, t\, l$$

$$1 \quad \delta_1 = \int \frac{M_1^2\, ds}{EI} + \int \frac{N_1^2\, ds}{EA} + \int \frac{Q_1^2\, ds}{GA'}$$

in which

M_o, N_o & Q_o are the bending moment, normal force and shearing force of the main system due to the given loads, and M_1, N_1 and Q_1 are the B M, N F and S F of the main system due to H = 1

$\alpha t l$ = horiz displ of main system at b due to a temperature increase of t if α is the coef of linear expansion

The displacements δ_o and δ_1 due to shear stresses are very small compared to those due to normal stresses and are generally neglected furthermore only in structures where the normal forces are relatively big and govern the design (e g in arches), the bending moments only are considered when calculating δ_o and δ_1 so that

$$1 \quad \delta_o = \int \frac{M_o M_1 ds}{EI} + \alpha t l$$

and

$$1 \quad \delta_1 = \int \frac{M_1^2 ds}{EI}$$

So that

$$H = -\frac{\delta_o}{\delta_1} = -\frac{\int \frac{M_o M_1 ds}{I} + E \alpha t l}{\int \frac{M_1^2 ds}{I}}$$

When using this method of virtual work, it has to be noted that H is to be chosen in the direction satisfying the condition of elasticity It can however be chosen in any direction (inwards or outwards), in this case the bending moments M_o and M_1 are to be drawn on the tension side taking the sense of H in consideration $\int M_1^2 ds/I$ is always positive and $\int M_o M_1 ds/I$ is to be assumed positive if the M_o and the corresponding M_1 diagrams are on the same side of the axis of the frame and negative if the two diagrams are on oppsite sides If the sign of H according to the previous equation is positive, then the assumed direction is correct and if it is negative, the assumed direction is to be reversed $\int M_o M_1 ds$ is the area of the M_o diagram along a certain element miltiplied by the ordinate of the corresponding M_1 diagram at the position of the center of gravity of the M_o diagram that is $\int M_o M_1 ds$ for the inclined member of length s (fig IV-7) is given by $A_o y$ whereas $\int M_1^2 ds$ for the same element is equal to the area of the trapezoidal M_1 - diaram multiplied by the ordinate of the same diagram lying at its center of gravity

Valu s f $\int M_j M_k \, ds$ s t rms g n n table

Nº	s	k k	k	k k_2	k k	k $\frac{k}{2}$	αs βs k	$\int j^2 ds$
1	j j	jk	$1/2\, jk$	$1/2\, j(k_1+k_2)$	0	$1/4\, jk$	$1/2\, jk$	j^2
2	j	$1/2\, jk$	$1/3\, jk$	$1/6\, j(k_1+2k_2)$	$1/6\, jk$	0	$1/6\, jk(1+\alpha)$	$1/3\, j^2$
3	j	$1/2\, jk$	$1/6\, jk$	$1/6\, j(2k_1+k_2)$	$1/6\, jk$	$1/4\, jk$	$1/6\, jk(1+\beta)$	$1/3\, j^2$
4	j_1 j_2	$1/2\, k(j_1+j_2)$	$1/6\, k(j_1+2j_2)$	$1/6[j_1(2k_1+k_2)$ $+j_2(k_1+2k_2)]$	$1/6\, k(j_1\ j_2)$	$1/4\, j_1 k$	$1/6\, k[j_1(1+\beta)$ $+j_2(1+\alpha)]$	$1/3(j_1^2+j_1 j_2$ $+j_2^2)$
5	j j	0	$1/6\, jk$	$1/6\, j(k_1\ k_2)$	$1/3\, jk$	$1/4\, jk$	$1/6\, jk(1-2\alpha)$	$1/3\, j^2$
6	j $\frac{j}{2}$	$1/4\, jk$	0	$1/4\, jk_1$	$1/4\, jk$	$1/4\, jk$	$1/12\, jk(1\ \alpha+2\beta)$	$1/4\, j^2$
7	$\frac{j}{2}$ j	$1/4\, jk$	$1/4\, jk$	$1/4\, jk_2$	$1/4\, jk$	$1/8\, jk$	$1/12\, jk(1+2\alpha-\beta)$	$1/4\, j^2$
8	$s/2$ j	$1/2\, jk$	$1/4\, jk$	$1/4\, j(k_1+k_2)$	0	$1/8\, jk$	$\frac{jk}{2\beta}(3\ 4\alpha^2)$	$1/3\, j^2$
9	αs δs j	$1/2\, jk$	$1/6\, jk(1+\gamma)$	$1/6\, j[k_1(1+\delta)$ $+k_2(1+\gamma)$	$1/6\, jk(1\ 2\gamma)$	$1/2\, jk(1+2\delta\gamma)$	$\frac{jk}{6\beta\gamma}(2\gamma\ \gamma^2\alpha^2)$ $\gamma \geq \alpha$	$1/3\, j^2$
10	j sq ∩	$2/3\, jk$	$1/3\, jk$	$1/3\, j(k_1+k_2)$	0	$1/6\, jk$	$1/3\, jk(1+\alpha\beta)$	$8/15\, j^2$
11	j j sq ∩	$1/3\, jk$	$1/6\, jk$	$1/6\, j(k_1+k_2)$	0	$1/12\, jk$	$1/6\, jk(1\ 2\alpha\beta)$	$1/5\, j^2$

Values of $\int M_j M_k \, ds = s \times$ terms given in table

Nr	s	k ▭ k	◺ k	k ▭ k_2	k ⋈ k	k ⋈ $\frac{k}{2}$	αs, β, k	$\int j^2 ds$
12	j, sq A	$2/3\, jk$	$1/4\, jk$	$1/12\, j(5k+3k_2)$	$1/6\, jk$	$7/24\, jk$	$1/12\, jk(5\ \alpha\ \alpha^2)$	$8/15\, j^2$
13	j, sq A	$2/3\, jk$	$5/12\, jk$	$1/12\, j(3k_1+5k_2)$	$1/6\, jk$	$1/24\, jk$	$1/12\, jk(5\ \beta\ \beta^2)$	$8/15\, j^2$
14	j, sq A	$1/3\, jk$	$1/4\, jk$	$1/12\, j(k+3k_2)$	$-1/6\, jk$	$-1/24\, jk$	$1/12\, jk(1+\alpha+\alpha^2)$	$1/5\, j^2$
15	j, sq A	$1/3\, jk$	$1/12\, jk$	$1/12\, j(3k+k_2)$	$1/6\, jk$	$5/24\, jk$	$1/12\, jk(1+\beta+\beta^2)$	$1/5\, j^2$
16	j, $s/2$, sq A	$1/6\, jk$	$1/6\, jk$	$1/6\, j\, k_2$	$1/6\, jk$	$-1/12\, jk$	$1/6\, jk[1\ \beta(1\ 2\alpha)]$	$1/5\, j^2$
17	j, $s/2$, sq A	$1/6\, jk$	0	$1/6\, j\, k_1$	$1/6\, jk$	$1/6\, jk$	$1/6\, jk[1-\alpha(1\ 2\beta)]$	$1/5\, j^2$
18	j_1, j_2, j_3, $s/2$, sq A	$1/6\, k(j_1+4j_2+j_3)$	$1/6\, k(2j_2+j_3)$	$1/6[j\, k_1+2j_2(k+k_2)+j_3 k_2]$	$1/6\, k(j_1\ j_3)$	$1/12\, k(2j+2j_2-j_3)$	$1/6\, k[j\ \beta+2j_2+j_3\ \alpha\ \alpha\beta(j_1-2j_2+j_3)]$	–
19	j, cu A	$1/4\, jk$	$1/5\, jk$	$1/20\, j(k_1+4k_2)$	$-3/20\, jk$	$-1/10\, jk$	$1/20\, jk(1+\alpha)(1+\alpha^2)$	$1/7\, j^2$
20	j, cu A	$1/4\, jk$	$1/20\, jk$	$1/20\, j(4k_1+k_2)$	$3/20\, jk$	$7/40\, jk$	$1/2\, jk(1+\beta)(1+\beta)$	$1/7\, j^2$
21	j, A	$1/4\, jk$	$2/15\, jk$	$1/6\, j(7k+8k_2)$	$-1/6\, jk$	$1/80\, jk$	$1/2\, jk(1+\alpha)(\frac{7}{3}\ \alpha^2)$	$8/15\, j^2$
22	j, A	$1/4\, jk$	$7/6\, jk$	$1/6\, j(8k+7k_2)$	$1/6\, jk$	$3/4\, jk$	$1/2\, jk(1+\beta)(\frac{7}{3}\ \beta^2)$	$8/105\, j^2$

Tables on page 44 & 45 give the values of $\int M_j M_k\, ds$ as a general form for $\int M_o M_1\, ds$ and $\int M_1^2\, ds$ along an element of length s for constant moment of inertia and different diagrams of M_j and M_k

The magnitude of the horizontal thrust H and the corresponding internal forces depend on the exact form of the axis of the frame and the moment of interia of the different sections both depend on the dimensions of the sections which in turn depend on the internal forces This means that the final dimensions of a frame depend on the perliminary dimensions for which the axis and the moments of inertia have been determined The solution can be assumed correct when the final results are the same as the preliminary Such a coincidence can be achieved if we proceed as follows (Fig IV-8)

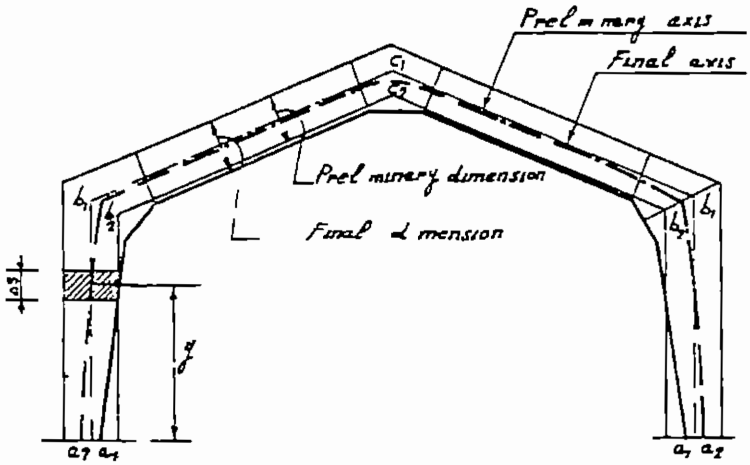


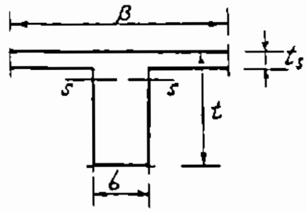
Fig IV-8 Axis of a two-hinged frame

1) For any frame of general layout assume preliminary dimensions and axis based on previous experience or similar structures

2) Calculate the own weight of the different elements and determine the dead and live loads

3) Compute the moments of inertia of the elements of the frame for the assumed dimensions The moment of inertia of girders monolithically cast with the slabs may be calculated for a plain concrete T-section with breadth of flange $B = 6t_s + b_o$ Such values are given in the following table

Moment of Inertia of T-Sections



$$I_s = \mu B t^3$$

Values of $10^4 \mu$

b_o / B	t_s/t 05	10	15	20	25	30	35	40	50	55	60
05	97	109	111	111	112	115	122	132	169	196	231
06	110	125	129	129	129	132	137	147	181	207	241
07	122	140	145	146	146	148	152	161	193	218	251
08	133	154	161	162	162	163	167	175	205	229	260
09	143	167	176	178	178	182	182	189	217	240	270
10	154	179	190	192	192	193	196	202	228	250	279
11	164	192	203	206	207	207	209	215	240	260	288
12	173	204	216	220	221	221	223	227	251	271	298
13	182	215	229	233	234	234	236	240	262	281	307
14	191	226	241	246	247	247	248	252	272	290	316
15	200	236	252	258	260	260	261	264	283	300	324
16	209	245	263	270	272	272	273	276	293	310	333
17	217	255	273	282	284	284	285	287	304	319	342
18	225	265	284	293	296	296	[illegible]	298	314	329	350
19	234	274	295	304	307	308	307	309	324	338	359
20	242	283	304	314	318	319	319	320	333	347	367
22	258	301	323	334	339	340	340	341	353	365	384
24	275	318	342	354	359	360	360	361	371	382	400
26	291	334	360	373	378	380	380	381	389	399	417
28	306	350	376	390	397	399	399	400	407	416	431
30	320	366	392	407	415	417	418	418	424	432	446
32	336	380	408	424	432	435	435	435	441	448	461
34	352	396	424	440	448	452	452	452	457	464	[illegible]
36	367	410	438	455	464	468	468	469	473	479	490
38	382	426	453	470	480	484	485	485	488	497	504
40	397	441	468	485	495	499	500	500	503	508	517
42	412	454	482	499	509	514	515	515	518	522	530
44	427	468	496	513	523	528	530	530	532	536	544
46	441	482	509	527	537	542	544	544	546	549	557
48	456	496	523	540	551	556	558	558	560	563	[illegible]
50	470	509	533	553	564	569	571	572	573	576	582
55	505	544	567	585	596	601	604	604	605	607	612
60	544	575	599	616	626	631	634	635	636	637	641
65	581	609	630	645	655	660	663	664	664	665	668
70	616	642	660	674	683	688	691	691	692	692	695
75	652	675	691	702	709	714	717	718	718	718	720
80	689	706	720	729	736	740	742	743	743	743	744
90	761	770	779	782	786	788	789	790	790	790	791
1 0	833	833	833	833	833	833	833	833	833	8 3	833

4) Calculated the statically indeterminate value and the corresponding internal forces The results of some simple orms of frames are given at the end of this chapter they simplify these calculations

5) Determine the dimensions required to resist the calculated internal forces, which may be modified to suit the expected final form of the frame

6) For the new dimensions draw the new axis and calculate the moment of inertia of the different sections In this stage, it is recommended to divide the frame to a convenient number of strips of length Δs and to determine H from the relation

$$H = - \frac{\sum \frac{M_o M_1 \Delta s}{I} + E \alpha t l}{\sum \frac{M_1^2 \Delta s}{I}} = - \frac{\sum \frac{M_o y \Delta s}{I} + E \alpha t l}{\sum \frac{y^2 \Delta s}{I}}$$

The calculations may be put in table form as follows

Element No	Δs	b orB	t	I	y	y^2	$y^2 \Delta s/I$	M_o	$M_o y \Delta s/I$
1									
2									
3									
4									
							$\Sigma =$		$\Sigma =$

7) The calculations are to be repeated until the final dimensions are the same as the preliminary After some experience each step can be done once only

In big frames the real axis and the variation of the moment of inertia are to be considered in the calculations otherwise big errors are liable to take place as shown in fig IV-9 in which the corner moments will be increased by values that may amount to 25% in which case the field moments are decreased by about 20% due to the increase of I towards the corners (Fig IV-9)

The steps of the design will be shown in the following simple example

It is required to design the main supporting element of a hall which is to be covered by a reinforced concrete flat roof, if it is 20 ms wide, 40 ms long and 70 ms clear height

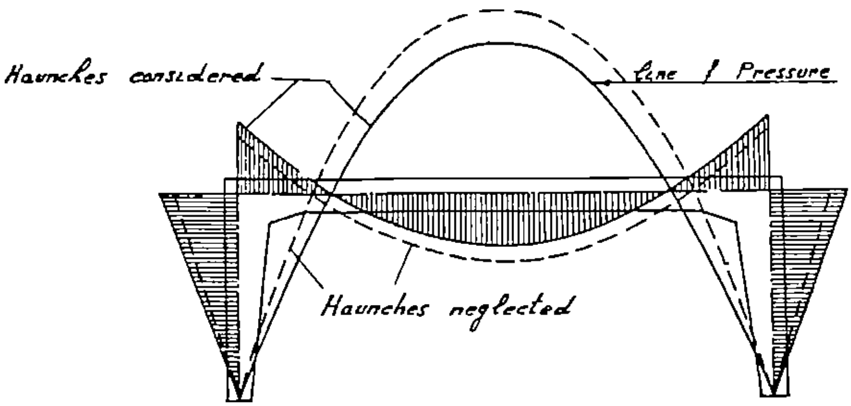


Fig IV-9 Effect of haunches on internal forces

The main supporting element can be chosen as a rectangular frame, 20 ms span arranged every 5 0 ms In order to get a resonable slab ~ 10 cms thick, secondary beams 20 x 40 cms will be arranged every ~ 4 0 ms Accordingly, the general layout of the different supporting elements will be as shown in fig IV-10

As a first estimate assume the depth of the main girder of the frame ~ 1/14 of the span i e ~ 1 40 ms the column may also be assumed 1 40 m at the top and 0 6 m at the bottom The breadth of both girder and columns b_o can be chosen = 0 4 m For the preliminary calculations the load on the frame from the roof may be assumed as uniformly distributed

Slab load =	own weight	+	roof cover	+	live load		
=	250	+	150	+	100		= 500 kg/m²

Equivalent uniform load due to own weight of secondary beams

= 0 2 x 0 3 x 2500/4 = 150/4 = 38

Roof load = (500 + 38) 5 = 538 x 5 ≃ 2700 kg/m

Own weight = 0 4 x 1 3 x 2500 ≃ 1300

total = 4000

Breadth of flange of main girder $B = 6\ t_s + b_o = 6 \times 10 + 40 = 100$ cm

Area of cross-section $A = 100 \times 10 + 40 \times 130 = 6200\ cm^2$

Arm of center of gravity from bottom $y_o = \frac{1000x135+40x130x65}{6200} = 70$ cms

The moment of inertia I_2 can be determined according to table given on page 47 hence

for $b_o/B = \frac{40}{100} = 0.4$ $t_s/t = 10/140 = 0.071$ $\mu = 0.0415$

Therefore

$I_2 = \mu B t^3 = 0.0415 \times 1.0 \times 1.4^3 = 0.114$ m⁴

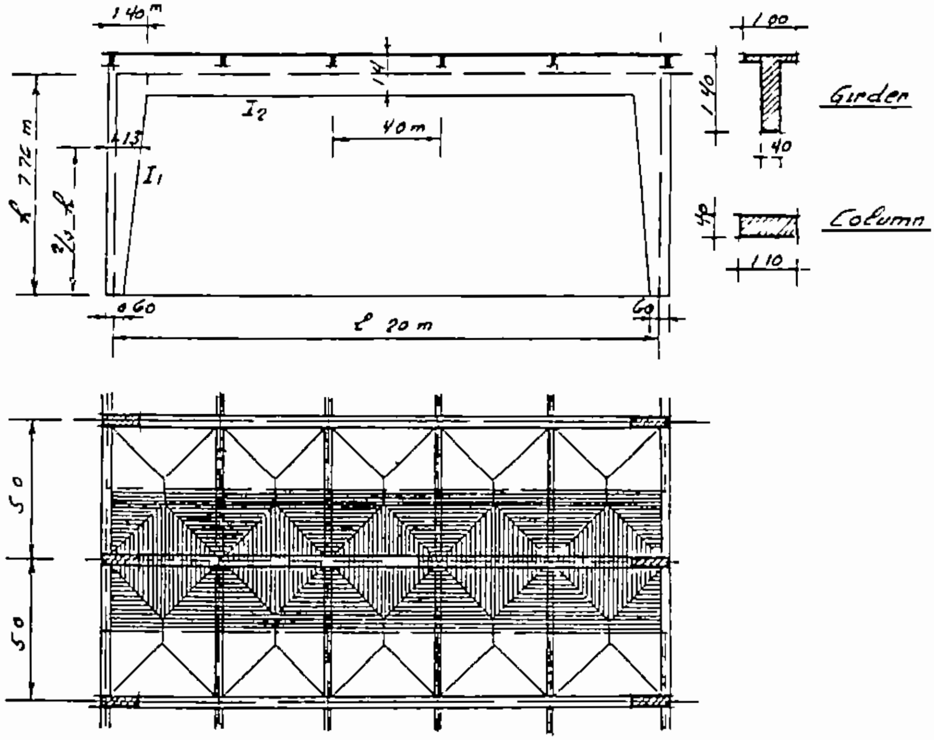


Fig IV-10 Example of a two-hinged frame

For the preliminary calculations one may assume the moment of inertia of the column as constant and compute it for an average section at 2/3 h i e 0.40 1.13^m , thus

$I_1 = b\, t^3/12 = 0.4 \times 1.13^3/12 = 0.048$ m⁴

The relative stiffness is therefore given by

$$\varkappa = \frac{h}{l} \frac{I_2}{I_1} = \frac{7.76}{20} \times \frac{0.114}{0.048} = 0.925$$

The connecting moment M_c is therefore given by

$$M_c = -\frac{p\,l^2}{4(3+2\varkappa)} = -\frac{p\,l^2}{4(3+2x0\;925)} = -\frac{pl^2}{19\;4} = -\frac{4x20^2}{19\;4} = -82\;5\text{ mt}$$

Due to the increase of the depth of the columns towards the upper corner M_c will be increased by say 10% Such an increase is however allowed due to the possible redistribution of the maximum moments Hence $M_c \approx 1\;1 \times 82\;5 \approx 90$ mt

The horizontal thrust H is therefore given by

$$H = M_c/h = \frac{90}{7\;76} = 11\;6 \text{ tons}$$

The field moment M_m is

$$M_m = \frac{p\,l^2}{8} - M_c = \frac{4x20^2}{8} - 90 = 110 \text{ mt}$$

The final check will be done for the dimensions computed to resist the following internal forces

1) Middle section of girder $M_m = 110$ mt $N = 11\;6^t$ (comp)
2) Sections at sup $M_c = 90$ mt $N = 11\;6^t$
3) Top section of column $M_c = 90$ mt $N = 4x10 = 40^t$ comp

The chosen dimensions are just sufficient

For the final calculations of the internal forces, the dimensions, the reinforcements the stresses etc will be done for the real axis taking the varition of the moment of inertia in consideration M_o is to be calculated for the direct uniform load from the slab plus its own weight and the concentrated reactions of the secondary beams (Fig IV-10&11) It is preferable to divide the frame into strips of

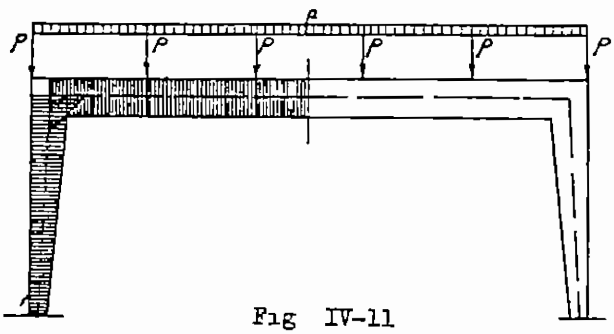


Fig IV-11

convenient length and to tabulate the results as shown on page 48

The direct uniform load is given by

p = 0 5 x 4 x 500 + 0 4 x 1 3 x 2500 = 1000 + 1300 = 2300 kg/m

The average load breadth on the secondary beams is $\frac{1+5}{2}\,\frac{4}{5}$ = 2 4 ms

So that the concentrated loads are given by

P = (2 4x500 + 0 2x0 3x2500)5 = (1200+150)5 = 1350 x 5 = 6750 kgs

The moments of inertia of the girder and columns at the corner can be computed for the enlarged section shown in fig IV-11

Assuming that the required tension reinforcement at the middle of the girder is 10 φ 25 and at the corner is 7 φ 25 then , the typical details can be done as shown in fig IV-12

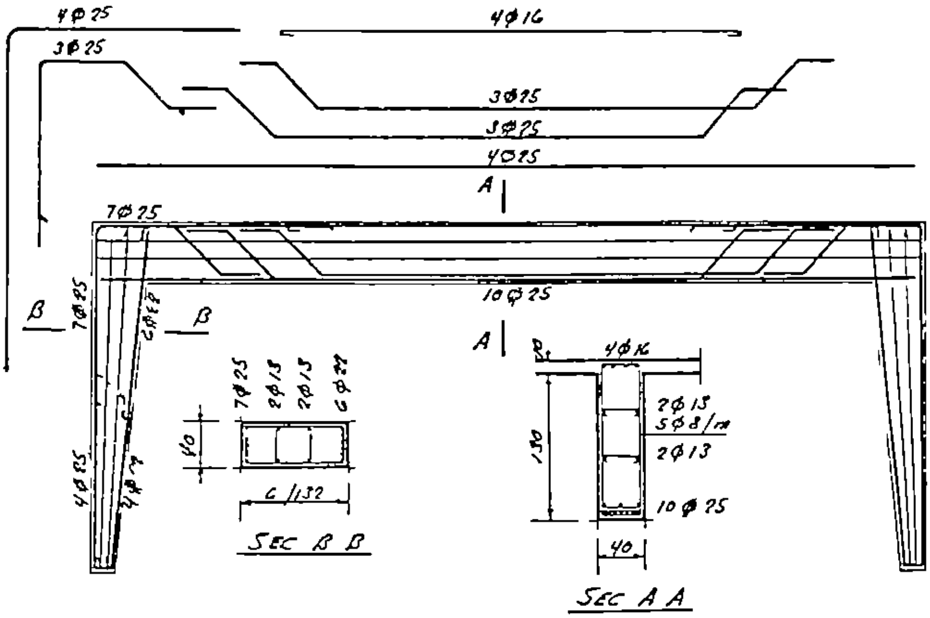


Fig IV-12 Reinforcements of a two-hinged frame

It is recommended to resist the corner moment of the frame by the tension reinforcements of the column and not to introduce the bent bars of the girder in the columns It is also not necessary to bend the tension reinforcements in the columns - shown dotted - because the normal and diagonal tensile stresses are generally low due to the existance of big normal compressive forces moreover such an arrangement causes undesirable movements which may loosen the reinforcement during concreting operations

Shrinkage reinforcements ϕ10 mm@ 30 cms or ϕ13 mm @40 cms are essential to prevent the formation of vertical craks between the stirrups

The stirrups at the hinges in a height equal to the depth of the column foot are to be doubled* to resit the tensile horizontal splitting force due to the concentration of the reaction in the hinge

We give in the following , examples of some frames used as main supporting elements in relatively big halls

1) The rectangular frame of the main factory hall of the Paints and Chemical Industries at Matariah , shown in figure IV-13

The frames are 24 ms span spaced every 4 ms and support a saw tooth roof in the form shown in figure VII-16 In order to have a convenient slope for the roof slab and sufficient height for the windows the depth of the main girder was chosen 2 0 ms which is much bigger than 1/16 of the span Such a choice led to relatively thin columns due to small connecting moments

2) The main frames supporting the roof of the main studio at the T V building Cairo, (fig IV-14) the span and spacing are 36 0 and 4 9 ms respectively The columns of the frames were designed to support the vertical reactions of a simple roof steel truss for this reason they were 0 5 x 1 6 m only and no special provisions were made to resist big horizontal forces After executing the cast in place plain concrete mechanical pile foundations for the previous condition it was decided to cover the studio by a reinforced concrete roof The space available for the main girders was 3 40 ms which is the height between the upper two floors It was also required to make the necessary provisions for constructing a suspended ceiling at the level of the lower floor to be used for lighting mounting and control purposes Provisions for big air conditioning ducts were specified

Due to these special conditions, the girders were assumed as partially fixed to the columns The connecting moments were limited by the maximum moment of resistance of the columns and the maximum moment and horizontal forces that can be resisted by the existing pile foundations

The span being relatively big all possible provisions were taken to

---

* The determination of the required area will be given later

reduce the dead loads of the roof slab and the own weight of the main girders For this purpose the main girders were chosen such that they give maximum resistance and minimum weight by choosing a web 25 cms thick the enlarged width at the bottom was necessary to have sufficient space for the tension steel The openings at the middle of the span are arranged for the air-conditioning ducts and to reduce the internal forces due to the own weight of the girder The big haunches between the web and the roof slab were necessary to resist the compressive stresses of the girder Due to the big moment of inertia of the roof slab at the girders its field moments were very small and it was possible to construct it as one way slab 8 cms thick In order to prevent the lateral buckling of the web vertical stiffeners were arranged The breadth of the main girder is increased on the two sides at the zones of high shear stresses and to have a smooth gradual transition from column to girder The lower connecting cross-beams are arranged to simplify the construction of the required suspended ceiling

The fixing moments of the girder are resisted by the tension column reinforcement these being 20 Φ 25 mm then (See fig IV-15)

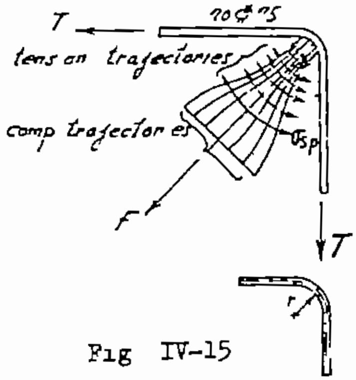


Fig IV-15

$A_s = 100\ cm^2 \qquad \sigma_s = 2\ t/cm^2$

$T = A_s\sigma_s = 100 \times 2 = 200\ t$

Resultant F is given by

$F = T\sqrt{2} = 200\sqrt{2} = 282\ t$

This value is relatively big and acts on a length = $1/4(2\pi r)$, the smaller the radius r the higher is the concentration of the force F and the bigger the splitting tensile stresses σ_{sp} In order to resist σ_{sp} the diagonal corner bars shown in fig IV-14 are arranged

2) Fig IV-16 shows the details of a polygonal frame used as the main supporting element of a factory at Helwan The arrangement of the reinforcements follows the same general simple principles stated before, that is, the tension in the outside fiber of the corners between the column and the girder is resisted by the reinforcements of the column and no reinforcements are introduced from the girder in the columns

The maximum field moment of this frame takes place at the crown causing maximum tensile stresses at the lower fiber of the section These tensile stresses are resisted by the main tension reinforcements c,d,e and f which must be sufficiently anchored in the compression zone with a minimum anchorage length of 40 Φ

Fig IV-17 shows that if the tension reinforcements are continuous over a sharp corner as shown in (a), failure of the lower cover will

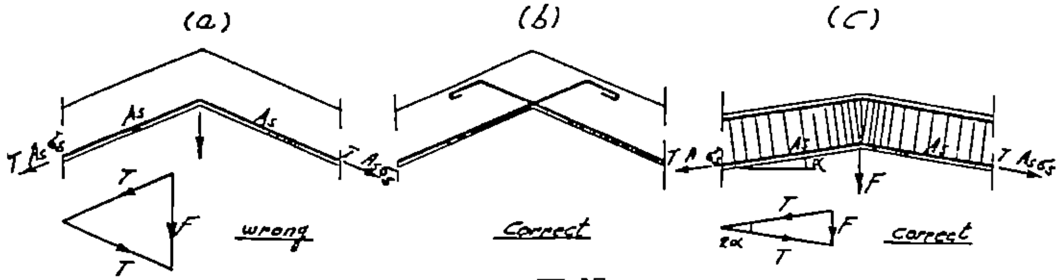


Fig IV-17

take place due to the action of the resultant force F If α is small as in (c), then F is small and can be resisted by additional stirrups placed at the crown of the girder Their area of cross-section A_{st} can be calculated as follows

$$F = 2\ T \sin\alpha = 2\ A_s\ \sigma_s \sin\alpha$$

If the allowable tensile stress in the stirrups is σ_{st} , then

$$A_{st}\ \sigma_{st} = 2\ A_s\ \sigma_s \sin\alpha$$

or

$$A_{st} = A_s\ 2\ \frac{\sigma_s}{\sigma_{st}} \sin\alpha$$

However, if the tension reinforcements change their direction on an arc of a circle with radius r , then every stirrup must be able to resist a tensile force F due to this change Assuming that $\sin\alpha \simeq \tan\alpha$, then , according to fig IV-18, we get

$F/T = s/r$ but $T = A_s \sigma_s$

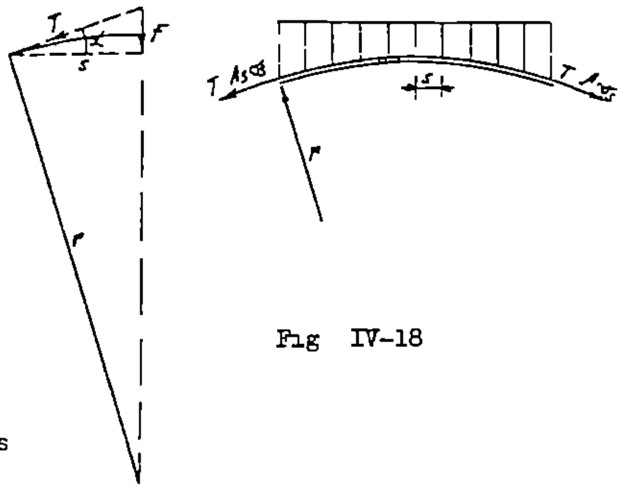


Fig IV-18

then

$$\Gamma = A_{st}\,\sigma_{st} = A_s\,\sigma_s\, s/r$$

and

$$A_{st} = \frac{A_s\, s}{r}\;\frac{\sigma_s}{\sigma_{st}}$$

In frames with sharp corners at positions of maximum field moment for example point c of figure IV-19, many designers arrange a haunch as that shown in figure IV-19, but as the moments at points c have nearl

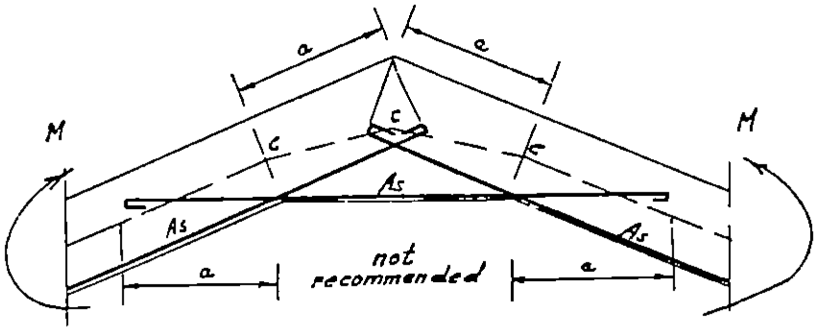


Fig IV-19

the same magnitude as that at c then the horizontal and the inclined reinforcements A_s and A_s will be of the same order and need sufficient anchorage lengths a as shown Such an arrangement includes a big waste in the reinforcements and therefore not recommended A simple corner as that shown in figure IV-17b is more convenient

However in corners where the internal compressive forces C give an outward resultant F, special stirrups of area A_{st} must be arranged to resist it Assuming the allowable stress of the stirrups σ_{st} , then according to figure IV-20, we get

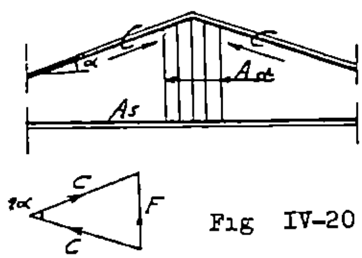


Fig IV-20

$$A_{st} = A_s \;\; 2\,\frac{\sigma_s}{\sigma_{st}}\,\sin\alpha$$

Two Hinged Frame with A Tie

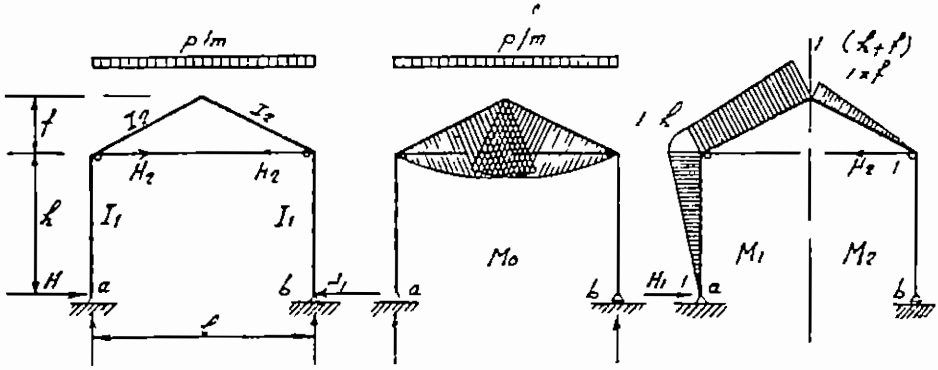


Fig IV-21 Analysis of a two-hinged frame with a tie

A two hinged frame with a tie as shown in figure IV-21 is twice statically indeterminate If the statically determinate simple frame with a roller at b and tie cut is chosen as main system, the conditions of elasticity will be

1) The horizontal displacement δ_1 of point b equals zero, or

$$\delta_1 = 0$$

2) The relative horizontal displacement of the two ends at the cut section of the tie δ_2 must be equal to the elongation of the tie under the effect of the load H_2, or

$$\delta_2 = - H_2 l / A_t E_t$$

in which A_t and E_t are the effective area of cross-section and modulus of elasticity of the tie If the tie is made of reinforced concrete and is subject to tensile stresses that cause cracks in the concrete then A_t and E_t are the area of cross-section and the modulus of elasticity of the steel in the tie The negative sign is introduced in the equation because H_2 acts in a direction opposite to δ_2

The statically indeterminate values are H_1 at the lower hinges and H_2 in the tie

The equations of elasticity are therefore

$$\delta_1 = 0 = \delta_{10} + H_1 \delta_{11} + H_2 \delta_{12}$$

$$\delta_2 = - H_2 l / A_t E_t = \delta_{20} + H_1 \delta_{21} + H_2 \delta_{22} \quad \text{or}$$

$$0 = \delta_{20} + H_1 \delta_{21} + H_2 (\delta_{22} + l / A_t E_t)$$

in which

$\delta_{10} = \int M_o M_1 \, ds/EI$

= the horizontal displacement of the main system at the level of the lower hinges due to load

$\delta_{20} = \int M_o M_2 \, ds/EI$

= the horiz displ of the main system at the level of the tie due to load

$\delta_{11} = \int M_1^2 \, ds/EI$

= the horiz displ of the main system at the level of the lower hinge due to $H_1 = 1$

$\delta_{22} = \int M_2^2 \, ds/EI$

= the horiz displ of the main system at the level of the tie due to $H_2 = 1$

$\delta_{12} = \delta_{21} = \int M_1 M_2 \, ds/EI$

= the horiz displ of the main system at the level of the lower hinges due to $H_2 = 1$

= the horiz displ of the main system at the level of the tie due to $H_1 = 1$

The values of the elastic displacements δ are very small and the magnitude of H_1 and H_2 may be subject to serious errors if the values of δ are not exactly calculated and if the equations of elasticity are solved by approximate methods (e g by the slide rule)

In a frame adceb hinged at a & b and with a rigid tie de, figure IV-22, the elastic deformation δ_2 of the tie is equal to zero and H_2 is much bigger than H_1 Moreover , if the elastic deformation of dc and ce is neglected, then c will be fixed in space relative to d and e In this manner, the frame adceb with the rigid tie de subject to symmetrical load can be considered as a continuous beam of spans ad = h, dc = ce = s and eb = h in which the two spans dc and ce are loaded The bending moments at corners d,c and e are negative and the magnitude of the maximum moments is much smaller than in a frame without tie Fig IV-22 shows the limiting cases of the horizontal thrusts and bending moments of a polygonal frame case a shows the bending moments and line of pressure of a frame without a tie whereas case b shows a frame with a rigid tie Bending moments of frames with elastic ties lie between the shown two limiting cases

Fig IV-23 shows the details of a frame with a tie The reinforcements of the tie must be carefully anchored to the corner of the frame, the anchorage length, measured beyond the point of intersection of column, girder and tie, must satisfy the requirements of the code (min 40 ϕ) In order to prevent the saging of the tie, hangers at convenient distances (3-4 ms) must be arranged

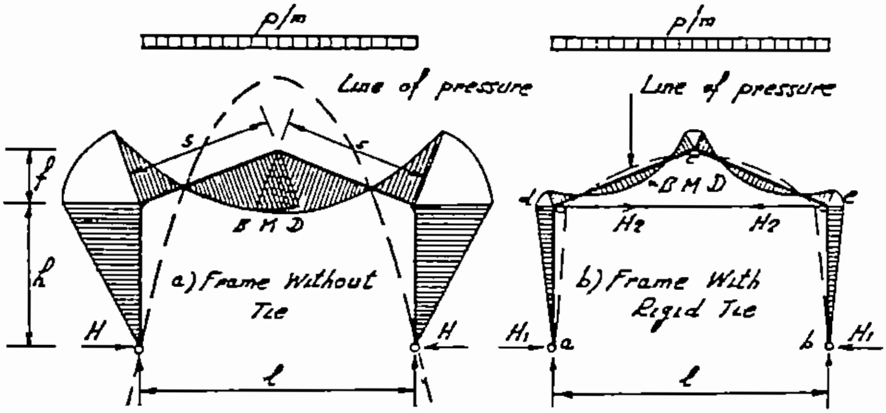


Fig IV-22 Effect of ties on two-hinged frames

Because of the small horizontal thrust at the lower hinges, the columns may be directly connected to their footing

A rigid tie at the top of the columns gives a better distribution of the internal forces in the columns and the girder although this does not necessaily mean a more economic solution because of the complicated form work and the big amount of steel in the tie and hangers It gives however a smaller horizontal thrust on the foundations

A frame without a tie is simpler and architecturally more acceptable than a frame with a tie and hangers

If the foundations of a frame cannot resist its horiz thrust a tie may be arranged at the bottom hinges to resist the thrust (Fig IV-24) In this case, the frame is once statically, indeterminate and the horizontal thrust H can be determined from the equation of elasticity

$$\delta = - H l / A_t E_t = \delta_o + H \delta_1$$

so that

$$H = - \frac{\delta_o}{\delta_1 + l / A_t E_t} = - \frac{\int \frac{M_o M_1 \, ds}{EI}}{\int \frac{M_1^2 \, ds}{EI} + \frac{l}{A_t E_t}}$$

For elastic ties at foundation level, H is smaller than that of two hinged frames without ties or with rigid ties, the result is that the corner moment is smaller and the field moment is bigger

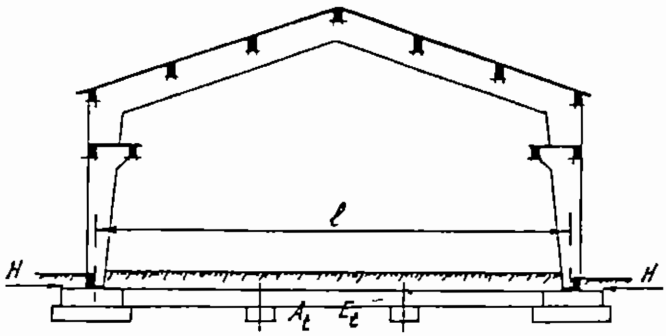


Fig IV-24 Two-hinged frame with a tie at foundation level

Fixed Frames

In a fixed frame (fig IV-25) the horizontal displacement the vertical dsiplacement and the angles of rotation at the points of supports a and b are equal to zero It is three times statically indeterminate The nature and sense of the statically indeterminate values statisfying the conditions of elasticity depend on the main system which may be a cantilever double cantilevers a simple beam, a three hinged frame etc The conditions of elasticity and the statically indeterminate values for three different main systems of a rectangular frame are shown in figure IV-26

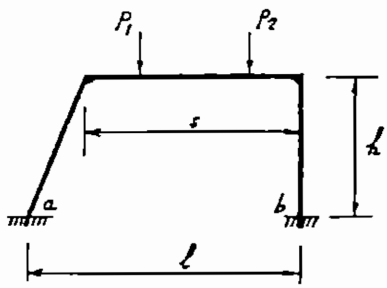


Fig IV-25 Fixed frame

The equations of elasticity in the three cases are

$$\delta_1 = 0 = \delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} + X_3 \delta_{13}$$

$$\delta_2 = 0 = \delta_{20} + X_1 \delta_{21} + X_2 \delta_{22} + X_3 \delta_{23}$$

$$\delta_3 = 0 = \delta_{30} + X_1 \delta_{31} + X_2 \delta_{32} + X_3 \delta_{33}$$

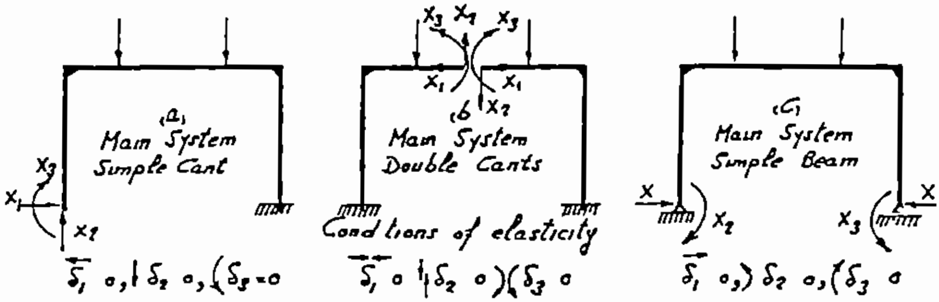


Fig IV-26 Different main systems of a fixed frame

In order to explain the problem, we discuss the solution using the simple cantilever shown in a as main system In this case, the displacements δ_1 , δ_2 and δ_3 with single indeces give the three components of the movement at the free end due to the loading namely

δ_1 = horizontal displacement,

δ_2 = vertical ' ,

δ_3 = angle of rotation

While in the displacements with double indeces, the first index indicates the sense and the second the cause, e g

δ_{10}= horiz displ of free end of main system due to load

δ_{22}= vert ' $X_2 = 1$

δ_{31}= angle of rot ' " ' " ' $X_1 = 1$ and so on

According to figure IV-27, the theory of vitual work gives

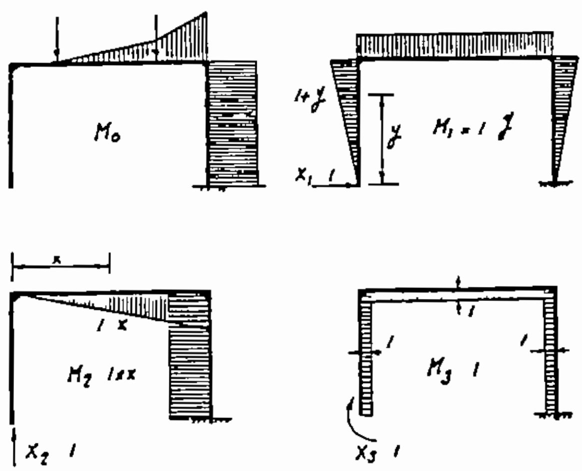


Fig IV-27 Analysis of a fixed frame

$$E\ \delta_{10} = \int M_o y\ ds/I\ , \quad E\ \delta_{20} = \int M_o x\ ds/I\ , \quad E\ \delta_{30} = \int M_o ds/I$$

$$E\ \delta_{11} = \int y^2\ ds/I\ , \quad E\ \delta_{22} = \int x^2\ ds/I\ , \quad E\ \delta_{33} = \int\ ds/I$$

$$E\delta_{12} = E\delta_{21} = \int xy\ ds/I\ , E\ \delta_{23} = E\delta_{32} = \int x\ ds/I\ , E\ \delta_{31} = E\ \delta_{13} = \int y\ ds/I$$

If the free end of the main system is connected to a rigid member with its other end at a point O which is chosen such that (Fig IV-28)

$$\int xy\ ds/I = 0\ , \quad \int x\ ds/I = 0 \quad \text{and} \quad \int y\ ds/I = 0$$

then, the displacements of the edge at O will be identical with those of the free end and if the statically indeterminate forces X_1 (= H) and X_2 (= V) and X_3 (= M) are assumed acting at point O (called the center of elasticity of the frame) then the displacements

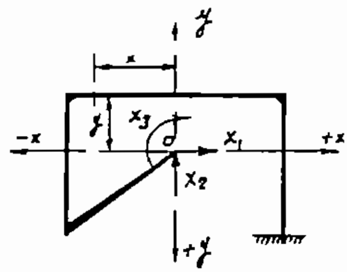


Fig IV-28

$$\delta_{12} = \delta_{21} = \frac{1}{E}\int xy\ ds/I\ ,$$

$$\delta_{13} = \delta_{31} = \frac{1}{E}\int y\ ds/I \text{ and}$$

$$\delta_{23} = \delta_{32} = \frac{1}{E}\int x\ ds/I$$ are all equal to zero and the equations of elasticity will be reduced to

$$0 = \delta_{10} + X_1\ \delta_{11} \quad \text{or} \quad X_1 = -\ \delta_{10} / \delta_{11}$$

$$0 = \delta_{20} + X_2\ \delta_{22} \quad \text{or} \quad X_2 = -\ \delta_{20} / \delta_{22}$$

$$0 = \delta_{30} + X_3\ \delta_{33} \quad \text{or} \quad X_3 = -\ \delta_{30} / \delta_{33}$$

So that

$$X_1 = H = -\frac{\int M_o y\ ds/I}{\int y^2\ ds/I}$$

$$X_2 = V = -\frac{\int M_o x\ ds/I}{\int x^2\ ds/I}$$

$$X_3 = M = -\frac{\int M_o\ ds/I}{\int ds\ /I}$$

The M_o M_1 (= 1 y), M_2 (= 1 x) and M_3 (due to X_3 = 1) diagrams drawn on the tension side are shown in their final form in fig IV-29

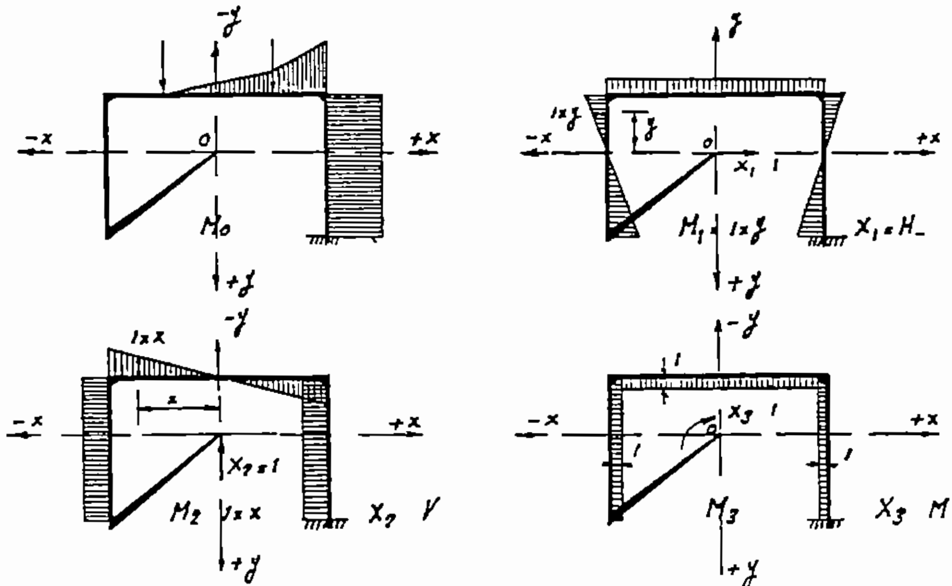


Fig IV-29

In this manner the determination of the statically indeterminate values is much simplified The bending moment in any section of the frame at x and y from the principal axes of elasticity passing through O is given by

$$M_{x,y} = M_o + X_1 M_1 + X_2 M_2 + X_3 M_3$$

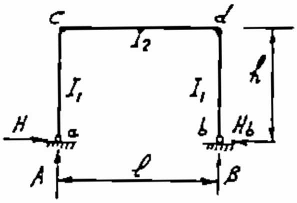
$$= M_o + X_1 y + X_2 x + X_3$$

$$= M_o + H y + V x + M$$

In order to simplify the determination of the internal forces, we give in the following, the reactions of 6 types of frames of extensive use in reinforced concrete, namely

1) Two hinged symmetrical rectangular frames
2) " frames with parabolic girder
3) " polygonal frames without & with ties
4) Fixed symmetrical rectangular frames
5) " " frames with parabolic girders
6) " polygonal frames

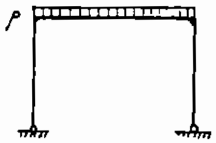
I- TWO HINGED SYMMETRICAL RECTANGULAR FRAMES



$\varkappa \quad \frac{I_2}{I_1} \quad \frac{h}{l}$ $\qquad\qquad \mu \quad 2\varkappa + 3$

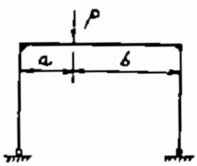
in case of unloaded columns

$M_c \quad -H_a \; h$ $\qquad\qquad M_d \quad -H_b \; h$



$H_a \quad H_b \quad \frac{p l^2}{4 h \mu}$

$A \quad B \quad pl/2$

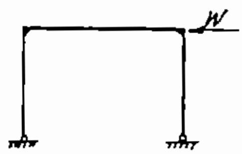


$H_a \quad H_b \quad \frac{3}{2} \, \frac{P a b}{h \, l \, \mu}$

$A \quad Pb/l$ $\qquad\qquad B \quad Pa/l$

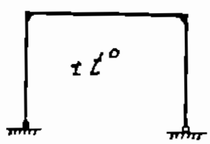
$H_a \quad \frac{wh}{8} \quad \frac{5\varkappa + 6}{\mu}$ $\qquad\qquad H_b \; - \frac{wh}{8} \quad \frac{11\varkappa + 18}{\mu}$

$A \quad B \quad wh^2/2l$



$H_a \quad -H_b \quad W/2$

$A \quad B \quad Wh/l$



Temperature change of $t°$

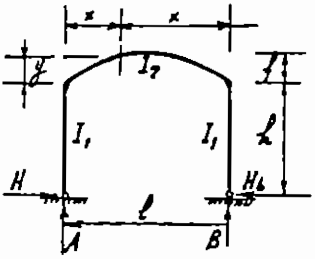
$H_a \quad H_b \quad \frac{3 \alpha t}{\mu} \quad \frac{E I_2}{h^2}$ $\qquad\qquad A \quad B \quad 0$

α coeff of linear expansion $\quad = 0.00001$

Horizontal displacement of support δ

$H_a \quad H_b \quad - \frac{3 \delta E I_2}{h^2 \, l \, \mu}$ $\qquad\qquad A \quad B \; = 0$

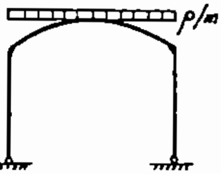
2_TWO HINGED SYMMETRICAL FRAME WITH PARABOLIC GIRDER



Equation of Parabola $y = 4f\xi\xi'$

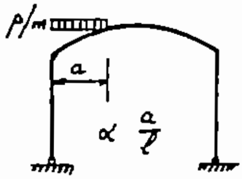
$\xi = x/l \qquad \xi' = x'/l \qquad \gamma = f/h$

$\alpha = \frac{I_2}{I_1} \cdot \frac{h}{l} \qquad \mu = 5(2\alpha+3)+4\gamma(5+2\gamma)$



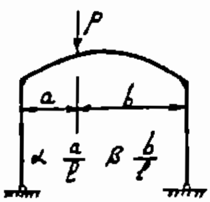
$H_a = H_b = \frac{pl^2}{4h} \cdot \frac{5+4\gamma}{\mu}$

$A = B = pl/2$



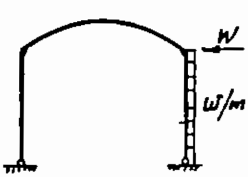
$H_a = H_b = \frac{pa^2}{4h} \cdot \frac{5(3-2\alpha)+2\gamma(5-5\alpha^2+2\alpha^3)}{\mu}$

$A = pa(2-\alpha)/2 \qquad B = pa\alpha/2$



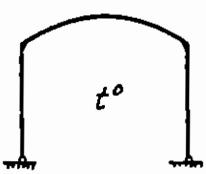
$H_a = H_b = \frac{5Pab}{2lh} \cdot \frac{3+2\gamma(1+\alpha\beta)}{\mu}$

$A = P\beta \qquad B = P\alpha$



$H_a = \left[\frac{5wh}{8} \cdot \frac{5\alpha+6+4\gamma}{\mu}\right] + \left[\frac{5W}{2} \cdot \frac{2\alpha+3+2\gamma}{\mu}\right]$

$H_b = wh + W - H_a \qquad A = B = \frac{wh^2}{2l} + \frac{Wh}{l}$

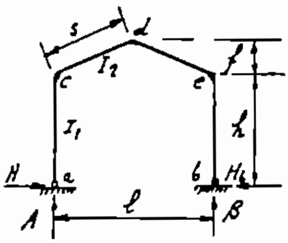


Temperature change of $t°$

$H_a = H_b = \pm \frac{15\alpha t E I_2}{h^2 \mu}$

$A = B = 0$

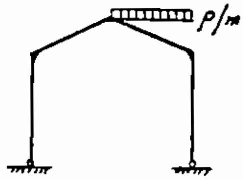
3. TWO HINGED SYMMETRICAL POLYGONAL FRAME



$\varkappa \quad \frac{I_2}{I_1}\,\frac{h}{s} \qquad \eta \quad f/h \qquad \mu \quad (\varkappa+3)+\eta(\eta+3)$

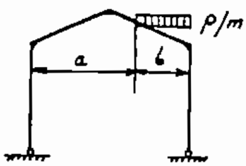
In case of unloaded columns

$M_c \quad -H_a\, h \qquad M_e \quad -H_b\, h \qquad M_d \quad M_0 \quad H_a(h+f)$



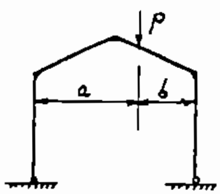
$H_a - H_b \quad \frac{pl^2}{64h} \quad \frac{5\eta+8}{\mu}$

$A \quad pl/8 \qquad\qquad B \quad 3pl/8$



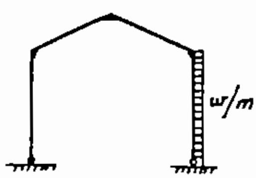
$H_a \quad H_b \quad \frac{pb^2}{8h} \quad \frac{2l(3l-2b)+\eta(3l^2-2b^2)}{l^2\,\mu}$

$A \quad pb^2/2l \qquad\qquad B \quad \frac{pb}{l}(l \quad 0.5b)$



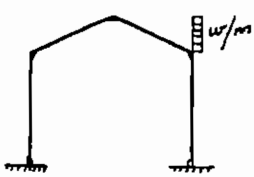
$H_a \quad H_b \quad \frac{Pb}{4h} \quad \frac{6a \quad l+\eta(3l^2-4b^2)}{l^2\,\mu}$

$A \quad Pb/l \qquad\qquad B = Pa/l$



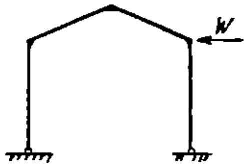
$H_a \quad \frac{wh}{16} \quad \frac{5\varkappa+6(\eta+2)}{\mu} \qquad H_b \quad H_a \quad wh$

$A \quad -B = wh^2/2l$



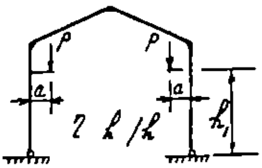
$H_a - \frac{wf}{16} \quad \frac{8(\varkappa+3)+5\eta(\eta+4)}{\mu} \qquad H_b \quad H_a - wf$

$A \quad -B \quad wf(2h+f)/2l$



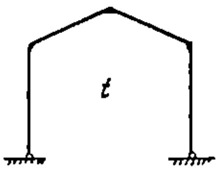
$H_a \quad \frac{W}{4} \quad \frac{2\varkappa+3(\eta+2)}{\mu}$ $\qquad$ $H_b \quad H_a - W$

$A = -B \quad Wh/l$



$H = H_a \quad H_b \quad \frac{3a}{4h}(P+P)\,\frac{\varkappa(1-\eta_1^2)+(\eta+2)}{\mu}$

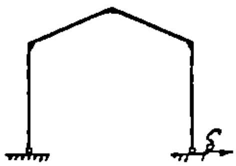
$A \quad P\frac{a}{l}+P\frac{(l-a)}{l}$ $\qquad$ $B - P\frac{a}{l}+P\frac{l-a}{l}$



Temp change of t°

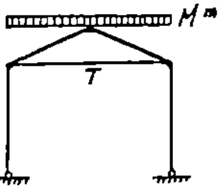
$H_a = H_b = \frac{3\alpha t l}{2\mu} \quad \frac{EI_2}{h^2 s}$ $\qquad$ $A \quad B \quad 0$

α = coeff of linear expansion $\qquad$ 0 00001



Horizontal displacement of support δ

$H_a = H_b = -\frac{3\delta}{2\mu} \quad \frac{EI_2}{h^2 s}$ $\qquad$ $A \quad B = 0$

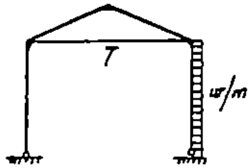


Frame with rigid tie

$4\varkappa+3 - \mu$

$H = H_a = H_b \quad \frac{pl^2}{16h\mu}$ $\qquad$ $A \quad B = \frac{wl}{2}$

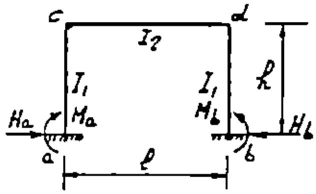
$T = \frac{pl^2}{16h} \quad \frac{10\varkappa+6\eta}{\eta\mu}$



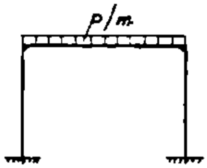
$H_a \quad wh \quad \frac{\varkappa+\mu}{4\mu}$ $\qquad$ $H_b = wh - H_a$

$T = wh \quad \frac{(3+\varkappa)+2\eta(\varkappa+\mu)}{8\mu\eta}$ $\qquad$ $A = -B = wh^2/2l$

4- FIXED SYMMETRICAL RECTANGULAR FRAME



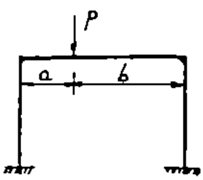
$\varkappa = \frac{I_2}{I_1} \quad \frac{h}{l}$ $\qquad \mu_1 \; \varkappa+2 \qquad \mu_2 \; 6\varkappa+1$



$H \quad H_a - H_b \quad \frac{pl^2}{4h\mu_1}$ $\qquad A \; B \; \frac{pl}{2}$

$M_a \quad M_b \quad \frac{pl^2}{12\mu_1} \quad H\, h/3$

$M_c \quad M_d - \frac{pl^2}{6\mu_1} \quad -2H\, h/3$

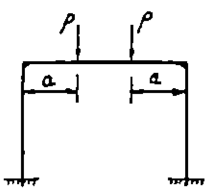


$H = H_a \quad H_b \quad \frac{3Pl\alpha\beta}{2h\mu_1}$, $A \quad P\beta\left[1+\frac{\alpha(\beta\;\alpha)}{\mu_2}\right] \quad B \; P \; A$

$M_a \quad \frac{Pl\alpha\beta}{2} \quad \frac{5\varkappa-1+2\alpha\mu_1}{\mu_1\mu_2}$ $\qquad M_c \quad M_a - H\,h$

$M_b \quad \frac{Pl\alpha\beta}{2} \quad \frac{7\varkappa+3 \;\; 2\alpha\mu_1}{\mu_1\mu_2}$ $\qquad M_d \quad M_b - H\,h$

$\alpha - \frac{a}{l} \quad \beta \quad \frac{b}{l}$

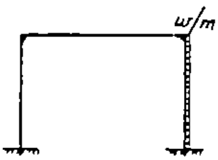


$H \quad H_a \quad H_b = \frac{3Pl\alpha\beta}{h\mu_1}$ $\qquad A \quad B \quad P$

$M_a \quad M_b = \frac{Pl\alpha\beta}{\mu_1} = H\, h/3$

$M_c \quad M_d - \frac{2Pl\alpha\beta}{\mu_1} \;--2H\, h/3$

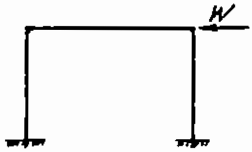
$\alpha \quad \frac{a}{l}$



$H_a \quad \frac{wh}{8} \; \frac{2\varkappa+3}{\mu_1}$ $\qquad H_b \quad H_a - wh$ $\qquad A - B \quad \frac{wh^2\varkappa}{l\mu_2}$

$M_a \quad \frac{wh^2}{24}\left(\frac{5\varkappa+9}{\mu_1} \quad \frac{12\varkappa}{\mu_2}\right)$ $\qquad M_c \quad M_a - H_a\, h$

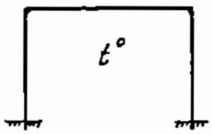
$M_b \; - \frac{wh^2}{24}\left(12 - \frac{5\varkappa+9}{\mu_1} - \frac{12\varkappa}{\mu_2}\right)$ $\qquad M_d \quad M_b - H\,h + \frac{wh^2}{2}$



$H_a = -H_b = W/2$ $\qquad A - B \; \frac{3Wh\varkappa}{\ell\mu_2}$

$M_a \; -M_b = \frac{Wh}{2} \; \frac{3\varkappa+1}{\mu_2}$

$M_c \; -M_d = -\frac{Wh}{2} \; \frac{3\varkappa}{\mu_2}$



$\alpha = 0.00001$

Temp change t

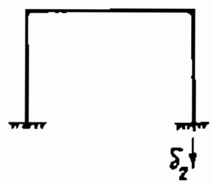
$H - H_a = H_b \; 3\alpha t \frac{EI_2}{h^2} \; \frac{2\varkappa+1}{\varkappa\mu_1}$ $\qquad A \; B \; 0$

$M_a \; M_b \; H \frac{h\varkappa+1}{2\varkappa+1}$ $\qquad M_c \; M_d \; -H \frac{h\varkappa}{2\varkappa+1}$

Horizontal displacement δ_1

$H - H_a - H_b \; -3\delta_1 \frac{EI_2}{h^2\ell} \; \frac{2\varkappa+1}{\varkappa\mu_1}$ $\qquad A - B = 0$

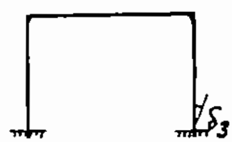
$M_a - M_b \; -Hh \; \frac{1+\varkappa}{\mu_1}$ $\qquad M_c \; M_d = -H \frac{h}{\mu_1}$



Vertical displacement δ_2

$H_a = H_b = 0$ $\qquad A \; -B = 6\delta_2 \frac{EI_2}{\ell^2\mu_2}$

$M_a \; M_c \; M_b = M_d \; -3\delta_2 \frac{EI_2}{\ell\mu_2} = -A \; \ell/2$



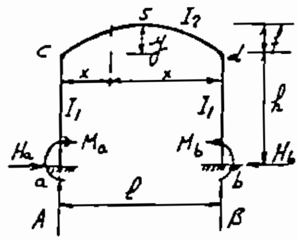
Rotation of support δ_3

$H = H_a \; H_b = \frac{3\delta_3 EI_2}{2h\ell\mu_1}$ $\qquad A = -B \; \frac{6\delta_3 EI_2}{\ell^2\mu_2}$

$M_a = \frac{3\delta_3 EI_2}{\ell}\left(-\frac{1}{\mu_2} + \frac{1}{2\varkappa} + \frac{1}{6\mu_1}\right)$

$M_b \; \frac{3\delta_3 EI_2}{\ell}\left(\frac{1}{\mu_2} + \frac{1}{2\varkappa} \; \frac{1}{6\mu_1}\right)$

5- FIXED SYMMETRICAL FRAME WITH PARABOLIC GIRDER

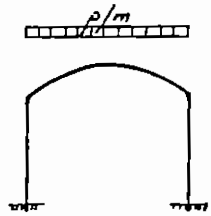


Equation of parabolic girder $y \; 4f \, \xi \, \xi'$

$\xi \; x/l \qquad \xi' \; x/l \qquad \eta \; f/l$

$\varkappa \; \frac{I_2}{I_1} \, \frac{h}{l} \qquad \mu_2 \; 6\varkappa + 1$

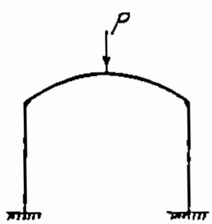
$\mu_1 \; 15\varkappa(\varkappa+2) + 12\varkappa\eta(5+4\eta) + 4\eta^2$



$H_a \; H_b \; \frac{pl^2}{4h} \, \frac{2\eta(12\varkappa+1)+15\varkappa}{\mu_1} \qquad A \; B \; pl/2$

$M_a - M_b \; \frac{pl^2}{4} \, \frac{2\eta\mu_2+5\varkappa}{\mu_1}$

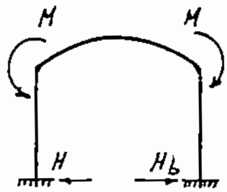
$M_c \; M_d \; -H\,h + M_a \qquad M_s \; \frac{pl^2}{8} - H(h+f) + M_a$



$H_a - H_b \; \frac{15Pl}{16h} \, \frac{\eta(10\varkappa+1)+6\varkappa}{\mu_1} \qquad A \; B \; P/2$

$M_a - M_b \; \frac{Pl}{16} \, \frac{\eta(2\eta+15)+15\varkappa(5\eta+2)}{\mu_1}$

$M_c - M_d \; -H\,h + M_a \qquad M_s \; \frac{Pl}{4} \; H(h+f) + M_a$

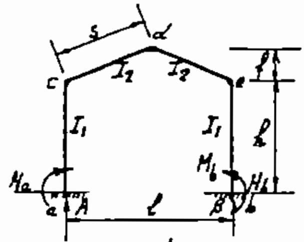


$H_a \; H_b \; \frac{15M\varkappa}{h} \, \frac{3+4\eta}{\mu_1} \qquad A \; B \; 0$

$M_a \; M_b \; -M \, \frac{15\varkappa(1+2\eta)-4\eta^2}{\mu_1}$

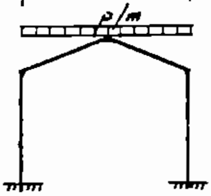
$M_c \; M_d \; Hh + M_a \qquad M_s \; -M + H(h+f) + M_a$

6- FIXED SYMMETRICAL POLYGONAL FRAME



$\varkappa - \frac{I_2}{I_1} \quad \frac{h}{s}$ $\qquad$ $\varphi \;\; f/h$

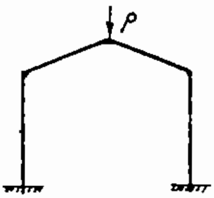
$\mu_1 \;\; \varkappa(\varkappa+4)+2\varkappa\varphi(3+2\varphi)+\varphi^2$ $\qquad$ $\mu_2 \;\; 3\varkappa+1$



$H_a = H_b = \frac{pl^2}{8h} \;\; \frac{4\varkappa+5\varkappa\varphi+\varphi}{\mu_1}$ $\qquad$ $A \;\; B \;\; pl/2$

$M_a \;\; M_b \;\; \frac{pl^2}{48} \;\; \frac{\varkappa(8+15\varphi)+\varphi(6 \;\; \varphi)}{\mu_1}$

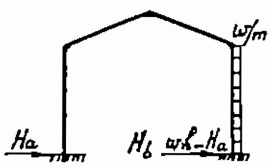
$M_c \;\; M_e = -H_a h + M_a$ $\qquad$ $M_d \;\; p\frac{l^2}{8} + M_a - H_a(h+f)$



$H_a - H_b = \frac{Pl}{4h} \;\; \frac{3\varkappa+4\varkappa\varphi+\varphi}{\mu_1}$ $\qquad$ $A \;\; B = P/2$

$M_a - M_b \;\; \frac{Pl}{4} \;\; \underline{\varkappa+2\varkappa\varphi+\varphi}$

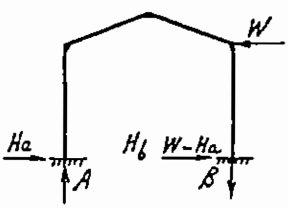
$M_c \;\; M_e \;\; -H_a h + M_a$ $\qquad$ $M_d \;\; \frac{Pl}{4} + M_a \;\; H_a(h+f)$



$H_a \;\; \frac{wh\varkappa}{4} \;\; \frac{\varkappa+3+2\varphi}{\mu_1}$ $\qquad$ $A \;\; -B \;\; \frac{wh^2}{2l} - \frac{M_b \; M_a}{l}$

$\left.\begin{matrix} M_a \\ M_b \end{matrix}\right\rangle \frac{wh^2}{24}\left[\frac{\varkappa(\varkappa+6)+\varkappa\varphi(15+16\varphi)+6\varphi^2}{\mu_1} \pm \frac{6(2\varkappa+1)}{\mu_2}\right]$

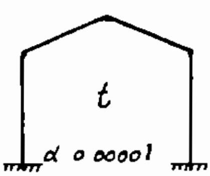
$M_c \;\; -H_a h + M_a$ $\qquad$ $M_e \;\; \frac{wh^2}{2} + M_a \;\; H_b h$



$H_a \;\; \frac{P\varkappa}{2} \;\; \frac{\varkappa+4+3\varphi}{\mu_1}$ $\qquad$ $A \;\; -B \;\; \frac{3Ph\varkappa}{2l\mu_2}$

$\left.\begin{matrix} M_a \\ M_b \end{matrix}\right\rangle \frac{Ph}{4}\left[-\frac{2\varphi(\varkappa+2\varkappa\varphi+\varphi)}{\mu_1} \pm \frac{3\varkappa+2}{\mu_2}\right]$

$M_c = -H \; h + M_a$ $\qquad$ $M_e \;\; H_b h + M_a$



$H - \frac{6EI_2\alpha t l}{sh^2} \;\; \frac{\varkappa+1}{\mu_1}$ $\qquad$ $A \;\; B \;\; 0$

$M_a \;\; M_b \;\; + \;\; \frac{3EI_2\alpha t l}{sh} \;\; \frac{\varkappa+2+\varphi}{\mu_1}$

$M_c - M_e \;\; M_a \;\; H h$ $\qquad$ $M_d = M_a \;\; H(h+f)$

Continuous Frames

Continuous frames are generally high grade statically indeterminate The degree of indeterminancy depends on the number and type of supports Using the method of virtual work, the determination of the internal forces in the continuous frame shown in fig IV-30 can be done in the following manner

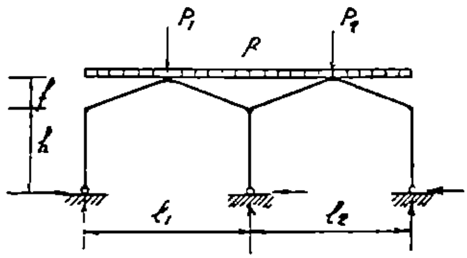


Fig IV-30 Continuous frame

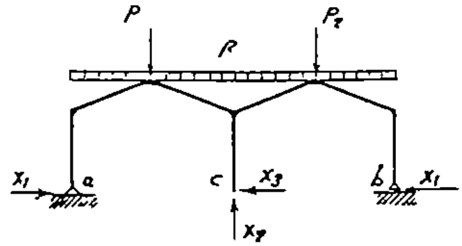


Fig IV-31 Main system

This frame has 6 reaction components, the conditions of equilibrium being 3, then the frame is 3 times statically indeterminate Choosing the statically determinte frame, hinged at a and supported on a roller at b as a main system, fig IV-31 then due to the loading, support b moves horizontally a distance δ_1 while point c moves vertically a distance δ_2 and horizontally a distance δ_3 The conditions of elasticity are therefore $\delta_1 = 0$ $\delta_2 = 0$ and $\delta_3 = 0$

To satisfy these conditions 3 statically indeterminate values X_1, X_2 and X_3 are required

The equations of elasticity are given by

$$\delta_1 = 0 = \delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} + X_3 \delta_{13}$$

$$\delta_2 = 0 = \delta_{20} + X_1 \delta_{21} + X_2 \delta_{22} + X_3 \delta_{23}$$

$$\delta_3 = 0 = \delta_{30} + X_1 \delta_{31} + X_2 \delta_{32} + X_3 \delta_{33}$$

in which

$EI\,\delta_{10} = \int M_o M_1\, ds$, $EI\,\delta_{20} = \int M_o M_2\, ds$, $EI\,\delta_{30} = \int M_o M_3\, ds$

The M_o , M_1 , M_2 and M_3 diagrams drawn on the tension side and taking the assumed sense of the statically indeterminate values X_1 , X_2 and X_3 in consideration are shown in fig IV-32

If the frame is symmetrical in shape and loading as is generally the case in roof structures, then X_3 equals zero and the intermediate column is subject to axial forces only. In such a case the frame may

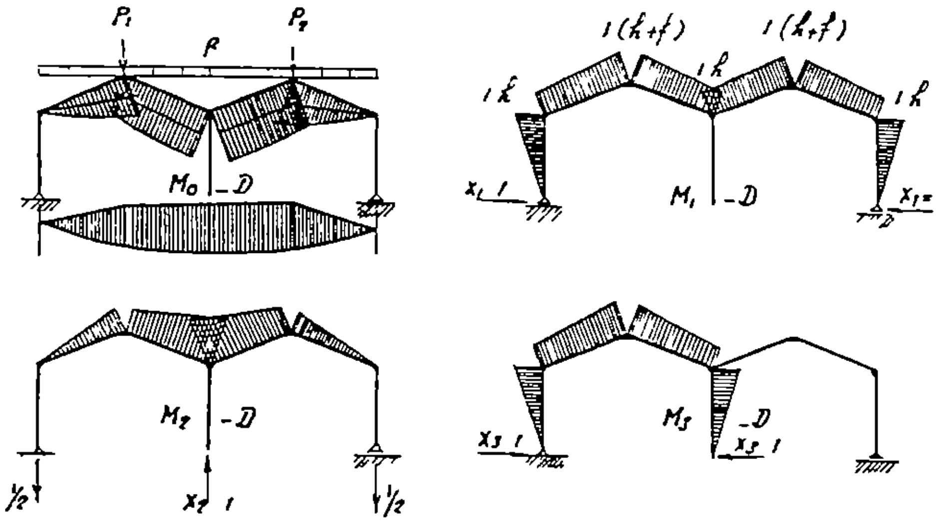


Fig IV-32 Analysis of a continuous frame

be constructed with a slender intermediate column which can be assumed as a pendulum. The system in this form is only twice statically indeterminate (Fig IV-33)

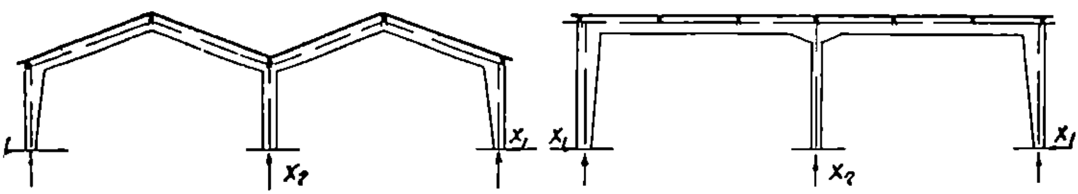


Fig IV-33 Continuous frames with a central pendulum

Assuming the same main system (as shown in fig IV-31), the equations of elasticity will be reduced to

$$\delta_1 = 0 = \delta_{10} + X_1\,\delta_{11} + X_2\,\delta_{12}$$
$$\delta_2 = 0 = \delta_{20} + X_1\,\delta_{21} + X_2\,\delta_{22}$$

Figs IV-34 and IV-35 give the general layout, main dimensions and details of the main continous frames of the printing, dying and bleeching halls of the Misr spinning and weaving company at Mehalla

As a result of the various industrial processes accomplished in these halls a big amount of hot chemicals evaporate This hot vapour must find it's way outside the hall It is absolutely essential to take the necessary provisions so that the vapour does not accumulate or condensate inside the hall In order to satisfy this requirement and to facilitate the movement of the vapour, the roof slab below the top windows was chosen circular and arranged at the inner side of the main girders The vapour being hot it moves upwards and, due to the cross ventilation created by the upper windows it is driven outwards without having the possibility to accumulate or condensate

During construction the main tension reinforcements were replaced by the equivalent amount of cold twisted steel

The upper monitor of the frame has been cancelled in the end panels of the halls at the end gables to get a better side view

The Slope - Deflection Method

This method is one of the oldest known methods used for determining the connecting moments of statically indeterminate beams and frames composed of a series of straight members The study of one single span elastically restrained at both ends is evidently of prime importance (Fig IV-36)

The deformation angles α and β can be determined according to law of superposition from the following relations

$$\alpha = \alpha_o + M_1 \alpha_1 + M_2 \alpha_2$$

$$\beta = \beta_o + M_1 \beta_1 + M_2 \beta_2 \qquad \text{in which}$$

α_o and β_o = deformation angles of main system due to given loads

$\alpha_1 \quad \beta_1$ = $\qquad M_1 = 1$

$\alpha_2 \quad \beta_2$ = $\qquad M_2 = 1$, hence

$$\alpha_1 = \int_0^l \frac{x'^2}{l^2} \frac{dx}{EI} \qquad \text{for } I = \text{constant} \qquad \alpha_1 = \frac{l}{3EI}$$

$$\alpha_2 = \int_0^l \frac{xx'}{l^2} \frac{dx}{EI} = 1 \qquad \alpha_2 = \frac{l}{6EI} = \beta_1$$

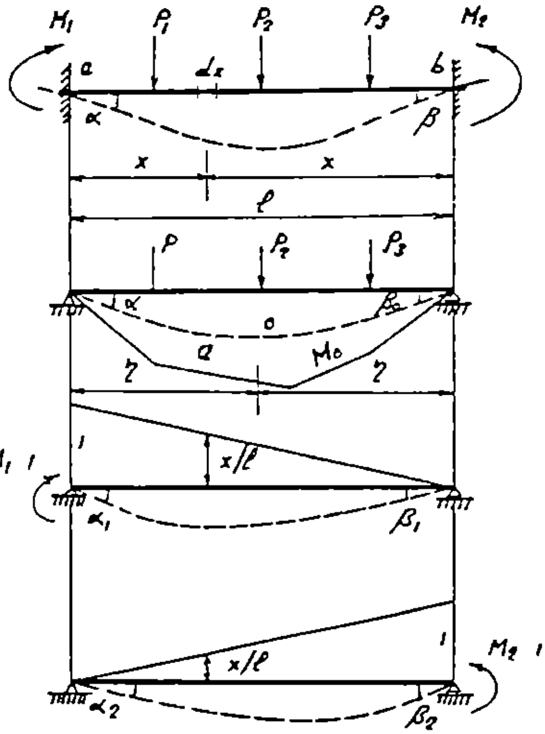
$$\beta_2 = \int_0^l \frac{x^2}{l^2} \frac{dx}{EI} \qquad \beta_2 = \frac{l}{3EI}$$

Refer to Theory of Elastically Restrained Beams by H Hilal

According to theory of Maxwell, we have

$$\alpha_o = \int_o^l \frac{M_o x'}{l} \frac{dx}{EI} \qquad \text{for } I = \text{constant} \qquad \alpha_o = \frac{a_o \eta}{l\,EI}$$

$$\beta_o = \int_o^l \frac{M_o x}{l} \frac{dx}{EI} \qquad\qquad \beta_o = \frac{a_o \eta}{l\,EI}$$



– Fig IV-36

where

a_o = area of B M D of main system due to given loads

η & η' = the distances of the center of gravity O of the M_o – diagram from the verticals through a and b

We consider in the following (fig IV-37) a frame 1-2-3 with a rigid joint at 2

The condition of elasticity at joint 2 is $\beta = -\alpha'$

So that

$$\beta + \alpha = 0 = \beta_o + M_1\beta_1 + M_2\beta_2 + \alpha_o + M_2\alpha_1 + M_3\alpha_2$$

or $\quad 0 = M_1\beta_1 + M_2(\beta_2 + \alpha_1) + M_3\alpha_2 + (\beta_0 + \alpha_0)$

For constant moments of inertia I and I' , we get

$$M_1 \frac{s}{6EI} + M_2\left(\frac{s}{3EI} + \frac{s'}{3EI'}\right) + M_3\frac{s'}{6EI'} = -\left(\frac{a_0\eta}{sEI} + \frac{a'_0\eta}{s'EI}\right)$$

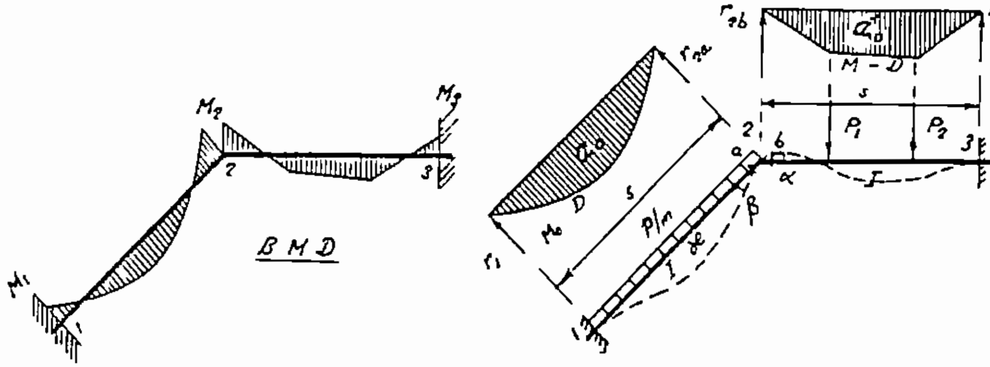


Fig IV-37

Assuming

$$\varkappa = \frac{I}{s'} \quad \frac{s}{I}$$

$$r_{2a} = a_0\,\eta/s$$

$$r_{2b} = a'_0\,\eta/s'$$

and multiplying the two sides of the equation by $\frac{6EI}{s}$, we get

$$M_1\varkappa + 2M_2(\varkappa + 1) + M_3 = -6\left(\frac{r_{2a}}{s}\varkappa + \frac{r_{2b}}{s'}\right)$$

This equation of three moments can be used for determining the moments at the corners of two hinged and fixed frames with unmovable joints

The following system of equations can be written for the frame shown in fig IV-37

$$1-2 \qquad 2M_1\varkappa + M_2\varkappa = -6\frac{r_1}{s}\varkappa$$

$$1-2-3 \quad M_1\varkappa + 2M_2(\varkappa+1) + M_3 = -6\left(\frac{r_{2a}}{s}\varkappa + \frac{r_{2b}}{s'}\right)$$

$$2-3 \quad M_2 + 2M_3 = -6\frac{r_3}{s}$$

They are sufficient to determine the moments M_1, M_2 and M_3

Frames symmetrical in shape and loading have generally unmovable joints and the connecting moments can be determined by the given equation of three moments if the members in any rigid joint are not more than two

<u>Examples</u>

1) Closed frame subject to internal pressure p/m as shown in fig IV-38

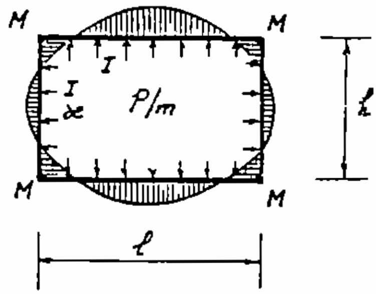


Fig IV-38

$$\varkappa = \frac{I}{l} \quad \frac{h}{I_1}$$

The connecting moment M can be determined from the equation

$$M\varkappa + 2M(\varkappa + 1) + M = -6\left(\frac{r_1}{h}\varkappa + \frac{r}{l}\right) = -6\left(p\frac{h^2}{24}\varkappa + p\frac{l^2}{24}\right)$$

$$M = -p\frac{(h^2\varkappa + l^2)}{12(\varkappa + 1)}$$

2) Two-vent closed frame subject to external pressure p as shown in fig IV-39

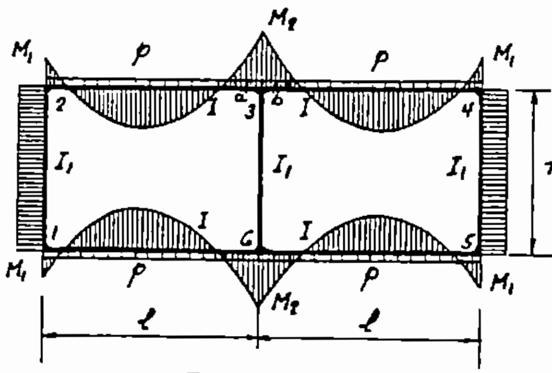


Fig IV-39

$$\varkappa = \frac{I}{l} \quad \frac{h}{I_1}$$

Due to symmetry the intermediate wall will not be subject to any moments and the only unknown values are the connecting moments M_1 and M_2 Thus

1 - 2 - 3 $\quad M_1\varkappa + 2M_1(\varkappa + 1) + M_2 = -6\frac{r_2}{l}$

or $\quad M_1(3\varkappa + 2) + M_2 = -p\frac{l^2}{4}$

2 - 3 - 4 $\quad M_1 + 2M_2 + 2M_2 + M_1 = -6\left(\frac{r_{3a}}{l} + \frac{r_{3b}}{l}\right)$

or $\quad 2M_1 + 4M_2 = -p\frac{l^2}{2}$

These two equations give

$$M_1 = - p\frac{l^2}{12(2\varkappa + 1)} \qquad M_2 = - p\frac{l^2}{12}\;\frac{3\varkappa + 1}{2\varkappa + 1}$$

3) Fixed frame subject to uniform load p/m (fig IV-40) Due to symmetry, the corners c and d will not move horizontally hence

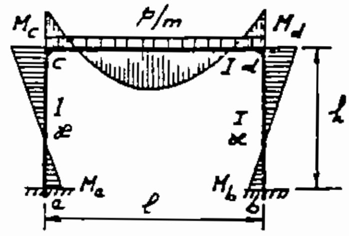


Fig IV-40

$$a - c \qquad 2M_a\varkappa + M_c\varkappa = 0 \qquad \varkappa = \frac{I}{l}\;\frac{h}{I_1}$$

$$\text{or} \qquad M_a = -\tfrac{1}{2} M_c \qquad \text{(a)}$$

$$a\text{-}c\text{-}d \qquad M_a\varkappa + 2M_c(\varkappa + 1) + M_d = -6\, r_c/l$$

but $M_c = M_d$ and $r_c = p\,l^2/24$ then,

$$M_a\varkappa + M_c(2\varkappa + 3) = - p\,l^2/4 \qquad \text{(b)}$$

Equations a and b give

$$M_a = M_b = p\frac{l^2}{12(\varkappa + 2)} \qquad M_c = M_d = - p\frac{l^2}{6(\varkappa + 2)}$$

Continuous frames with unmovable joints can be treated in a similar manner as follows (Fig IV-41)

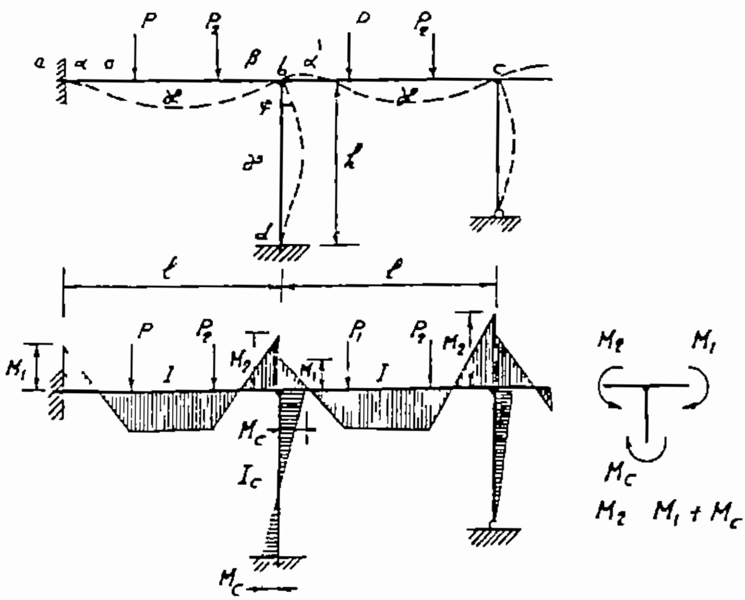


Fig IV-41

Consider two successive spans l and l subject to the loads P_1, P_2 and P_1', P_2 etc The elastic line and the bending moments will be as shown in figure IV-41

The condition of elasticity at the middle support is

$$\alpha' = -\beta = \varphi$$

According to the law of superposition, the angles of rotation β, α' and φ can be expressed as follows

$$\beta = \beta_o + M_1\beta_1 + M_2\beta_2$$

$$\alpha' = \alpha_o + M_1\alpha'_1 + M_2\alpha'_2$$

$$\varphi = \varphi_o + M'_c\varphi_1 + M_c\varphi_2$$

Due to the equilibrium of the middle joint, we must have

$$M_2 = M'_1 + M_c$$

Therefore, we get

$$\alpha + \beta = 0 = \beta_o + \alpha_o + M_1\beta_1 + M_2\beta_2 + M_1\alpha_1 + M'_2\alpha_2$$

$$\varphi + \beta = 0 = \beta_o + \varphi_o + M_1\beta_1 + M_2\beta_2 + M_c\varphi_1 + M'_c\varphi_2$$

This system of equations gives a number of equations equal to that of the unknown fixing moments

Assuming that the moment of inertia of each span or column is constant, and introducing the factors χ such that

$$\chi = \frac{I}{l}\quad\frac{l}{I} = 1 \qquad \chi' = \frac{I}{l}\quad\frac{l}{I'} \qquad \chi_c = \frac{I}{l}\quad\frac{h}{I_c}$$

we get

for a-b-c $\quad M_1 + 2M_2 + 2M_1\chi' + M_2\chi = -6\left(\frac{r}{l} + \frac{r}{l}\chi\right)$

a-b-d $\quad M_1 + 2M_2 + 2M_c\chi_c + M'_c\chi_c = -6\left(\frac{r}{l} + \frac{r_c}{h}\chi_c\right)$

This system of equations of four moments is sufficient for determining the values of the unknown moments as can be seen in the following examples

Examples

1) It is required to determine the corner moments in the continuous frame shown in figure IV-42 due to a uniform load p acting on l_1

Assume

$\gamma_c = \frac{I}{l} \quad \frac{h}{I_c}$ and $l_1 = l_2 = l$ so that $\varkappa_1 = \varkappa_2 = 1$

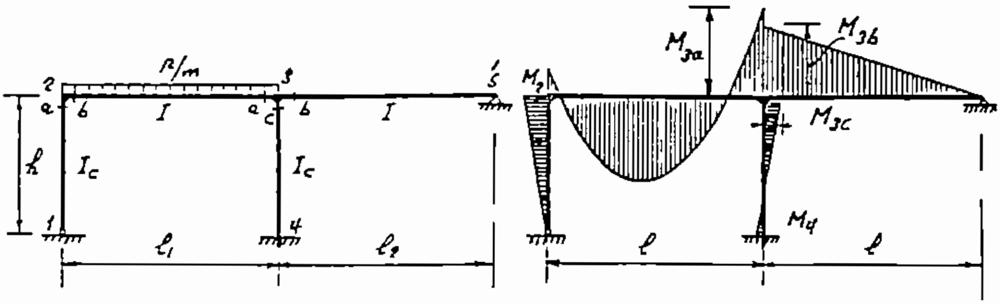


Fig IV-42

As the girder is hinged at 5, no vertical or horizontal movement of the supports will take place hence the equation of four moments can be applied in the form given before as follows

1-2-3 $\quad 2M_{2a}\varkappa_c + 2M_{2b} + M_{3a} = -6\, r_{2b}/l = -p\,l^2/4$

2-3-4 $\quad M_{2b} + 2M_{3a} + (2M_{3c} + M_4)\varkappa_c = -6\, r_{3a}/l = -p\,l^2/4$

2-3-5 $\quad M_{2b} + 2M_{3a} + 2M_{3b} = -6\, r_{3a}/l = -p\,l^2/4$

in which $\quad r_{2b} = r_{3a} = p\,l^3/24$

We have further

$$M_{2a} = M_{2b} = M_2 \quad \text{and} \quad M_{3a} - M_{3b} = M_{3c}$$

The column 3-4 being fixed at its base, then the fixed point lies at h/3 as shown in figure IV-42 and $M_4 = -M_{3c}/2$

These equations are sufficient to determine the unknown moments the give

$M_2 = -0\ 019\ p\,l^2 \qquad M_{3a} = -0\ 0662\ p\,l^2\ , \quad M_{3b} = -0\ 0496\ p\,l^2$

$M_{3c} = M_{3a} - M_{3b} = -0\ 0166\ p\,l^2$ and $M_4 = -M_{3c}/2 = +0\ 0083\ p\,l^2$

The bending moment diagram drawn on the tension side, is shown in figure IV-42

2) The multiple frame shown in figure IV-43 is symmetrical and symmetrically loaded If the change in length due to normal forces and temperature changes are neglected, the joints can be assumed to remain in position, and the equation of four moments can be applied

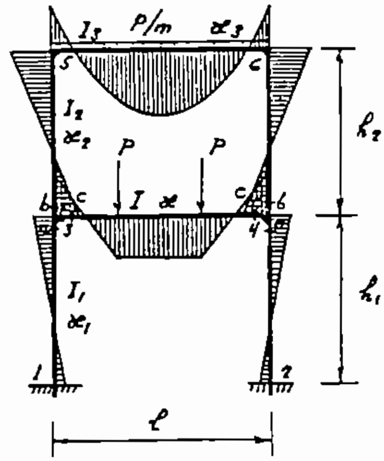


Fig IV-43

Assuming

$\varkappa = \frac{I}{l} \; \frac{l}{I} = 1$,

$\varkappa_1 = \frac{I}{l} \; \frac{h_1}{I_1}$,

$\varkappa_2 = \frac{I}{l} \; \frac{h_2}{I_2}$ and $\varkappa_3 = \frac{I}{l} \; \frac{l}{I_3}$

then the equations required to determine the unknown corner moments are

1-3 $\quad 2M_1\varkappa_1 + M_{3a}\varkappa_1 = 0 \quad$ or $\quad M_1 = -M_{3a}/2$

1-3-5 $\quad M_1\varkappa_1 + 2M_{3a}\varkappa_1 + 2M_{3b}\varkappa_2 + M_5\varkappa_2 = 0$

1-3-4 $\quad M_1\varkappa_1 + 2M_{3a}\varkappa_1 + 2M_{3c} + M_{4c} = -6\,r_{3c}/l$

3-5-6 $\quad M_{3b}\varkappa_2 + 2M_5(\varkappa_2 + \varkappa_3) + M_6\varkappa_3 = -6\,r_5\,\varkappa_3/l$

joint 3 $\quad M_{3a} - M_{3b} - M_{3c} = 0$

Due to symmetry $\quad M_{3c} = M_{4c} \quad$ and $\quad M_5 = M_6$

Figure IV-44 shows the details of reinforcements of El Masaa supporting multiple main frames at Mecca, it can be considered as a typical example for this system

Equation of three moments for frames with moving joints

The corners of frames are generally not subject to horizontal or vertical movements if there are fixed supports to prevent this movement or if they are symmetrical and symmetrically loaded, otherwise

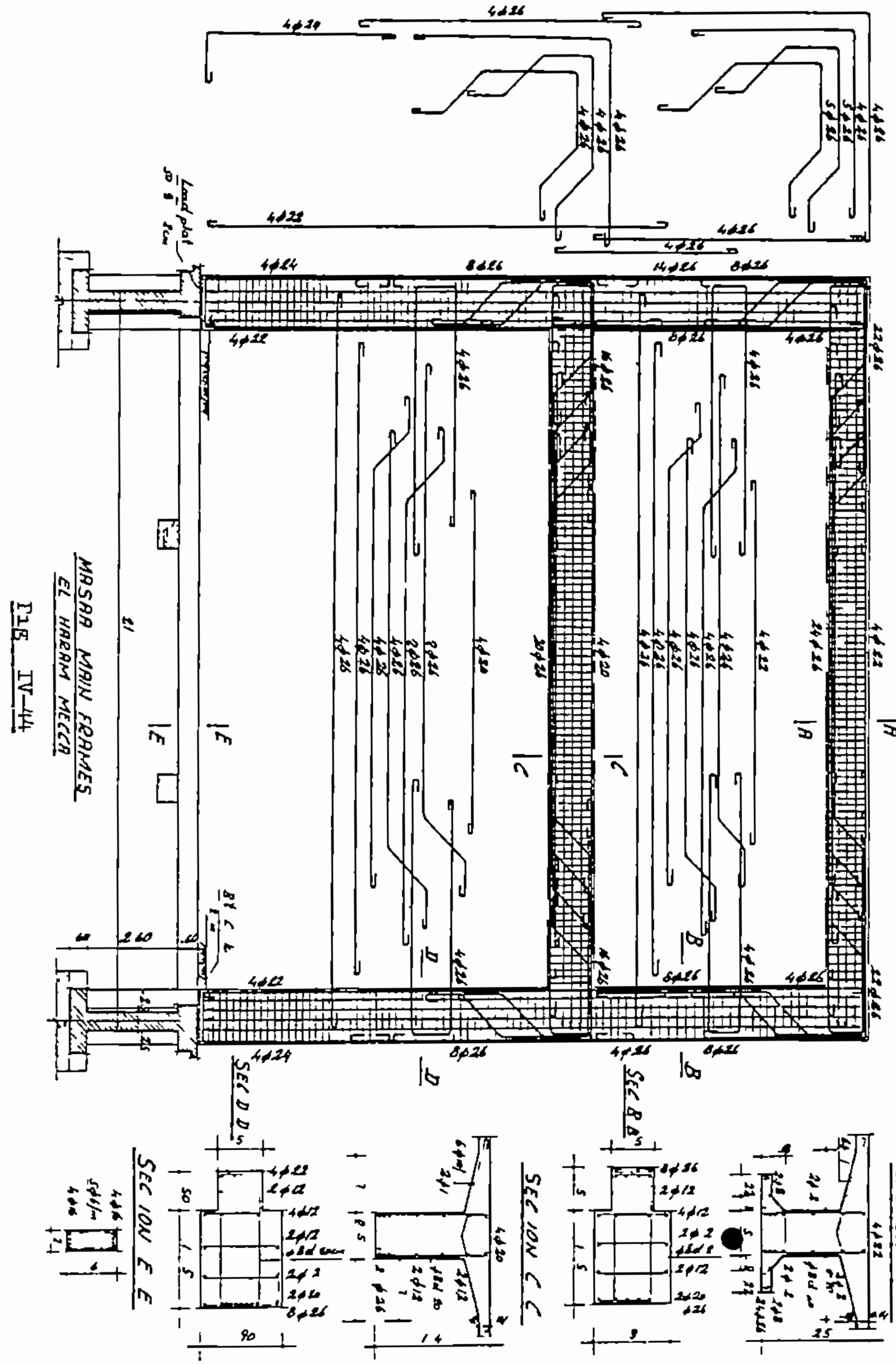

MASAA MAIN FRAMES
EL HARAM MECCA
SEC D D
SEC B B
SECTION A A
SECTION C C
SECTION E E

Fig IV-44

the corners move from their position Such frames can be treated as follows (Fig IV-45)

$$\varkappa = \frac{I_o}{s_o} \quad \frac{s}{I}$$

$$\varkappa' = \frac{I_o}{s_o} \quad \frac{s'}{I'}$$

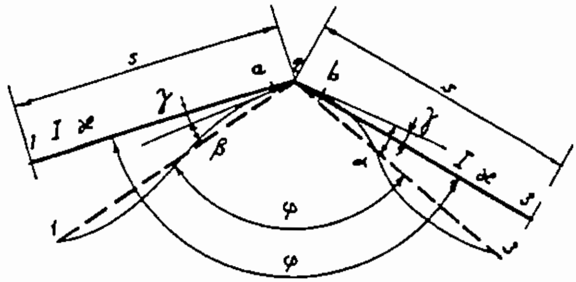


Fig IV-45

Assume that the two members 1-2 of length s and 2-3 of length s' of a frame are rigidly connected at 2 and that the angle between their axes is φ After deformation, joints 1 and 3 move to 1' and 3', the angle between the new positions 1'-2 and 2-3' is φ Due to the rigidity of the joint, the angle between the tangents to the elastic line at 2 does not change before and after deformation so that

$$\varphi' = \varphi - \beta - \gamma + \gamma - \alpha$$

or $$\varphi - \varphi = \overline{\Delta\varphi} = \beta + \alpha$$

Hence, according to the law of superposition, we get

$$\overline{\Delta\varphi} = \beta + \alpha' = \beta_o + M_1\beta_1 + M_2\beta_2 + \alpha'_o + M_2\alpha_1 + M_3\alpha_2$$

or $$\overline{\Delta\varphi} = M_1\beta_1 + M_2(\beta_2 + \alpha_1) + M_3\alpha'_2 + (\beta_o + \alpha_o)$$

The right side of this equation has the same general form of the equation of three moments given before

If we assume further that the moments of inertia I and I of the members s and s are constant, and

$$\varkappa = \frac{I_o}{s_o} \quad \frac{s}{I} \qquad\qquad \varkappa' = \frac{I_o}{s_o} \quad \frac{s'}{I}$$

where s_o and I_o are the length and moment of inertia of a reference member, we get

$$\frac{6EI_o}{s_o} \overline{\Delta\varphi} = M_1\varkappa + 2M_2(\varkappa + \varkappa') + M_3\varkappa + 6\left(\frac{r_{2a}}{s}\varkappa + \frac{r_{2b}}{s'}\varkappa\right)$$

Calling $\frac{6EI_o}{s_o}\overline{\Delta\varphi} = \Delta\varphi$, then

$$\Delta\varphi = M_1\varkappa + 2M_2(\varkappa+\varkappa') + M_3\varkappa + 6\left(\frac{r_{2a}}{s}\gamma + \frac{r_{2b}}{s}\varkappa\right)$$

This equation gives the modified form of the equation of three moments for frames with moving joints

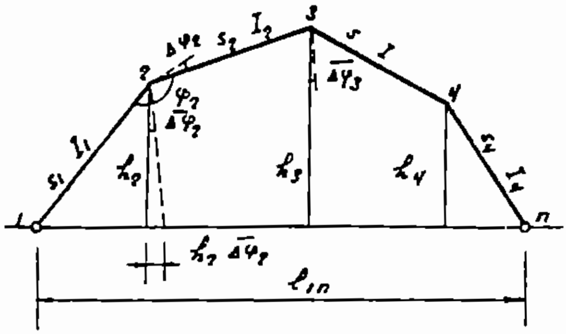


Fig IV-46

Figure IV-46 shows a frame with rigid moving joints, the change $\overline{\Delta\varphi}$ of the angles φ can be determined from the given general equation of three moments Due to $\overline{\Delta\varphi_2}$ we get a change in the span l_{1-n} equal to $h_2\overline{\Delta\varphi_2}$ so that the change in l_{1n} due to the change in the angles of all the joints is given by

$$\Delta l_{1n} = h_2\overline{\Delta\varphi_2} + h_3\overline{\Delta\varphi_3} + h_4\overline{\Delta\varphi_4}$$

Multiplying both sides of the equation by $6EI_o/s_o$, we get

$$\frac{6EI_o}{s_o}\Delta l_{1n} = h_2\Delta\varphi_2 + h_3\Delta\varphi_3 + h_4\Delta\varphi_4 = \sum h\Delta\varphi$$

If 1 and n are fixed in position, then $\Delta l_{1n} = 0$, and

$$\sum h\Delta\varphi = 0$$

If h is also constant, then

$$\sum\Delta\varphi = 0$$

The application of the method will be shown when solving the rectangular frame subject to wind pressuse shown in figure IV-47a

$$\sum \Delta\varphi = 0 \qquad \text{or} \qquad \Delta\varphi_c + \Delta\varphi_d = 0$$

hence

$$\left[2M_c\ (\varkappa + 1) + M_d + \frac{wh^2}{4}\varkappa\right] + \left[M_c + 2M_d\ (\varkappa + 1\ )\right] = 0$$

So that

$$M_c + M_d = -\frac{wh^2\varkappa}{4(\ 2\varkappa + 3)}$$

But $\qquad M_c = wh^2/2 - Hh \qquad$ and $\qquad M_d = -Hh$

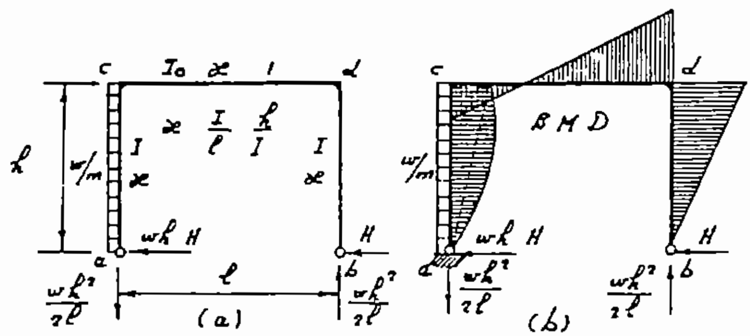


Fig IV-47

Therefore

$$\frac{wh^2}{2} - 2Hh \qquad = -\frac{wh^2\varkappa}{4\ (2\varkappa + 3)} \qquad \text{and}$$

$$H = \frac{wh}{8}\ \frac{5\varkappa + 6}{2\varkappa + 3}$$

Further $\qquad M_d = -Hh = -\dfrac{wh^2}{8}\ \dfrac{5\varkappa + 6}{2\varkappa + 3}$

and $\qquad M_c = \dfrac{wh^2}{2} - Hh = \dfrac{3wh^2}{8}\ \dfrac{\varkappa + 2}{2\varkappa + 3}$

The bending moment diagram is shown in figure IV-47b

If the unsymmetrical wind load w is replaced by symmetrical and anti-symmetrical loads w/2 as shown in fig IV-48b & c, the frame can be solved as follows

For case1 Symmetrical load w/2 hence $M_c = M_d$

$$2M_c\,(\varkappa+1) + M_c = -\frac{w}{2}\,\frac{h^2}{4}\varkappa \qquad \text{or} \qquad M_c = -\frac{wh^2\varkappa}{8\,(2\varkappa+3)}$$

For case 2 Anti-symmetrical load w/2, hence $H = wh/2$, and

$$M_c = -M_d = \pm\frac{wh}{4}$$

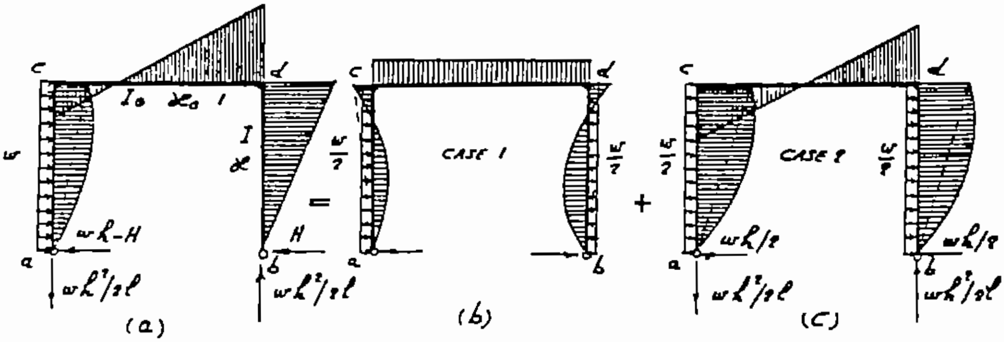


Fig IV-48

By superposition we get

$$M_d = -\frac{wh^2\gamma}{8\,(2\varkappa+3)} - \frac{wh^2}{4} = -\frac{wh^2}{8}\;\frac{5\varkappa+6}{2\varkappa+3} \qquad \text{and}$$

$$M_c = -\frac{wh^2\varkappa}{8\,(2\varkappa+3)} + \frac{wh^2}{4} = \frac{3wh^2}{8}\;\frac{\varkappa+2}{2\varkappa+3}$$

Continuous frames with moving joints

The same principles can be applied to continuous frames with moving joints using a modified four moment equation The solution can be much simplified if unsymmetrical loads can be replaced by symmetrical and antisymmetrical loads as shown in the following example of a continuous frame subject to wind pressure Fig IV-49

$$\varkappa_1 = \frac{I}{l}\;\frac{h}{I_1}$$

$$\gamma_2 = \frac{I}{l}\;\frac{h}{I_2}$$

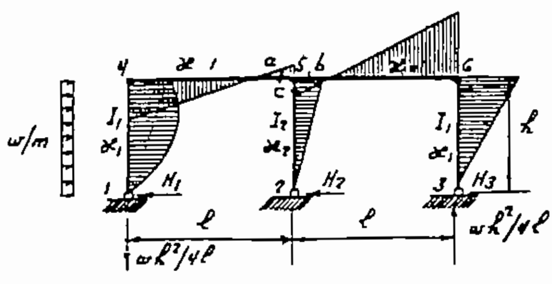


Fig IV-49

This case of loading can be replaced by the symmetrical case shown in figure IV-50 and the antisymmetrical case shown in figure IV-51

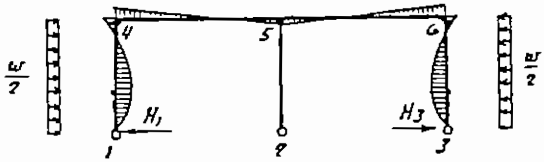


Fig IV-50

Case 1 Symmetrical case Fig IV-50

Due to symmetry $M_{5c} = 0$, $M_{5a} = M_{5b} = M_5$

$M_4 = M_6$, $H_1 = H_3$ & $H_2 = 0$

Hence

1-4-5 $$2M_4 (\varkappa_1 + 1) + M_5 = - \frac{w}{2} \frac{h^2}{4} \varkappa_1 \qquad (a)$$

4-5-6 $$M_4 + 4M_5 + M_6 = 0 \quad \text{or} \quad M_4 = - 2M_5 \qquad (b)$$

Equations (a) and (b) give

$$M_4 = - \frac{wh^2\varkappa_1}{4(4\varkappa_1 + 3)} \quad \text{and} \quad M_5 = + \frac{wh^2\varkappa_1}{8(4\varkappa_1 + 3)}$$

But $$H_1 h - wh^2/4 = M_4 \quad \text{so that} \quad H_1 = M_4/h + wh/4$$

and $$H_1 = H_3 = \frac{3wh}{4} \frac{\varkappa_1 + 1}{4\varkappa_1 + 3}$$

Case 2 Anti-symmetrical case of loading Fig IV-51

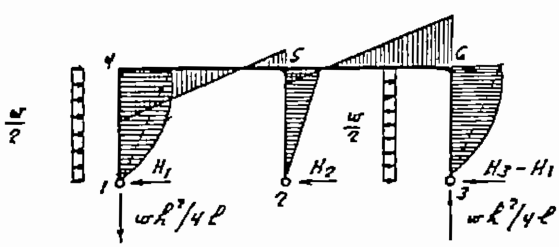


Fig IV-51

Due to anti-symmetry $M_6 = - M_4$, $M_{5a} = - M_{5b}$, $M_{5a} - M_{5b} - M_{5c} = 0$

So that $M_{5c} = 2M_{5a}$

Further $H_1 = H_3$ and $H_2 = wh - 2H_1$

e have $\Delta\varphi_4 + \Delta\varphi_{5(a-c)} = 0$ hence

$$\left[2M_4 (\varkappa_1 + 1) + M_{5a} + \frac{w}{2} \frac{h^2}{4}\varkappa_1\right] + \left[(M_4 + 2M_{5a} + 2M_{5c}\varkappa_2)\right] = 0 \quad (a)$$

Replacing M_{5a} by ½ M_{5c} equation a can be given in the form

$$2M_4 (2\varkappa_1 + 3) + M_{5c} (4\varkappa_2 + 3) + \frac{wh^2}{4}\varkappa_1 = 0 \quad (b)$$

Refering to figure IV-51 we find that

$M_4 = H_1 h - wh^2/4$, $M_{5c} = - H_2 h = - (wh - 2H_1) h$

Substituting these values in equation b , we get

$$H_1 = \frac{wh}{16} \frac{3\varkappa_1 + 16\varkappa_2 + 18}{\varkappa_1 + 2\varkappa_2 + 3}$$

As $H_2 = wh - 2H_1$ we get further

$$H_2 = \frac{wh}{8} \frac{5\varkappa_1 + 6}{\varkappa_1 + 2\varkappa_2 + 3}$$

By superposition of cases 1 and 2 we get

$$H_1 = \frac{wh}{16} \left[\frac{3\varkappa_1 + 16\varkappa_2 + 18}{\varkappa_1 + 2\varkappa_2 + 3} + \frac{12(\varkappa_1 + 1)}{4\varkappa_1 + 3}\right]$$

$$H_2 = \frac{wh}{8} \frac{5\varkappa_1 + 6}{\varkappa_1 + 2\varkappa_2 + 3}$$

$$H_3 = \frac{wh}{16} \left[\frac{3\varkappa_1 + 16\varkappa_2 + 18}{\varkappa_1 + 2\varkappa_2 + 3} - \frac{12(\varkappa_1 + 1)}{4\varkappa_1 + 3}\right]$$

Assuming $I = 2I_1 = 8I_2$ and $l = 2h$ then

$\varkappa_1 = 1$ $\varkappa_2 = 4$ and

$H_1 = 0.058\ wh \qquad H_2 = 0.115\ wh \quad \text{and} \quad H_3 = 0.227\ wh$

The bending moment diagrams are shown in figures IV-49 50 & 51

The following tables give the values of the bending moments at the corners of continuous frames having two three and four equal spans subject to uniform and wind loads The moments of inertia of the girder (I_o) and of the columns (I) are assumed to be constant The relative stiffness $\varkappa$ is given by

$$\varkappa = \frac{I_o}{I} \cdot \frac{h}{l}$$

The bending moments are given in the form

$M = \alpha\ (\pm\ pl^2/4)$ for uniform vertical load p/m

$M = \alpha\ (\pm\ wh^2)$ for horizontal wind load w/m, and

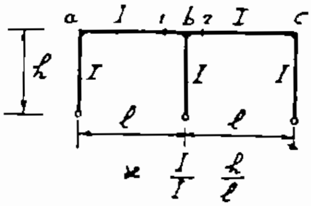
$M = \alpha\ (\pm\ Wh)$ for concentrated wind load W

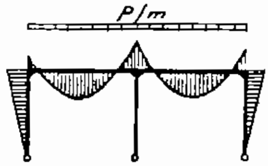
It has to be noticed here that

In case of roofs supported on continuous frames of the form shown in the following tables the live and wind loads are generally small relative to the dead loads so that for the main vertical superimposed loads, the frames are symmetrical and symmetrically loaded and the bending moments on the intermediate columns are either null or nil In such cases, the intermediate columns may be chosen slender and act as pendulums capable of resisting vertical reactions only as shown in figure IV-33

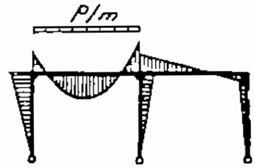
Symmetrical cases of loading can however be calculated by the equation of three moments given on page 79 while unsymmetrical cases can be replaced by symmetrical and antisymmetrical cases

Intermediate rigid columns are to be used in cases of unsymmetrical frames, heavy live or wind loads



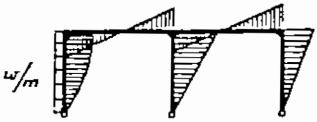


CASE 1

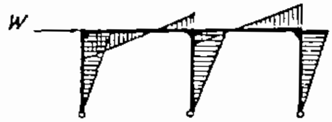


CASE 2

κ	Case 1		Case 2			
	$M_a = M_c$	$M_{b1} = M_{b2}$	M_a	M_{b1}	M_{b2}	M_c
0 05	0 3125	0 3438	0 3150	0 3306	0 0132	0 0025
0 10	0 2941	0 3529	0 2986	0 3280	0 0250	0 0045
0 20	0 2632	0 3684	0 2705	0 3231	0 0453	0 0073
0 30	0 2381	0 3810	0 2473	0 3187	0 0623	0 0092
0 33	0 2315	0 3843	0 2411	0 3174	0 0668	0 0096
0 40	0 2174	0 3913	0 2277	0 3147	0 0766	0 0104
0 50	0 2000	0 4000	0 2111	0 3111	0 0889	0 0111
0 60	0 1452	0 4074	0 1968	0 3079	0 0995	0 0116
0 75	0 1667	0 4167	0 1786	0 3036	0 1131	0 0119
1 00	0 1429	0 4286	0 1548	0 2976	0 1310	0 0119
1 25	0 1250	0 4375	0 1366	0 2928	0 1447	0 0116
1 50	0 1111	0 4444	0 1222	0 2889	0 1556	0 0111
2 00	0 0909	0 4545	0 1010	0 2828	0 1717	0 0101
2 50	0 0769	0 4615	0 0861	0 2784	0 1832	0 0092
3 00	0 0667	0 4667	0 0750	0 2750	0 1917	0 0083
3 50	0 0588	0 4706	0 0665	0 2723	0 1983	0 0076
4 00	0 0526	0 4737	0 0597	0 2702	0 2075	0 0070
5 00	0 0435	0 4783	0 0495	0 2669	0 2114	0 0060
6 00	0 0370	0 4815	0 0423	0 1646	0 2169	0 0053
	Multiplicator $p l^2 / 4$					



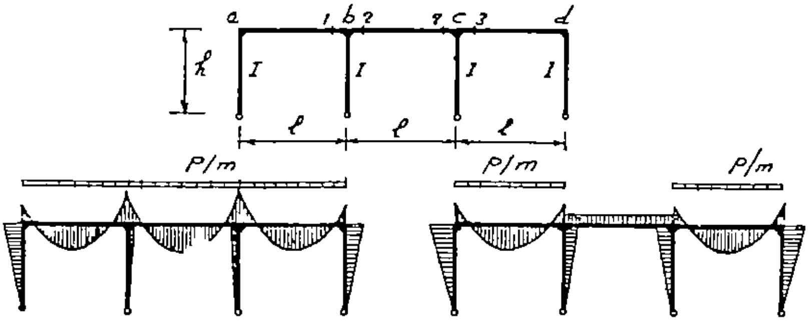
CASE 3



CASE 4

$\varkappa$	Case 3				Case 4	
	M_a	M_{b1}	M_{b2}	M_c	$M_a = -M_c$	$M_{b1} = -M_{b2}$
0 05	0 1221	0 1221	0 1260	0 1299	0 2540	0 2460
0 10	0 1196	0 1194	0 1268	0 1343	0 2570	0 2424
0 20	0 1153	0 1149	0 1281	0 1417	0 2639	0 2361
0 30	0 1120	0 1113	0 1291	0 1477	0 2692	0 2308
0 33	0 1111	0 1103	0 1294	0 1488	0 2707	0 2293
0 40	0 1093	0 1081	0 1299	0 1527	0 2737	0 2262
0 50	0 1069	0 1056	0 1306	0 1569	0 2778	0 2222
0 60	0 1050	0 1033	0 1311	0 1606	0 2813	0 2188
0 75	0 1027	0 1004	0 1317	0 1652	0 2857	0 2143
1 00	0 0997	0 0967	0 1324	0 1711	0 2917	0 2083
1 25	0 0975	0 0939	0 1329	0 1757	0 2963	0 2037
1 50	0 0958	0 0917	0 1333	0 1792	0 3000	0 2000
2 00	0 0934	0 0884	0 1338	0 1844	0 3056	0 1944
2 50	0 0918	0 0861	0 1341	0 1880	0 3095	0 1905
3 00	0 0906	0 0844	0 1344	0 1906	0 3125	0 1875
3 50	0 0897	0 0831	0 1345	0 1927	0 3148	0 1852
4 00	0 0891	0 0820	0 1346	0 1943	0 3167	0 1831
5 00	0 0880	0 0805	0 1348	0 1967	0 3194	0 1806
6 00	0 0872	0 0793	0 1349	0 1984	0 3214	0 1786
	Multiplicator			$w h^2$	$W h$	

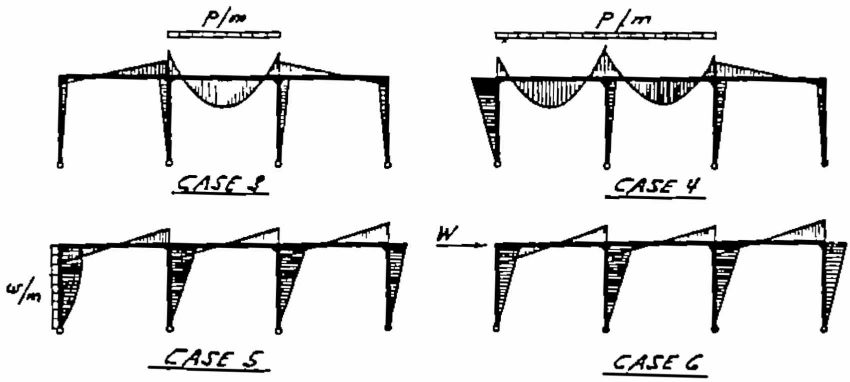
Bending Moments in Continuous Frames with Three Equal Spans



CASE 1 CASE 2

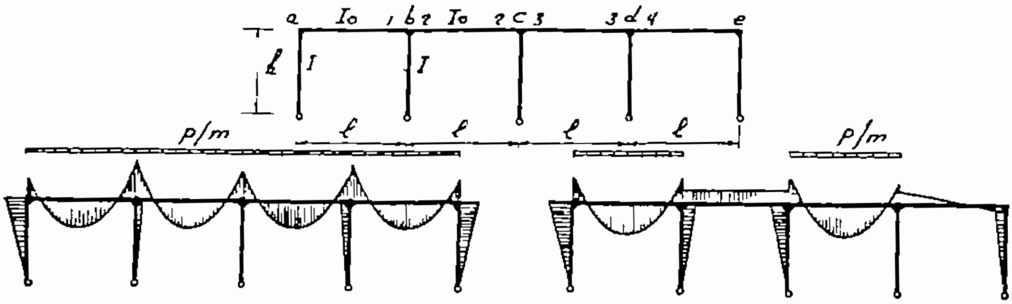
$\kappa = \frac{I_o}{I} \cdot \frac{h}{l}$		0 05	0 10	0 20	0 30	0 33	0 40	0 50	0 60	0 75
Case 1	$M_a = M_d$	313	295	266	242	230	223	207	193	175
	$M_{b1} = M_{c3}$	343	351	362	370	372	375	376	383	386
	$M_{b2} = M_{c2}$	334	334	337	339	340	342	345	347	351
Case 2	$M_a = M_d$	322	312	291	273	268	256	241	228	211
	$M_{b1} = M_{c3}$	323	315	301	291	288	283	276	167	263
	$M_{b2} = M_{c2}$	010	020	035	049	052	060	069	077	088
Case 3	$M_a = M_d$	009	016	025	030	031	033	035	035	035
	$M_{b1} = M_{c3}$	020	036	061	079	083	093	104	112	122
	$M_{b2} = M_{c2}$	323	315	301	291	288	283	276	270	263
Case 4	M_{b1}	348	300	378	391	394	401	408	414	421
	M_{b2}	339	344	352	363	365	371	377	383	390
Case 5	M_a	080	077	073	069	068	066	064	061	059
	M_{b1}	082	080	078	077	076	075	075	074	073
	M_{b2}	083	083	082	082	082	081	081	081	081
	M_{c2}	083	082	081	079	079	078	077	076	075
	M_{c3}	085	087	089	090	091	091	092	093	093
	M_d	088	092	098	103	105	108	111	114	118
Case 6	$M_a = -M_d$	172	177	184	191	192	196	200	204	208
	$M_{b1} = -M_{c3}$	167	167	167	167	167	167	167	167	167
	$M_{b2} = -M_{c2}$	162	157	149	143	141	138	133	130	125

Bending Moments in Continuous Frames with Three Equal Spans



1 00	1 25	1 50	2 00	2 50	3 00	3 50	4 00	5 00	6 00	Multiplicator
153 390 356	135 392 360	121 394 364	101 396 369	086 397 373	075 398 376	067 398 379	060 3o9 380	050 399 384	043 399 386	$pl^2/4$
186 254 102	167 248 113	152 242 121	128 235 134	110 230 144	097 226 151	086 223 156	078 221 160	065 217 167	056 215 172	
034 136 254	032 145 248	030 152 242	027 161 235	024 168 227	022 172 226	020 176 223	018 178 221	015 182 217	013 185 215	"
429 400	435 408	439 414	445 423	449 430	452 434	454 438	455 441	457 445	459 449	"
055 073 082 073 094 123	053 072 082 072 094 127	051 072 082 071 095 130	047 072 083 069 095 135	045 071 083 068 095 138	043 071 083 067 095 140	042 071 083 066 096 142	041 071 084 0oo 096 143	039 071 084 065 096 146	038 071 084 064 096 147	$w h^2$
214 167 119	219 167 115	222 167 111	227 167 106	231 167 103	233 167 100	235 167 098	237 167 097	239 167 094	241 167 093	W h

Bending Moments in Continuous Frames with Four Equal Spans

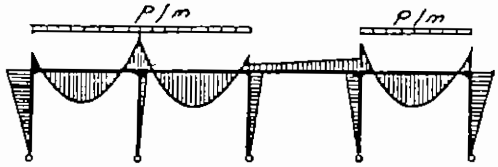


CASE 1

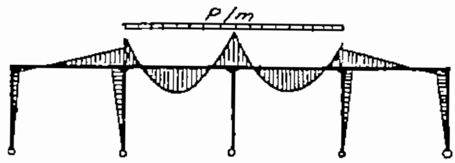
CASE 2

$\varkappa = \frac{I_o}{I} \frac{h}{l}$		0 50	0 10	0 20	0 30	0 33	0 40	0 50	0 60	0 75
Case 1	$M_a = M_a$	313	29[illegible]	266	242	236	222	206	192	174
	$M_{b1} = M_{d4}$	343	351	363	371	373	377	383	386	391
	$M_{b2} = M_{d3}$	334	335	339	344	346	349	353	357	363
	$M_{c2} = M_{c3}$	333	332	330	328	327	326	324	322	319
Case 2	M_a	319	305	281	261	255	243	227	213	195
	M_{b1}	327	321	310	302	300	29[illegible]	289	283	277
	M_{b2}	007	014	028	040	044	051	061	070	082
	M_{c2}	014	025	043	055	059	065	073	080	087
	M_{c3}	319	307	288	273	269	260	250	242	232
	M_{d3}	327	321	312	304	302	297	292	287	281
	M_{d4}	016	030	052	069	074	083	094	103	115
	M_b	006	010	016	019	019	020	021	021	021
Case 3	M_{b1}	344	353	367	379	382	388	396	402	410
	M_{b2}	344	352	367	378	381	387	395	401	409
Case 4	$M_{c2} = M_{c3}$	343	351	363	371	373	377	382	387	391
Case 5	M_a	059	055	050	045	044	042	039	036	033
	M_{b1}	061	059	057	055	055	054	053	053	052
	M_{b2}	063	063	064	065	065	065	066	067	068
	M_{c2}	063	063	063	064	064	064	064	065	065
	M_{c3}	062	062	062	062	061	061	061	060	060
	M_{d3}	062	062	061	061	060	060	059	058	057
	M_{d4}	064	066	068	070	070	071	072	073	073
	M_e	066	070	075	080	081	083	086	089	092
Case 6	$M_a = -M_a$	130	134	141	147	148	151	155	158	162
	$M_{b1} = -M_{d4}$	126	127	127	128	128	128	128	128	128
	$M_{b2} = -M_{d3}$	122	120	113	112	111	109	106	104	101
	$M_{c2} = -M_{c3}$	122	120	117	114	113	112	111	110	108

Bending Moments in Continuous Frames with Four Equal Spans

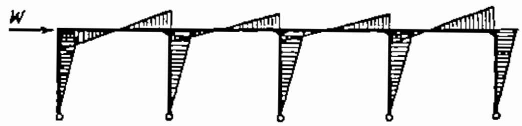


case 3



case 4

case 5



case 6

1 00	1 25	1 50	2 00	2 50	3 00	3 50	4 00	5 00	6 00	Multiplicator
151	133	119	099	084	073	065	058	049	041	
397	402	405	409	412	415	416	418	420	421	$pl^2/4$
370	376	381	389	394	398	402	404	408	411	
315	312	310	306	303	301	299	298	296	294	
171	152	137	114	098	086	076	069	057	049	
268	262	257	249	244	240	237	235	231	229	
098	111	121	136	147	156	162	167	175	181	
097	103	108	115	120	123	126	128	130	132	
218	209	201	191	183	178	174	170	166	162	
272	265	260	253	247	243	240	237	233	231	
129	140	148	160	168	175	179	183	188	192	
020	019	018	016	014	013	011	010	009	008	
421	428	434	443	449	453	457	459	463	466	
419	427	433	442	448	452	456	459	463	465	
397	402	405	409	412	415	416	418	420	421	
028	025	022	019	016	014	012	011	009	008	
051	050	049	048	048	047	047	047	046	046	
069	071	071	073	074	075	075	076	077	077	
066	067	067	068	068	069	069	069	070	070	$w\ h^2$
059	059	058	057	057	056	056	056	056	055	
056	055	054	052	051	050	050	050	048	048	
075	075	076	077	077	078	078	078	079	079	
097	100	103	107	109	111	113	114	116	117	
167	171	174	179	182	184	186	188	190	191	
128	128	128	128	127	127	127	127	126	126	$W\ h$
098	095	093	090	088	086	085	084	082	081	
107	105	105	104	103	103	102	102	102	101	

Multiple frames with moving joints

Joints of multiple frames subject to unsymmetrical loads (e g wind pressure) move horizontally The equations of elasticity are (Fig IV-52a)

$$\Delta\varphi_1 + \Delta\varphi_2 = 0 \quad , \quad \Delta\varphi_1 + \Delta\varphi_{3ac} = 0 \quad \Delta\varphi_2 + \Delta\varphi_{4ac} = 0 \quad \text{etc}$$

Their number is 10 and are sufficient to determine the 10 unknown moments although they need a lot of time

The solution can be much simplified if the unsymmetrical wind load shown in a is replaced by the symmetrical load shown in b and the anti-symmetrical load shown in c

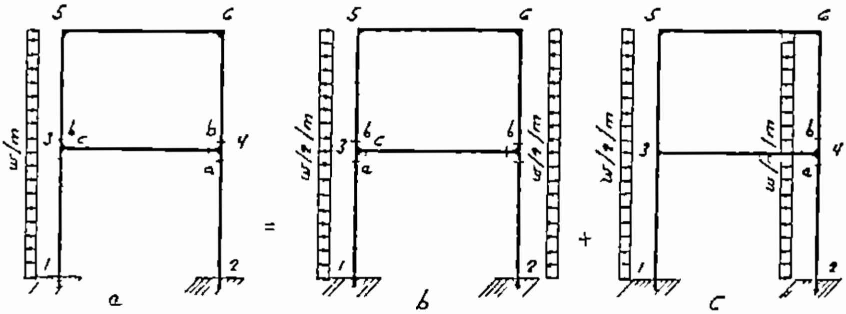


Fig IV-52

Due to symmetry of case b, we have

$$M_1 = M_2 \quad , \quad M_{3a} = M_{4a} \quad M_{3b} = M_{4b},$$

$$M_{3c} = M_{4c} \quad \text{and} \quad M_5 = M_6$$

whereas due to anti-symmetry of case c, we have

$$M_1 = -M_2, \quad M_{3a} = -M_{4a}, \quad M_{3b} = -M_{4b}$$

$$M_{3c} = -M_{4c} \quad \text{and} \quad M_5 = -M_6$$

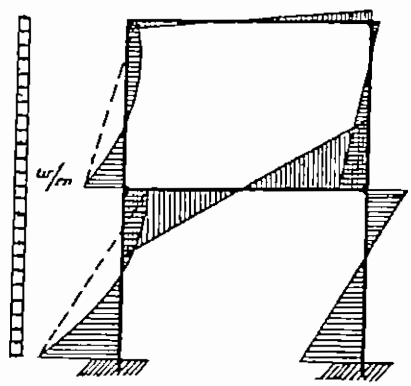


Fig IV-53

The bending moment diagram is shown in figure IV-53

The distribution of the bending moments in building frames depends mainly on the relative stiffness of the members* as shown in figure IV-54

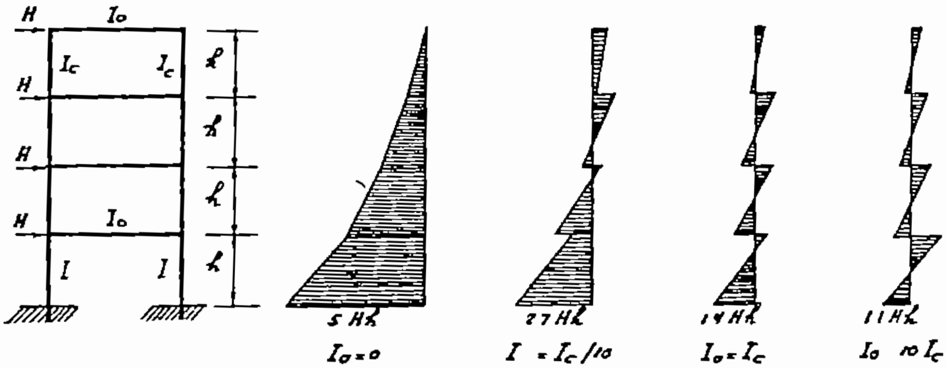


Fig IV-54

In practical cases of building frames, the bending moments can be estimated in the manner shown in figure IV-55

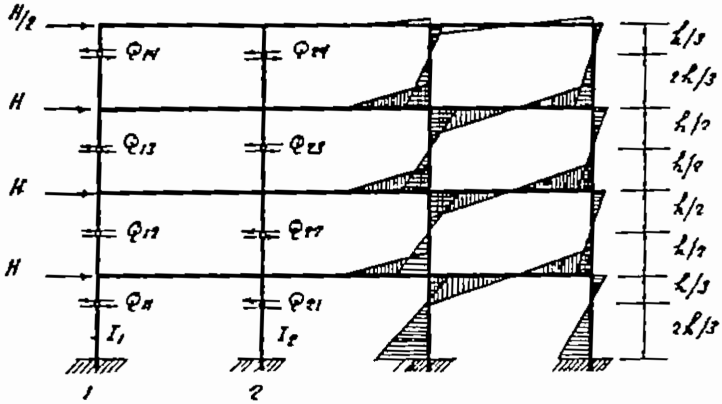


Fig IV-55

One may assume that the total shearing force $\sum H$ acting on any of the floors is distributed on the columns in proportion to their moment of inertia* Hence

$$Q_1 \approx \sum H \frac{I_1}{2(I_1 + I_2)} \quad , \quad Q_2 \approx \sum H \frac{I_2}{2(I_1 + I_2)}$$

The bending moments in the girders at inner supports may be assumed equal, they must keep the equilibrium of the joints as shown in fig IV-56 which shows the equilibrium of the joints at exterior and

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* G Franz Konstruktionslehre des Stahlbetons Springer-Verlag Berlin

interior columns

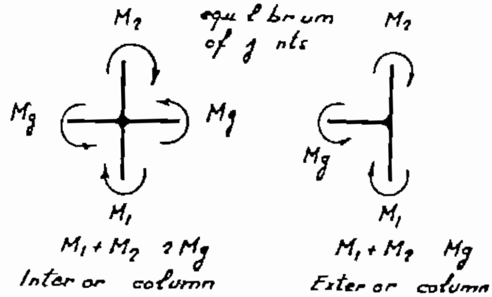


Fig IV-56

Cantilever Frames

Cantilever frames are extensively used in simple statically determinate plane forms as shown in figures IV-57 , 58 and 59

Figure IV-57 shows the cross-section of an exhibition hall at Munich The hall is 40 ms wide and 12 ms high, each of the main supporting elements is composed of two double-cantilever statically determinate frames The upper cantilever arm is 15 ms long while the intermediate one is 10 ms only The intermediate 10 ms in the roof are covered by crytal glass windows The soil being rocky it was possible to use isolated footings of the form shown in figure

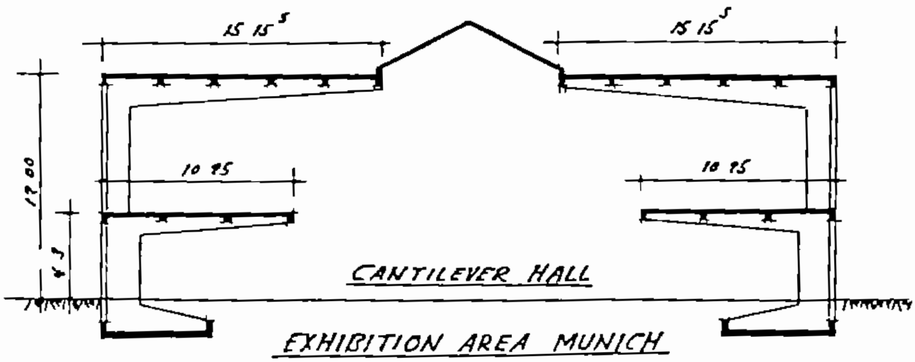


Fig IV-57

Figure IV-58 shows the main stand of the parade area at Cairo The main cantilevers are 21 5 ms long and 3 0 ms between centers Due to the architectural and structural requirements, the folded slab of the roof is located at the bottom of the main cantilever girders which have a maximum depth of 2 87 ms and a breadth of 30 cms(Sec 2-2) Due to the big amount of tension steel required in the outside fiber at the joint between the cantilever and the main column, it was necessary to add diagonal corner bars of the form shown in figure IV-14 to resist the possible splitting tensile stresses

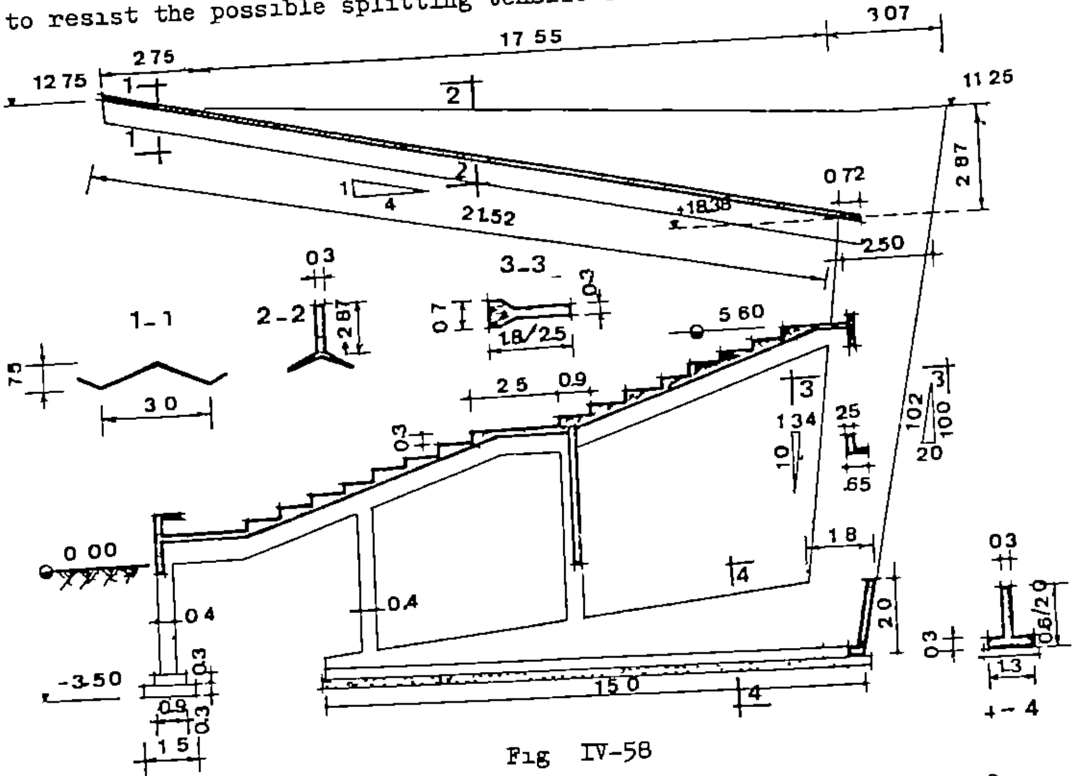


Fig IV-58

Figure IV-59 shows the layout, main dimensions and details of reinforcements of the television and broadcasting cabin in the parade area at Cairo The main supporting element of the cabin is composed of two curved simple cantilever frames at 1 7 ms between centers The max cross-section of each frame is 30x250 cms reinforced by 30 Ø 1 on the tension side During construction, the main tension reinforcement was replaced by 21 Φ 25 mm cold-twisted tor-steel of wh ch, not more than two bars were spliced in any section Due to the high compressive stresses in the girders, each one was provided with an unsymmetrical compression flange as shown in figure The reinforced concrete landings of the staircase together with the foundation slab and cross-rib, and the top slab at level 18 75 join the two girders and prevent any lateral buckling Due to the concentration of the main loads in the cabin, it was necessary to extend the combined footing of the frames to 15 ms The breadth of the footing, necessary to limit the stress on the sandy layers in the site to 1 5 kg/cm^2 was 8 ms

Special Forms of Frames

Simple and continuous frames can further be used in various forms to satisfy certain requirements as shown in the following structures

1) Unsymmetrical simple or continuous saw-tooth frames as shown in figure IV-60

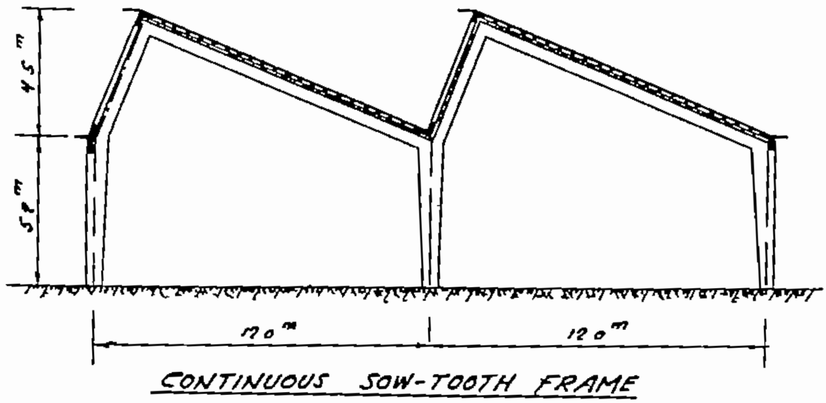


Fig IV-60

2) Continuous frames as shown in figure IV-61

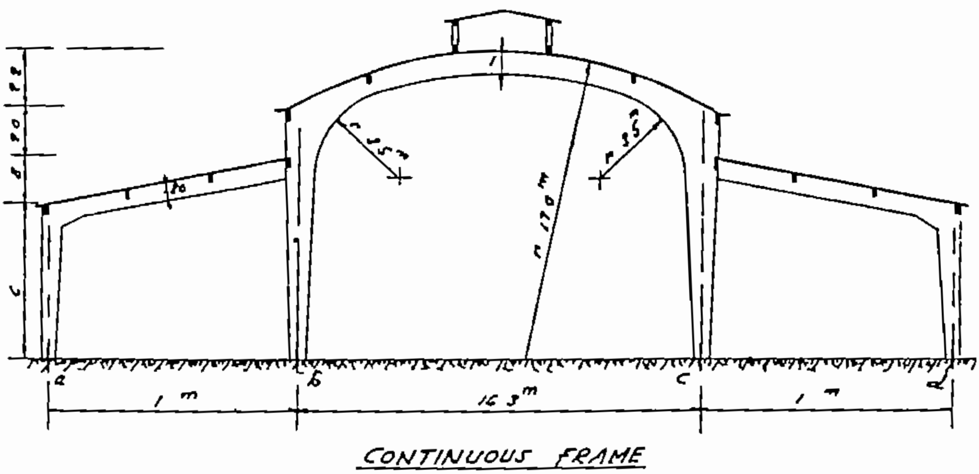


Fig IV-61

The system snown in figure is provided with hinges at the foot of the columns and hence it is five times statically indeterminate for general cases of loading and three times statically indeterminate for symmetrical cases of loading if the main system is chosen statically determinate for example, by removing the hinges at a & d and replacing the hinge at c by a roller

The statically indeterminate values and the required equations of elasticity for both cases are as follows

a) General case of loading e g wind loads (Fig IV-62)

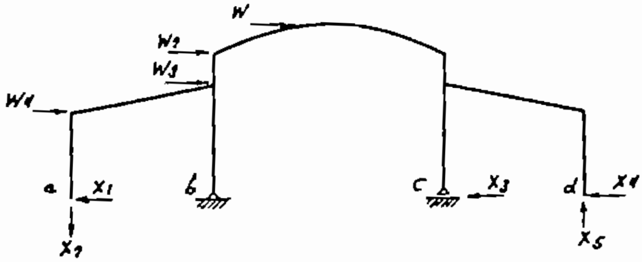


Fig IV-62

$$\delta_1 = 0 = \delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} + X_3 \delta_{13} + X_4 \delta_{14} + Y_5 \delta_{15}$$

$$\delta_2 = 0 = \delta_{20} + X_1 \delta_{21} + X_2 \delta_{22} + X_3 \delta_{23} + X_4 \delta_{24} + Y_5 \delta_{25}$$

$$\delta_3 = 0 = \delta_{30} + X_1 \delta_{31} + X_2 \delta_{32} + X_3 \delta_{33} + X_4 \delta_{34} + X_5 \delta_{35}$$

etc

b) Symmetrical case of loading e g dead loads (Fig IV-63)

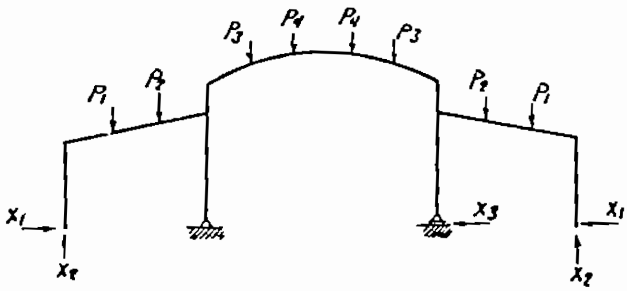


Fig IV-63

$$\delta_1 = 0 = \delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} + X_3 \delta_{13}$$

$$\delta_2 = 0 = \delta_{20} + X_2 \delta_{21} + X_2 \delta_{22} + X_3 \delta_{23}$$

$$\delta_3 = 0 = \delta_{30} + X_1 \delta_{31} + X_3 \delta_{32} + X_3 \delta_{33}$$

The displacements with double indeces δ_{jk} can be determined according to the theory of virtual work from the general equation

$$1 \quad \delta_{jk} = \int M_j M_k \, ds \; / \; EI$$

The general case of loading given under a can however be simplified if it is replaced by a symmetrical case of loading and an antisymmetrical case of loading

For the symmetrical case, figure IV-64, we have

$X_1 = X_4$

and

$X_2 = X_5$

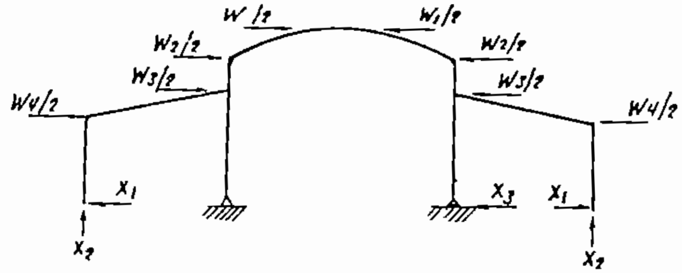


Fig IV-64

Whereas, for the antisymmetrical case, figure IV-65, we have

$X_1 = -X_4$

and

$X_2 = -X_5$

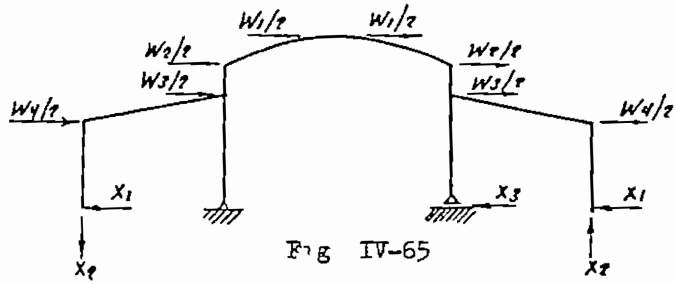


Fig IV-65

It is also possible to choose a statically indeterminate main system by dividing the frame into three units and introducing rollers at the joints of the side halls with the intermediate columns as shown in figure IV-66

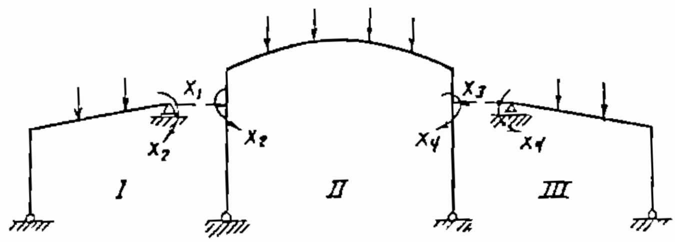


Fig IV-66

In this manner, the main system of the two units I and III is statically determinate while that of unit II is once statically indeterminate

For general unsymmetrical cases of loading the unknowns are X_1, X_2, X_3 and X_4 For symmetrical frames symmetrically loaded, $X_1 = X_3$ and $X_2 = X_4$, while for antisymmetrical cases of loading $X_1 = - X_3$ and $X_2 = - X_4$ If unsymmetrical loads are replaced by symmetrical and antisymmetrical loads, the unknowns are generally not more than two and the problem is much simplified

If a tie is allowed at the foot of the arched girder, the system shown in figure IV-67 may be used

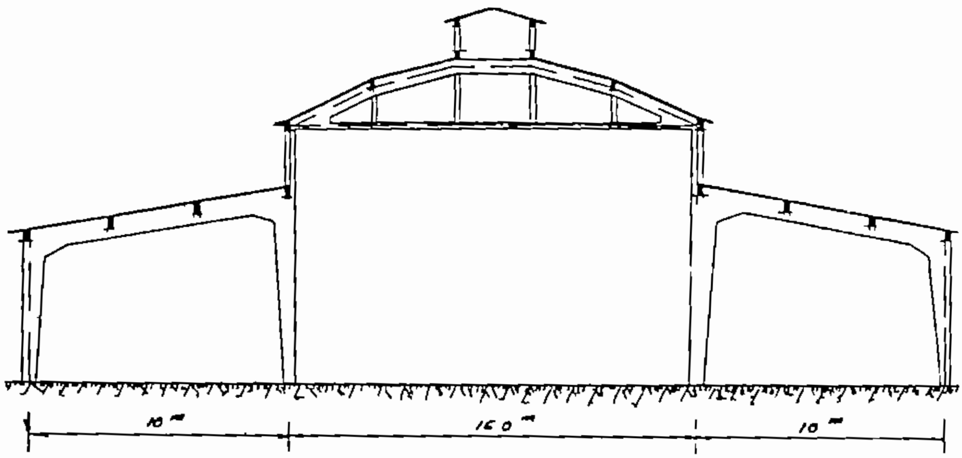


Fig IV-67

Each of the two main side elements may be considered as a two hinged frame while the intermediate polygonal girder with its tie may be considered as freely supported on the two upper slender columns

3) Frames supporting sports stands as shown in figure IV-68

It is recommended in such a simple two-hinged frame to choose its form such that the bending moment along d-e is partly negative at d and e and partly positive at m and that the maximum positive and negative bending moments are approximately equal

Figures IV-69 and 70 give the general layout and details of reinforcement of the arena at the Cairo International Fair-city It shows an ingenious application of a simple idea to give a masterpiece in the in the field of structural engineering In order to give the required form for the main girders, it was necessary to increase their breadth at the outside support as shown

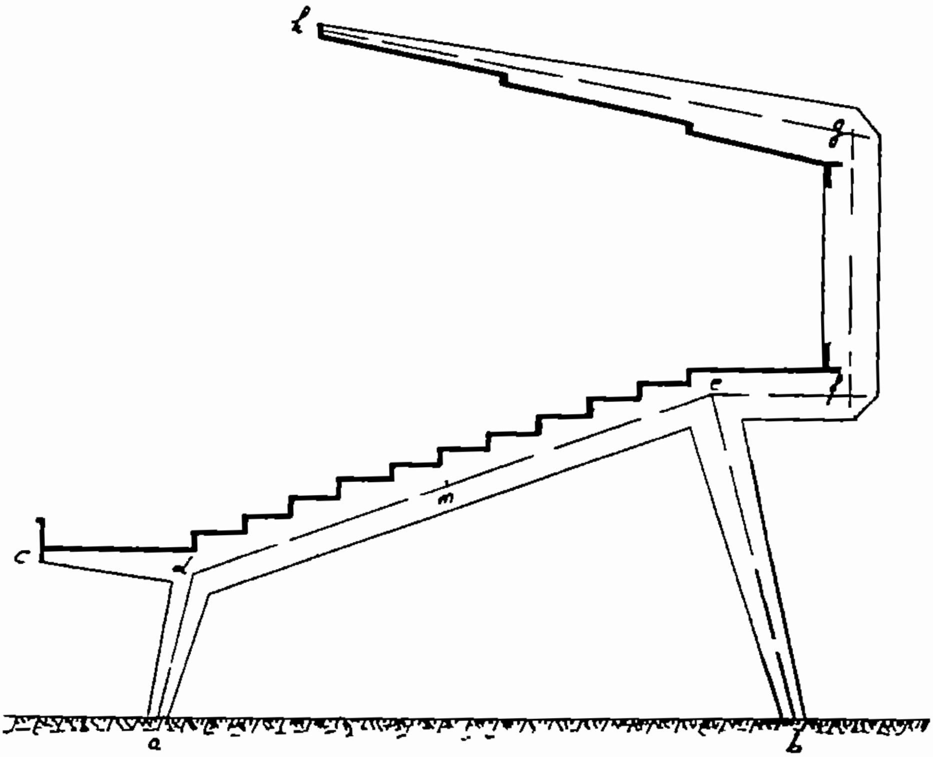


Fig IV-68

4) Radial frames supporting the Planetarium at Cairo **Figure IV-71**

It was required to construct a floor for the Planetarium inside one of the main buildings in the exhibition land at Gezireh. It was further specified that the new structure must be completely separated from the original building and to arrange the supporting columns in such a way that the use of the ground area for exhibition purposes is possible in a convenient manner The hall reserved for this purpose was circular, 30 ms diameter at ground level and 23 5 ms diameter at floor level

In order to satisfy these requirements, it was decided to support the floor on 10 columns arranged radially along the rays from the center of the hall to the main intermediate columns of the existing buil-

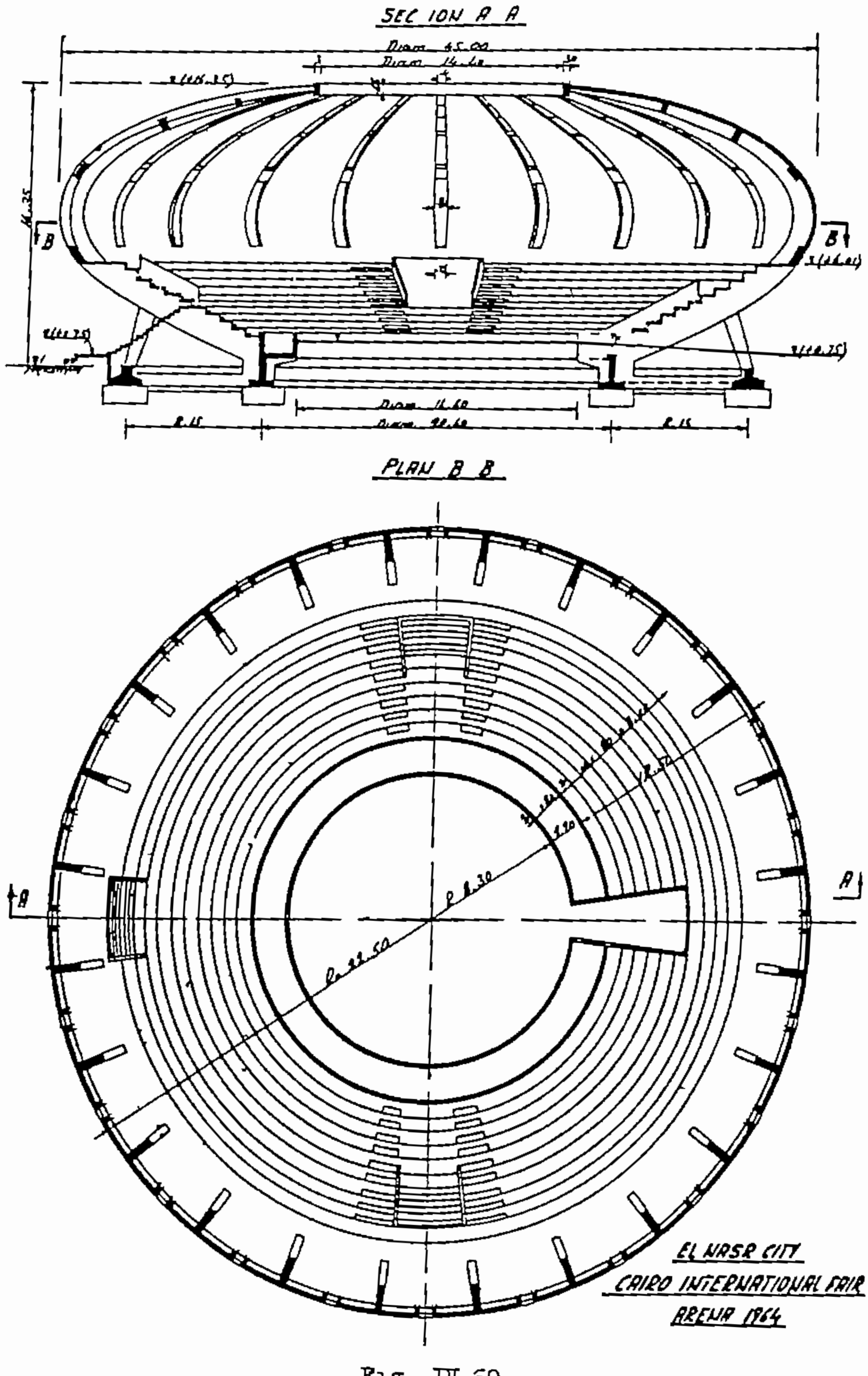

SEC ION A A
PLAN B B
EL NASR CITY
CAIRO INTERNATIONAL FAIR
ARENA 1964

Fig IV-69

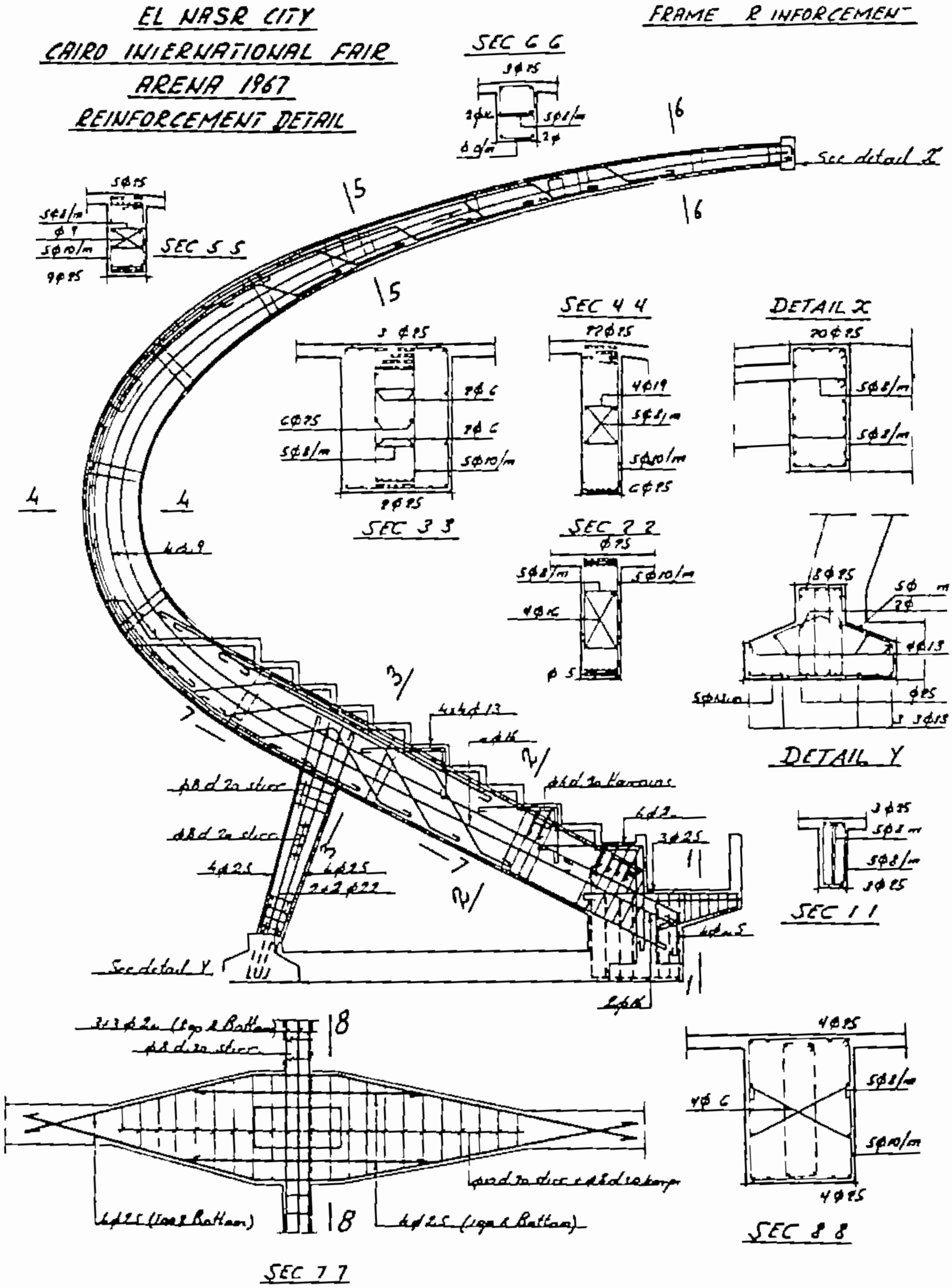

EL NASR CITY
CAIRO INTERNATIONAL FAIR
ARENA 1967
REINFORCEMENT DETAIL
FRAME REINFORCEMENT
SEC 6 6
See detail X
SEC 5 5
SEC 4 4
DETAIL X
SEC 3 3
SEC 2 2
DETAIL Y
SEC 1 1
See detail Y
SEC 8 8
SEC 7 7

Fig IV-70

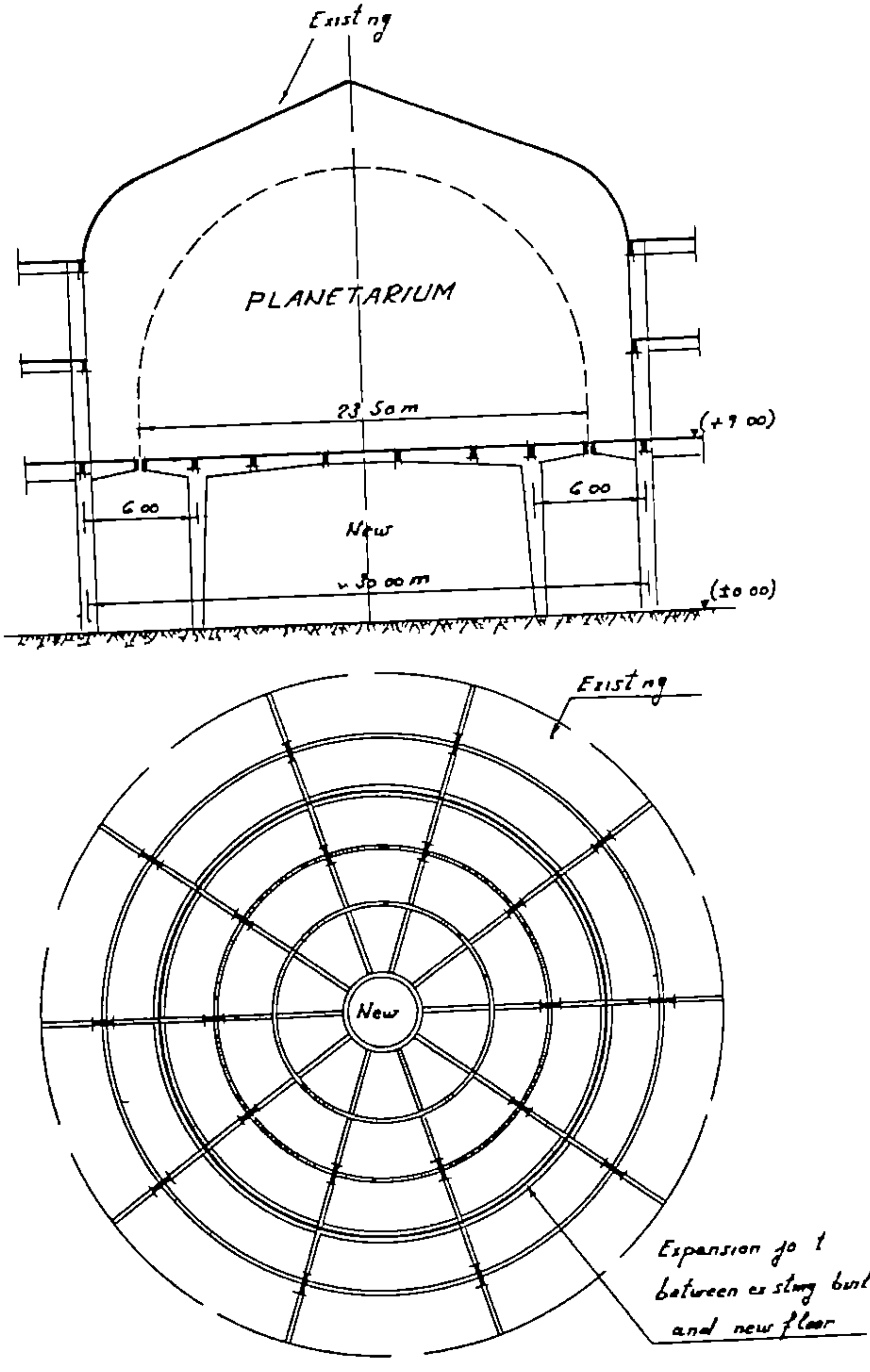

Existing
PLANETARIUM
23.50 m
(+9.00)
6.00
6.00
New
30.00 m
(±0.00)
Existing
New
Expansion joint
between existing building
and new floor

Fig. IV-71

ding and at 6 ms from them in order to have adequate space between the two rows of columns Figure IV-71 shows the general layout of the new floor and its supporting circular beams and radial frames

Due to the loading, the frame ab has the tendency to rotate around the lower hinge b pressing the inner circular ring beam at a ,

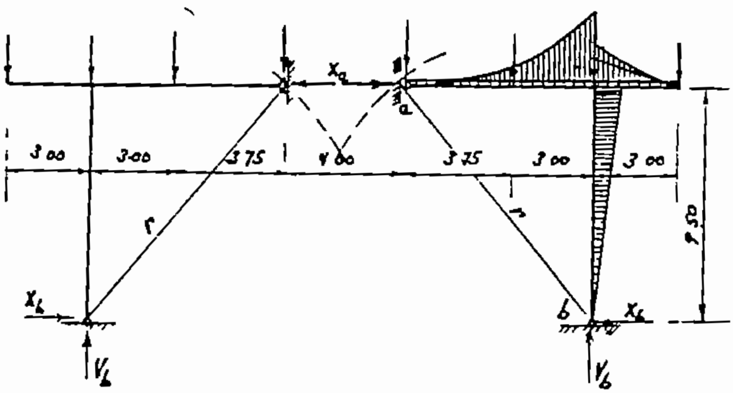


Fig IV-72

so that each of the frames can be assumed as hinged at b and supported on the circular beam at a as shown in figure IV-72

The concrete dimensions and details of reinforcements are shown in figure IV-73

5) Continuous frames with ties as shown in figure IV-74

The systems shown in figure IV-74 represent economic solutions for halls of moderate spans because the slabs and secondary beams are arranged in such a way that the axis of the polygonal girder coincides approximately on the line of pressure of the loads If the spans are equal to or smaller than 10 ms, the effect of the elongation of the tie on the columns is small and may be neglected In this case, the polygonal girder with its tie may be assumed as a shed giving for vertical loads on the girder, vertical reactions on the columns

The internal forces can be determined for one single span both for vertical loads and wind pressure However, adequate top reinforcement must be arranged and well anchored at the supports to resist the connecting moments that are liable to take place

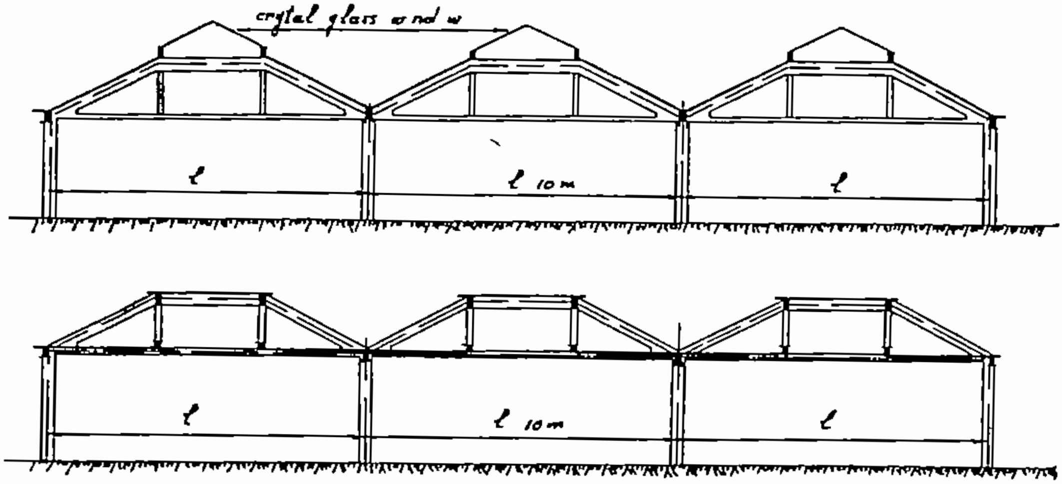


Fig IV-74

For vertical roof loads, the girder may be treated as a two hinged frame with a tie, externally statically determinate and internally once statically indeterminate as shown in figure IV-75 The bending moments due to this case of loading are generally very small giving relatively small concrete dimensions and reinforcements

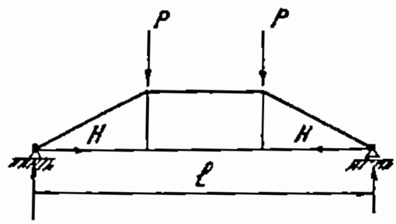


Fig IV-75

For wind loads, the internal forces may be determined for one single polygonal frame without a tie as shown in figure IV-76

The details of reinforcements can be done as shown in figure IV-77

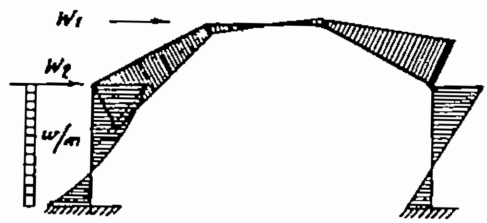


Fig IV-76

DETAILS OF REINFORCEMENTS

OF

A POLYGONAL SHED

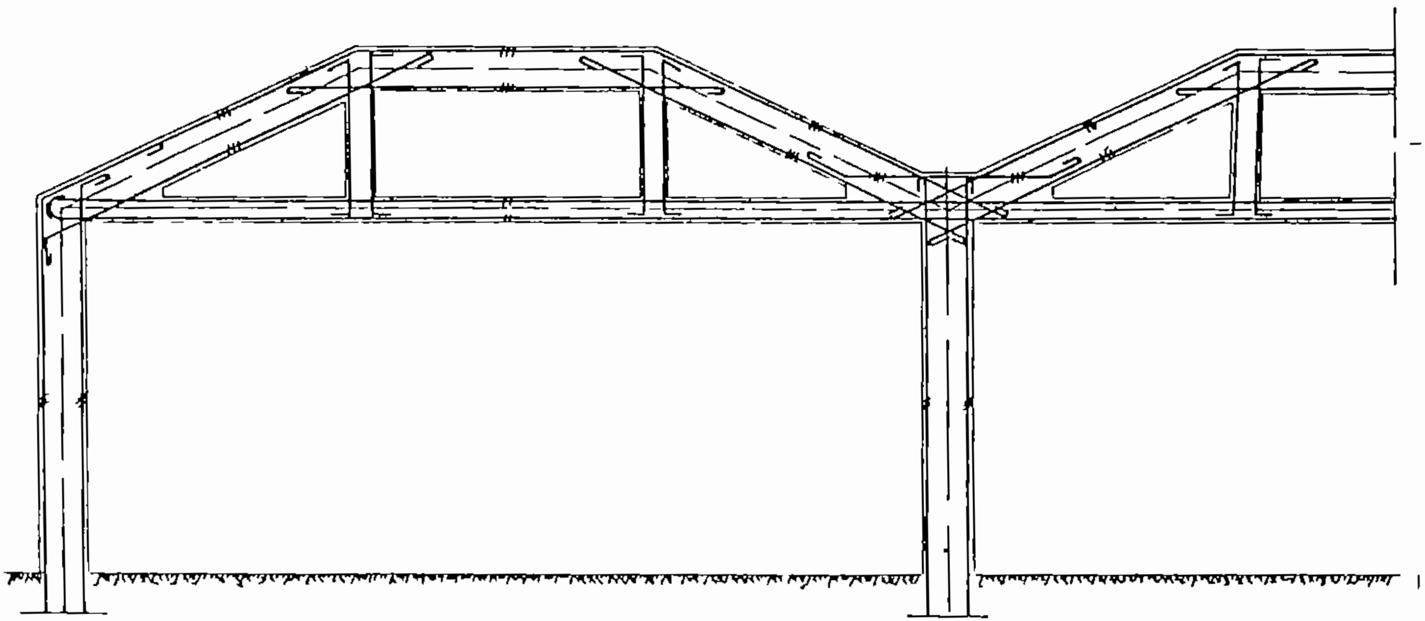


Fig. IV-22

V- VIERENDEEL GIRDERS

If the depth of the main supporting girder is relatively big, a Vierendeel girder as shown in figure V-1 may in some cases be used

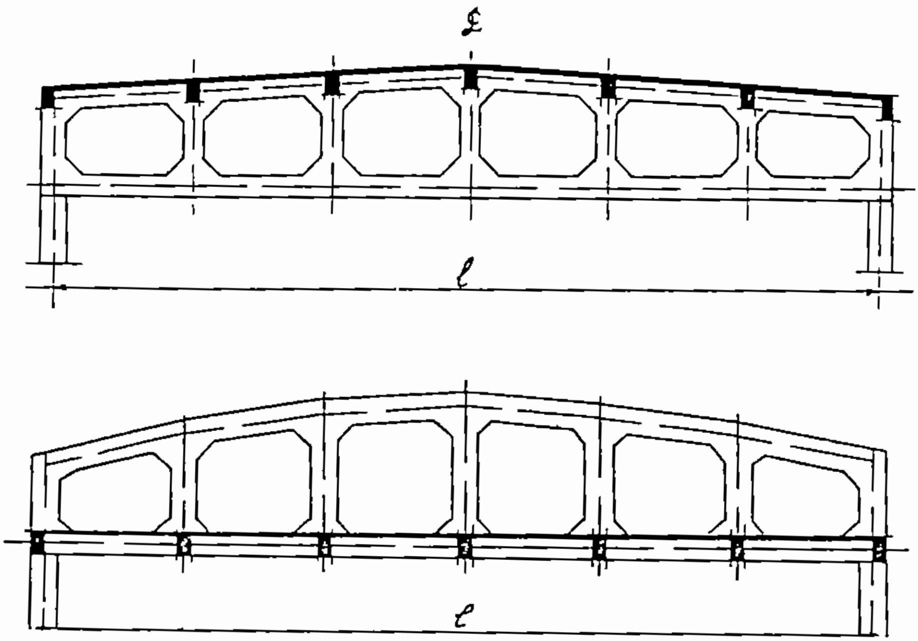
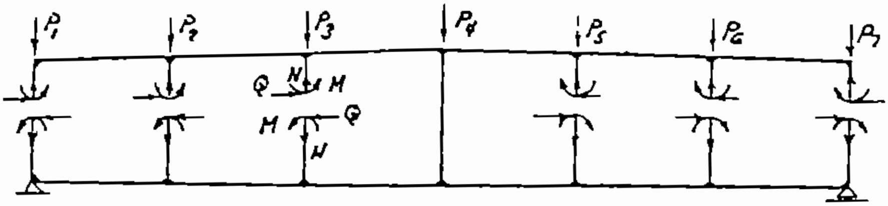


Fig V-1

A Vierendeel girder is a high grade statically indeterminate structure composed of a top chord, a bottom chord and verticals only Internally, it is 3 n times statically indeterminate n being the number of the panels (fig V-2), whereas, externally, it may be statically determinate as in simply supported girders, or indeterminate as in continuous girders

The exact solution of a Vierendeel girder is relatively complicated but essential f the members are thin compared to the height of the girder In reinforced concrete the dimensions of the different members chords and verticals are generally big and the following proposed approximate solution is simple and gives acceptable results for normal Vierendeel girders with parallel chords and verticals having equall stiffness The method can however be applied if the top chord is polygonal slightly curved or inclined (Fig V-1)



Statically determinate main system and statically indeterminate forces

n = 6 hence system is 18 times internally statically indeterminate

Fig V-2

Let us consider a Vierendeel girder subject to concentrated load p_t and p_b acting at the joints of the top and bottom chords (fig V-3a) These loads cause the external bending moments M_I , M_{II} etc at the joints (fig V-3b) and the shearing forces Q_1 Q_2 etc in the panels a_1 a_2 etc (fig V-3c)

The bending moments in the different members - being not directly loaded - are linear (fig V-3d) and can be determined from the external bending moments if the points of zero bending moments in the members are known

In case of symmetrical Vierendeel girders with equally stiff chords and verticals subject to symmetrical loads, the point of zero bending moments in any of the panels of the top and bottom chords and in the verticals may be assumed at the middle If we imagine that hinges are introduced at the above mentioned points of zero bending moments the girder will be internally statically determinate and the internal forces can be calculated as follows

Assuming

$I_t = I_b$ and $I_1 = I_2 =$ etc are the moments of inertia of the top chord the bottom chord and the verticals 1 2 etc

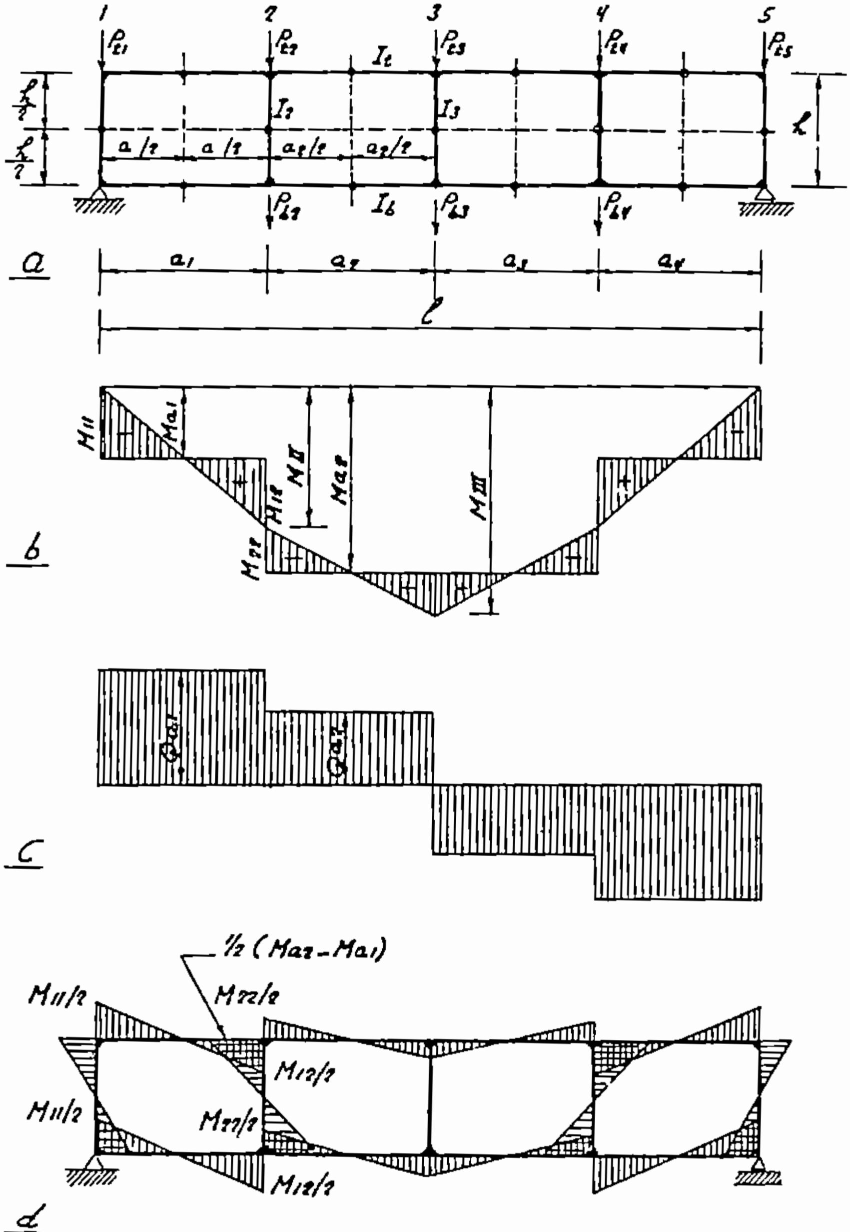


Fig V-3

N_t, N_b, N_1, N_2 etc are the normal forces in chords and verticals

Q_t Q_b Q_1 Q_2 etc shearing forces

M_t M_b M_1 M_2 etc bending moments "

Then (fig V-4)

The normal forces in any of the panels say a_1 are given by

$N_{b1} = - N_{t1} = M_{a1} / h$

tension in the bottom chord and compression in the top chord The shearing force Q_{a1} will be equally resisted by the two chords,

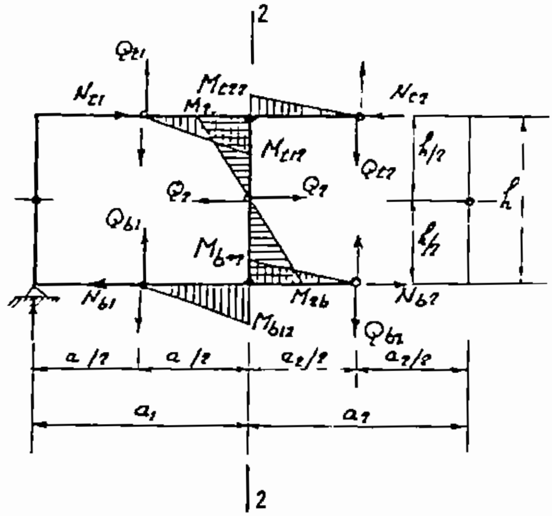


Fig V-4

therefore

$Q_{b1} = Q_{t1} = Q_{a1} / 2$

It has to be noted here that $Q_{b1} = Q_{t1}$ shown in figure express the vertical component of the resultant of the loads and reactions to the left of the vertical through the middle of a_1 and hence their sense is upwards whereas $Q_{b2} = Q_{t2}$ give the vertical component of the resultant of the loads and reactions to the right of the vertical through the middle of a_2 and hence their sense is downwards

The bending moments to the left of vertical 2 are given by

$M_{b12} = Q_{b1}\, a_1/2$ and $M_{t12} = Q_{t1}\, a_1/2$

Therefore

$$M_{b12} = M_{t12} = Q_{a1} \quad a_1 /4$$

Similarly

$$M_{b22} = M_{t22} = Q_{a2} \quad a_2 /4$$

It is recommended to draw the bending moment diagrams on the tension side as shown

The normal force in vertical 2 can be calculated according to figure V-5 from the relation

$$N_2 = Q_{t1} - Q_{t2} - P_{t2}$$

$$= \frac{1}{2} (Q_{a1} - Q_{a2}) - P_{t2}$$

$$= \frac{1}{2} (P_{b2} + P_{t2}) - P_{t2} \qquad \text{i e}$$

$$N_2 = \frac{1}{2} (P_{b2} - P_{t2})$$

The bending moments in the verticals can be determined from the equilibruim of the joints as shown in figure V-6 thus

$$M_{2t} = M_{t12} + M_{t22}$$

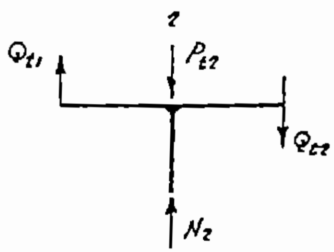


Fig V-5

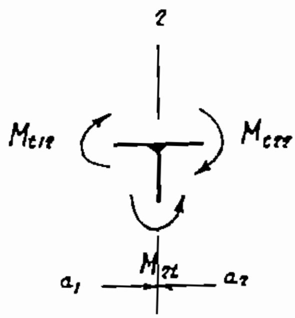


Fig V-6

It has to be noted here that the known moments M_{t12} and M_{t22} are expressed in figure V-6 by arrows giving their sense i e M_{t12} is clockwise (positive) causing tension at the lower fiber and compression at the upper fiber to the left of joint 2, it will therefore be expressed by a clockwise arrow from the tension side to the compression side Whereas M_{t22} is also clockwise but causing tension at the upper fiber and compression at the lower fiber to the right of the same joint The unknown moment M_{2t} at the upper joint of vertical 2 must keep the equilibruim of the two moments M_{t12} and M_{t22} , therefore its magnitude must be equal to their sum and its sense must be anti-clockwise i e causing tension on the left side and compression on the right side of vertical 2

The final bending moment diagrams drawn on the tension side are shown in figure V-4

The shearing forces at the points of zero bending moments in the verticals can be determined by dividing the bending moment at any of the corresponding joints by h/2

The shearing forces Q_2 and the bending moments M_{2t} and M_{2b} acting on vertical 2 can also be determined as follows

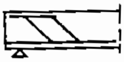
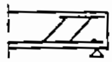
Referring to figure V-4, we find that

$$Q_2 = N_{t1} - N_{t2} = N_{b1} - N_{b2} = -(M_{a2} - M_{a1}) / h$$

$$M_{2t} = - M_{2b} = - Q_2 \, h/2 = \frac{1}{2} (M_{a2} - M_{a1})$$

The bending moments in the chords can be determined graphically by drawing vertical lines through the points of zero bending moments of the chords to meet the sides of the external bending moment diagram Through the points of intersection draw horizontal lines as shown in figure V-3 the diagrams enclosed between these horizontals and the sides of the external bending moment diagram, hatched diagrams give the bending moments to be resisted by the two chords If the chords are of equal stiffness, then each chord will resist half the bending moment

Having determined the bending moments in the chords, the bending moments in the verticals can be determined from the equilibruim of the joints shown in figure V-6

The final bending moment diagram drawn on the tension side for the whole Vicrendeel girder is shown in figure V-3d Such an illustration is very convenient as it gives the tension side and respectively the position of the longitudinal reinforcement in every part of the girder It has been further found that the diagonal tension in a beam is in the direction of the sides of the bending moments diagram if it is drawn on the tension side. Hence, the direction of the diagonal tension corresponding to the bending moment shown in figure V-3d will be in the chords and verticals to the left in the left half , and to the right in the right half , see fig V-9.

If the moments of inertia of the chords or the verticals are not equal, the points of zero bending moments cannot be assumed at the middle of the chords or the verticals and can approximately be determined as follows.

Assume I_t , I_b , I_{v1} , I_{v2} are the moments of inertia of the top chord, the bottom chord and the verticals 1,2 etc , and

$\varkappa_{gt} = \frac{a}{I_t} \cdot \frac{I_t}{a} = 1$ is the relative rigidity of the top chord

$\varkappa_{gb} = \frac{a}{I_b} \cdot \frac{I_t}{a} = \frac{I_t}{I_b}$ " bottom chord

$\varkappa_{v} = \frac{h}{I_v} \cdot \frac{I_t}{a}$ " verticals

If we consider the panel 2-3, length a_2, and assume that the point of zero bending moment in vertical 2 lies at a distance y_2 from the top chord, and that the point of zero bending moment in the chords 2-3 lies at a distance x_2 from vertical 2 fig V-7 it is possible to prove that

$$-\frac{M_{2t}}{M_{2b}} = \frac{y_2}{h-y_2} = \frac{3\varkappa_v + \varkappa_{gb}}{3\varkappa_v + \varkappa_{gt}} = C_t$$

and

$$-\frac{M_{t22}}{M_{t23}} = \frac{x_2}{a_2-x_2} = \frac{3\varkappa_g + \varkappa_{v3}}{3\varkappa_g + \varkappa_{v2}} = C_2$$

So that

$$y_2 = \frac{C_t}{C_t + 1} h = k_t h$$

and

$$x_2 = \frac{C_2}{C_2 + 1} a_2 = k_2 a_2$$

The external shearing force acting in panel a_2 is given by Q_{a2} and is distributed between the top and bottom chords according to the ratios

$$Q_{t2} = k_t \quad Q_{a2}$$

and

$$Q_{b2} = (1 - k_t) \, Q_{a2}$$

Having known Q_t and Q_b in the different panels the other internal forces can be easily determined as shown before

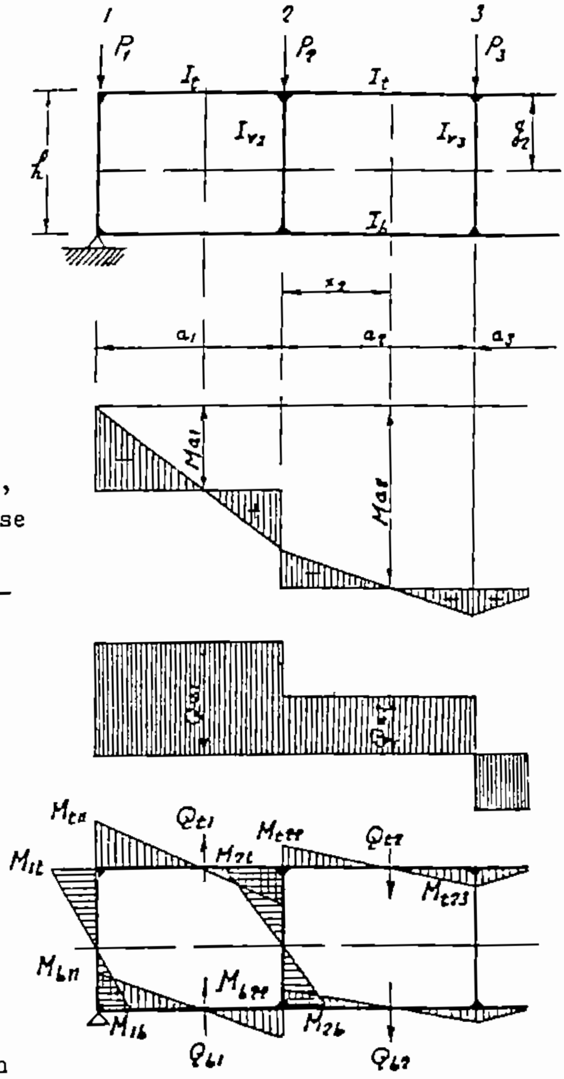


Fig V-7

If the loads act directly on the top or bottom chords, the bending moments due to these direct loads are to be determined for each chord as a continuous beam supported on the verticals Such bending moments are to be added to those due to the concentrated loads acting at the joints Fig V-8

In this case, P_t and P_b are the sum of the direct loads on the joints plus the reactions of the top and bottom chords as continuous beams

The corners of a Vierendeel girder are subject to high secondary stresses so that girders free from corner cracks can only be obtained by careful study of the reinforcements and very good execution The use of haunches in the corners is in this respect , where possible recommended

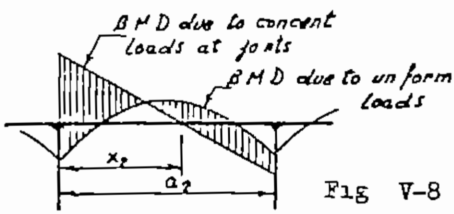


Fig V-8

The bending moments of this system change their sign in every panel so that a simple arrangement of the reinforcements can only be obtained by careful study Its deflection is generally bigger than that of solid girders

It is generally recommended to use this system when it gives the only convenient solution, especially because it is relatively expensive

Example

It is required to determine the internal forces in the Vierndeel girder shown in figure V-9

As the moments of inertia of the top and bottom chords are equal (= I) and the moments of inertia of the verticals are equal (= I_v), then the points of zero bending moments can be assumed at the middle of every panel or vertical

In order to determine the internal forces

1) Draw the external bending moment and shearing force diagrams

2) Draw vertical lines through the points of zero bending moment in the chords (middle points) extend them to meet the sides of the external B M D

3) Through the points of intersection draw horizontal lines The hatched diagrames between these horizontals and the sides of the external B M D give the bending moments in the two chords Eash of the chords is subject to half these values

4) The bending moments at the upper and lower joints of the verticals can be determined from the equalibrium of the joints

5) The shearing forces in the chords and verticals can be determined by diving the moment at the joint by half the length of the corresponding panel

6) The normal forces in any of the chords can be determined by dividing the ordinats of the external B M D at the middle of the chord by the height of the Vierendeel $h = 4^m$

Figure V-10 shows the details of a Vierendeel girder designed by the auther in 1968 It gave the only convenient solution for the control area of the main studio in the television building at Cairo In spite of the change of the sign of the bending moment in every panel and post and the relatively high diagonal stresses, it was possible by a careful study of the reinforcement to get the shown relatively simple formes for the bars

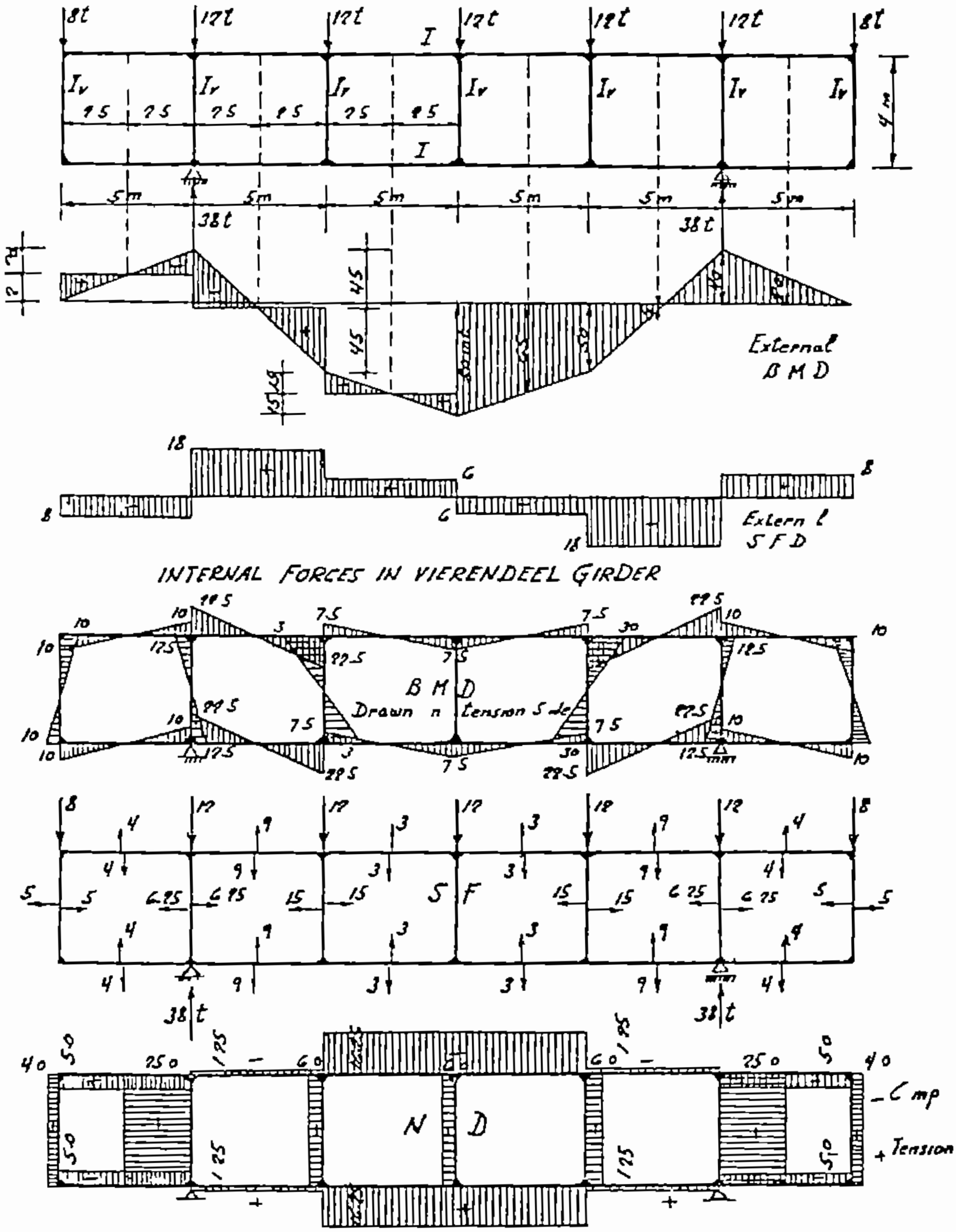

External BMD
External SFD
INTERNAL FORCES IN VIERENDEEL GIRDER
BMD Drawn on Tension Side
SF
ND
- Cmp
+ Tension

Fig V-9

VI- TRUSSES

Trusses in reinforced concrete are seldome used, their shape is generaly chosen similar to those constructed in steel (Fig VI-1)

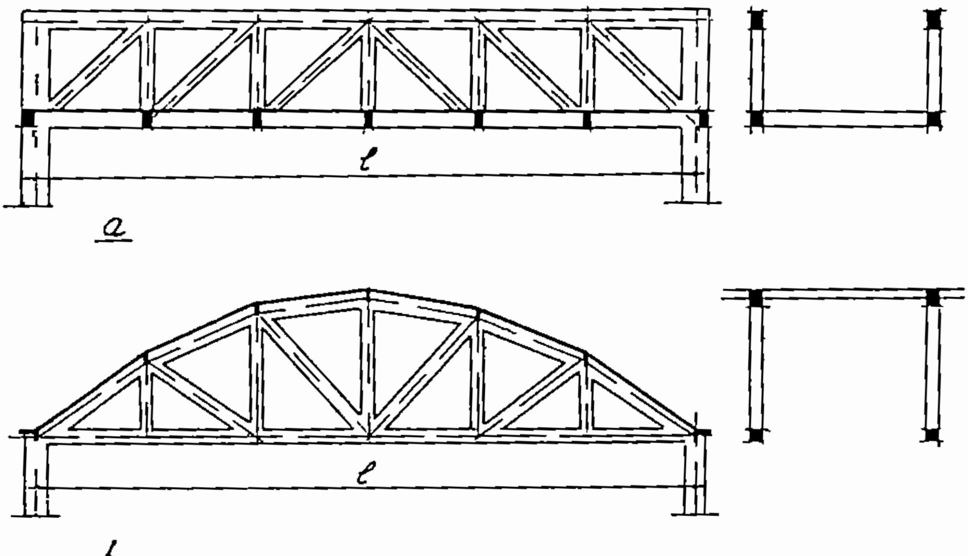


Fig VI-1

Figure VI-1a shows a reinforced concrete truss bridge with parallel chords whereas figure VI-1b shows a reinforced concrete roof truss with polygonal top chord In both trusses, the members are mainly subjected to axial forces and small bending moments due to the rigid joints If, in case a, we dispense with the diagonals, we get a Vierendeel girder with parallel chords, in which case all members must be sufficiently stiff to resist the bending moments, shearing forces and thrusts acting on them as was shown in the previous article If in case b, we cancel the diagonals, choose a stiff polygonal top chord and a slender bottom chord, the main girder may be treated as an arch (or polygonal girder) with a tie, a system which is generally well adapted for reinforced concrete structures as will be shown later

In special cases of saw-tooth roofs in which the north is parallel to the span of the hall the truss may give a convenient solution which adapts very well to the case under consideration (Fig VI-2)

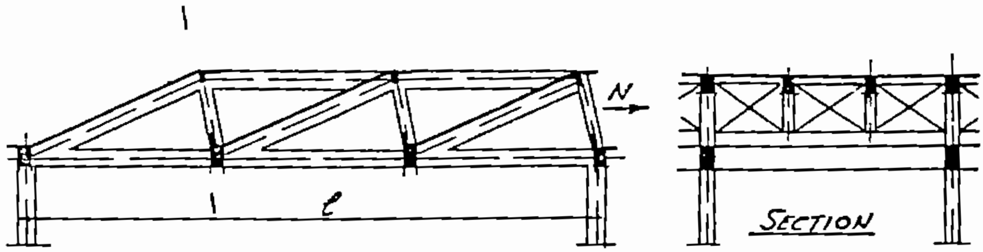


Fig VI-2

It is possible to get a convenient distribution of the forces in the members if the compression diagonals of the truss are chosen such that they bisect the angle between the tension diagonals and the bottom chord as shown in figure VI-3 in which case the force in the tension diagonal is equal to the difference between the tension forces in the bottom chord at the connecting joint Accordingly, the steel reinforcement in the tension diagonal will be equal to the difference

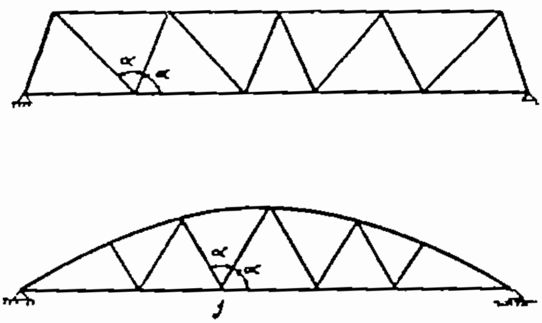


Fig VI-3

between the reinforcements of the bottom chord at the joint as shown in figure VI-4 which gives the details of joint J

The disadvantages of reinforced concrete trusses are

1) Formwork of concrete and form of reinforcements are complicated and hence they are relatively expensive

2) Safety against cracking is low

In order to avoid cracking of the tension members, Finsterwalder proposes to concrete the compression members only first after hardenening and removal of shuttering, the truss can be artificially loaded so as to stress the steel reinforcement in the tension members to high tensile stresses With the truss loaded, the tension members are to be concreted and when hardened, the load can be removed

The internal forces in the members of a truss are

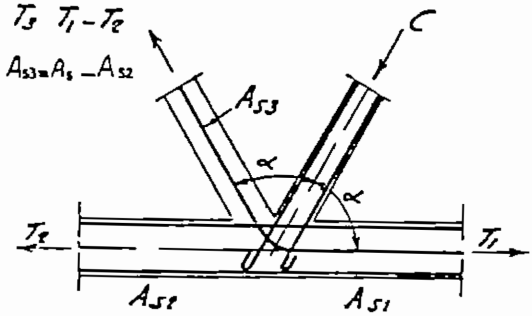


Fig VI-4

1) The axial forces N due to the concentrated loads at the joints

2) The bending moments and shearing forces due to the eventual direct loads on some members of the truss, and

3) The bending moments and shearing forces due to the rigidity of the joints

The axial forces N due to the concentrated loads at the joints may be determined by any of the known methods of trusses with hinged joints

The bending moments due to the direct load on any of the members can be determined by the moment distribution method in which the member under consideration is first assumed fixed at both ends and the fixed-end-moments due to the direct load on the member are determined, then the unbalanced moments in the joints are distributed in the usual way

The shearing force Q acting on any member is given by

$$Q = Q_o + \frac{M_r - M_l}{l}$$

in which

Q_o = the shearing force due to the direct load on the member assumed as a simple beam

M_r and M_l are the final connecting moments at the right and left ends of the member

l = the span of the member under consideration

Based on the assumption of Mohr which states that the deformation of a pin joined truss is not so far from the deformation of the corresponding truss with rigid joints the bending moments and the corresponding shearing forces due to the rigidity of the joints can approximately be determined as follows *

a) Due to the loads on the truss determine the axial forces N in the members assuming hinged joints

b) Draw the Williot-Mohr displacement diagram then determine for the various members, the relative displacements of the ends Δ that are normal to the members

c) Assume that the members are fixed at their ends, and compute the fixed-end-moments $\bar{M}$ due to Δ from the relation

$$\bar{M} = \pm \frac{6EI}{l^2} \Delta$$

The signs may be taken according to the known Grinter's notations in which the clockwise direction around the joint is assumed negative and the anti-clockwise is assumed positive as shown in figure VI-5

Fig VI-5

d) Applying the moment distribution method, the unbalanced moments in the joints can be distributed in the normal way, leading to the final connecting moments in the different members of the truss

Example 1

To illustrate the application of the method, the axial forces and bending moments in the members of the main trusses supporting the saw-tooth roof of the machine workshop in the forging plant at Helwan shown in figure VII-17 are given for the following data

---

* Refer to plane Analysis of Indeterminate Structures by Prof Dr A Shaker Published by the Arab Writer Cairo

span of truss = 18 ms   height = 3 ms   spacing = 6 ms

The inclined slabs are 9 cms thick They are supported by the diagonals of the truss and inclined secondary beams arranged midway between the trusses, so that the slabs are one-way with a span of 3 ms

In this manner, the diagonals of the truss are directly loaded from the slab by a direct uniform vertical load equal to 1 25 t/m'

Due to the own weight of the roof elements and slab loads, each of the lower intermediate joints of the truss is subject to a concentrated load of 32 tons The normal forces in the members of the truss N due to these concentrated loads are shown in figure VI-7a The Williot-Mohr diagram giving the displacement of the joints is shown in figure VI-6 The bending moments due to the displacement of the joints, the direct loads on the diagonals and the final bending moments are shown in figures VI-7b, c and d

It can be seen that the values of the bending moments in the members of such a truss are not big and it may be allowed to design its members for their axial forces plus a bending moment equal to

$$M - \pm (N\ t/10 + 0\ 8\ M_o) \qquad \text{in which}$$

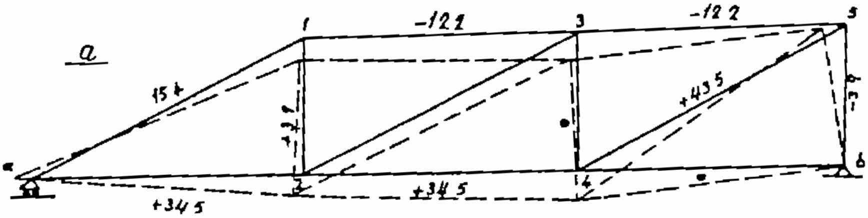
t = the thickness of the member in the plane of the truss

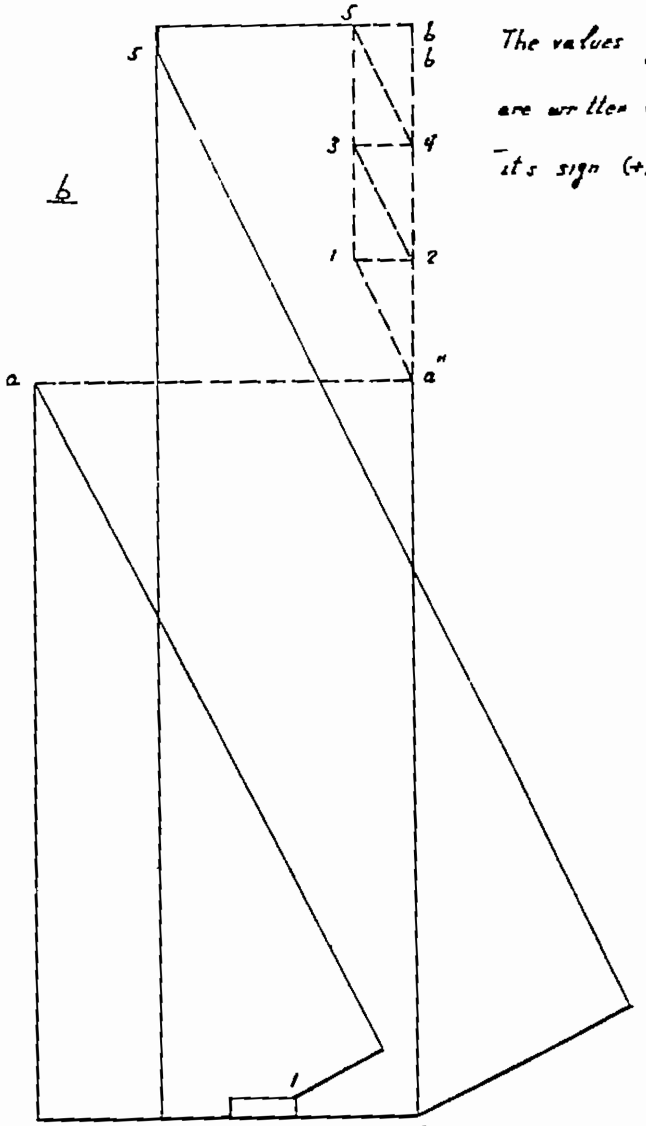
M_o = The biggest maximum bending momentin the diagonals considered as simple beams under their direct load

Figure VI-8 shows the general layout and main dimensions of the workshop and figure VI-9 snows the details of the saw-tooth trusses used as main supporting elements of the roof of the structure

In order to have a convenient shape for the members of the truss and a relatively low percentage of the steel reinforcement in the tension members, the top and bottom chords, the diagonals, and the verticals have been cheosen with the same concrete dimensions for each of tne three groups

Due to the direct loads of tne slabs on the diagonals and the rigid joints of the truss, the members are subject to bending moments M and shearing forces Q These bending moments being small relative to the axial forces N, the members are reinforced with symmetrical straight bars only (i e no bent bars are used) This is explained in the following





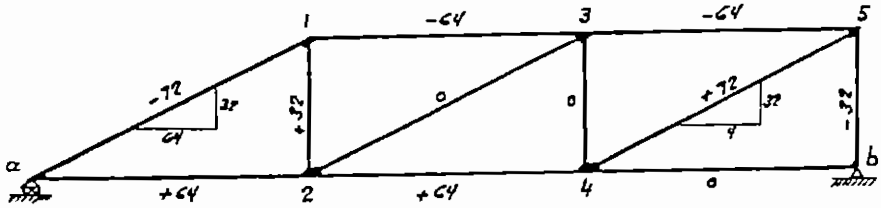
The values f δ multipled by 10^2 are wr tten on each member with its sign (+) means elongat on

Pont (2) fixed point

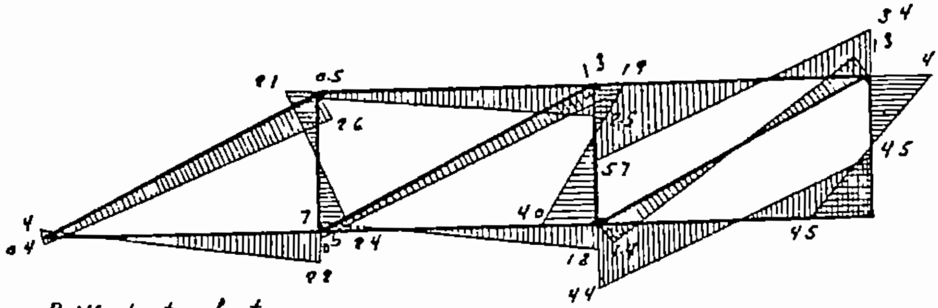
Member (2-4) fixed d rect on

Scale 1cm 0 1 cm

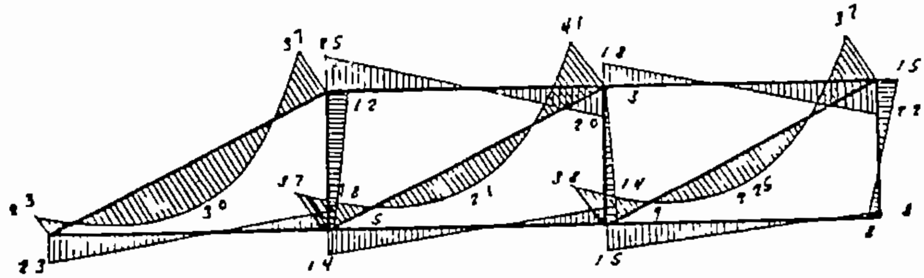
Fig VI-6



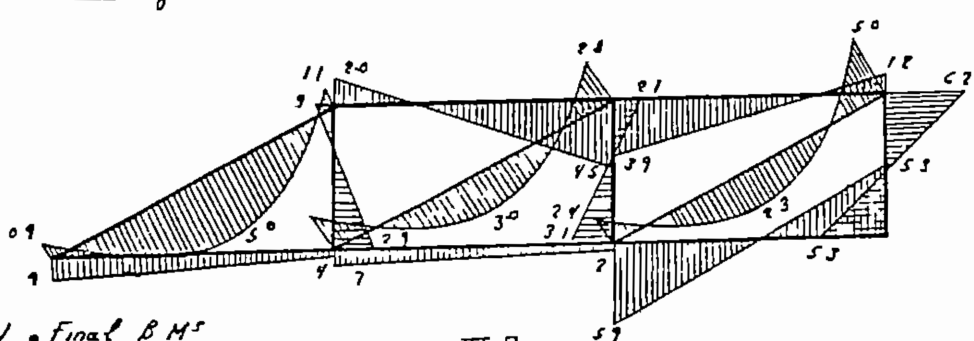
a Axial forces



b B Ms due to elastic deformations



c B Ms due to direct loading



d - Final B Ms

Fig VI-7

If N is compressive and the normal stress σ due to M and N is high say 50 kg/cm^2 while the corresponding shear stress τ calculated from the relation

$$\tau = Q\,S \,/\, I\,b \qquad \text{where}$$

S = statical moment of area of the part of the section above the plane under consideration about the c g axis,

I = moment of inertia of section about c g axis, and

b = breadth of section under consideration

is equal to say 15 kg/cm^2 , then the principal diagonal tensile stress σ_1 is given by

$$\sigma_1 = -\frac{\sigma}{2} + \sqrt{\frac{\sigma^2}{4} + \tau^2} = -\frac{50}{2} + \sqrt{\frac{2500}{4} + 225} = -25 + 29\ 2$$

$$= 4\ 2 \ \text{kg/cm}^2$$

which is relatively low and does not need any diagonal reinforcement

If N is tensile and the normal stress due to M and N is high, say 80 kg/cm^2, and the shear stress τ is say 12 kg/cm^2 then the principal tensile stress σ_1 is

$$\sigma_1 = \frac{80}{2} + \sqrt{\frac{6400}{4} + 144} = 40 + 42 = 82 \ \text{kg/cm}^2$$

its inclination to the axis of the member is given by

$$\tan 2\alpha = -2\tau/\sigma - 2 \times 12/80 = 0\ 3$$

which means that

$$2\alpha \cong 20 \qquad \text{and} \qquad \alpha \cong 10^\circ$$

i e the principal tensile stress is nearly parallel to the axis of member and again here no bents are required

It has to be further noted that the tension bars are anchored in the direction forming an obtuse angle with the bar because if the are anchored so that an acute ang is created, undesirable high splitting tensile stresses σ_{sp} are developed (Fig VI-10)

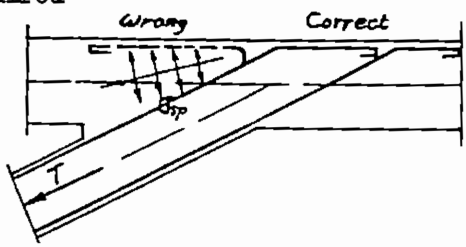


Fig VI-10

Example 2

In the following example we show the use of a truss, rigidly connected to the columns, as the main supporting element of a heavily loaded girder of relatively big span

Fig VI-11a shows the general layout of the stage-roof of Heliopolis Cinema and Theater It is about 42 ms long and 12 5 ms maximum width The reinforced concrete roof slab, 12 cms thick, is supported on cross-girders of 12 5 ms maximum span and having cantilever arms of 7 0 ms maximum projection of the form shown in figure VI-11b the distance between the center-lines of the cross-girders is 5 ms The slab in the projecting part is arranged at the lower fiber of the cantilevers so that they act as T-beams The cross-girders are supported at their outer edges on two columns creating a couple to fix the girders and reduce their bending moments their details of reinforcement are shown in figure VI-11b

This stage was originally constructed for the national theater of the U A R The main supporting element was a frame, 31 ms clear span and 11 25 ms clear height as shown in figure VI-12 In order to reduce the bending moments due to the own weight of the girder of the frame, 3 openings 1 6 ms high and 10 20 ms total length were arranged at the middle third of the span The effect of the possible concentration of the stresses in the upper outside corner of the frame due to the change of direction of the tensile forces has been reduced by bending the tension reinforcements around relatively big radii increasing gradually from row to row as shown in the details given in figure VI-12

During construction and after executing foundations, it was decided to raise the level of the roof by 3 5 ms in order to have the possibility of hanging timber floor for operation and control purposes Due to this change and because of the big loads on the main girder , it was decided to replace it by a truss rigidly connected to the columns in the form shown in figure VI-13 which gives full concrete dimensions and details of reinforcements

In order to reduce the required area of reinforcing steel and to have a good distribution of the tension cracks high grade steel was chosen for the tension reinforcements of the cross-girders, the main frame and the truss

To avoid splices, the main reinforcing bars have been supplied with their full required length although they were relatively big

VII - SAW - TOOTH ROOF STRUCTURES

In big covered halls where a uniform distribution of natural light, that is not possible from windows in outside walls, is required, the saw tooth roofs in which the light from the windows is directly reflected by the roof inside the hall, gives a convenient solution, fig VII-1

For industrial buildings, it is generally recommended not to allow

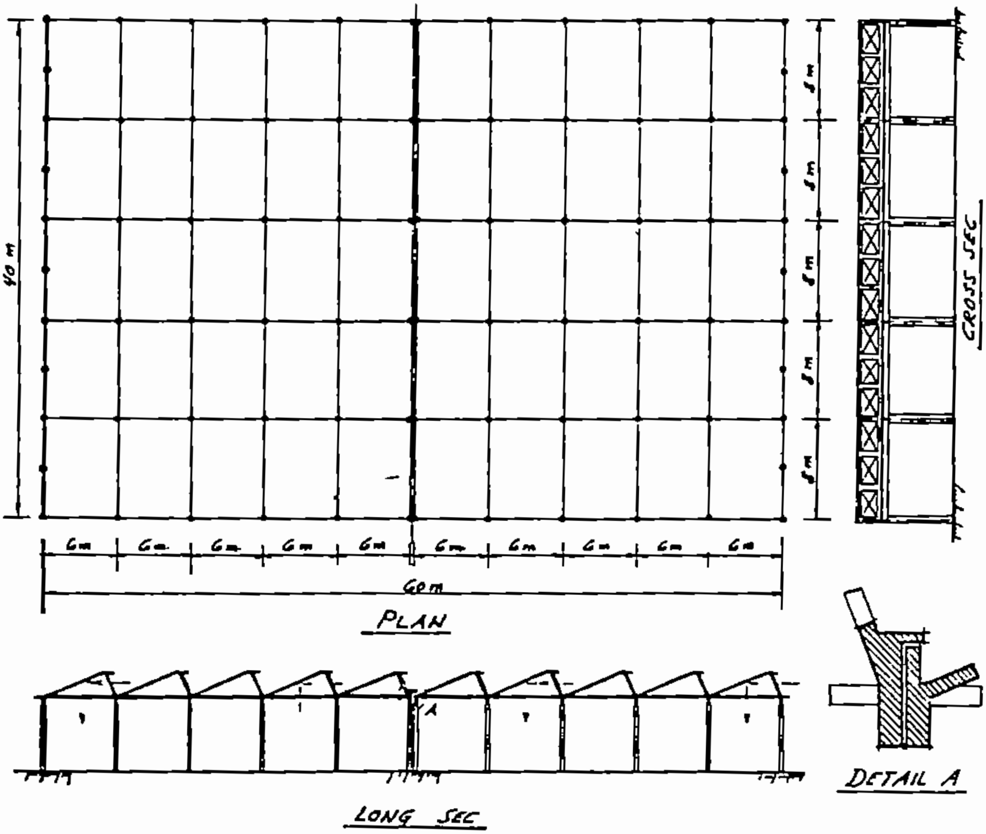


Fig VII-1

any direct sunrays to fall inside the hall for which purpose, saw - tooth north light roofs, in which the windows are arranged to face the north, are used

The convenient slope of the roof slab lies between 20° and 30° with the horizontal so that the concreting can be done without double shuttering The windows are either vertical or inclined with a maximum angle of c a 15° with the vertical

In order to have sufficient uniform light inside a hall, the area of the windows must be bigger than 20% of the horizontal area and the spacing between them to be maximum 10 ms The steel sections required for the windows being proportional to their area, it is recommended to divide the windows by reinforced concrete posts at 2 to 3 ms between centers

The saw-tooth slab type shown in figure VII-2 gives the simplest

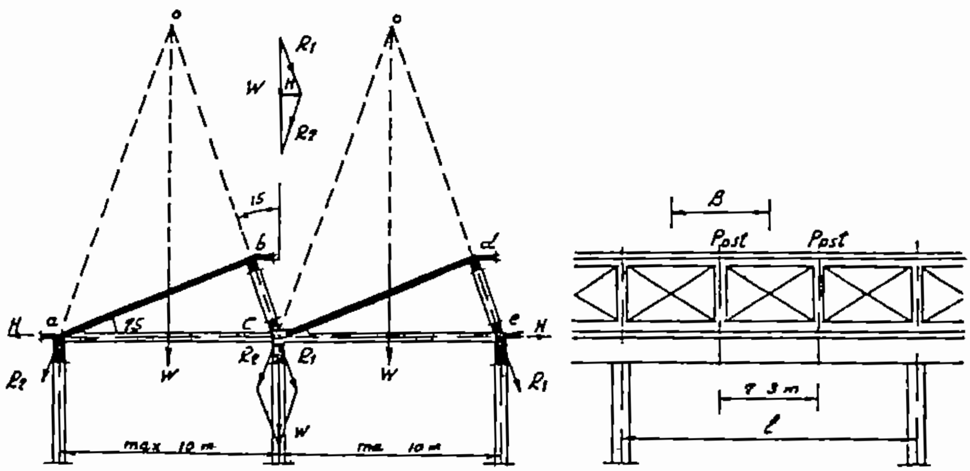


Fig VII-2

form of this system Its structural elements are

The inclined slab supported at its top edge by the ridge beam and at its bottom edge by the Y -beam, the upper continuous ridge beam supported by the inclined posts which are again supported on the Y - beam and, the continuous Y - beam supported by the main columns If the inclination of the posts is big, a tie is to be arranged to resist the horizontal thrust of the system

As the rain water is accumulated at the lowest point of the slab, it is essential to choose the form of the Y - beam so that there is sufficient space for the rain water and the necessary slopes of the gutter (Fig VII-3)

The statical behavior of the system can be shown if we consider the equilbruim of a frame composed of a slab of breadth B hinged at a and supported at b on the post bc (Fig VII-2) Assuming that the slab thickness

$$t = 16 \text{ cms}$$

the post bc 20 x 16 cms

the distance between centers of posts

$$B = 800/3 = 266 \text{ cms}$$

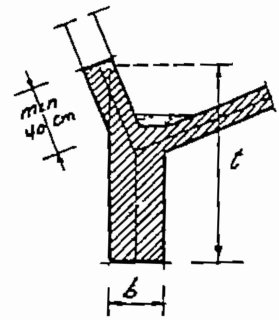


Fig VII-3

the span of the slab l_s = 2 5 length of post l_p

then, the moment of inertia of the slab is

$$I_s = 266 \times 16^3/12 = 91130 \text{ cm}^4$$

and, the moment of inertia of the post is

$$I_p = 20 \times 16^3/12 = 6830 \text{ cm}^4$$

The relative stiffness $\varkappa$ of slab to post is therefore given by

$$\varkappa = \frac{I_s}{l_s} \quad \frac{l_p}{I_p} = \frac{91130}{2\ 5 \times 6830} = 5\ 35$$

This means that the post is very slender relative to the slab and can be assumed as a pendulum that can resist axial forces only i e hinged at both ends b and c Therefore any of the units of the saw-tooth may be assumed as three hinged

Assuming further that the dead weight plus the superimposed loads on slab ab for a breadth B is W, then the reaction at b is equal to R_1 acting in the direction of cb and the reaction at a is equal to R_2 acting in the direction of ao where o is the point intersection of W and R_1

Considering now two adjacent panels of the saw-tooth , we find that the reactions at the intermediate support are equal to R_1 from panel abc and R_2 from panel cde, their resultant is again vertical and equal to W The horizontal components of R_2 at a and R_1 at e are the same and equal to H If the posts are vertical then H = o and if their inclination with the vertical is small ($< 15^\circ$), then H is small and can be easily resisted by the outside columns If the inclination of the posts is big, H is big and arranging a tie is recommended

The previous investigation shows that the inclined slab may be considered as simply supported The reaction at its top edge can be assumed in the direction of the supporting posts (Fig VII-4)

The ridge beam is continuous over the posts It carries in addition to its own weight, the reaction R_1 of the slab acting along its axis

The posts carry the reactions of the ridge beam they can be assumed as axially loaded

The intermediate Y-beam is continuous over the main columns, it carries in addition to its own weight the reactions of the slab, R_2 at its lower edge as a uniform load and R_1 as concentrated loads transmitted through the posts If the intermediate posts in a span are two or more, the load on the beam may be assumed uniform and vertical For resisting the field moments of the Y-beam, the section may be assumed as rectangular with breadth b and depth t (fig VII-3), whereas, for resisting the connecting moments, the breadth is b and the theoretical depth over the columns is equal to the distance from the center of the tension steel over the supports to the lower surface of the compression zone

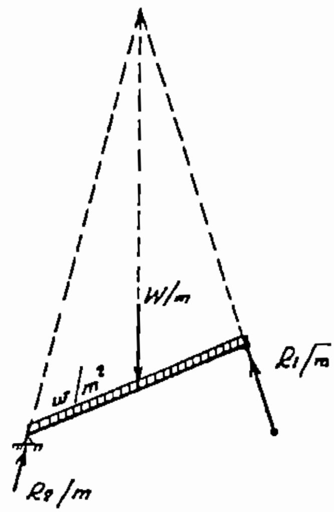
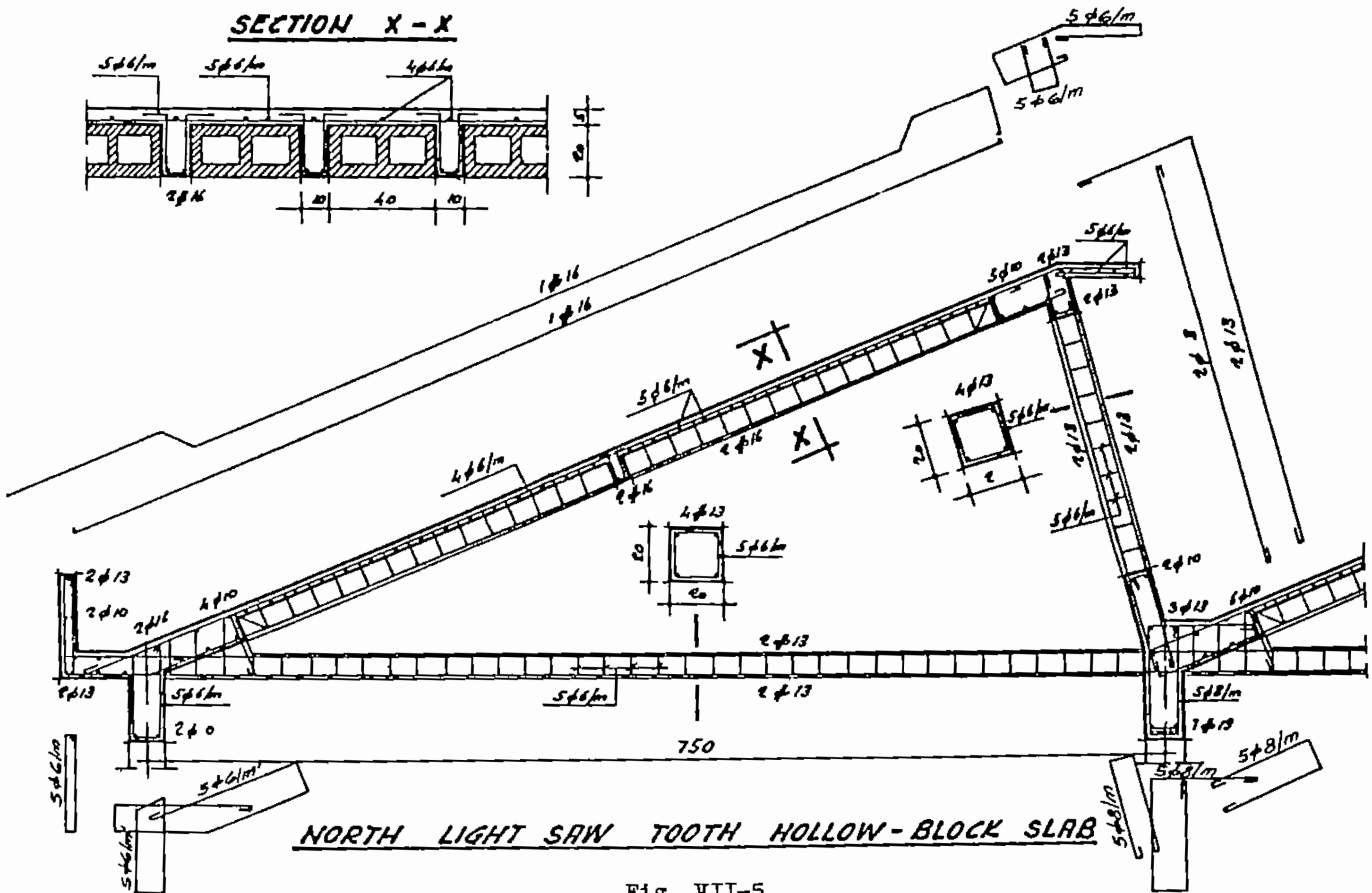


Fig VII-4

The outside beams are continuous over the outside columns, the load on the beam at a (fig VII-2) is uniform, inclined and equal to R_2, the loads on the beam at e are concentrated, equal to the loads on the posts and act in their direction If the inclination of the post is big it is recommended to arrange a horizontal beam as shown

From the economic point of view a one way solid slab as that shown in figure VII-2 may be recommended if its thickness is smaller than or equal to ~16 cms which corresponds to a distance between the windows $\ll 5$ ms For bigger spans, a one way hollow-block or ribbed slab may be used Cross ribs having the same cross-section and reinforcement as the main ribs must be arranged for a slab span < 8 ms one cross rib



NORTH LIGHT SAW TOOTH HOLLOW-BLOCK SLAB

Fig VII-5

and for 8 m $< l <$ 10 ms two ribs (Fig VII-5)

For big saw-tooth spans, it is possible to get very economic solutions by arranging secondary inclined beams @ 2-3 ms, supported at their upper edge over the posts and at their lower edge over the Y-beams, giving a one way slab whose main supporting direction is normal to the inclined direction (refer to fig III-15) In this manner, the thickness of the slab is 8 to 9 cms For the mentioned structure, the span of the saw-tooth is 10 ms, the spacing between the secondary inclined beams is 2 5 ms which means that the slab is one way and continuous over the secondary beams Its span is 2 5 ms and its total load normal to the roof is max 350 kg/m^2 giving a thickness of 8 cms

The inclined secondary beams may be assumed simply supported As the posts are vertical, then the reactions are also vertical and the max bending moment is given by (Fig VII-6)

$$M_{max} = wll/8$$

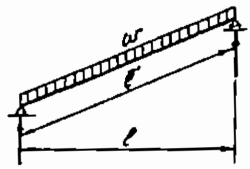


Fig VII-6

where

w = total vertical load per meter beam

l = horizontal span and l = inclined span of beam

In our case l = 10 ms and the beam is 20 x 63 cms only because it behaves as a T-beam

The average thickness of the slab and secondary beams is therefore

$$8 + 20 \times 55/250 = 8 + 4\ 4 = 12\ 4 \text{ cms only}$$

In some cases, e g dusty and smoky halls, a plane roof is required in such cases the beams may be inverted (refer to figs VII-17 & VII-21) , but becasue the slab lies in the tension zone of the beam, the sections behave as rectangular and need bigger depths In some cases it may be convenient to choose a beam of variable depth as shown in figure VII-7

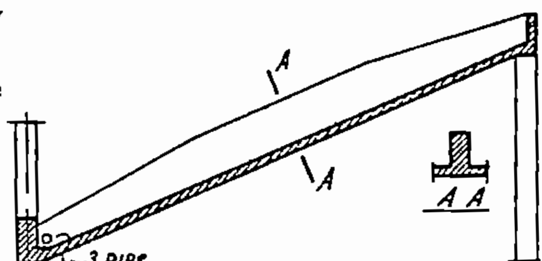


Fig VII-7

In order to be able to drain the rain water, a steel pipe of 3 min diameter must be concreted inside the web

The main columns are to be calculated for the vertical reactions of the Y-beams plus the bending moments due to wind, these may approx be assumed as shown in fig VII-8a, if the columns are of equal moment of inertia and the wind loads are assumed concentrated

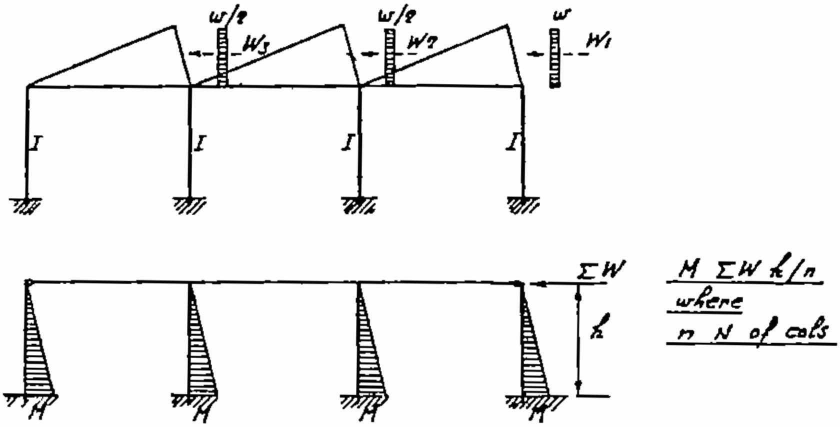


Fig VII-8a

If the wind loads w are assumed uniformly distributed we get

For two columns of equal moment of inertia (Fig VII-8b)

$$X = \frac{3}{16}\,wh\ ,$$

$$M = -\frac{5}{16}\,wh^2\ ,$$

$$M_1 = +\frac{3}{16}\,wh^2\ ,$$

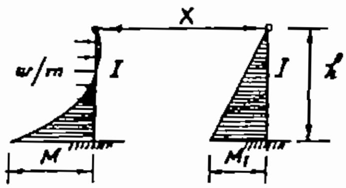


Fig VII-8b

For three columns where the moments of inertia are not equal we get (Fig VII-8c)

$$X_1 = \frac{3}{8}\,wh\,\frac{1+k}{2+k} \quad \text{and}$$

$$X_2 = \frac{3}{8}\,wh\,\frac{1}{2+k} \quad \text{where } k = I_1/I$$

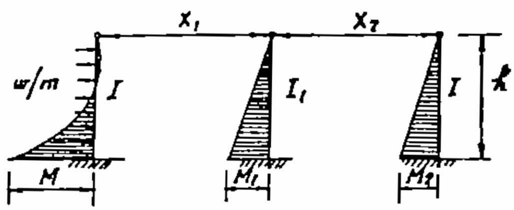


Fig VII-8c

Therefore

$$M = \frac{wh^2}{8}\,\frac{5+k}{2+k}\ ,\quad M_1 = \frac{3wh^2}{8}\,\frac{k}{2+k} \quad \text{and} \quad M_2 = \frac{3wh^2}{8}\,\frac{1}{2+k}$$

For four columns , we get (Fig VII-8d)

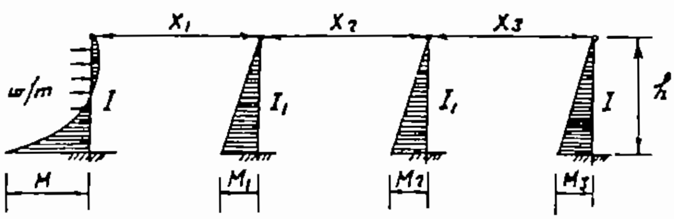


Fig VII-8d

$$X_1 = \frac{3}{16}\, wh\, \frac{1 + 2k}{1 + k} \qquad X_2 = \frac{3}{16}\, wh$$

$$X_3 = \frac{3}{16}\, wh\, \frac{1}{1 + k}$$

and

$$M = \frac{wh^2}{16}\, \frac{5 + 2k}{1 + k} \qquad M_1 = \frac{wh^2}{16}\, \frac{3k}{1 + k} \qquad M_2 = M_1$$

$$M_3 = \frac{3wh^2}{16}\, \frac{1}{1 + k}$$

It is however possible to get reasonable dimensions for the saw-tooth roof elements if triangular frames are arranged at convenient distances varying between 5 and 6 ms as shown in figure VII-9 This system is generally used for distances between windows varying between 7 and 10 ms

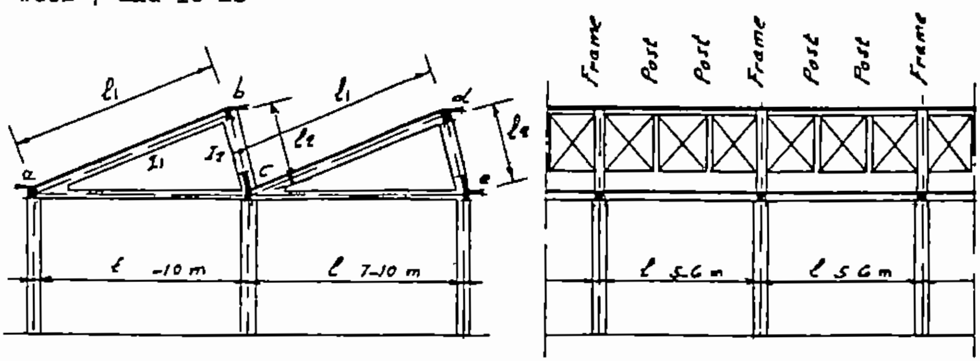


Fig VII-9

In this case the different elements of the saw-tooth roof are

1) The slab

It is generally two way its rectangularity is to be determined for the side lengths l_1 and l measured in the plane of the slab. The component of the load normal to the slab only causes the bending moments and shearing forces (Fig VII-10)

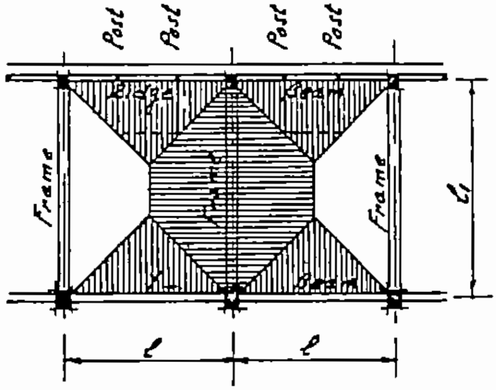


Fig VII-10

2) The ridge beam

It is continuous over the posts and frames, it carries in addition to its own weight the triangular load shown acting in the direction of the axis of the beam. The equivalent uniform loads on the different spans are not equal

3) The posts

These are axially loaded, carry the reactions of the ridge beam and transmit them to the Y-beam

4) The Y-beam

It behaves as a continuous beam, of spans l, supported by the main columns. It carries, in addition to its own weight the triangular load at the lower edge of the slab plus the concentrated loads from the intermediate posts. If these posts are two or more, the Y-beam may be calculated for an equivalent uniform vertical load in the usual way

5) The triangular frame

It carries the tapezoidal load from the slab plus its own weight, both may be replaced by an equivalent uniform load in the usual way. The internal forces can be determined by one of two methods

First method: as a continuous beam

Neglecting the elastic deformation of the tie ac, then points a, c and e may be assumed fixed in position and concequently points b and d can also be considered as fixed in space. In this manner, the continuous frame can be considered as a continuous beam abcde of unequal spans l_1 and l_2, unequal moments of inertia I_1 and I_2 and unequal loading w_1 and w_2. The moment of inertia I_1 is to be determined for a T-section with breadth of flange $B = 6t_s + b$ in which t_s is the thickness of the slab and b is the breadth of the web and I_2 to be determined for the rectangular section of l_2. The load w_2 on l_2 may be neglected without making an appreciable error (Fig VII-11)

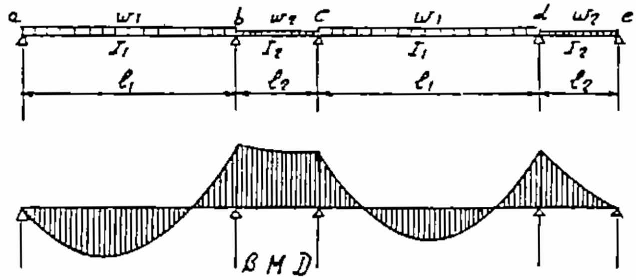


Fig VII-11

Second method: as isolated triangular frames

If the continuity of the two triangular units at c is neglected, each unit may be calculated as a triangular frame with a tie in the usual way as follows (Fig VII-12)

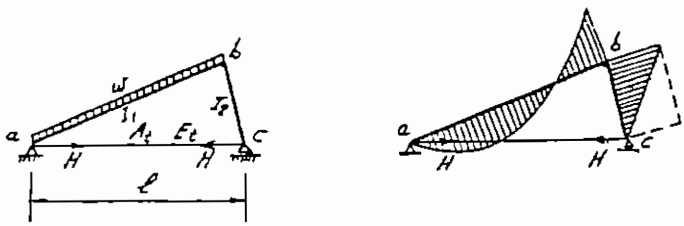


Fig VII-12

The tensile force H in the tie is $$H = -\frac{\delta_0}{\delta_1} = -\frac{\int \frac{M_0 M_1\, ds}{EI}}{\int \frac{M_1^2\, ds}{EI} + \frac{l}{A_t E_t}}$$

According to figure VII-11 the connecting moment at c may be estimated ~ equal to M_b

It is practically sufficient to determine the internal forces due to wind pressure for a single frame neglecting the tie (Fig VII-13)

The details of reinforcements are shown in figure VII-14

If north light is required in big span halls without intermediate supports, the choice of the system of the main supporting element depends on the dimensions of the hall and its disposition with respect to the north For a hall with the span parallel to the north, frames at 4-6 ms between centers spanning the shorter dimension of the hall supporting saw-tooth roofs arranged according to any of the forms shown in figure VII-15 may give a convenient solution

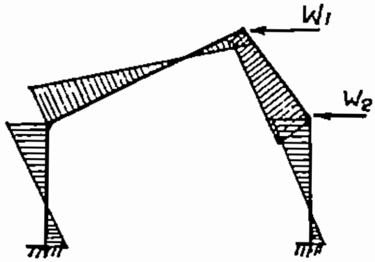


Fig VII-13

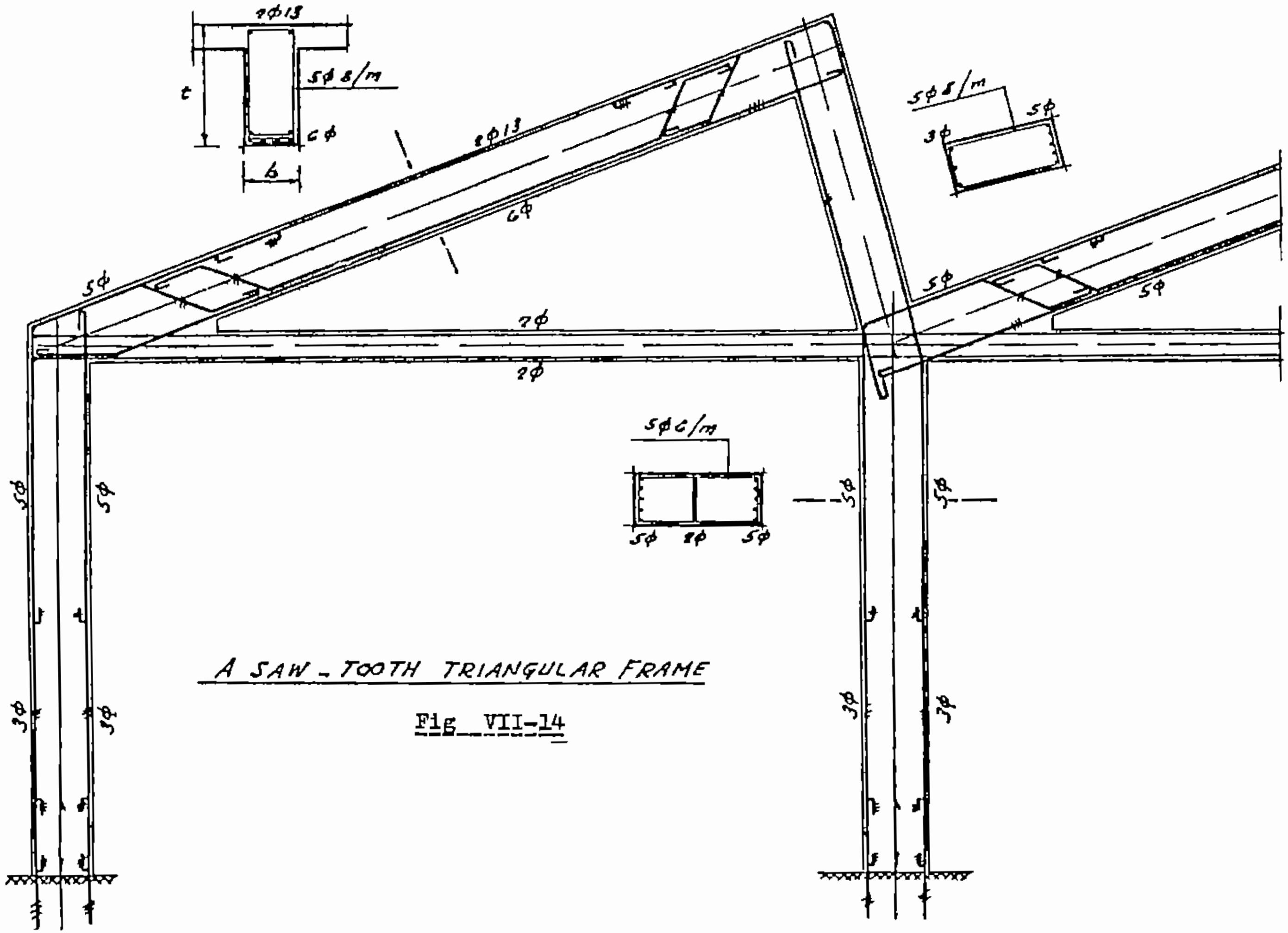
Figure VII-15b shows a saw-tooth roof arranged above the frames it gives a big construction height which may be reduced if the saw-tooth slabs are arranged in between the frames as shown in fig VII-15c This new solution may need a bigger depth for the main girder giving sufficient space for the windows and the beams supporting the slab The rain water can, in this case, be drained by inserting 3 steel pipes in the web of the main girder at the lower edge of the slab One can dispense with the pipes if the saw-tooth slabs are arranged as shown in figure VII-15d in which the rain water passes free below the main girders and the lower beams are hung to the frame by steel reinforcements sufficient to resist the reaction of the beam

The truss shown in figure VI-2 gives another typical solution for the hall shown in figure VII-15a The construction height of trusses (ca 1/6 - 1/7 l) being relatively big, this solution can be well adapted for bigger spans

We give in the following the details of two typical structures The first structure is the main factory hall of the Paints and Chemical Industries at Matarieh Figure VII-16 gives the general layout and the concrete dimensions of the main supporting frames of the factory hall, the details of reinforcements are shown in figure IV-13

The machine workshop of the Forging Plant at Helwan (Fig VI-8,9) gives the second typical solution for this type of structures

If the north is parallel to the longer side of the hall, i e , normal to the span, the following systems for the main supporting element may be used (Figure VII-17)



A SAW - TOOTH TRIANGULAR FRAME

Fig VII-14

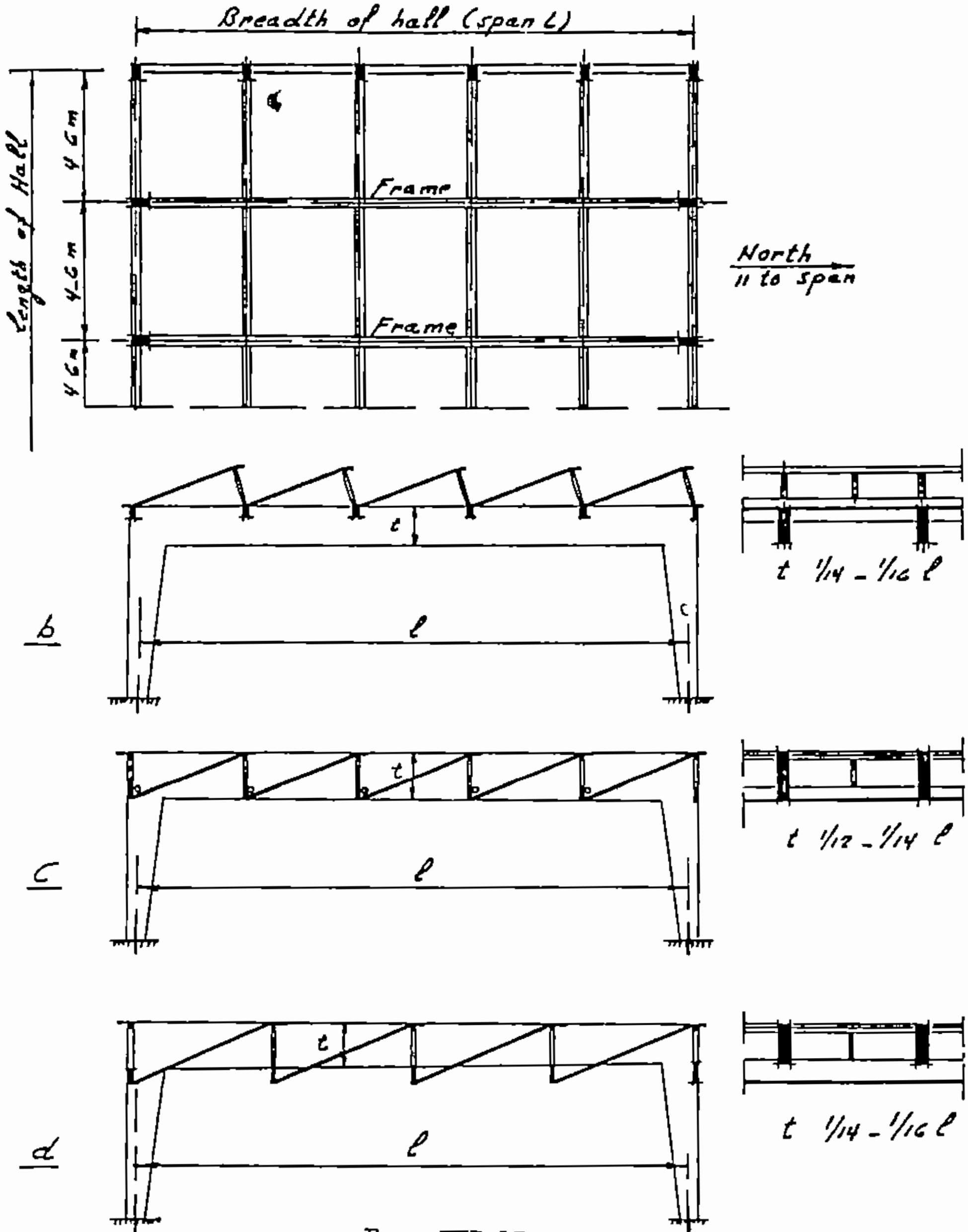
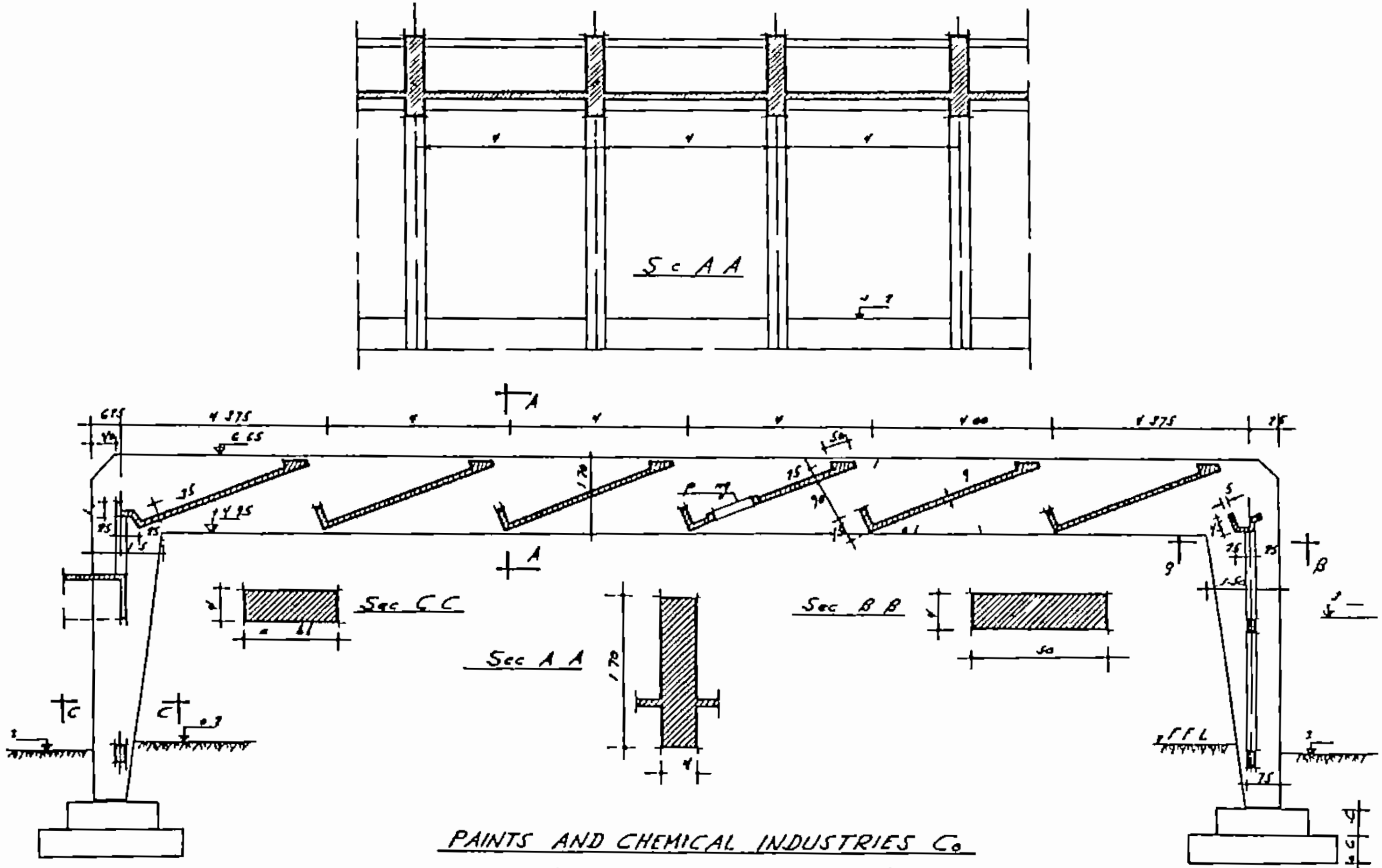

Breadth of hall (span l)
Length of Hall
4 6m
4 6m
4 6m
Frame
Frame
North
|| to span
b
t
l
t 1/14 - 1/16 l
c
t
l
t 1/12 - 1/14 l
d
t
l
t 1/14 - 1/16 l

Fig VII-15



PAINTS AND CHEMICAL INDUSTRIES Co

D ta ls of ma n supp rt ng elem nt

Fig VII-16

1) Two-hinged frames (Figure VII-17b) spaced every 4-6 ms the required depth for the main girders varies between 1/14 and 1/16 l

2) If the clear construction height necessary for the hall does not allow for the required depth of the main girder or for bigger spans where the use of a simple frame gives heavy cross-sections, a Vierendeel girder (figure VII-17c), in which the saw-tooth roof is supported at its upper edge on the top chord and its lower edge on the bottom chord of the girder, may be used The depth and spacing of the girder must be so chosen that they give a convenient slope for the saw-tooth slab

3) The two-hinged arch with a tie shown in figure VII-17d gives another convenient system in which the saw-tooth roof is supported at its upper edge on a horizontal beam, parallel to the main girder and supported by it, arranged at the crown of the arch and at its lower edge on the tie hung to the arched girder by hangers spaced every 2 5-4 ms It is recommended to arrange the hangers and the posts along the same line so that they form one element

The main workshops of 'El-Nasr Forging Plant at Helwan give a typical example for this last solution Figure VII-18 gives the general layout and main dimensions of the workshop and figure VII-19 shows the details of reinforcements of the parabolic arch used as main supporting element as well as the details of the crane girder carrying the rails of a 10^t crane giving the wheel loads shown in figure VII-18

This workshop is 24 ms wide 84 ms long and 12 5 ms clear height It is used for forging operations and includes a series of furnaces which necessitate sufficient uniformindirect light and good ventilation Because the hall is dusty and smoky, a plane roof surface is specified The hall is provided with a 10^t crane the clear height below the crane rail is 9 9 ms and is 2 5 ms above it The workshop is divided into 3 blocks, 28 ms each, by two expansion joints

To satisfy the specified requirements, a north-light saw-tooth roof with a plane bottom surface was chosen

The span of the hall being relatively big, an arch with a tie as the main supporting element for the roof was prefered To reduce the construction height of the structure and to get a reasonable distance between the arches, the minimum rise of $l/8$ = 3 ms was chosen To get a convenient slope for the saw-tooth roof, a minimum spacing of

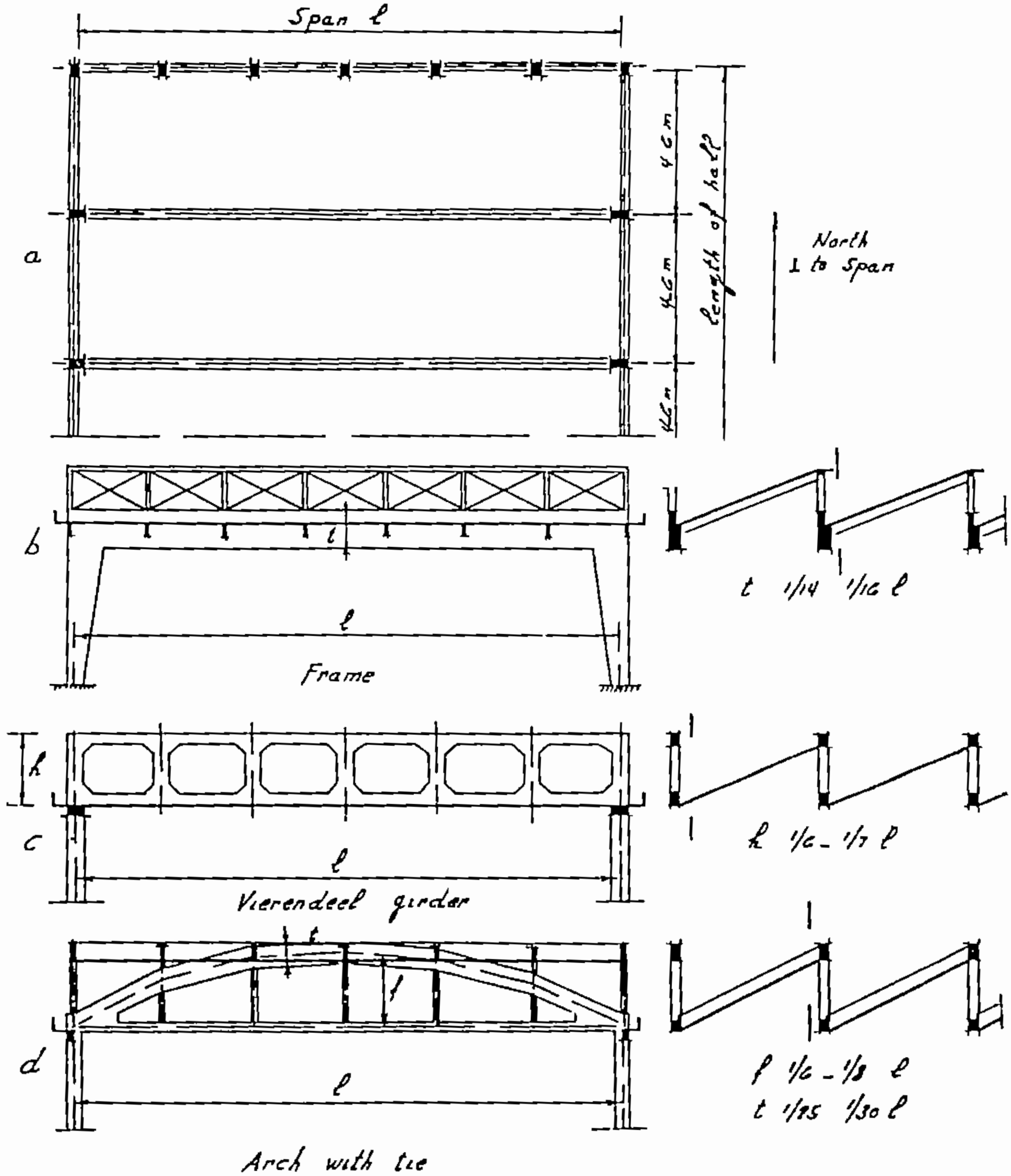

Span l
4 6m
4 6m
4 6m
Length of hall
North
⊥ to Span
a
b
t 1/14 1/16 l
l
Frame
c
h
h 1/6 – 1/7 l
Vierendeel girder
d
t
f
f 1/6 – 1/8 l
t 1/25 1/30 l
Arch with tie

Fig. VII-17

7 ms between the arches was necessary In order to reduce the thickness of the slab, inclined secondary beams @ 3 ms are arranged the beams are inverted to give the specified plane bottom surface of the roof

In this manner, the roof is composed of a one-way continuous slab 3 ms span and inclined simple beams 7 ms horizontal span The total vertical load being 350 kg /m^2 surface, a slab thickness of 8 cms and secondary beams 20 x 60 cms were sufficient

The arch carries its own weight plus the loads from the roof which are transmitted to the arch ad concentrated through the secondary beams and the supporting posts and ties In order to simplify the construction of the main girder and to reduce the bending moments to a minimum, its axis was chosen polygonal with the corners coinciding on a parabola following the equation (Figure VII-20)

$$y = 4 f \xi \xi$$

in which

$$\xi = x/l \quad \text{and} \quad \xi' = x'/l$$

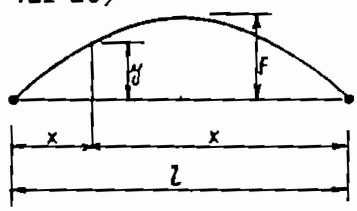


Fig VII-20

Assuming the loads as uniformly distributed, then the internal forces in the arch can be estimated in the following manner

Loads from slab	0 350 x 7 x 1 15	= 2 8 t/m'
Loads from secondary beams	0 2 x 0 52 x 2 5 x 7 x 1 15/3	= 0 7 t/m'
Own weight of the main girder		1 2 t/m'
Concrete for slopes of gutter+rain-water+windows		= 0 3 t/m'
Total weight		= 5 0 t/m'

Due to the elongation of the tie and the elastic deformation of the arch girder due to the normal force, the horizontal thrust H may be estimated by 0 95 of that of the corresponding three-hinged arch H_o, i e ,

For a three-hinged arch $\quad H_o = \dfrac{wl^2}{8 f} = \dfrac{5 \times 24^2}{8 \times 3} = 120$ t

For the two-hinged arch $\quad H = 0\ 95\ H_o = 0\ 95 \times 120 = 114$ t

$$\Delta H = H_o - H = 6 \text{ t}$$

The maximum bending moment M is therefore

$$\max M = Hf = 6 \times 3 = 18 \text{ mt}$$

Accordingly, the arch girder can be estimated for a normal force N equal to 114 t and a bending moment M equal to 18 mt A cross-section 35 x 80 cms (t = /30) reinforced by 0 8 to 1 2 % normal mild steel was sufficient The reinforcement is symmetrically arranged in the cross-section and the splices are staggered so that not more than two bars are spliced in any section

The tie can be estimated for a normal tensile force of 114 t Its tension bars are symmetrically arranged in a cross-section 35 x 40 cm without any splices It is however recommended to use high grade or cold-twisted steel in the tie as it is of higher strength and has bigger bond with the concrete

The arch was assumed externally statically determinate (giving vertical reactions only for vertical loads) due to the existance of the slender parts of the supporting columns

It is clear that the systems shown in figures VII-15 & VII-17 may be of one or more spans As an example, we show in figure VII-21 continuous frames spaced every 5 ms and supporting a saw-tooth roof with the north normal to the span

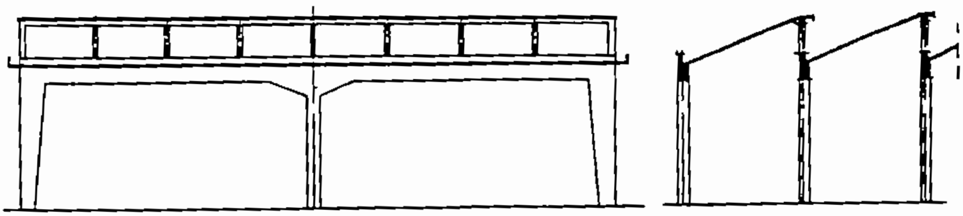


Fig VII-21 A symmetrical continuous frame with intermediate pendulum

Because of symmetry in shape and loading, the intermediate columns will be, due to roof loads, subject to axial loads only it is therefore recommended in such cases to choose a continuous frame with a slender pendulum support in the middle

Factory 135 at Helwan shown in figure VII-22 gives another typical solution In this factory the roof is saw-tooth 10 ms span supported on main girders (normal to the north) 15 ms span The details of the different elements are shown in figure III-17

Folded plates and shells have been recently extensively used as roof structures for halls in which indirect light is specified

VIII- ARCHED SLABS AND GIRDERS

In simple beams, commonly only one cross-section is subjected to the maximum design moment, and concequently, if the member is prismatic, only one cross-section of the beam is working at the maximum allawable stress at design load Knowing that the mentioned maximum stresses act at the extreme fibers only and that all other fibers are understressed one can directly observe that the simple beam is one of the least efficient of structural forms This situation is somewhat improved by continuity because the maximum field and connecting moments are commonly smaller in magnitude than of a simple beam of the same span and the beam can be designed such that the extreme fiber stress is equal to the allawable values at sections of maximum positive and negative moments

Arched girders of convenient form are mainly subject to high compressive forces and low bending moments and shearing forces so that nearly all sections of the arch are approximatly subjected to the same average compressive stress which means a high efficiency in the use of reinforced concrete as a building material

For these reasons, arched roofs give a convenient economic solution for long span roof structures without intermediate supports in cases where a plane roof surface is not necessary to meet the functional requirements of the structure, however plane roofs or floors can occasionally be supported on arched girders

According to the conditions at the supports, arches generally used in reinforced concrete structures are of three main types

a) Three hinged arches supported at each end by a hinge resting on the abutment and provided with an intermediate hinge, generally placed at the crown

b) If the intermediate hinge in the previous system is not arranged, the arch is two hinged.

c) If the arch is rigidly connected to the abuments in such a way that no rotation and no vertical or horizontal displacements are allowed then the arch is fixed

The best form for the axis of an arch is that which coincides on the line of pressure of the loads thus for uniform loads parabolic arches are most convenient

We give in the following, the theory and design of arches extensively used in reinforced concrete structures

a) <u>Three Hinged Arches</u>

A three hinged arch is statically determinate its horizontal thrust H can be determined from the condition that the bending moment at the intermediate hinge c is equal to zero (FigVIII-1)

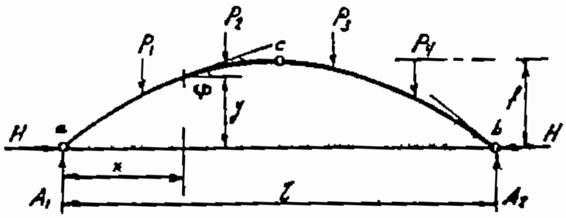


Fig VIII-1

Assuming that M_c is the bending moment at the position of the intermediate hinge c of the arch ab assumed as a simple beam then the horizontal thrust H can be calculated from the equation

$$H = M_c/f$$

The bending moment M, shearing force Q and thrust N in any section x,y are given by

$$M = M_o - Hy$$

$$Q = Q_o \cos\varphi - H \sin\varphi$$

$$N = H \cos\varphi + Q_o \sin\varphi$$

in which

M_o and Q_o are the bending moment and shearing force of ab assumed as a simple beam and

φ = angle between the tangent to the section under consideration and the horizontal

In flat arches one can approximately assume

$$N = H \sec\varphi$$

Parabolic Arches

As stated before, parabolic arches give the most convenient form for uniform loads

The equation of the axis of the arch according to figure VIII-2 is given by

$$y = \frac{4fx}{l^2}(l - x)$$

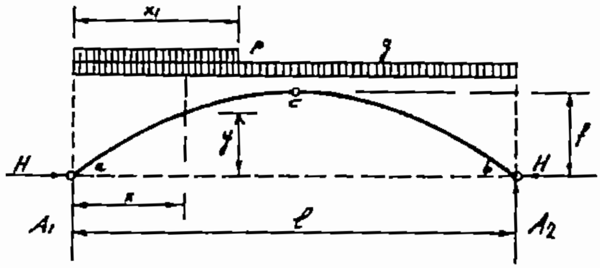


Fig VIII-2

Due to a uniform load g over the whole span and a live load p acting on $x_1 = 2l/5$ or $l/2$ (fig VIII-2) the approximate value of the thrust at the quarter points is given by

$$N = H\sqrt{1 + \left(\frac{2f}{l}\right)^2}$$

The vertical reactions A_1 and A_2, the horizontal thrust H and the bending moments M at the quarter points are given in the following table

Parabolic Arch

D L g acting over the whole span l		L L p acting on	
		$x_1 = 2l/5$	$x_1 = l/2$
Vert reactions	$A_1 = gl/2$	$8pl/25$	$3pl/8$
	$A_2 = gl/2$	$2pl/25$	$pl/8$
Horiz thrust	$H = gl^2/8f$	$pl^2/25f$	$pl^2/16f$
B M at quarter pts	$M = 0$	$\pm 3pl^2/160$	$\pm pl^2/64$

Flat arches subject to uniform loads may be constructed with a circular axis in which case we get the following internal forces

Due to a uniform load g over the whole span and a live load p acting on half the span ($x_1 = l/2$) , the reactons are

$$A_1 = (g/2 + 3p/8)l \qquad , \qquad A_2 = (g/2 + p/8)l$$

$$H = (g + p/2)l^2 /8f$$

The bending moments M and the approximate normal force N at the quarter points of the arch are given in the following table

Circular Arch

Ratio f /l	Dead Load - Mg	Live Load On ½ span - Mp	Live Load On ½ span + Mp	Approx N at ¼ points
0 10	0 0009	0 0161	0 0152	1 0198
0 15	0 0019	0 0166	0 0146	1 0440
0 20	0 0038	0 0175	0 0136	1 0771
0 25	0 0061	0 0184	0 0124	1 1180
0 30	0 0088	0 0196	0 0108	1 1662
0 35	0 0122	0 0210	0 0088	1 2207
0 40	0 0162	0 0225	0 0063	1 2806
0 45	0 0189	0 0241	0 0034	1 3454
0 50	0 0259	0 0259	0 0000	1 4142
	gl^2	pl^2	pl^2	H

Example Of An Arched Roof Slab With Ties

A hall 18 ms wide is to be covered by a reinforced concrete circuler slab with a tie as shown in figure VIII-3 At crown, the rise of the arch is 3 0 ms and its thickness t is 10 cms, at quarter points t = 12 5 cms and at foot t = 14 cms The distance between the ties is 5 ms and the developed length of the arch is about 19 ms

Because of the relatively big normal compressive force and small bending moments, such arched slabs are generally symmetrically reinforced with a minimum percentage of 0 8% of the average concrete section

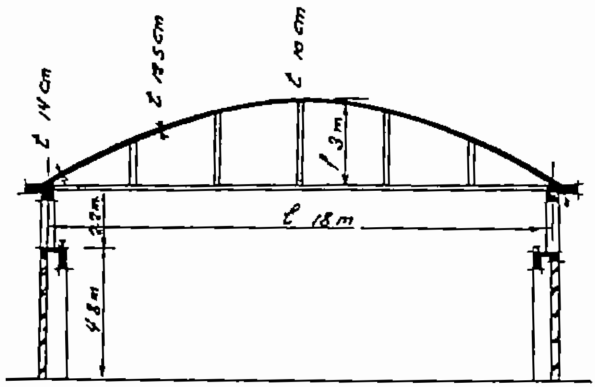


Fig VIII-3

With the help of this choice and due to the slenderness of the concrete section, it is generally allowed to calculate such slab arched roofs as three hinged

In order to simplify the calculations, the dead weight of the slab may be assumed as uniformly distributed The live load is equivalent to 50 kg/m^2 horizontal and the wind load is assumed equivalent to another 50 kg/m^2 horizontal acting on half the span

Accordingly the main steps of the statical calculation can be done in the following manner

1) Loads

Own weight of slab assumed uniformly distributed =

$$0\ 125 \times 2500 \times \frac{19}{18} = 330 \text{ kg/m}^2 \text{ horiz}$$

Roof cover = 70

Total dead load g = 400

Wind + live load p = 100

2) Maximum bending and axial force in circular arched slab

According to table given on page 152 ', we get

For f /l = 3/18 = 0 166

$M_{max} \simeq M_g + M_p = -(0\ 0025\ g + 0\ 0169\ p)l^2$

$= -(0\ 0025 \times 400 + 0\ 0169 \times 100) \times 18^2 = -870$ kgm

$N_{max} \simeq 1\ 055\ H = 1\ 055\ (g + p/2)l^2 / 8f$

$= 1\ 055\ (400 + 50)\ 18^2 / 8 \times 3 = 6400$ kgs

3) Check of stresses

Thickness of arched slab at quarter points t = 12 5 cms

Assume section symmetrically reinforced and $\mu = \mu = 0\ 4\%$

i e $A_s = A'_s = \frac{0\ 4}{100} \times 12\ 5 \times 100 = 5\ cm^2$ chosen $7\phi10$ mm/m

M = 870 kgm , N = 6400 kgs

Eccentricity e = M/N = 870/6400 = 0 135 ms

Eccentricity to tension steel $e_s = e + t/2 - 2 = 13\ 5 + 12\ 5/2 - 2$
$= 17\ 75$ cm

$e_s/d = 17\ 75/10\ 5 = 1\ 67$

Moment about tension steel $M_s = N\ e_s = 6400 \times 0\ 1775 = 1140$ kgm

Max compressive stress in concrete

$$\sigma_c = C_1\ M_s/bd^2 = 5\ 1 \times 1140 \times 100/100 \times 12^2 = 40\ kg/cm^2$$

Max tensile stress in steel

$$\sigma_s = C_2\ M_s/bd^2 = 140 \times 1140 \times 100/100 \times 12^2 = 1100 \quad "$$

The longitudinal reinforcement A'_s must be minimum 20% of main steel i e

min $A'_s = 0\ 2\ A_s = 0\ 2 \times 5 = 1\ 00\ cm^2/m$ chosen $5\phi6$ mm/m

4) The tie

Max horizontal thrust $H_{max} = (g + p)l^2 / 8f$

$= 500 \times 18^2 / 8 \times 3 = 6750$ kg/m

Max tension in tie $T_{max} = 5 \times 6750 = 33750$ kgs

Using high grade steel with a max allowable stress $\sigma_s = 2000$ kg/cm^2, then the required area of steel is

$$A_s = T/\sigma_s = 33750/2000 = 16\ 875\ \text{cm}^2$$

chosen 6 Φ 19 mm

5) <u>Vertical and horizontal beams</u> Span = 5 0 ms

The maximum vertical reaction A of the arch will be resisted by a vertical beam, thus

$A_{max} = (g + p)\, l\,/2 = 500 \times 18/2 = 4500$ kg/m' and

the maximum horizontal thrust $H_{max} = 6750$ kg/m' will be resisted by a horizontal beam (Fig VIII-4)

Assuming that the Vertical beam is 40x60 cms & the horiz " 20x80 cms

The load on the vertical beam is

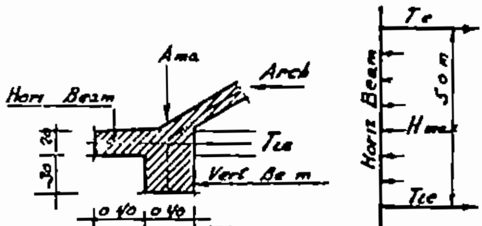


Fig VIII-4

$w = 4500 + (0\ 4 \times 0\ 3 + 0\ 2 \times 0\ 4)\ 2500 = 5000$ kg/m'

Its max bending moment ≈12000 kgm and
its max tension steel 6 Φ 19 mm

The load on the horizontal beam is H_{max} = 6750 kg/m

its max bending moment ≈16000 kgm and
its max tension steel 6 Φ 19 mm

It is recommended in such beams to resist the diagonal tensile stresses by stirrups and to use straight bars for the longitudinal reinforcement of the beams

6) <u>Details of reinforcements</u> as shown in fig VIII-5

7) <u>Wind pressure and crane loads</u>

For resisting the wind pressure (and eventual crane loads) the system may be assumed as composed of two stiff columns fixed at their bottom end and connected together by a rigid tie i e a simple once statically indeterminate system (Fig VIII-6) The main system can

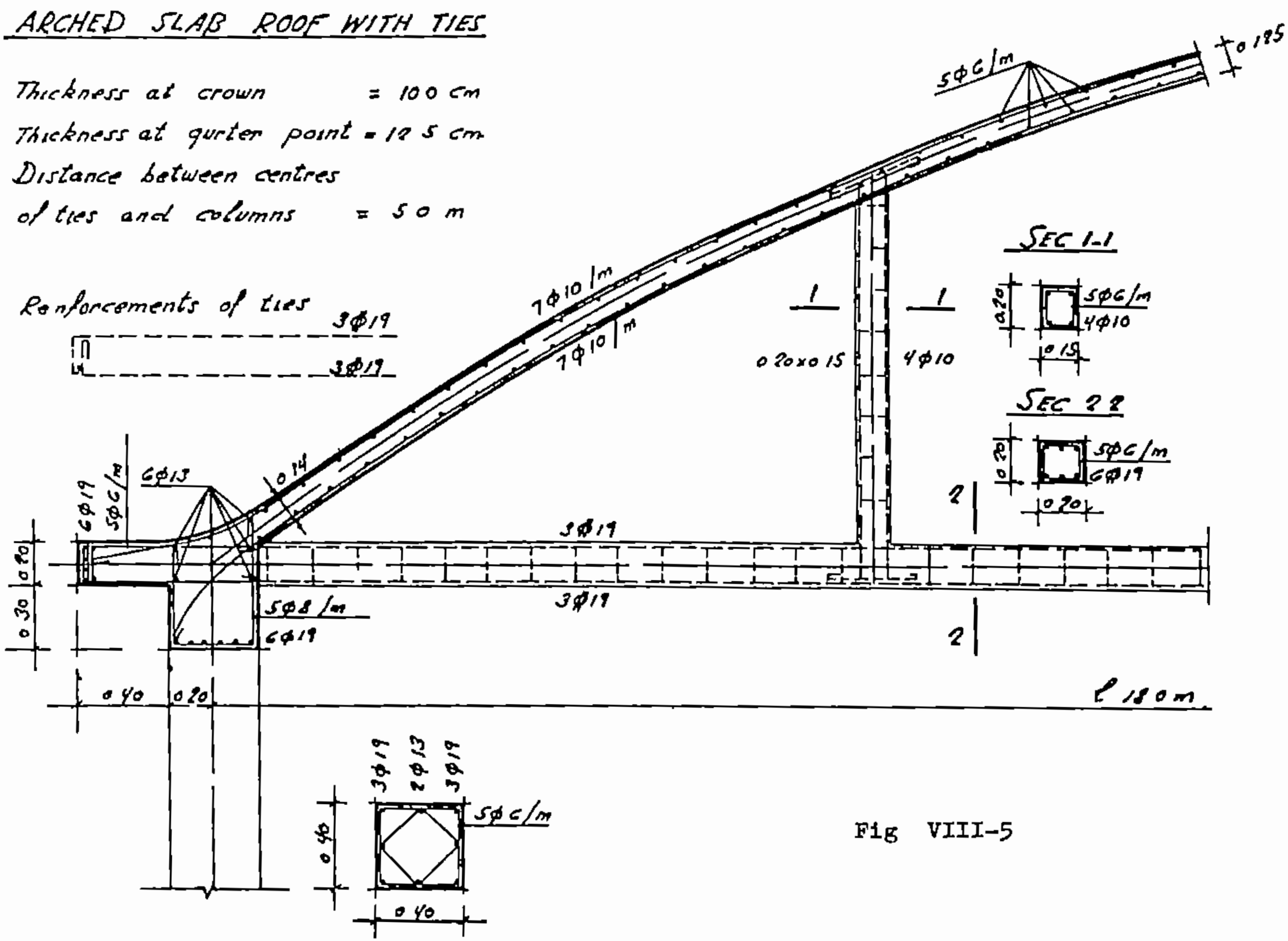

ARCHED SLAB ROOF WITH TIES
Thickness at crown = 100 cm
Thickness at qurter point = 12 5 cm
Distance between centres
of ties and columns = 5 0 m
Reinforcements of ties
3φ19
3φ19
5φ6/m
0 185
7φ10/m
7φ10/m
0 14
6φ13
6φ19
5φ6/m
3φ19
3φ19
5φ8/m
6φ19
0 20x0 15
4φ10
SEC 1-1
5φ6/m
4φ10
0 15
SEC 2 2
5φ6/m
6φ19
0 20
ℓ 18 0 m
0 40
0 20
0 30
3φ19
2φ13
3φ19
5φ6/m
0 40

Fig VIII-5

be obtained by cutting the tie , in which case, the statically indeterminate value is X and

$$X = - \int M_o \, M_1 \, ds / \int M_1^2 \, ds$$

in which M_o, M_1 and ds are the notations normally used in the theory of virtual work

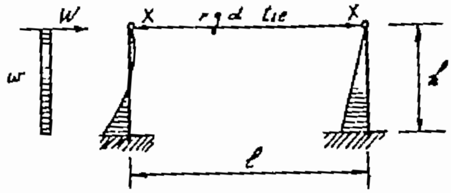


Fig VIII-6

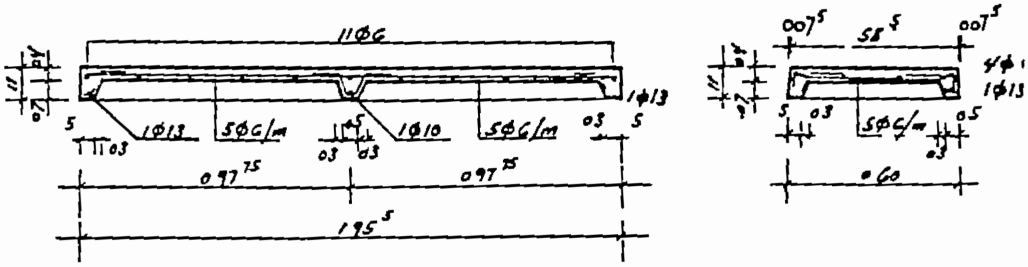
We give in the following some other examples showing the use of 3-hinged arches or arched frames in covering relatively big spans

Fig VIII-7 shows a project for prefabricated 3-hinged arched roof prepared to cover the ammonium nitrate limestone silo of El Nasr Fertilizer and Chemical industries at Suez The span of the arch is 26 5 ms, its rise is 10 375 ms, the distance between the center lines is 2 2 ms, the roof cover is composed of precast reinforced concrete slabs (Fig VIII-7a) In addition to the dead and live loads acting on the arch after construction, each half must be checked as a beam for the internal forces due to its own weight under transportation and lifting conditions before and during construction Fig VIII-7b The sections of the arch shown in this figure are chosen such that they give maximum resistance and minimum weight

Fig VIII-8 shows the 3-hinged superphosphate hangar at Kafr El-Zayat, its span is 26 ms and its rise is 13 ms This big rise is chosen in order to have minimum air volume between the roof and the stored superphosphate because of the bad undesired effect of the air humidity on the fertilizer The side cantilevering structure is used as a shed covering the raiway lines serving the hangar In order to reduce the loads on the arces, the roof cover is made of corrugated sheets weighing 20 kg/m^2 only

Fig VIII-9 shows the details of the main frames supporting the roof slab of the urea silo at Abu-Kir Fertilizers and Chemical Industries plant This silo is~ 50 ms span, 186 ms long and 20 ms high The max height of the stored urea is 14 3 ms The soil at the site is extremely weak and for this reason, the floor slab carrying the urea and the roof structure are supported on cast in-situ,~ 25 ms long piles A three hinged structure with ties is found to be the most convenient for this case The roof is chosen circular with inverted main frames @ 6 ms The one-way roof slab is chosen at the bottom surface of the main frames to enable the use of mechanically moving forms and to have the frames acting as T In this case, isolation of roof is essential

a) REINFORCED CONCRETE PRECAST SLABS



b)

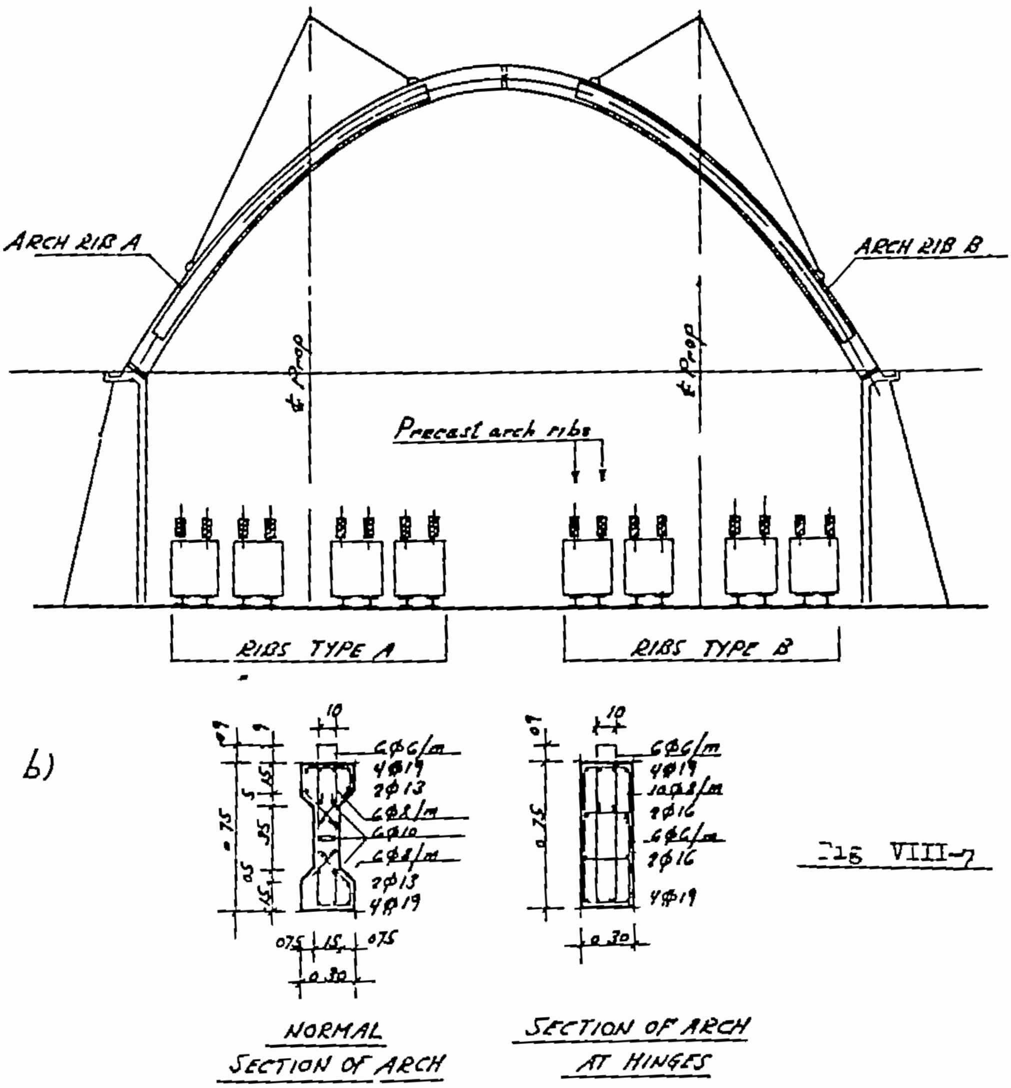


Fig VIII-7

b) Two Hinged Arches

A two hinged arch is once statically indeterminate Choosing the simple girder with a hinge at a and a roller at b as main system, then the equation of elasticity is (Fig VIII-10)

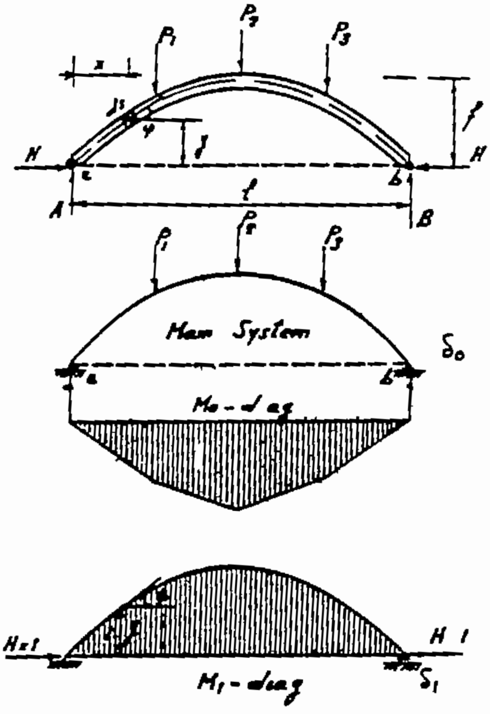


Fig VIII-10

$$\delta = 0 = \delta_o + H\delta_1$$

and

$$H = -\frac{\delta_o}{\delta_1}$$

Due to H = 1 , we have

$$M_1 = -1 \quad y$$

$$N_1 = 1 \quad \cos\varphi$$

$$Q_1 = -1 \quad \sin\varphi$$

According to theory of virtual work, we have

$$\delta_o = -\int \frac{M_o\ y\ ds}{EI} + \int \frac{\bar{N}_o \cos\varphi\ ds}{EA} - \int \frac{Q_o \sin\varphi\ ds}{GA'}$$

and

$$\delta_1 = \int \frac{y^2\ ds}{EI} + \int \frac{\cos^2\varphi\ ds}{EA} + \int \frac{\sin^2\varphi\ ds}{GA'}$$

In these equations, we have

1) The elastic deformation due to shear stresses is generally small compared to that due to normal stresses so that $\int \frac{Q_o \sin\varphi\ ds}{GA'}$ may be neglected without making an appreciable error

2) The normal force N_x in any section of a two hinged arch subject to vertical loads only can be calculated from the relation

$$N_x \cos\varphi = H \qquad \text{i e} \qquad N_x = H / \cos\varphi$$

But $\quad N_x = N_o + H \cos\varphi$

Therefore $\quad H/\cos\varphi = N_o + H \cos\varphi$

or $\quad H = N_o \cos\varphi + H \cos^2\varphi$

For flat arches $\cos^2\varphi \approx 1$ and accordingly $N_o \cos\varphi \approx 0$

3) It is known that $G = \dfrac{E}{2(1+v)}$ in which

v = Poissons ratio $\approx 0\ 2$ for reinforced concrete

So that $G \sim 0\ 4E$, and

$A' = \dfrac{5}{6} A$ for rectangular sections , hence

$$GA' \approx \frac{1}{3} EA$$

4) As $\cos^2\varphi = 1 - \sin^2\varphi$, we get

$$\int \frac{\cos^2\varphi\, ds}{EA} + \int \frac{\sin^2\varphi\, ds}{GA'} = \int (1 - \sin^2\varphi) \frac{ds}{EA} + \int 3 \frac{\sin^2\varphi\, ds}{EA}$$

$$= \int \frac{ds}{EA} + 2 \int \frac{\sin^2\varphi\, ds}{EA} \approx \int \frac{ds}{EA}$$

The second term has been neglected becase in flat arches φ is small and $\sin^2\varphi$ is a very small value

Accordingly, in normal cases, it is sufficiently accurate to assume that

$$\delta_o = - \int \frac{M_o\, y\, ds}{EI} ,$$

$$\delta_1 = \int \frac{y^2\, ds}{EI} + \int \frac{ds}{EA}$$

and

$$H = \frac{\int \frac{M_o\, y\, ds}{I}}{\int \frac{y^2\, ds}{I} + \int \frac{ds}{A}}$$

Due to an increase Δl of the span , we get

$$H_{\Delta l} = - \frac{E\, \Delta l}{\int \frac{y^2\, ds}{I} + \int \frac{ds}{A}}$$

and due to a temperature increase of t^o , we get

$$H_t = \frac{E\, \alpha\, t\, l}{\int \frac{y^2\, ds}{I} + \int \frac{ds}{A}}$$

<u>Symmetrical Parabolic Two Hinged Arches</u>

The axis of a symmetrical parabolic arch is given by the equation

$$y = \frac{4f}{l^2} \quad x \quad (l - x)$$

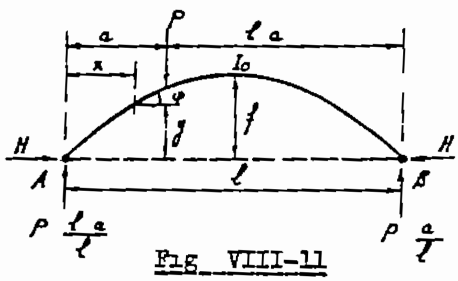


Fig VIII-11

If the moment of inertia I and the area A of any section are chosen such that

$$I \cos\varphi = I_o \qquad \text{and} \qquad A \cos\varphi = A_o$$

where I_o and A_o are the moment of inertia and the area of the cross-section at the crown, the integrals in the equations of H can be mathematically evaluated as follow (Fig VIII-11)

For a concentrated load P acting at a distance a from the left support, we get

$$\int \frac{M_o\, y\, ds}{I} = \int \frac{M_o\, y \cos\varphi\, ds}{I \cos\varphi} = \frac{1}{I_o} \int M_o\, y\, dx$$

$$= \frac{1}{I_o} \int_0^a \frac{P(l - a)}{l} \quad x \quad \frac{4f}{l^2} \quad x \quad (l - x)$$

$$+ \frac{1}{I_o} \int_a^l \frac{P\ a}{l} (l - x) \quad \frac{4f}{l^2} \quad y\ (l - x) \quad dx$$

$$= \frac{1}{I_o} \quad \frac{f}{3l^2} \quad P \quad a\ (l - a)\ (l^2 + la - a^2\ )$$

Further

$$\int \frac{y^2\, ds}{I} = \int_0^l \frac{y^2 \cos\varphi\ ds}{I \cos\varphi} = \frac{1}{I_o} \int_0^l \frac{16f^2}{l^4} \quad x^2\ (l - x)^2 \quad dx = \frac{8f^2\, l}{15\ I_o}$$

and

$$\int \frac{ds}{A} = \int_0^l \frac{ds \cos\varphi}{A \cos\varphi} = \frac{1}{A_o} \int_0^l dx = \frac{l}{A_o}$$

So that, for a series of concentrated loads P , we get

$$H = \frac{5 \sum_0^l P\ a\ (l - a)\ (l^2 + l\, a - a^2)}{8f\ l^3\ (\ 1 + \dfrac{15\ I_o}{8A_o\ f^2}\ )}$$

in which

$\frac{15\ I_o}{8A_o\ f^2} = \varepsilon$ is a correction factor due to the elastic deformation of the normal force, for a rectangular section of breadth b and depth t we have

$I_o = bt^3/12$, $A_o = bt$ and $\varepsilon = 0\ 155\ t^2/f^2$

Therefore

$$H = \frac{5\sum_{0}^{l} P\, a\, (l - a)\, (l^2 + l\, a - a^2)}{8 f^{3}\, (1 + \varepsilon)}$$

This equation can be used for determining the ordinates of the influence line of H if P is assumed equal to 1 and acts at a series of distances a

For an increase Δl of span l, we have

$$H_{\Delta l} = \frac{-\,15\, E\, I_o\, \Delta l}{8f^2\, l\, (1+\varepsilon)} \approx \frac{15\, E\, I_o\, \Delta l}{8f^2 l}$$

whereas for a temperature increase of t^o, we get

$$H_t = \frac{15\, E\, I_o\, \alpha\, t}{8f^2\, (1 + \varepsilon)} \approx \frac{15\, E\, I_o\, \alpha\, t}{8f^2}$$

For a uniform load g acting over the whole span, we get

$$H_g = \frac{g l^2}{8f\, (1 + \varepsilon)}$$

Assuming $\frac{1}{1 + \varepsilon} = k$, then

$$\underline{H_g = k\, \frac{g l^2}{8\, f}}$$

For different values of t/f, the magnitude of ε and k are accordingly as follows

t/f	ε	k
0 2	0 0062	0 994
0 3	0 014	0 986
0 4	0 025	0 976

Fig VIII-12 shows the general layout of a two hinged arch 55 ms span and 28 7 ms high used as the main supporting element of an airoplane Hangar

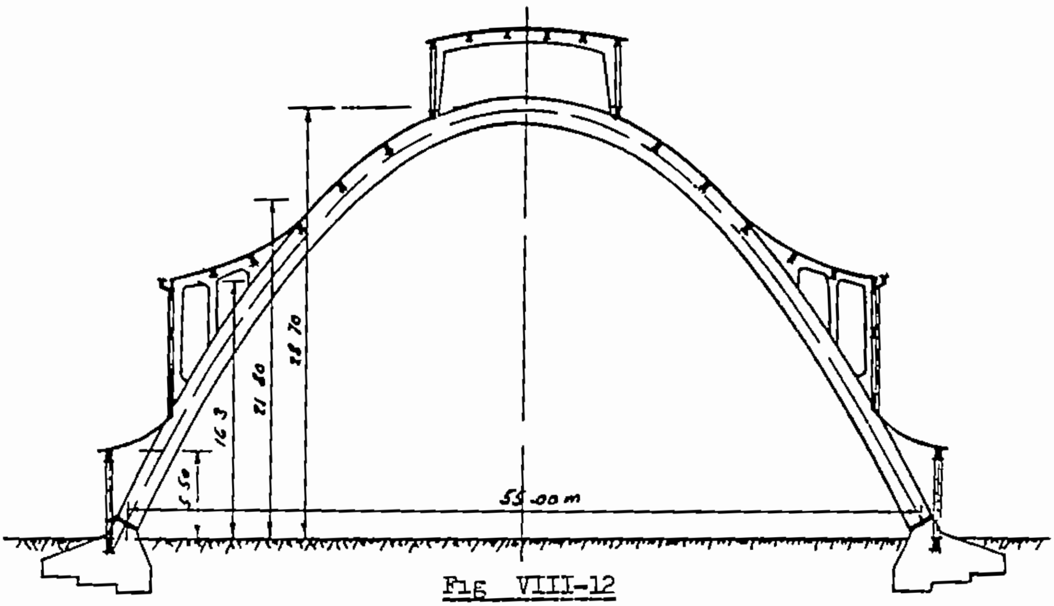


Fig VIII-12

Fig VIII-13 shows the general layout, concrete dimensions and details of reinforcements of a two hinged arch used as the main supporting element for one of the halls in the Copper Factory at Alexandria It has to be noticed here that the slabs are so arranged that the hall has sufficient light, and that the cantilever arms supporting the crane girders are smoothly connected to the arch in such a way that the bending moments transmitted to the arch are easily resisted The reinforcements are arranged with staggered splices and such that not more than 20% of the reinforcement bars are spliced in any section

c-Two hinged Arch with a Tie

This system is externally statically determinate and internally once statically indeterminate The main system is chosen by cutting the tie The equation

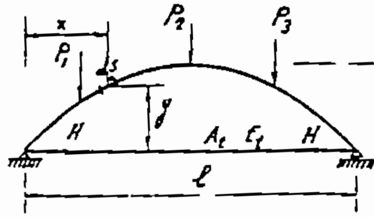


Fig VIII-14

of elasticity is given by Fig VIII-14

$$\delta = -\frac{Hl}{A_t E_t} = \delta_o + H\delta_1$$

so that $$H = -\frac{\delta_o}{\delta_1 + \frac{l}{A_t E_t}}$$

Proceeding in the same way as in the previous case we get

$$H = \frac{\int \frac{M_o y\, ds}{EI}}{\int \frac{M_1^2\, ds}{EI} + \int \frac{ds}{EA} + \frac{l}{A_t E_t}}$$

For a symmetrical parabolic arch with $I \cos\varphi = I_o$ and $A \cos\varphi = A_o$ subject to a series of concentrated loads P, we get

$$H = \frac{5\sum_{0}^{l} P\, a\,(l - a)\,(l^2 + la - a^2)}{8fl^3\,(1 + \varepsilon) + \frac{15 I_o}{f}\,\frac{l^3}{A_t}\,\frac{E}{E_t}}$$

For a uniform load g, we get

$$H = \frac{gl^2}{8f\,(1 + \varepsilon + \frac{15\, I_o E}{f^2 A_t\, E_t})}$$

Assuming that the section of the arch is rectangular with breadth b and depth t then $I_o = \frac{bt^3}{12}$ and as $\frac{E}{E_t} = \frac{1}{10}$ then,

$$\frac{15\, I_o\, E}{f^2\, A_t\, E_t} = \frac{bt^3}{80f^2\, A_t} = \varepsilon'$$

ε' is a factor expressing the effect of the elongation of the tie Therefore

$$H = \frac{gl^2}{8f\ (1 + \varepsilon + \varepsilon')}$$

Assuming $\frac{1}{1 + \varepsilon + \varepsilon'} = k'$, we can write

$$H = k' \frac{gl^2}{8\ f}$$

For an arch with b = 30 cms, t = 80 cms, f = 3 00 ms and 12 Φ 25 mm in the tie (A_t = 60 cm^2) we get

$$\varepsilon = \frac{0\ 155\ t^2}{f^2} = \frac{0\ 155 \times 80^2}{300^2} = 0\ 011$$

$$\varepsilon' = \frac{bt^3}{80f^2\ A_t} = \frac{30 \times 80^3}{80 \times 300^2 \times 60} = 0\ 0356$$

$$k' = \frac{1}{1 + \varepsilon + \varepsilon'} = \frac{1}{1\ 0466} = 0\ 956$$

So that

$$H = 0\ 956\ \frac{gl^2}{8\ f}$$

This means that the horizontal thrust of a two-hinged arch with a tie is generally 4 to 5 % smaller than that of the corresponding three-hinged arch The bending moment at the crown is therefore

$$M = 0\ 04 \text{ to } 0\ 05\ \frac{gl^2}{8}$$

The parabolic arch shown in figure IV-3 gives a typical example for this system The polygonal frame shown in the same figure gives another example in which the corners of the polygon coincide on the line of pressure of the loads The slabs are arranged in the middle third at the level of the arch and at the outside thirds at the level of the tie Such an arrangement is well adapted to halls containing smoke or fumes The cross ventilation between the upper windows do not allow the fumes to accumulate inside the hall

Figures VII-21 and VII-22 give the general layout and details of reinforcements of the main workshops of 'El Nasr Forging Plant at Helwan in which arched girders with ties were used as the main supporting element for the saw-tooth roof

Figure VIII-15 gives the general layout of the Covered Gymnasium at the Faculty of Police, El Nasr City Cairo The roof is flat at the middle part of the hall and is composed of curved inverted shells on the two sides, all are hung to the arched girders as shown The tie between the footings of the main arches is post-tensioned by two Freyssinet cables 12 ϕ 7 mm each

d) Two Hinged Arch with Polygonal Tie

Two hinged arches with parabolic ties as shown in figure VIII-16 may be used in roof structures as well as in bridges

The statically indeterminate value is the horiz component H of the forces in the tie

The horizontal thrust H due to a series of concentrated loads P is given by

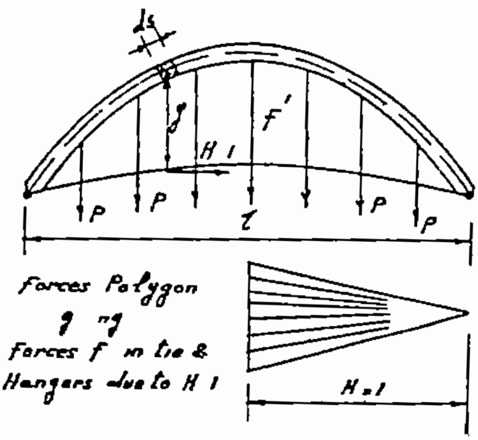


Fig VIII-16

$$H = \frac{\sum\limits_{h} \frac{F_o\ F'\ \ s}{E_h\ A_h} + \int \frac{M_o\ y\ \ ds}{EI}}{\sum\limits_{t} \frac{F'^2\ s}{E_t\ A_t} + \sum\limits_{h} \frac{F'^2\ s}{E_h\ A_h} + \int \frac{y^2\ ds}{EI} + \int \frac{ds}{EA}}$$

in which

$\sum\limits_{h}$, $\sum\limits_{t}$ = sum of forces in hangers and tie respectively

E_h , E_t = modulus of elasticity of hangers and tie respectively, generally = E_s

A_h , A_t = effective area of cross-section of hangers and tie respectively, generally equal to the area of steel in the member

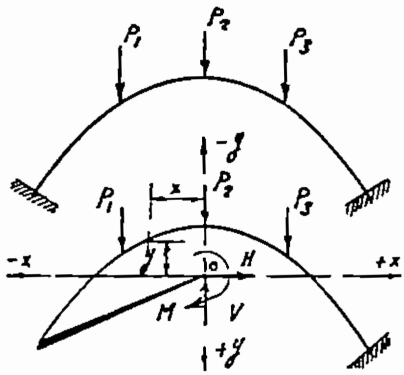
F_o = forces in hangers due to loading P and are equal to P if the loads act at the level of the tie and are equal to zero if the loads act on the arch

F = forces in tie and hangers due to H = 1 and can be determined from the force polygon shown in figure VIII-16

s = length of any of the hangers or elements of the tie

e) <u>Fixed Arches</u>

A fixed arch is three times statically indeterminate It can be treated in the same way as fixed frames Choosing the simple cantilever as main system and applying the statically indeterminate values H , V and M in the elastic center O, we get (Fig VIII-17)



<u>Fig VIII-17</u>

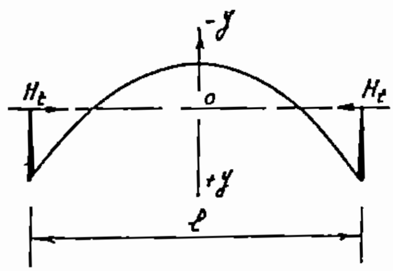
$$H = -\frac{\int \frac{M_o y \, ds}{I}}{\int \frac{y^2 \, ds}{I} + \int \frac{ds}{A}} \qquad V = -\frac{\int \frac{M_o x \, ds}{I}}{\int \frac{x^2 \, ds}{I}} \qquad M = -\frac{\int \frac{M_o \, ds}{I}}{\int \frac{ds}{I}}$$

The bending moment in any section can be determined by superposition, thus

$$M = M_o + H\,y + V\,x + M$$

Due to a change of temperture of t^o, we get only the reaction

$$H_t = \frac{E\,\alpha\,t\,l}{\int \frac{y^2 \, ds}{I} + \int \frac{ds}{A}}$$



This force acts at the level of the x - axis as shown in figure VIII-18

<u>Fig VIII-18</u>

Fixed arches or frames are to be used only if they are constructed on firm soils and the foundations are chosen such that they do not allow any displacement or rotation of the supports. Such displacements as well as temperature changes and shrinkage cause relatively high internal stresses and hence must be considered in the design. It is recommended to take all possible means to reduce the shrinkage stresses. For this purpose, a key ca 30 cms wide is concreted only when most of the shrinkage of the arch has taken place (say after 3 weeks).

f) Continuous Arches

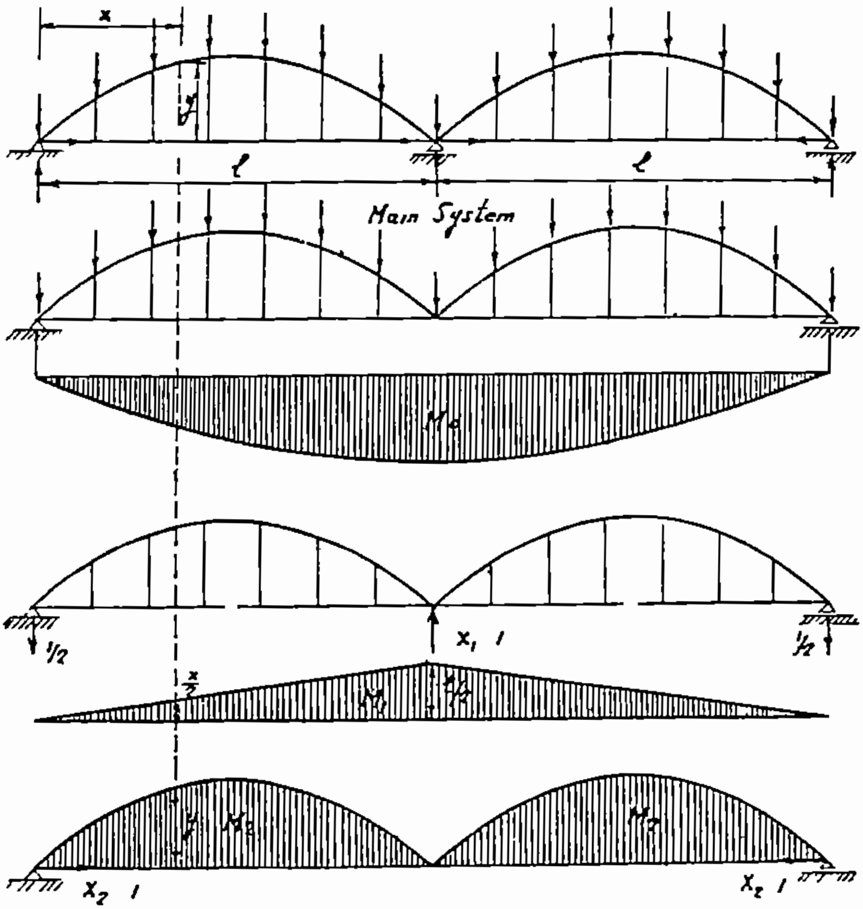


Fig VIII-19

A symmetrical continuous arch with a tie symmetrically loaded as shown in figure VIII-19 is twice statically indeterminate. The main system may be chosen by removing the intermediate support and cutting the ties. The statically indeterminate values are X_1 and X_2. The equations of elasticity are

$$1)\quad \delta_1 = 0 = \delta_{10} + X_1\delta_{11} + X_2\delta_{12}$$

$$2)\quad \delta_2 = -\frac{2X_2 l}{A_t E_t} = \delta_{20} + X_1\delta_{21} + X_2\delta_{22}$$

Neglecting the elastic deformations due to the normal and shearing forces we get.

$$1)\quad E\delta_{10} = \int M_o M_1 \frac{ds}{I} = 2\int_0^l M_o M_1 \frac{ds}{I} = 2\int_0^l M_o \left(-\frac{x}{2}\right)\frac{ds}{I} = -\int_0^l M_o x \frac{ds}{I}$$

$$2)\quad E\delta_{11} = \int M_1^2 \frac{ds}{I} = 2\int_0^l \left(-\frac{x}{2}\right)^2 \frac{ds}{I} = \frac{1}{2}\int_0^l x^2 \frac{ds}{I}$$

$$3)\quad E\delta_{12} = E\delta_{21} = 2\int_0^l M_1 M_2 \frac{ds}{I} = 2\int_0^l \left(-\frac{x}{2}\right)(-y)\frac{ds}{I} = \int_0^l xy \frac{ds}{I}$$

$$4)\quad E\delta_{20} = \int M_o M_2 \frac{ds}{I} = 2\int_0^l M_o(-y)\frac{ds}{I} = -2\int_0^l M_o y \frac{ds}{I}$$

$$5)\quad E\delta_{22} = \int M_2^2 \frac{ds}{I} = 2\int_0^l (-y)^2 \frac{ds}{I} = 2\int_0^l y^2 \frac{ds}{I}$$

For a continuous parabolic arch with $I\cos\varphi = I_o$ and subject to uniform load p/m', we get

$$y = 4\frac{fx}{l^2}(l - x)$$

and

$$M_o = p\frac{x}{2}(2l - x)$$

So that

$$E\delta_{10} = -\int_0^l M_o x \frac{ds}{I} = -\int_0^l M_o x \frac{ds\cos\varphi}{I\cos\varphi} = -\frac{1}{I_o}\int_0^l M_o x\,dx$$

$$= -\frac{1}{I_o}\int_0^l p\frac{x^2}{2}(2l - x)dx = -\frac{5pl^4}{24I_o}$$

$$E\delta_{11} = \frac{1}{2}\int_0^l x^2 \frac{ds}{I} = \frac{1}{2}\int_0^l x^2 \frac{ds\ \cos\varphi}{I\ \cos\varphi} = \frac{1}{2I_o}\int_0^l x^2\ dx = \frac{l^3}{6I_o}$$

$$E\delta_{12} = \int_0^l xy \frac{ds}{I} = \int_0^l xy \frac{ds\ \cos\varphi}{I\ \cos\varphi} = \frac{1}{I_o}\int_0^l xy\ dx$$

$$= \frac{1}{I_o}\int 4f \frac{x^2}{l^2}(l - x)\ dx = \frac{fl^2}{3I_o}$$

$$E\delta_{20} = -2\int_0^l M_o y \frac{ds}{I} = -2\int_0^l M_o y \frac{ds\ \cos\varphi}{I\ \cos\varphi} = -\frac{2}{I_o}\int_0^l M_o y\ dx$$

$$= -\frac{2}{I_o}\int_0^l p\frac{x}{2}(2l - x)\quad 4f\frac{x}{l^2}(l - x)\ dx$$

$$= -\frac{4pf}{I_o l^2}\int_0^l (2l^2 x^2 - 3l x^3 + x^4)\ dx = -\frac{7pfl^3}{15\ I_o}$$

$$E\delta_{22} = 2\int_0^l y^2 \frac{ds}{I} = 2\int_0^l y^2 \frac{ds\ \cos\varphi}{I\ \cos\varphi} = \frac{2}{I_o}\int_0^l y^2\ dx$$

$$= \frac{2}{I_o}\int_0^l 16\frac{f^2}{l^4}x^2(l - x)^2 = \frac{32f^2}{I_o l^4}\int x^2(l - x)^2\ dx = \frac{16f^2 l}{15I_o}$$

Substituting these values in the equations of elasticity we get

$$1)\quad 0 = -\frac{5pl^4}{24\ I_o} + X_1\frac{l^3}{6I_o} + X_2\frac{fl^2}{3I_o}$$

$$2)\quad -\frac{2X_2 l E}{A_t\ E_t} = -\frac{7pfl^3}{15\ I_o} + X_1\frac{fl^2}{3I_o} + X_2\frac{16f^2 l}{15\ I_o}$$

These two equations give

$$X_2 = \frac{pl^2}{8f\left(1 + \frac{5E\ I_o}{f^2 A_t E_t}\right)}$$

Assuming $\frac{5E\ I_o}{f^2\ A_t E_t} = \mu$ and $\frac{1}{1 + \mu} = k$, then

$$X_2 = \frac{pl^2}{8f(1 + \mu)} = k\ \frac{pl^2}{8\ f}$$ and

$$X_1 = \frac{pl}{4}\ \frac{4 + 5\mu}{1 + \mu} = \left(\frac{5 - k}{4}\right) pl$$

Example

Parabolic arch, continuous over two spans 24 ms each, rise 3 0 ms cross-section of arch at crown 30 x 80 cms, reinforcement of tie 12 ϕ 25 mm (A_t = 60 cm^2) , E = 210 t/cm^2, E_t = 2100 t/cm^2 and load p = 5 t/m

$$\mu = \frac{5\ I_o}{f^2 A_t}\ \frac{E}{E_t} = \frac{5 \times 30 \times 80^3}{12 \times 300^2 \times 60}\ \frac{1}{10} = 0\ 12$$

$$k = \frac{1}{1 + \mu} = \frac{1}{1\ 12} \approx 0\ 9$$

$$X_2 = k\ \frac{pl^2}{8\ f} = 0\ 9\ \frac{5 \times 24^2}{8 \times 3} = 0\ 9 \times 120 = 108 \text{ ton}$$

$$X_1 = \left(\frac{5 - k}{4}\right) pl = \left(\frac{5 - 0\ 9}{4}\right) 5 \times 24 = 1\ 025 \times 120 = 123 \text{ ton}$$

Reaction at outside supports

$$A = B = \frac{1}{2}\ (5 \times 48 - 123) = \frac{117}{2} = 58\ 5 \text{ ton}$$

Maximum bending moment M_m at crown

$$M_m = A\ \frac{l}{2} - \frac{pl^2}{8} - X_2\ f = \frac{58\ 5 \times 24}{2} - \frac{5 \times 24^2}{8} - 108 \times 3 = 18 \text{ mt}$$

Maximum bending moment M_s at intermediate support

$$M_s = A\ l - \frac{pl^2}{2} = 58\ 5 \times 24 - \frac{5 \times 24^2}{2} = -\ 36 \text{ mt}$$

For preliminary calculations one may assume here also that the maximum bending moment M_m at the crown and the maximum bending moment M_s at the support are given by

$$M_m \approx 0.05 \frac{pl^2}{8}$$

$$M_s \approx -0.10 \frac{pl^2}{8}$$

Applying these two equations on our example , we find that

$$M_m = 0.05 \times \frac{5 \times 24^2}{8} = 18 \text{ mt}$$

and

$$M_s = -0.10 \times \frac{5 \times 24^2}{8} = -36 \text{ mt}$$

<u>Tables of Internal Forces in Two Hinged and Fixed Parabolic Arches</u>

with $I \cos \varphi = I_o$

The axis of a parabolic arch fig VIII-20 is given by

$$y = \frac{4fx}{l^2} (l - x) \qquad \text{or}$$

$$y = 4f\xi\xi' \quad \text{where } \xi = \frac{x}{l} \text{ and } \xi' = \frac{x'}{l}$$

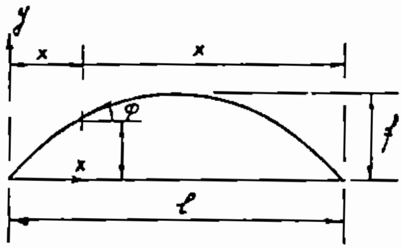


Fig VIII-20

The length of the axis s for parabolic arches is approximately given by

$$s = l\left(1 + \frac{8}{3} \eta^2\right) \qquad \text{where } \eta = \frac{f}{3}$$

The following tables[+] give the reactions and maximum moments in two hinged and fixed arches, they help much in the preliminary design

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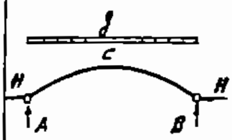
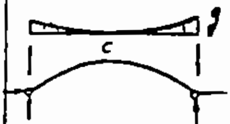
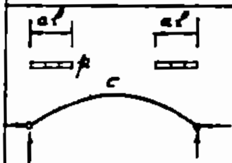
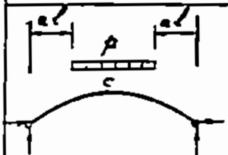
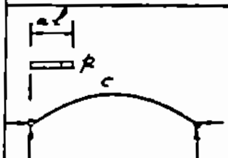
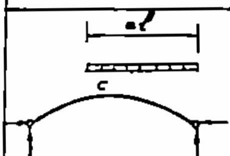
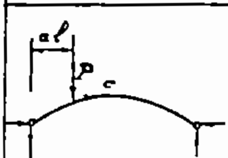
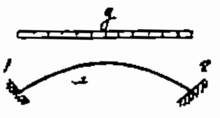
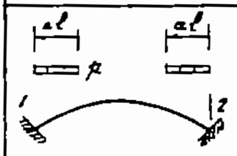
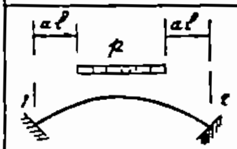
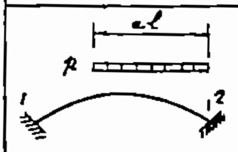
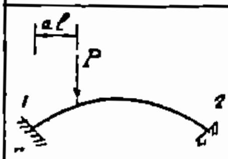
* " Design of Concrete Structures by Winter,Urquhart,O Rourke and Nilson, Published by McGraw-Hill Book Company New York

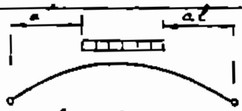
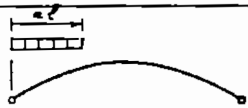
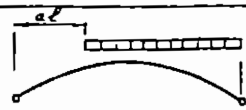
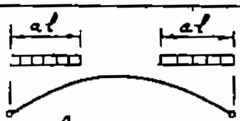
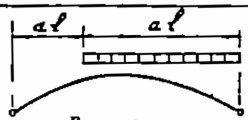
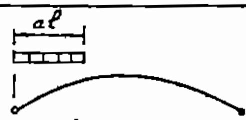
MOMENTS AND REACTIONS OF TWO HINGED PARABOLIC ARCHES

	$A = B = \frac{gl}{2}$ $M_c = 0$	$H = \frac{gl^2}{8f}$ $M_{l/4} = 0$
	$A - B = \frac{gl}{6}$ $M_c = -\frac{gl^2}{338}$	$H = \frac{gl^2}{47f}$ $M_{l/4} = \frac{gl^2}{234}$
	$A = B\ pal$ $M_c = \frac{-pl^2}{8} a^2(2a^3 - 5a^2 + 1)$	$H = \frac{pl^2}{8f} a^2(2a^3 - 5a^2 + 5)$
	$A = B = \frac{pl}{2}(1 - 2a)$ $M_c\ \frac{pl^2}{8} a^2(2a^3 - 5a^2 + 1)$	$H\ \frac{-pl^2}{8f}(2a^5 - 5a^4 + 5a^2 - 1)$
	$A = \frac{pal}{2}(2\ a)\quad B - \frac{pa^2l}{2}$ For $\frac{l}{4} \leq al \leq \frac{l}{2}$	$H - \frac{pl^2}{16f} a^2(2a^3 - 5a^2 + 5)$ $M_{l/4} = \frac{-pl^2}{64}(6a^5 - 15a^4 + 23a^2 - 16a + 2)$
	$A\ \frac{pal}{2}\quad B = \frac{pal}{2}(2 - a)$ For $\frac{l}{2} \leq al \leq \frac{3l}{4}$	$H = \frac{pl^2}{16f} a^2(2a^3 - 5a^2 + 5)$ $M_{l/4} - \frac{-pl^2}{64} a^2(6a^3 - 15a^2 + 7)$
	$A = P(1 - a)\quad B\ Pa$ $M_c\ \frac{Pl(1-a)}{8}(-5a^3 + 10a^2 - 1)$	$H = \frac{5Pl}{8f} a(a^3 - 2a^2 + 1)$
Uniform temp change of t	Coeff of linear expansion α $M_c = \frac{-15\ EI\alpha t}{8f}$	$A = \frac{15\ EI\alpha t}{8f^2}$ $M_{l/4} - \frac{-45\ EI\alpha t}{32f}$

MOMENTS AND REACTIONS OF FIXED PARABOLIC ARCHES

	$A \quad B \quad \frac{ql}{2}$ $M_1 \quad 0$	$H - \frac{ql^2}{8f}$ $M_2 \quad 0$
	$A \quad B \quad \frac{ql}{6}$ $M_1 \quad -\frac{ql^2}{210}$	$H \quad \frac{ql^2}{56f}$ $M_2 = -\frac{ql^2}{210}$
	$A \quad B \quad pal$ $M_1 \quad -\frac{pl^2}{2}a^2(1-a)^2(1-2a)$	$H = \frac{pl^2}{4f}a^3(6a^2 \; 15a+10)$ $M_2 \quad -\frac{pl^2}{2}a^2(1 \; a)(1-2a)$
	$A \quad B \quad \frac{pl}{2}(1-2a)$ $M_1 = \frac{pl^2}{2}a^2(1-a)^2(1 \; 2a)$	$H \quad -\frac{pl^2}{8f}(12a^5 \; 30a^4+20a^3-1)$ $M_2 \quad \frac{pl^2}{2}a^2(1-a)^2(1-2a)$
	$A = \frac{pl}{2}a(a^3 \; 2a^2+2)$ $B = pal - A$ $M_1 = -\frac{pl^2}{2}a^2(1 \; a)^3$	$H \quad \frac{pl^2}{8f}a^3(6a^2-15a+10)$ $M_2 \quad \frac{pl^2}{2}a^3(1-a)^2$
	$A \quad \frac{pl}{2}(1+a)(1-a)^3$ $B \quad pal - A$ $M_1 = \frac{pl^2}{2}a^2(1-a)^3$	$H = \frac{pl^2}{8f}(1 \; a)^3(6a^2+3a+1)$ $M_2 \quad -\frac{pl^2}{2}a^3(1-a)^2$
	$A = P(1-a)^2(1+2a) \quad B \quad PA$ $M_1 = \frac{-Pl}{2}a(1-a)^2(2-5a)$	$H \quad \frac{15}{4}\frac{Pl}{f}a^3(1-a)^2$ $M_2 = \frac{Pl}{2}a^2(1-a)(3-5a)$
Uniform temp change of t	$A \quad 0 \qquad B - 0$ $M_1 \quad \frac{15EI\alpha t}{2f}$	$H - \frac{45EI\alpha t}{4f^2}$ $M_2 \quad \frac{15EI\alpha t}{4f}$

Positions of loads and values of a which result with good accuracy in maximum moments at the crown, quarter point, or springing are given for both types of arches in the following table

	Crown	left quarter point	left springing
Positive Moment	 Two hinged a = 0.350 Fixed a = 0.375	 Two hinged a = 0.425 Fixed a = 0.400	 Fixed a = 0.40
Negative Moment	 Two hinged a = 0.350 Fixed a = 0.375	 Two hinged a = 0.575 Fixed a = 0.400	 Fixed a = 0.460

IX - CONSTRUCTIONAL DETAILS

IX-1 INSULATION AND ISOLATION OF ROOFS

In big span roof structures supporting mainly their own weight, it is recommended to choose the shape of the main supporting element giving the minimum dimensions The spacing between the main girders and the secondary longitudinal beams should be so chosen as to give a thickness of slab varying between 8 and 10 cms Such thin slabs are generally not water-tight and must be given a sufficient slope($\geqslant$1/100) so that rain water can be directly drained and is not allowed to gather on the roof An effective isolating material such as pluvex or asphaltoid in 2 or more layers may be used

Insulation of these thin slabs by the use of 5-8 cms celton, airocrete, thermocrete, no fine light concrete etc is also essential Cement tiles may also be used as a covering material for horizontal and lightly inclined roofs The details are shown in fig IX-1

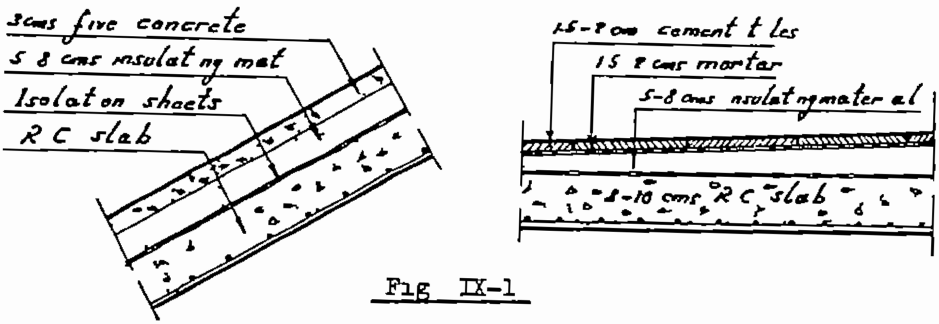


Fig IX-1

IX-2) SECONDARY BEAMS

Secondary beams are generally arranged normal to the main supporting frames or arches giving together a stable space structure and dividing the slab in convenient areas giving economic dimensions The spacing lies generally between 2 5 and 5 0 ms, while the span varies between 4 and 8 ms These are determined by economical and practical considerations They will be affected by the use to which the structure is to be put, the size and shape of the structure and the load which must be carried A comparison of a number of trial designs and estimates should be made and the most satisfactory arrangement selected

If the slope of the roof is smaller than 1 3 (3:1), the secondary beams are made vertical, for bigger slopes, the beams are made normal to the slab The arrangement of intermediate, edge, wall and crane beams can be chosen according to figure IX-2

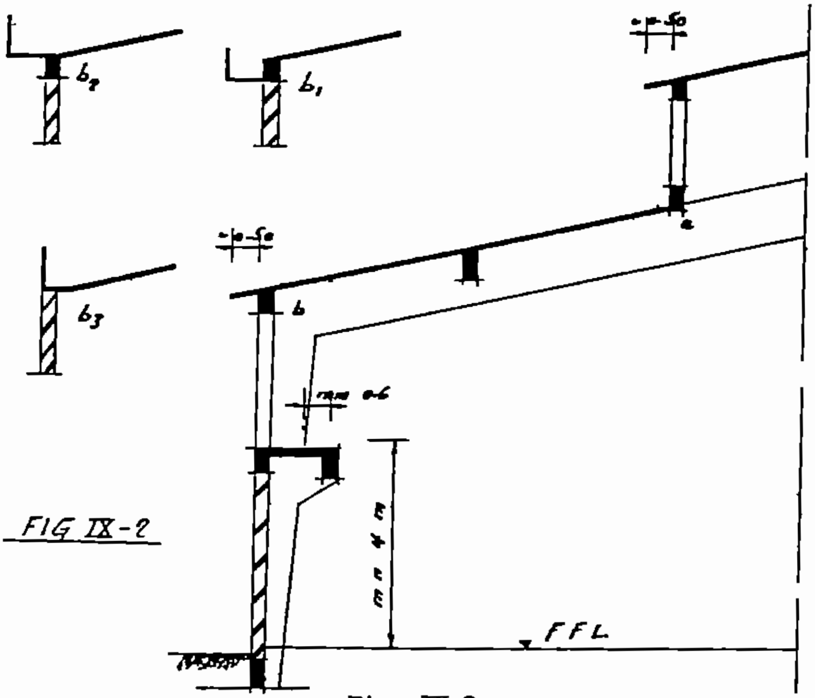


FIG IX-2

Fig IX-2

The slabs in the structure shown project ~ 0 5 ms outside the windows so that the rain water does not seep on them The rain water is either left to fall freely from the roof of the monitor to the roof of the hall and then to the free area outside the structure or it is collected in a gutter as shown in b_1 , b_2 or b_3 The beam at a is inverted in order to allow for a good fixation of the window crytal frame and to prevent seeping of the rain water at the lower edge of the window It gives more stiffness if the wall beam and the crane girder (if any) are arranged in the same horizontal plane and con - nected with a slab which may be used as a foot-path at the windows and helps in resisting the horizontal component of the crane load

IX-3) SHORT CANTILEVERS (Fig IX-3)

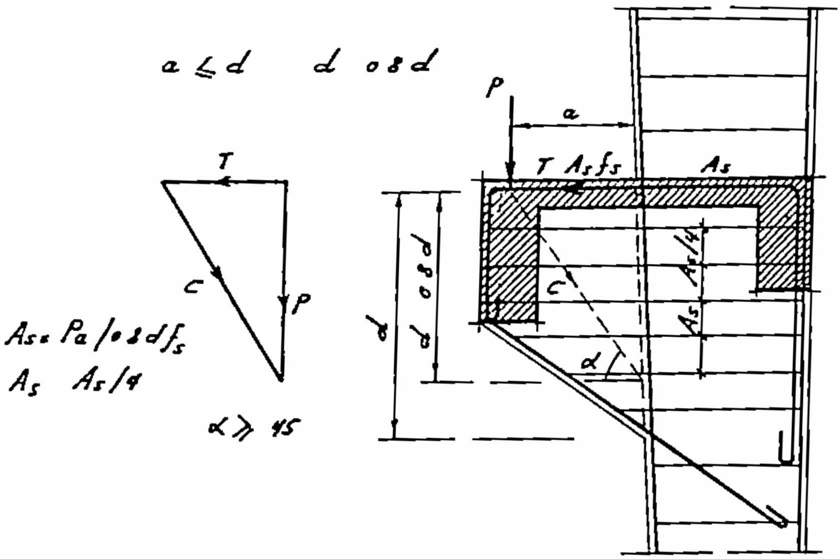


Fig IX-3

Crane girders are generally supported by short cantilevers as shown in figure IX-3 If the projection a is smaller than d' (=0 8 d) or equal to it so that $\alpha \geqslant 45$, the tension T can be determined by direct resolution of forces for the same reasoning given in (X-5) Hence

$T/P = a/d'$ but $T = A_s\ \sigma_s$ and $d' = 0\ 8\ d$

then $A_s = P\ a/0\ 8\ d\ \sigma_s$

Horizontal stirrups having an area $A_s' = A_s/4$ to resist the possible tensile forces normal to the direction of the compression C are essential

IX-4) EXPANSION JOINTS

Long structures are to be divided into blocks of maximum length or breadth equal to 40 ms by expansion joints according to one of the systems shown in figure IX-4

IX-5) END GABLES

In order to resist the wind pressure acting on the end cross - walls of big span halls one has to arrange a sieres of columns, generally framing with the longitudinal beams of the roof at distances of 4-6 ms To reduce the bending moments on such columns horizontal beams at convenient distances - 4 0 to 6 0 ms are essential As an example refer to section C-C of fig IV-34

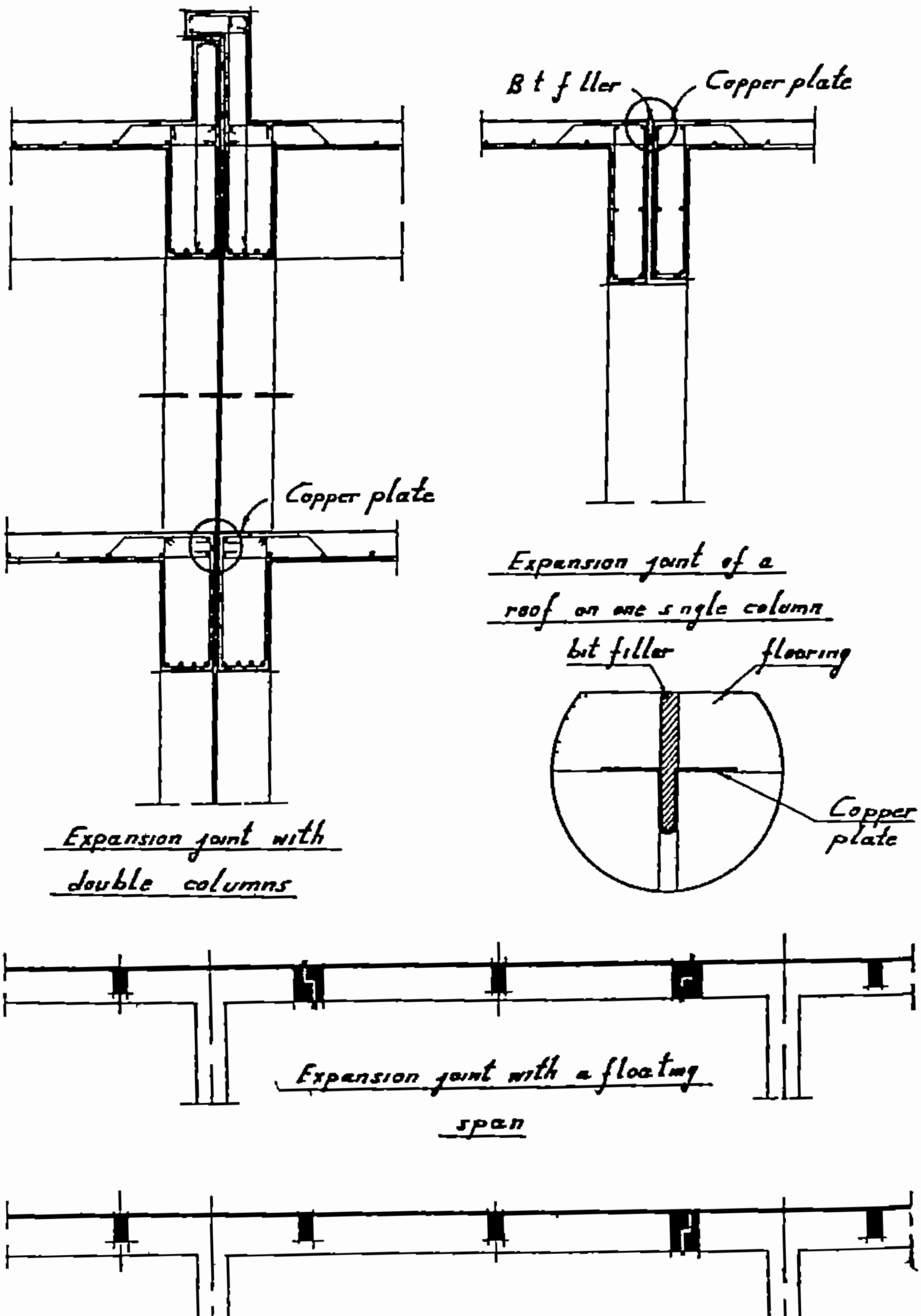


Fig IX-4

X - HINGED AND FREE BEARINGS

A free bearing is arranged to give a reaction at the point of support in a specified direction without any restraint It must therefore be free to rotate and slide normal to the specified direction of the reaction so that the reaction at a free bearing includes one unknown only

A hinged bearing is free to rotate without any restraint so that the point of application of the reaction is known but its direction and magnitude are unknown i e a reaction at a hinged support includes two unknowns as shown in figure X-1

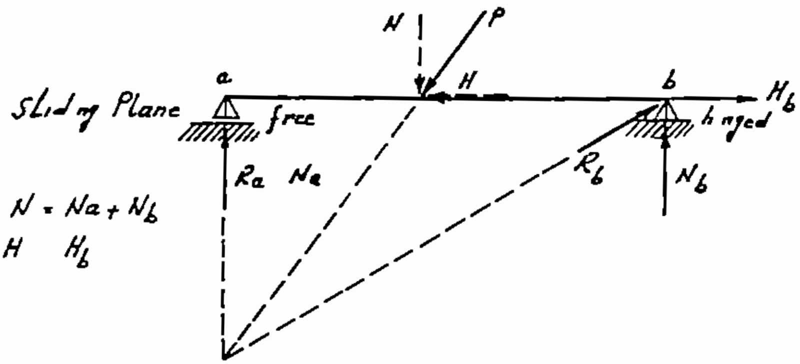


Fig X-1

Such ideal free and hinged bearings cannot be constructed , since friction or other inevitable restraining factors cannot be avoided

A HINGED BEARINGS

The main practical types of hinges used in concrete structures are

X-1) STEEL HINGES

The steel hinges e g as shown in figure X-2 result in more perfect hinge action than the concrete hinges, but are considerably more expensive Their use today, is restricted to bridges and unusually heavy concrete structures

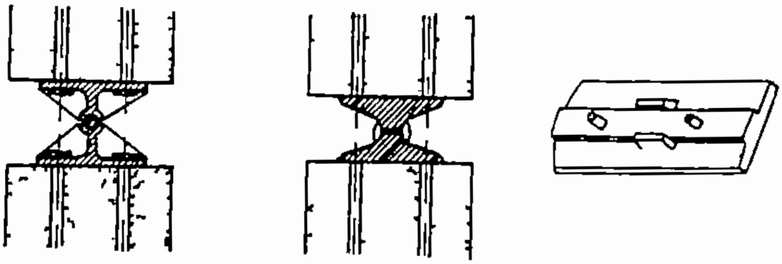


Fig X-2

X-2) MESNAGER HINGES

In this type of hinge, the reaction R is transmitted through the crossing bars A_{s1} and A_{s2} Their inclination with the free face of hinge lies between 30° and 60° They are protected from rusting by 2 5 cms oxidized asphalt, bituminous cork or bituminous felt arranged as shown in figure X-3

The crossing bars are in this manner subject to compressive stresses σ_s which must not exceed 0 3 σ_y This low stress is assumed because any rotation actually occuring at the hinge bends the bars and induces corresponding flexural stresses Such rotations actually do take place during the lifetime of the structure and are caused primarily by changes in live load and temperature These flexural stresses superpose on the computed compression and correspondingly increase the value of the actual maximum stresses in the bars Rather than to attempt computing these additional stresses, it is generally satisfactory to keep the compression stress σ_s' sufficiently low so that the bars will not be overstressed by superposed bending

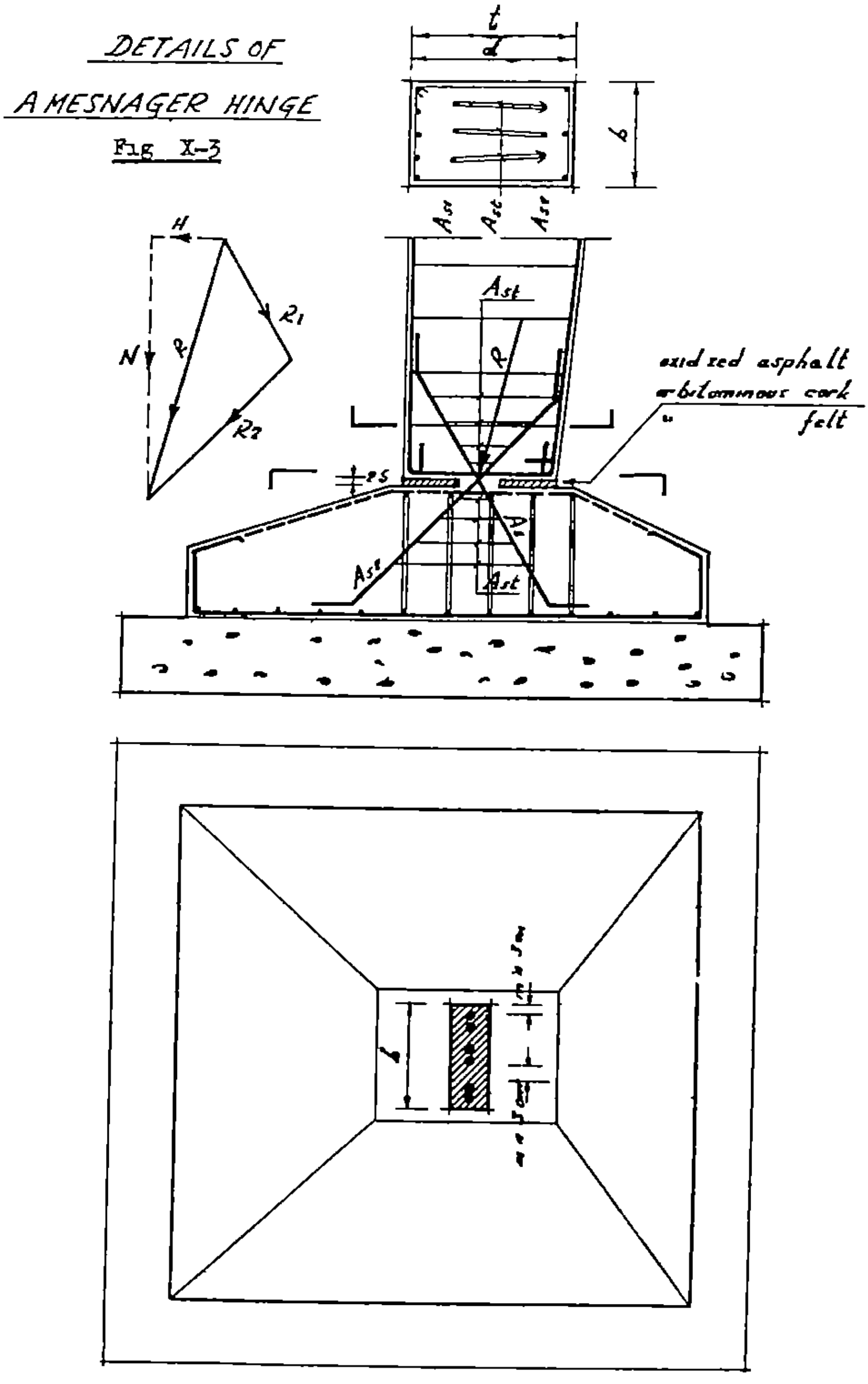
The crossing bars, transmitting their force to the concrete by bond along the embedded length, exert a bursting force which must be resisted by additional ties Only the part of the lateral reinforcement within a distance a = 8 Φ (Φ being the diameter of crossing bars) from the face of the concrete is considered effective in resisting the bursting force The tensile stress σ_{st} in the ties A_{st} can be computed according to figs X-3 & X-4 from the relation

$$\sigma_{st} = \frac{(N/2)\tan\theta + H\,a / y_{CT}}{0\ 005\ a\ b + A_{st}}$$

DETAILS OF

A MESNAGER HINGE

Fig X-3



where, in addition to previously defined quantities, A_{st} is the combined area of lateral ties located within a = 8 φ from the free face of the hinge and y_{CT} = 0 87 d

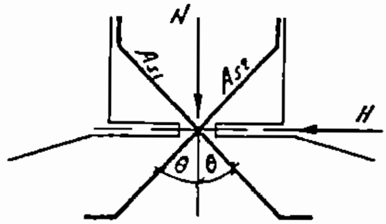


Fig X-4

Example

A Mesnager hinge of the form shown in figure X-3 is subject to an inclined reaction R whose component normal to the free face of the hinge N = 25t and its component parallel to it H = 5t The breadth b and depth t of the foot of the column are b = 40 cms and t = 60 cms Determine the reinforcements required for the hinge using mild steel with σ_y = 2700 kg/cm^2

Referring to figure X-3, assume

R_1 = 14 ton so that $A_{s1} = R_1 / \sigma_s'$ = 14/0 8 = 18 cm^2 4φ25

and

R_2 = 17 6 ton so that $A_{s2} = R_2 / \sigma_s'$ = 17 6/0 8 = 22 cm^2 4φ28

The stress in the lateral ties required to transmit the forces in these crossing bars by bond along a length a = 8 φ = 8x2 8 =22 4cm to the concrete is calculated in the following manner

Assuming the ties are 8 φ 6 spaced at ~ 7 5 cms , then we will have four rows of ties with 4 x 8 = 32 branch and having a total area A_{st} = 32 x 0 24 = 7 68 cm^2 and tan θ = 0 8

We have further d = 55 cms and y_{CT} = 0 87 d = 48 cms

So that

$$\sigma_{st} = \frac{12500 \times 0\ 8 + 5000 \times 22\ 4/48}{0\ 005 \times 22\ 4 \times 40 + 7\ 68} = 1015 \ \text{kg/cm}^2 < 1400 \ \text{Safe!}$$

This type of hinges can only be used for relatively small values of R (~ 30 tons) and is limited by the maximum amount of crossing bars that can be placed in the breadth of the foot of the column giving a clear distance of minimum 3 cms between the couples of crossing bars as shown in figure X-3

X-3) CONSIDERE HINGES fig X-5

In this type of hinge, the normal component N of the reaction

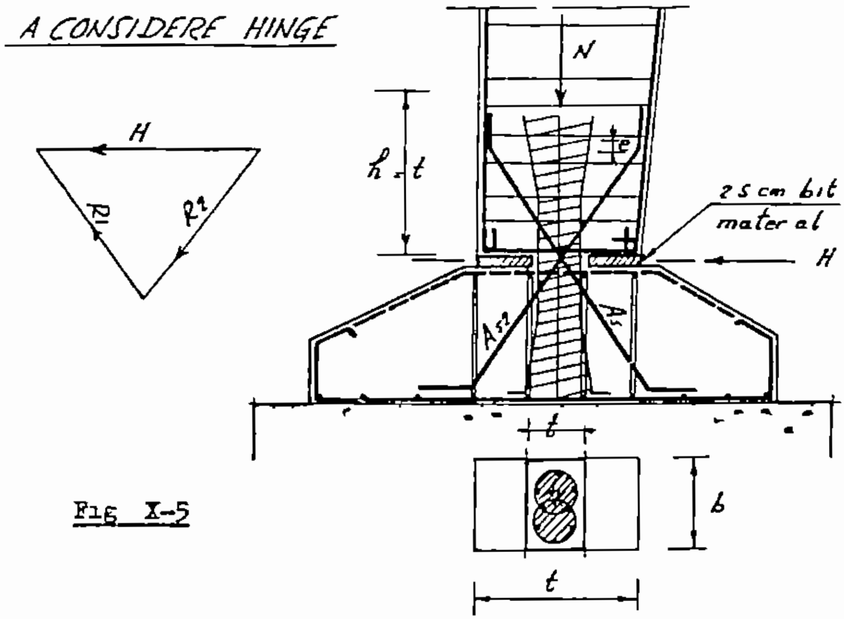


Fig X-5

is transmitted to the foundation by the short spirally reinforced concrete column arranged at the middle of the foot of the main column where as the thrust H is resisted by the crossing bars A_{s1} and A_{s2}

In order to have an acceptable hinge action t' must be $\leq t/3$ The spirally reinforced short column may be calculated from the known relation of the elastic theory

$$N = \sigma_b A_k + \sigma'_s A_s + 2.5 \quad \sigma_s A'_s \leq 2 (\sigma_b A_c + \sigma'_s A_s)$$

in which

A_k = the area of the core inside the spirals (hatched area)

A_c = b t'

A_s = area of cross-section of longitudinal reinforcements inside the spirals

A'_s = imaginary longitudinal reinforcements having the same volume of the spirals with cross-sectional area A_{sp} , pitch e ($\leq$8 cms) and diameter D i e $A'_s = A_{sp} \pi D/e$

$\sigma_b = \sigma_{co} \sqrt[3]{A/A'} \leq \sigma_{c28}/2$ the bearing stress of partially loa - ded areas

σ_{co} = allowable compressive stress of the concrete used

$A = b\ t =$ area of foot of column

$A' = A_k$ or A_c effective loaded area

σ'_s = allowable compressive stress of longitudinal reinforcement

σ_s = allowable tensile stress of spirals

$n = 15$ = modular ratio

If the ultimate strength theory is applied the corresponding equation with the specified load and reduction factors can be used

Due to the concentration of the stresses in the hinge horizontal splitting tensile forces are created at the foot of the column The stirrups along the height $h = t$ are generally increased to resist these splitting forces

X-4) LEAD HINGES fig X-6

In this type of hinges the normal component N of the hinge is transmitted to the foundation by bearing through a 2 cms thick lead plate arranged at the middle of the column foot The horizontal component H is resisted by the shear resistance of the connecting bars A_s which are protected from rusting by 2cms thick bituminous cork, bituminous felt or oxidized asphalt arranged as shown in fig X-6

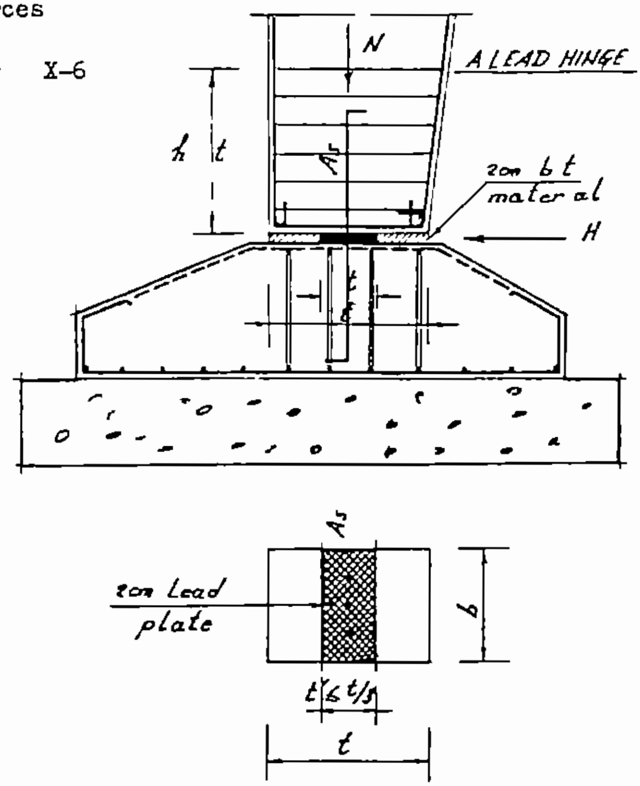


Fig X-6

In order to have acceptable hinge action, the length of the lead plate t' must be smaller than or equal to one third of the depth of the column at the position of the hinge measured in the direction of the required rotation

The lead hinge can accordingly be calculated in the following manner

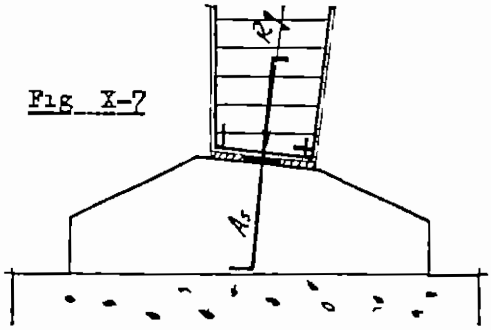
$$N/bt' < \sigma_b = \sigma_{co} \sqrt[3]{A/A'} \leqslant \sigma_{c28}/2$$

$A_s = H/\tau_s$ where $\tau_s = 0\ 8\ \ \sigma_s$ = allow shear stress of steel

If it is required to reduce the amount of the connecting steel A_s to a minimum, the lead plate is to be arranged normal to the direction of the reaction due to dead loads as shown in figure X-7 in which case, A_s can be calculated to resist the thrust due to live loads only

Due to the concentration of the stresses at the position of the hinge transverse tensile splitting forces are created These can be explained as follows (fig X-8)

Fig X-7



The transmission of the normal compressive stresses is assumed to take place from the breadth of the column t to the breadth of the lead plate t' in a height h approximately equal to the breadth t

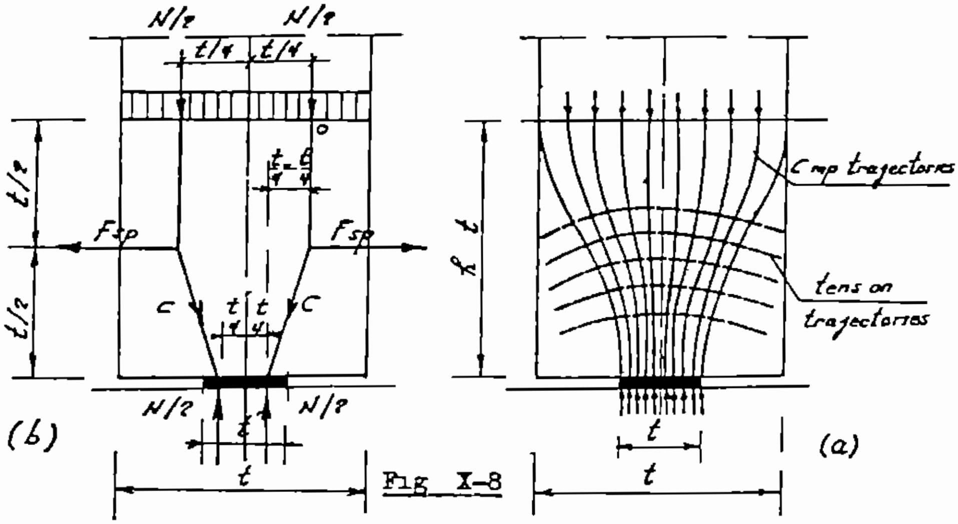


Fig X-8

of the column so that the compression trajectories and the corres - ponding tension trajectories will be as shown in fig X-8a

Considering the equilibrium of any half of the concrete block, height h = t, at the hinge, we can see that the two forces N/2 are equal and opposite but not colinear and equilibrium is only possible if a transverse force F_{sp} is assumed acting as shown in figure X-8b Its magnitude can be determined, according to Moersch, if we take moments of the forces acting on say the right half about point O

Hence

$$\frac{N}{2} \left(\frac{t}{4} - \frac{t'}{4} \right) = F_{sp} \frac{t}{2}$$

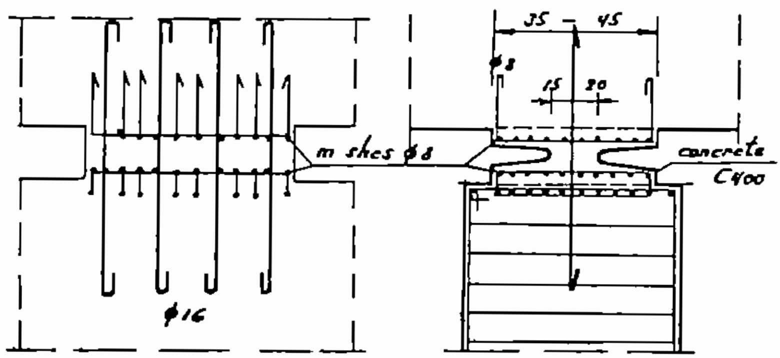
So that

$$F_{sp} = N (t - t') / 4t \simeq N/4$$

F_{sp} is tension and called the transverse splitting tensile force, it assumes its max value of N/4 for t' = 0 It must be resis - ted by horizontal stirrups of area $A_{st} = F_{sp} / \sigma_s$ arranged at the foot of the column in a height h = t Refer to figures X-5, 6 & 7

X-5) CONCRETE HINGES

Fig X-9



This type can only be used if the hinge is subjected to small angles of rotation The vertical component of the reaction is transmitted to the supporting element by bearing on concrete with a stress $\sigma_b = \sigma_{co} \sqrt[3]{A/A'} \leq \sigma_{c28}/2 = 150$ to 200 kg/cm^2 The horizontal component of the reaction is taken by the shear resistance of the vertical connecting bars

It is recommended to arrange reinforcing meshes ϕ 8 mm on both sides of the hinge as shown in figure X-9 It is not allowed to make a construction joint through the hinge and to make it of high grade concrete

For this reason precast hinges (or rockers) as shown in figure X-10 may be used

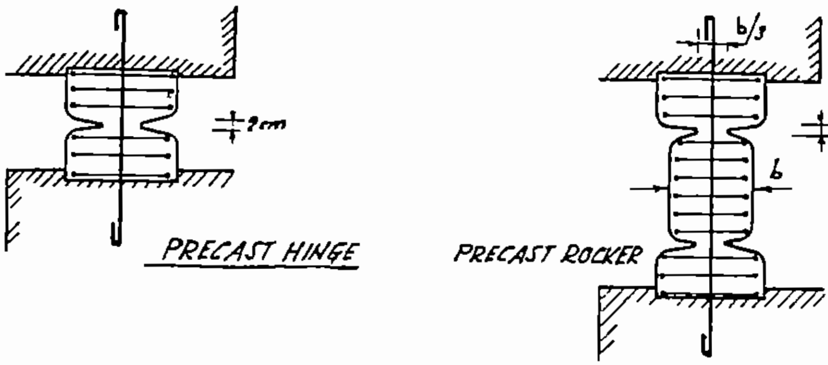


Fig X-10

It is however essential to arrange sufficient cross-reinforcement to resist the possible splitting forces

In all previous types the minimum anchorage length of crossing or longitudinal bars must be $\geqslant$ 30 ϕ and $\geqslant$ 50 cms on each side of the hinge

B FREE BEARING

The main types of free bearings used in concrete structures are

X-6) STEEL BEARINGS

The different types of steel bearings may be used in concrete structures Types of low cost and low construction height are recommended

X-7) ROCKER BEARINGS (Fig X-11)

A rocker is a short reinforced concrete element between two hinges as shown in figure X-11 As the rocker is not laterally loaded, the reaction supported by a rocker will be in the direction of the line connecting the two hinges Its height h depends on the required displacement and rotation It supports the reaction by the combined action of the concrete , the longitudinal reinforcement and eventually the cross reinforcements which if placed in layers at a small spacing e $<$ 8 cms , they increase the resistance of the rocker by an amount equal to 2.5 n A'_s σ_s similar to that of spiral reinforcement. A'_s is in this case equal to the volume of the cross reinforcements per unit height i.e. total length of cross reinforcements multiplied by their cross-section and divided by their pitch e.

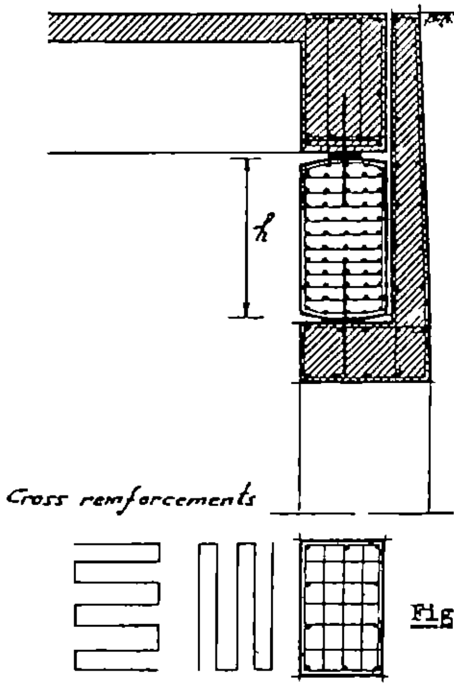


Fig. X-11

X-8) RUBBER BEARINGS (Fig. X-12)

A rubber bearing is composed of steel plates or wire meshes embedded in a special material - the neoprene - made of artificial synthetic rubber. It possesses very high resistance to clymatic changes and chemical actions of acids or alkalies that are liable to exist in air or water. The steel in these bearings is completely protected against rusting. A neoprene bearing possesses the high resistance of steel and the elasticity of rubber. It can resist high vertical forces without appreciable vertical displacements. It allows for horizontal displacements δ and rotations α in two normal

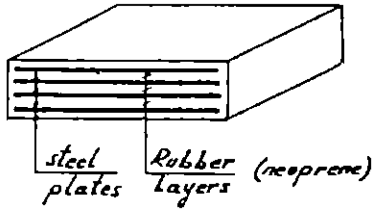


Fig. X-12

directions as shown in figure X-13

Fig X-13

This kind of bearings is to be assumed as free allowing rotations and horizontal displacements in any direction

The exact stress distribution in a neoprene bearings is very complicated but for practical designs, the following method may be used One has to check

1) The axial compressive stress

$$\max \sigma = \max N/A \leqslant \text{allow } \sigma = 100 \text{ kg/cm}^2$$

2) Sliding angle

$$\tan \gamma = \delta / h_o < \text{allow } \tan\gamma$$

3) Horizontal force

$$H = G \; A \; \tan\gamma$$

4) Angle of friction between neoprene bearing and concrete

$$\max \mu = \max H/\min N \leqslant \text{allow } \mu$$

5) Angle of rotation of every rubber layer

$$\beta < \text{allow } \beta$$

in which

max N and min N are the maximum and minimum normal components of the reaction

A is the area of the bearing in plan

δ the horizontal displacement of the point of support

h_o the net height of the bearing (sum of all rubber layers)

h the height of the bearing

n number of rubber layers

G bulk modulus of rubber

The following two cases of loading must be recognised

Case I - Permanent Loads

(dead loads, prestressing, creep shrinkage, temperature)

Case II - <u>Live Loads</u> (acting for a short time)
(tractive & wind forces)

The following values must not be exceeded

<u>Table 1</u>

Case of loading	tan γ	allow normal σ in kg/cm^2					allow β in minutes			G kg/cm^2
		20	40	60	80	100	150x200	200x300	300X400	
I	0 7	0 50	0 45	0 40	0 35	0 30	10 0	7 5	5 0	13
II	0 3	0 30	0 26	0 22	0 18	0 15	4 0	3 0	2 0	20
I + II	0 9	0 50	0 45	0 40	0 35	0 30	14 0	10 5	7 0	

all values are related to the effective depth h_o

The compressibility of the neoprene bearing due to vertical forces is very small and has no effect on the internal forces in statically indeterminate structures

The standard dimensions are

150 x 200 mm for max N = 30 t with E = 3500 kg/cm^2

200 x 300 mm for max N = 60 t with E = 7000 kg/cm^2

300 x 400 mm for max N =120 t with E =14500 kg/cm

Its height is chosen according to the statical requirements and should be ≤ 1/5 of its breadth The rubber layers are 5mms thick

<u>Table II</u>

No of rubber layers	n=	2	3	4	5	6	7	8	9	10	11	12	
effective depth	h =	10	15	20	25	30	35	40	45	50	55	60	mm
total depth	h=	14	21	28	35	42	49	56	63	70	77	84	
for case of load I	w=	7 0	10.5	14.0	17 5	21 0	24 5	28.0	31 5	35 0	38 5	42 0	
for bearing 150 x 200	max V=	30	30	30	30	30	-	-	-	-	-	-	t
for bearing 200 x 300		60	60	60	60	60	60	60	-	-	-	-	t
for bearing 300 x 400		-	-	-	-	120	120	120	120	120	120	120	t

Example

Design the required neoprene bearings for the shown simple prestressed foot bridge (Fig X-14)

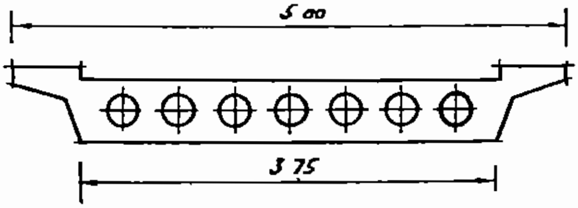


Fig X-14

data

Cross-section	$A = 2\ 8\ m^2$	I_c	$= 0\ 08\ m^4$
Initial prestressing force		F_o	$= 900$ t
Span		l	$= 20$ ms
Temperature change		Δt	$= \pm 20°$
Creep strain $\varepsilon_{cr} = 3$ times elastic strain		ε_c	
Shrinkage strain 0 20 mm/m		ε_{sh}	$= 0\ 2 \times 10^{-3}$
Modulus of elasticity of concrete		E_c	$= 300\ 000$ kg/cm^2
Bulk modulus of bearing		G	$= 13$ kg/cm^2
Dead load		g	$= 7\ 5$ t/m
Live load		p	$= 2\ 5$ t/m

Average concrete stress σ_c due to $F_o + g$ is given by

$$\sigma_c = F_o / A = 900/2\ 8 = 320\ t/m^2 = 32\ kg/cm^2$$

The horizontal displacements of the bridge are

Due to prestress $\varepsilon_o = \sigma_c/E_c = 32/300\ 000 = 0\ 11 \times 10^{-3}$

Due to creep $\varepsilon_{cr} = 3 \times \varepsilon_o = 3 \times 0\ 11 \times 10^{-3} = 0\ 33 \times 10^{-3}$

Due to shrinkage $\varepsilon_{sh} = 0\ 2$ mm/m $= 0\ 20 \times 10^{-3}$

Due to temperature $\varepsilon_t = 20 \times 10^{-5} = 0\ 20 \times 10^{-3}$

total $\varepsilon = 0\ 84 \times 10^{-3}$

Horizontal displacement of point of support

$$w = l\ \varepsilon/2 = 20\ 000 \times 0\ 84 \times 10^{-3}/2 = 8\ 4\ mm$$

Reactions

$$\max N = (g + p) l /2 = (7\ 5 + 2\ 5) \times 20/2 = 100 \text{ t}$$

$$\min N = g l /2 = 7\ 5 \times 20/2 = 75 \text{ t}$$

For each side of the bridge, we choose
2 neoprene bearings each 200 x 300 x 20 mm

Bearing area $A = 2 \times 20 \times 30 = 1200 \text{ cm}^2$

Height of bearing h_o = 20 mm (4 rubber layers 5 mms each)

Normal stress $\sigma_{max} = \max N/A = 100\ 000/1200 = 83 \text{ kg/cm}^2 < 100$

Displacement angle $\tan \gamma = w/h_o = 8\ 4/20 = 0\ 42 < 0\ 7$

Max horizontal force $H_{max} = G\ A \tan\gamma = 13 \times 1200 \times 0\ 42 = 6500 \text{ kg} = 6\ 5\text{t}$

Coeff of friction $\mu_{max} = \max H/\min N = 65/75 = 0\ 087 < 0\ 40$

Note = 0 40 is the allowed value for $\sigma_{min} = \min N/A = 75\ 000/1200 = 62 \text{ kg/cm}^2$)

Angle of rotation β per layer of rubber

$$= \frac{1}{4}\ \frac{pl^3}{24\ E_c\ I_c} = \frac{1}{4}\ \frac{2\ 5 \times 20^3}{24 \times 300\ 000 \times 0\ 80} = 0\ 86\ \%_0 = 2\ 95' < 3\ 0'$$

Neoprene bearings do not generally need any sort of fixation and are loosely placed on the bearing surface They can be used both for steel and concrete structures It is, however, of utmost importance that the bearing surface is smooth clean, plane and dry

For the construction of neoprene supports in concrete structures, they are to be encased in such a way that the shuttering can be easily removed For supports of small height, cork plates are to be used The surface of the neoprene support must be flush with the shuttering

After the removal of the shuttering (or cork), the lateral surfaces of the support must be entirely free so as to guarantee a smooth functioning of the support (Fig X-15)

For small bridges and buildings in which the reactions are generally small neoprene supports without steel plates may be used The max normal stress is to be chosen smaller than 25 kg/cm^2 and $\tan\gamma \leq 0\ 5$

Neoprene strips can be conveniently used as supports for cylindrical tanks with sliding base where they transmit the load of the wall (and eventually the roof) to the floor and, allow for the free movement of the wall and give a watertight joint

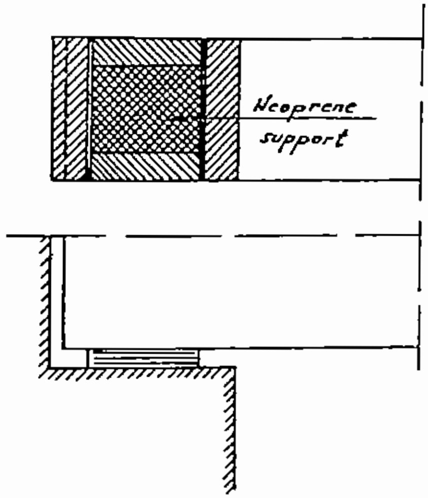
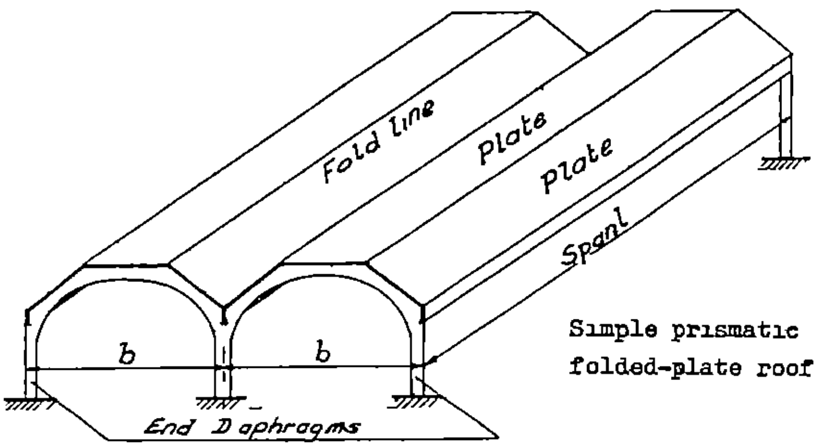


Fig X-15

XI - FOLDED PLATE STRUCTURES

XI-1) DEFINITION AND TYPES

Folded or hipped-plate structures consist of an assembly of flat plate strips intersecting at fold lines and arranged such that they form a stable three-dimensional structure Fig XI-1



Simple prismatic folded-plate roof

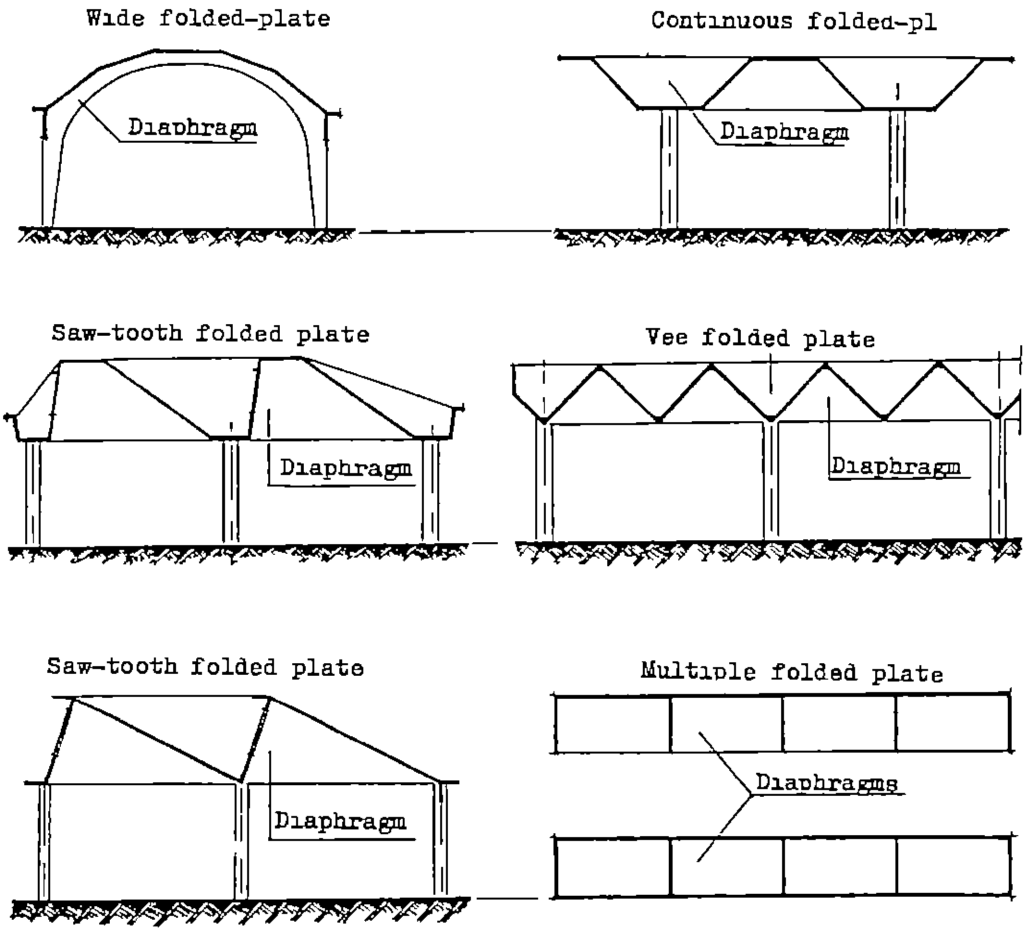
Fig XI-1

They may be prismatic, prismoidal, pyramidal, etc , they find application as roofs, coal bunkers, cooling towers, bridges, stair - cases, etc Fig XI-2

Prismatic structures must be stiffened by diaphragms in at least two cross-sections

The plate elements of a folded structure being straight, they are subject to bending moments between the fold lines and hence, they consume a little more material than continuously curved cylindrical shells, but the extra cost on this account is much smaller than the saving in the forms

Examples of prismatic folded-plates



Examples of non-prismatic folded-plates

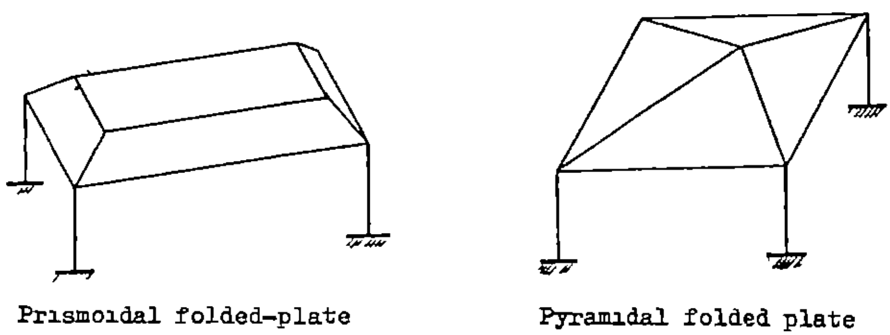
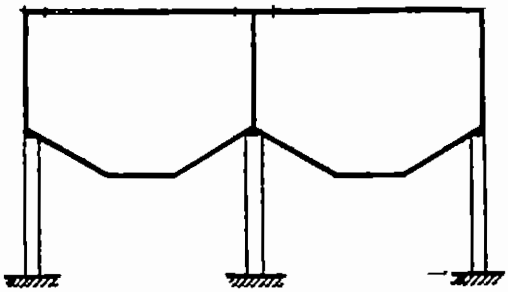
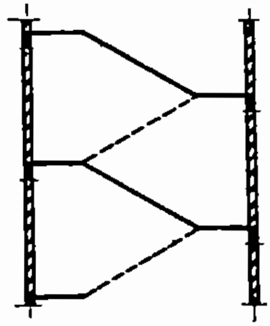


Fig XI-2

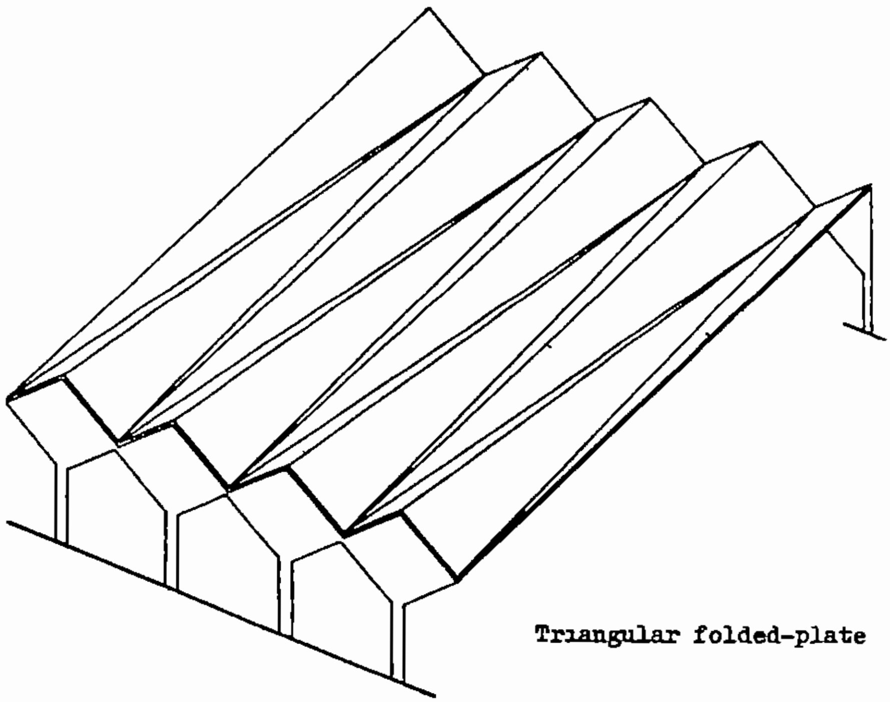
Examples of non-prismatic folded-plates (continued)



Bunker



Stair case



Triangular folded-plate

Fig XI-2
(continued)

XI-2) ASSUMPTIONS AND STRUCTURAL BEHAVIOR

Being primarily interested in folded plates as roofs, our study will be restricted to prysmatic folded plates consisting of rectangular plates, each plate being of uniform thickness

The following assumptions are usually made in the computations

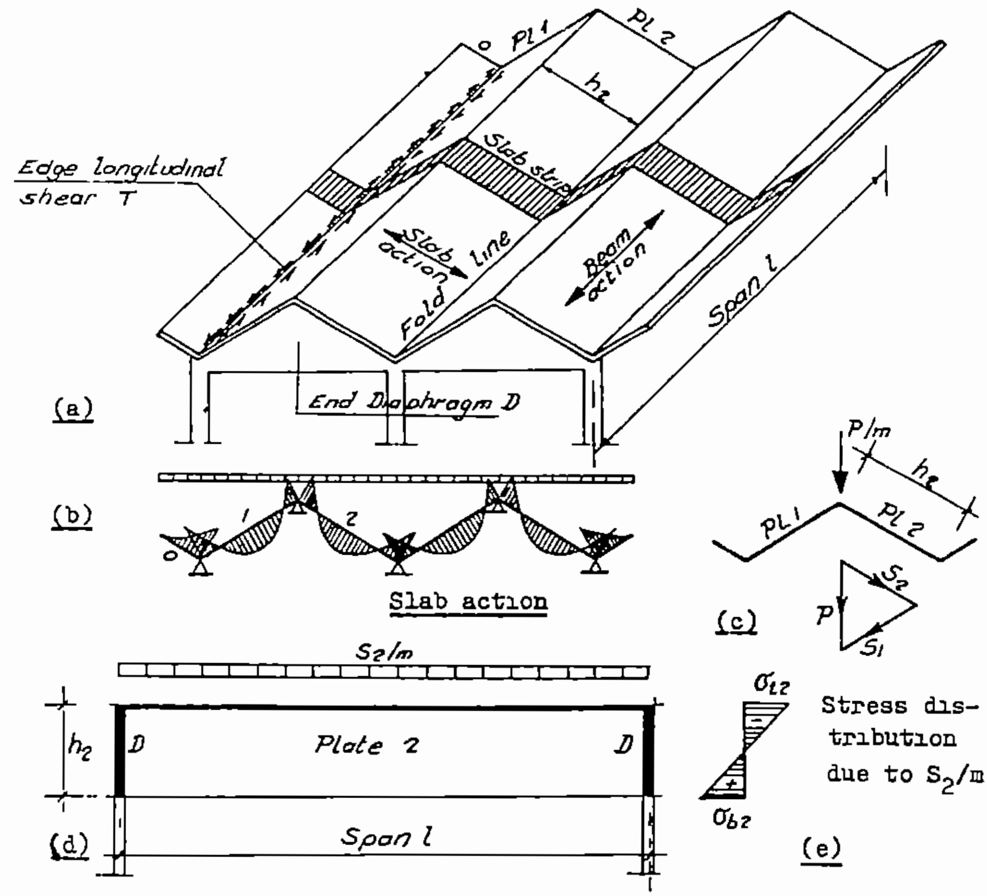
i) The structure is monolithic and the joints are rigid

ii) The material is elastic, homogeneous and isotropic

iii) The length of each plate is more than twice its width

iv) In all plates, plane sections remain plane after deformation (It is, however, to be carefully noted that a plane cross section of the entire structure does not necessarily remain plane after deformation)

XI-3) SLAB AND BEAM ACTION



Beam action

Fig XI-3

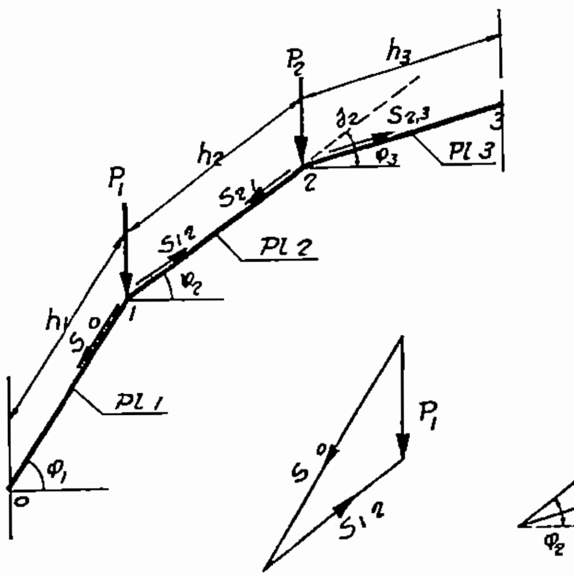
A) Slab Action

In prismatic folded plates, the length of each plate is more than twice its width (assumption iii) that the surface loads are carried by each plate as a one way slab supported on the fold lines. This is termed as slab action. A typical transverse strip is shown in Fig. XI-3 a. The corresponding bending moments as a one-way continuous slab are shown in Fig. XI-3 b.

B) Beam Action

The reactions P/m from such slab strips are applied as line loads to the fold lines. The only direction in which each plate can apply a reactive force to resist this line load is parallel to its own surface. The resultant line load P/m (see Fig. XI-3 c) therefore resolves into components parallel to the two adjacent plates. The plates in turn carry this edge loading longitudinally between the end diaphragms 'D' by beam action as shown in Fig. XI-3 d.

a) Ridge and plate loads



$$P_1 = \frac{1}{2}(p_1 h_1 + p_2 h_2) \quad XI\text{-}1$$

$$P_2 = \frac{1}{2}(p_2 h_2 + p_3 h_3) \quad XI\text{-}2$$

$$S_2 = S_{2,1} - S_{1,2} \quad XI\text{-}3$$

Fig. XI-4

Loads such as P_2 applied at the ridge 2 are known as ridge loads. Considering a unit length of the folded plate, then $P_2 = ½ (p_2 h_2 + p_3 h_3)$. P_2 may be resolved into plate loads $S_{2,1}$ and $S_{2,3}$ lying respectively in the planes of the second and third plates by means of a triangle of forces as shown in Fig. XI-4.

In the designation of plate loads such as $S_{2,1}$ the first subscript stands for the joint at which the load acts and the second subscript indicates the joint toward which the plate load is directed Thus the plate load $S_{2,1}$ is directed from joint 2 to joint 1 at ridge 2 The plate loads S may include the effect of the connecting moments in the cross direction It is clear that the net plate load carried as a beam by e g the second plate is

$$S_{2,1} - S_{2,3} \qquad \text{XI-3}$$

From the triangle of forces given in Fig XI-4, it is easy to prove that

$$S_{2,1} = \frac{P_2 \cos\phi_3}{\sin\delta_2} \qquad \text{and} \qquad S_{2,3} = \frac{P_2 \cos\phi_2}{\sin\delta_2} \qquad \text{XI-4}$$

b) Free edge stresses and compatibility at the ridges

The distribution of the normal stresses in any section of a free plate subject to a bending moment M_o is given by

$$\sigma_{b,t} = \mp M_o / Z$$

where Z is the section modulus of a plate, $Z = b\,h^2/6$ for rectangular sections of breadth b and height h as shown for plate 2 in Fig XI-5 d and e

If conditions are such that the free edge stresses are the same on both sides of all fold lines, then they are identical with the final stresses This would be the case, for example, at the edges of an interior unit of a roof consisting of many identical units (e g intermediate joints of the Vee folded plate of Fig XI-2) If, on the other hand, the initial plate analysis indicates a stress difference on either side of a fold line, an incompatibility is indicated which cannot actually exist, because the strains on either sides of a given fold line must be equal This indicates the presence of longitudinal shears acting along the joint as shown in Fig XI-3 a

The stress due to an edge shear T can be calculated as follows

Fig XI-5

$$\sigma_{t,b} = T/A \pm M/Z \qquad \text{where}$$

$M = Th/2$ and $Z = bh^2/6$, so that

$$\sigma_{t,b} = \frac{T}{A} \pm \frac{T\,h}{2}\;\frac{6}{b\,h^2} = \frac{T}{A} \pm \frac{3\,T}{A} \quad \text{or}$$

$$\sigma_t = + \frac{4\,T}{A} \quad \text{and} \quad \sigma_b = - \frac{2\,T}{A}$$

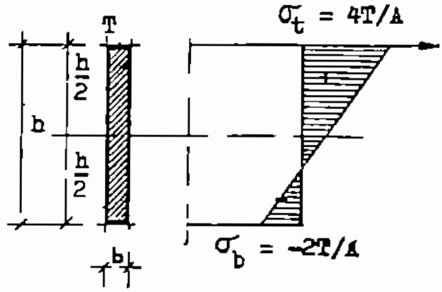


Fig XI-5

XI-4) DETERMINATION OF EDGE SHEARS AND FINAL STRESSES

A) Theorem of Three Edge Shears

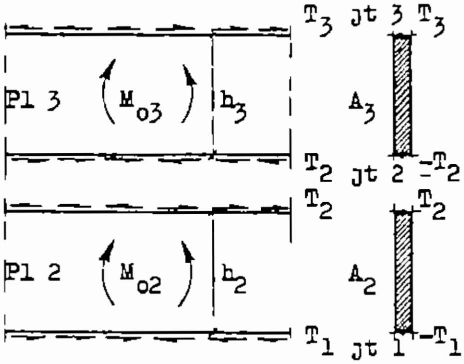


Fig XI-6

Stress at joint 2 in plate 2 is given by Fig XI-6

$$\sigma_2 = -\frac{M_{o2}}{Z_2} + \frac{4\,T_2}{A_2} + \frac{2\,T_1}{A_2}$$

Stress at joint 2 in plate 3 is given by

$$\sigma_2 = +\frac{M_{o3}}{Z_3} - \frac{4\,T_2}{A_3} - \frac{2\,T_3}{A_3}$$

But the fiber stress at joint 2 from plates 2 and 3 has to be the same as the plates are monolithically connected, we may equate the two previous expressions to get

$$\frac{T_1}{A_2} + 2\left(\frac{T_2}{A_2} + \frac{T_2}{A_3}\right) + \frac{T_3}{A_3} = \frac{1}{2}\left(\frac{M_{o2}}{Z_2} + \frac{M_{o3}}{Z_3}\right) \qquad \text{XI-5}$$

This relation is called the theorem of three edge shears similar to the known theorem of three moments

Having determined the edge shears T, the final stresses can be computed by superposition

B) Stress Distribution Method

The final stresses in a folded plate can be determined by the stress distribution method in the following manner Fig XI-7

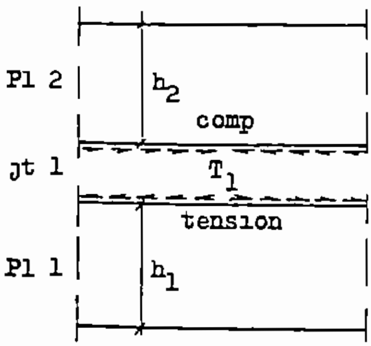
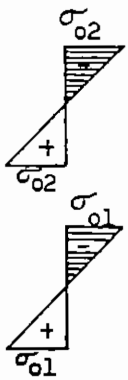
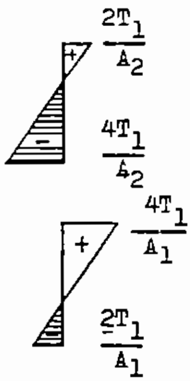


Fig XI-7



Stresses in individual pl for M_o



Stresses in individual pl for T_1

The compatibility of strains at joint 1 necessitates that the stress σ at both sides of joint 1 be the same Hence

$$\sigma_{o2} - \frac{4\,T_1}{A_2} = \sigma_{o1} + \frac{4\,T_1}{A_1} \qquad \text{or} \qquad \sigma_{o2} - \sigma_{o1} = 4\,T_1\left(\frac{1}{A_1} + \frac{1}{A_2}\right) \quad \text{i e}$$

$$\underline{\frac{4\,T_1}{A_1} = (\sigma_{o2} - \sigma_{o1})\,\frac{A_2}{A_1 + A_2}} \quad \text{and} \quad \underline{-\frac{4\,T_1}{A_2} = -\,(\sigma_{o2} - \sigma_{o1})\,\frac{A_1}{A_1 + A_2}} \qquad \text{XI-6}$$

This means that the correction of the stress σ_{o1} due to T_1, which is $4\,T_1/A_1$, is equal to the difference $(\sigma_{o2} - \sigma_{o1})$ multiplied by the distribution factor $A_2/(A_1+A_2)$ and the correction of the stress σ_{o2} due to T_1, which is $-4T_1/A_2$ is equal to $-(\sigma_{o2} - \sigma_{o1})$ multiplied by the distribution factor $A_1/(A_1+A_2)$.

These relations show that the difference of stresses in a joint can be distributed on the plates 1 and 2 meeting in joint 1 in the following ratio

Distribution factor for plate 1 equals $A_2 / (A_1 + A_2)$, and

" " " " 2 " $A_1 / (A_1 + A_2)$

It is clear that the carry-over-factor is - 1/2

This method can be directly used instead of the three shears equation XI-5

XI-5) <u>SHEAR STRESSES IN FOLDED PLATES</u>

The shear stresses due to slab action are generally very small and need not to be considered

The main shear stresses are due to the beam action of the different plates they are caused by the shearing forces of the plate loads S and the edge shears T

Our study will be restricted to simple prismatic folded plates only

Due to the force S/m acting on a plate of breadth b, depth h i e (area A = b h) and span l, the maximum shearing force Q_{max} at the diaphragms is given by

$$Q_{max} = S\ l\ /$$

It causes parabolic shear stresses τ_o, with a maximum value at the middle height of the section of the plate equal to Fig XI-8

$$\tau_{o\,max} = \frac{3}{2}\ \frac{Q_{max}}{A} \qquad \text{XI-7}$$

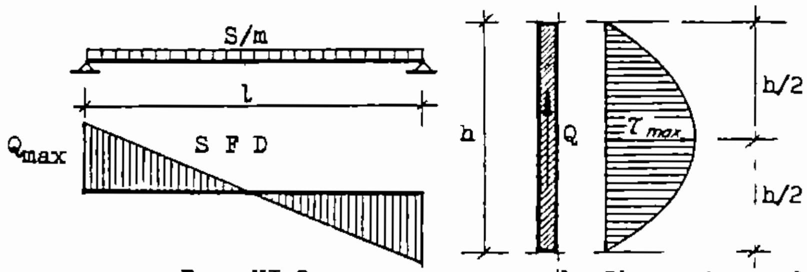


Fig XI-8 Shear stress diag

When calculating the shear stresses τ due to the edge shears T, one has to notice that the edge shear diagram at any fold line is similar to that of the bending moment due to S, i e parabolic with maximum value at midspan and zero at the supports in case of simple folded plates It is therefore easy to prove that the edge shear T at a distance x from the support of a plate is given by

$$T = T_{max}\ \frac{4\ x}{l}\ (1 - \frac{x}{l}) \qquad \text{XI-8}$$

where T_{max} is the maximum edge shear at midspan , it may be calculated from equation XI-5

The shear stress distribution on a section at a distance x from the support due to an edge shear T at one of the edges of a plate is also parabolic as shown in Fig XI-9 Its maximum value τ at the edge

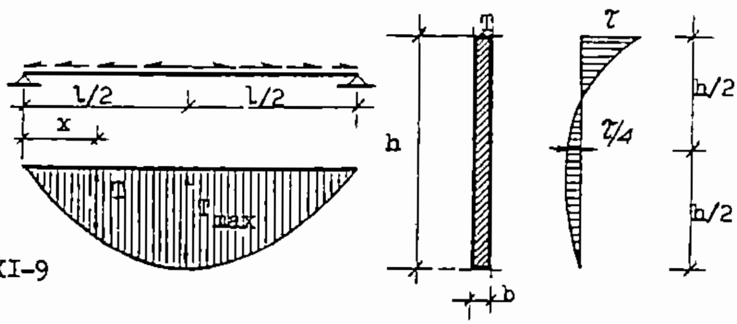


Fig XI-9

where T is applied is given by

$$\tau = \frac{4\ T_{max}}{b\ l}\left(1 - \frac{2\ x}{l}\right) \qquad \text{XI-9}$$

The shear stress at midheight of plate is equal to $\tau/4$ Fig XI-9.

If the plate is subject to two edge shears T_1 and T_2 at its top and bottom edges, the distribution of shear stresses will be as shown in Fig XI-10

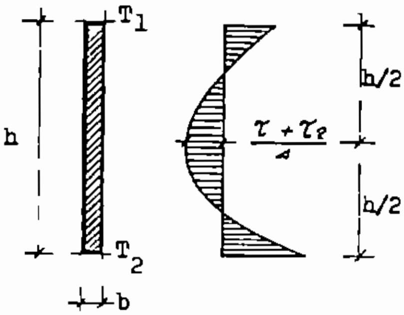


Fig XI-10

At the end diaphragms of a plate (x = 0), the shear stress due to an edge shear T is therefore

$$\tau = \frac{4\ T_{max}}{b\ l} \qquad \text{XI-10}$$

and the value at midheight of plate is

$$\frac{\tau}{4} = \frac{T_{max}}{b\ l} \qquad \text{XI-11}$$

In simple folded plates, the sense of the shear stresses due to edge shears is positive at the top and bottom edges of the plate and negative at midheight and hence, these last values are to be substracted from τ_{omax}

The determination of the longitudinal and transverse reinforcements as well as the web reinforcements will be shown in the following examples

XI-6) ILLUSTRATIVE EXAMPLES

1) It is required to cover an area 32.8 x 20.0 ms by a folded plate roof Columns are allowed in the outside perimeter only Fig XI-11

The arrangement shown in Fig XI-11 gives a convenient simple solution for the following reasons

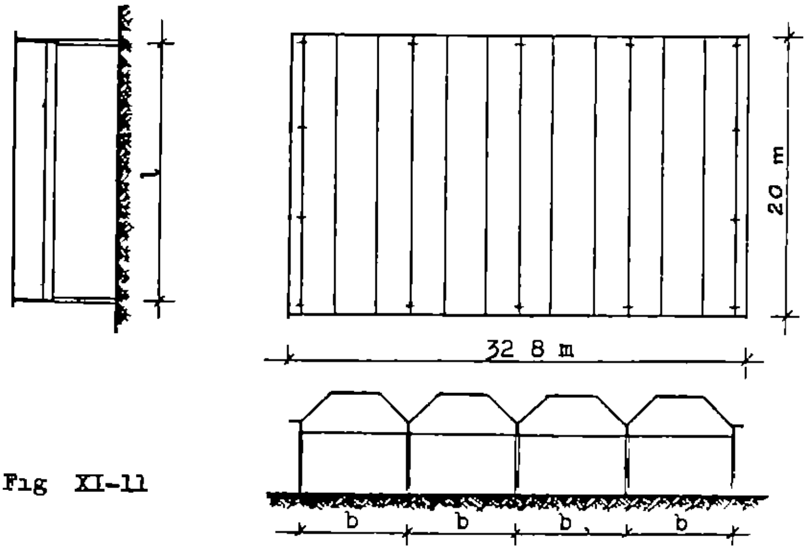


Fig XI-11

a) The span of the folded plate roof is chosen in the shorter direction

b) The span being relatively big, the system is chosen such that the compressive stresses can be easily resisted by the horizontal part of the folded roof and the big amount of the tension steel can be easily arranged in the edge beams

c) The height of the folded slab is chosen 1 8 ms and that of the edge beam is -theoretically- 0 70 ms so that the total height of 2 5 ms is @ 1/7 5 the span l of 18 4 ms

d) All interior folded plates have the same length of 3 0 ms

The chosen concrete dimensions are shown in details in Fig XI-12

<u>Design of interior bay of roof in transverse direction (slab action)</u>

Assume slab thickness = 12 cms Load/m^2 surface = 0 12 x 2500 = 300 kg/m^2

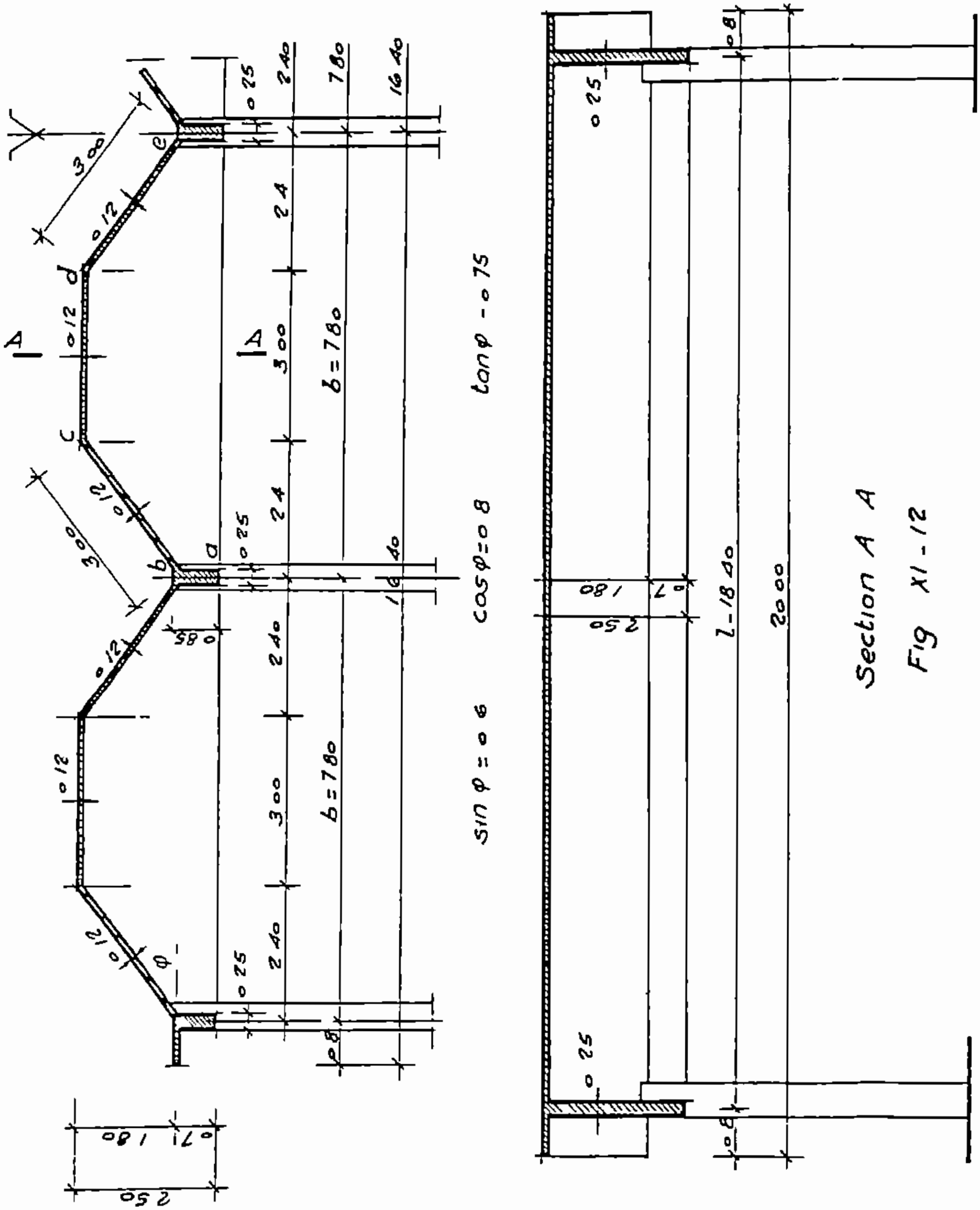
" superimposed load (L L, cover & plaster) / m^2 surface = 100

Total load on horizontal slab w = 400

Total load, causing bending moments, on inclined slabs

$w' = w \cos\varphi$ = 320

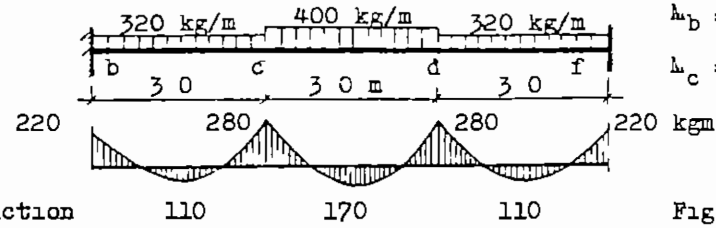
The slab may be assumed as fixed at b and e, so that the bending mo-



Section A A

Fig XI - 12

ment in the transverse direction as follows (Fig XI-13)



$M_b = M_f$ &

$M_c = M_d$

Slab action Fig XI-13

$$2 M_b \times 3\ 0 + M_c \times 3\ 0 = -6 \times \frac{320 \times 3\ 0^3}{24} \quad \text{or} \quad 2 M_b + M_c = -720$$

$$M_b \times 3\ 0 + 2 M_c \times 6\ 0 + M_c \times 3\ 0 = -6 \left(\frac{320 \times 3\ 0^3}{24} + \frac{400 \times 3\ 0^3}{24}\right)$$

$$\text{or} \quad M_b + 5 M_c = -1620$$

giving $M_b = -220$ kgm and $M_c = -280$ kgm

Hence, the bending moment diagram is as shown in Fig XI-13 However, the field moment in panels b c and d f should not be smaller than

$$M+ = \frac{320 \times 3\ 0^2}{24} = 120 \text{ kgm}$$

The bending moments are very small for a slab thickness of 12 cms, hence, a minimum main steel of 6 Ø 8 mm/m may be used The distributers are chosen 5 Ø 6 mm/m

Design of interior bay of roof in longitudinal direction (beam action)

Since each of the 25 cms wide beams is common for two bays of the folded roof, the beams belonging to one bay are to be introduced with a breadth of 12 5 cms

Ridge loads per bay

P_b = own weight of ridge beam + ½ load of plate b c + weight of slope concrete plus rain water in gutter, or

P_b = 0.125 x 0 85 x 2500 + 1 5 x 400 + 135 = 265 + 600 + 135 = 1000 kg/m

P_c = 3 0 x 400 = 1200 kg/m

Resolution of ridge load (Fig XI-14)

Load on plate 1 = P_b = 1000 kg/m

" " " 2 = $S_{c,b}$ = 2000 "

" 3 = $S_{c,d} - S_{d,c}$ = 0

because of symmetry

$$S_{c,d} = S_{d,c}$$

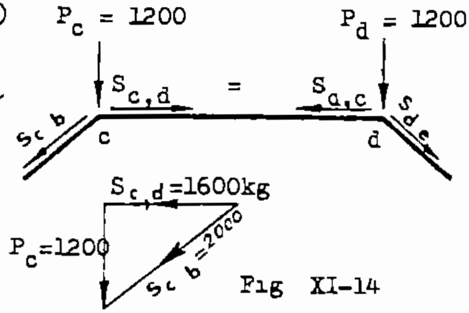


Fig XI-14

Moments and properties of individual plates

Plate	b m	h m	A = b h m^2	$Z = \frac{b h^2}{6}$ m^3	S kg/m	$M_o = S \frac{18\ 4^2}{8}$ kg m
1	0 125	0 85	A_1=0 106	Z_1=0 0151	1000	M_{o1}=42320
2	0 120	3 00	A_2=0 360	Z_2=0 180	2000	M_{o2}=84640
3	0 120	3 00	A_3=0 360	Z_3=0 180	0	M_{o3}= 0

Edge shears

The application of equation XI-5 to joints b and c respectively gives

Joint b $$0 + 2\left(\frac{T_b}{A_1} + \frac{T_b}{A_2}\right) + \frac{T_c}{A_2} = \frac{1}{2}\left(\frac{M_{o1}}{Z_1} + \frac{M_{o2}}{Z_2}\right)$$ or

$$0 + 2\left(\frac{T_b}{0\ 106} + \frac{T_b}{0\ 36}\right) + \frac{T_c}{0\ 36} = \frac{1}{2}\left(\frac{42320}{0\ 0151} + \frac{84640}{0\ 180}\right)$$ or

$$8\ 72\ T_b + T_c = 589 \times 10^3$$ and

Joint c $$\frac{T_b}{A_2} + 2\left(\frac{T_c}{A_2} + \frac{T_c}{A_3}\right) + \frac{T_d}{A_3} = \frac{1}{2}\left(\frac{M_{o2}}{Z_2} + 0\right)$$ but $T_d = -T_c$

then $$\frac{T_b}{0\ 36} + 2\left(\frac{T_c}{0\ 36} + \frac{T_c}{0\ 36}\right) - \frac{T_c}{0\ 36} = \frac{1}{2} \times \frac{84640}{0\ 180}$$ or

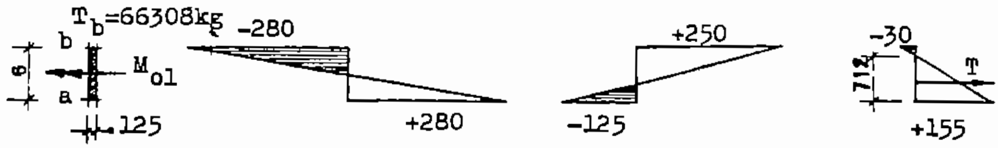
$$0\ 33\ T_b + T_c = 28\ 213 \times 10^3$$

Therefore T_b = 66308 kgs and T_c = 6113 kgs

The final normal stresses in the different plates will be determined by superposition as follows

<u>Normal stresses in the different plates due to beam action</u>

<u>Plate 1</u> Fig XI-15



M_{o1} = 42320 kgm

Fig XI-15

Stresses due to M_{o1} = 42320 kgm $\quad \sigma_{\substack{b\\a}} = \mp \frac{M_{o1}}{Z_1} = \mp \frac{4232000}{15100} = \mp 280$ kg/cm^2

T_b = 66308 kgs $\quad \sigma_b = + \frac{4T_b}{A_1} = + \frac{4x66308}{1060} = + 250$ "

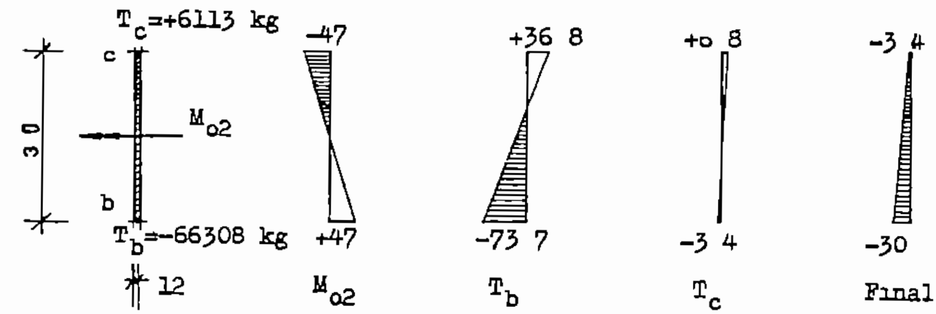
$\sigma_a = - \frac{2T_b}{A_1} = - \frac{2x66308}{1060} = - 125$

Final stresses $\quad \sigma_a = + 280 - 125 \quad = \underline{+ 155}$

$\sigma_b = - 280 + 250 \quad = \underline{- 30}$ "

Height of tension zone $= 0\ 85 \times \frac{155}{155 + 30} = 0\ 712$ ms

<u>Plate 2</u> Fig XI-16



M_{o2}=84640 kgm

Fig XI-16

Stresses due to M_{o2} = 84640 kgm $\quad \sigma_{\substack{b\\c}} = \pm \frac{M_{o2}}{Z_2} = \pm \frac{8464000}{1800\ 00} = \pm 47$ kg/cm^2

T_b = -66308 kgs $\quad \sigma_b = + \frac{4T_b}{A_2} = - \frac{4x66308}{3600} = - 73\ 7$

$\sigma_c = - \frac{2T_b}{A_2} = + \frac{2x66308}{3600} = + 36\ 3$,

Plate 2 (contd)

Stresses due to $T_c = + 6113$ kgs $\sigma_c = + \frac{4T_c}{A_2} = + \frac{4x6113}{3600} = + 6\ 8$ kg/cm^2

$$\sigma_b = - \frac{2T_c}{A_2} = - \frac{2x6113}{3600} = - 3\ 4 \quad "$$

Final stresses $\sigma_b = + 47 - 73\ 7 - 3\ 4 = \underline{- 30}$

$$\sigma_c = - 47 + 36\ 8 + 6\ 8 = \underline{- 3\ 4}$$

Plate 3

Stresses due to $T_c = - 6113$ kgs $\sigma_c = - \frac{4T_c}{A_3} = - \frac{4x6113}{3600} = - 6\ 8$

$T_d = - 6113$ kgs $\sigma_c = + \frac{2T_c}{A_3} = + \frac{2x6113}{3600} = + 3\ 4 \quad "$

Final stresses $\sigma_c = \sigma_d = \quad - 6\ 8 + 3\ 4 \quad = \underline{- 3\ 4}$

The final normal stress distribution in the folded plate is shown in Fig XI-18

The final stresses calculated above can be determined using the stress-distribution method in the following manner

Areas $A_1 = 1060$ cm^2 $A_2 = 3600$ cm^2 $A_3 = 3600$ cm^2

D F $D_1 = \frac{1060}{1060 + 3600} = 0\ 227$ $D_2 = \frac{3600}{3600 + 3600} = 0\ 50$

Plate	a 1		b 2		c 3
Dist Factor		0 773	0 227	0 500	0 500
	+280	-280	+47 0	-47 0	0
Distribution		+252	-75 0	+23 5	-23 5
Carry over	-126	0	-11 75	+37 5	+11 25
Distribution		-9 10	+ 2 65	-13 13	+13 13
Carry over	+4 6	0	+ 6 57	-1 33	- 6 56
Distribution		+5 07	- 1 50	-2 62	+ 2 62
Carry over	-2 54	0	+ 1 31	+0 57	- 1 31
Distribution		+1 01	- 0 30	-1 03	+ 1 03
Final stresses	+156	-31	-31	-3 4	-3 4

The results are approximately the same as in the previous solution

Determination of longitudinal tension steel

The longitudinal tension steel is chosen such that it resists all the tensile force T in the section Accordingly, the tension in any intermediate edge beam 25 x 85 cms is given by Fig XI-17

$$T = 155 \times \frac{71.2}{2} \times 25 = 138\ 000 \text{ kgs}$$

Using high grade tension steel with

$f_y = 3600$ kg/cm^2 and $\sigma_s = 0.9 \times 2000$

i e $\sigma_s = 1800$ kg/cm^2, we get

$$A_s = \frac{138\ 000}{1800} = 76.5 \text{ cm}^2 \qquad 13\phi 28$$

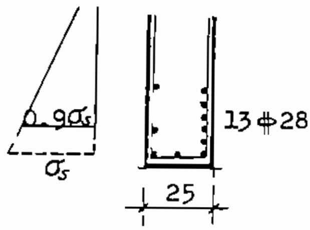


Fig XI-17

Shear stresses in plate 1

Max shearing force at diaphragms $Q_{max} = \frac{P_b\ l}{2} = \frac{1000 \times 18.4}{2} = 9200$ kgs

Shear stress at midheight of plate $\tau_{omax} = \frac{3}{2}\frac{Q_{max}}{A} = \frac{3}{2}\frac{9200}{1060} = 13$ kg/cm^2

Shear stress at diaphragms due to $T_{b\ max} = 66308$ kgs is

at top edge $\tau = \frac{4T_{max}}{b\ l} = \frac{4 \times 66\ 308}{12.5 \times 1840} = \underline{11.6}$

at mid-height $\tau/4 = 11.6 / 4 = 2.9$

Total shear stress at mid-height $= 13 - 2.9 = \underline{10.1}$,

In plate 2

Max shearing force at diaphragms $Q_{max} = \frac{S_{c,b}l}{2} = \frac{2000 \times 18.4}{2} = 18400$ kgs

Shear stress at midheight of plate $\tau_{omax} = \frac{3}{2}\frac{Q_{max}}{A} = \frac{3}{2}\frac{18400}{3600} = 7.7$ kg/cm^2

Shear stress at b due to $T_{b\ max} = 66308$ kgs is $\frac{4 \times 66308}{12 \times 1840} = \underline{12.0}$,

Shear stress at c due to $T_{c\ max} = 6113$ kgs is given by

$$\tau_c = \frac{4 \times 6113}{12 \times 1840} = \underline{1.10 \text{ kg/cm}^2}$$

Shear stress at middle of b c $= 7\ 7 - \frac{12}{4} - \frac{1\ 1}{4} = \underline{4\ 42\ kg/cm^2}$

Accordingly, the shear stresses may be assumed as shown in Fig XI-18 a in which the maximum value of 12 kg/cm^2 can be assumed constant in the tension zone of the section

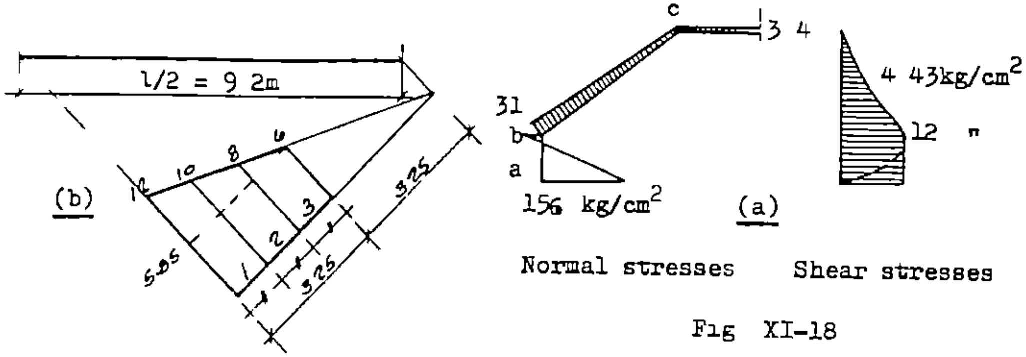


Fig XI-18

The required diagonal web reinforcement can be calculated as in beams in the following manner Fig XI-18 b

In zone 1 (1 08 m) $A_{sb} = \frac{12 + 10}{2} \times 12 \times \frac{325}{3} \times \frac{1}{1400} = 10\ 2\ cm^2$ 8 Ø 13

2 $A_{sb} = \frac{10 + 8}{2} \times 12 \times \frac{325}{3} \times \frac{1}{1400} = 8\ 35\ cm^2$ 4 Ø 13 + 4 Ø 10

3 " $A_{sb} = \frac{8 + 6}{2} \times 12 \times \frac{325}{3} \times \frac{1}{1400} = 6\ 5\ cm^2$ 8 Ø 10

It is however possible to resist a part of the diagonal tension at the diaphragms by the cross reinforcement 6 Ø 8 mm/m in which case it is advisable to have symmetrical reinforcement in top and bottom fibers of the slab as shown dotted in Fig XI-19 In such a case, one proceeds as in beams by determining first the diagonal tension (τ_{st}) resisted by the cross reinforcements, where $\tau_{st} = A_s\ \sigma_s / b\ s$, and the rest of the diagonal tension diagram is to be resisted by bent bars

Hence $$\tau_{st} = \frac{2 \times 0\ 5 \times 1400}{12 \times 20} = 5\ 85\ kg/cm^2$$

and the area of bent bars in each zone is therefore

In zone 1 $A_{sb} = (\frac{12+10}{2} - 5\ 85) \times 12 \times \frac{325}{3} \times \frac{1}{1400} = 4\ 8\ cm^2$ 6 Ø 10

Similarly, one needs in zone 2, 6 Ø 8, and in zone 3, 5 Ø 8 / 1 08 ms

The details of reinforcements are shown in Fig XI-19

2) It is required to design an interior panel of the simple saw-tooth folded plate roof (20 0 x 32 0 ms) shown in Fig XI-20 for its own weight plus a superimposed load of 100 kg/m^2 surface

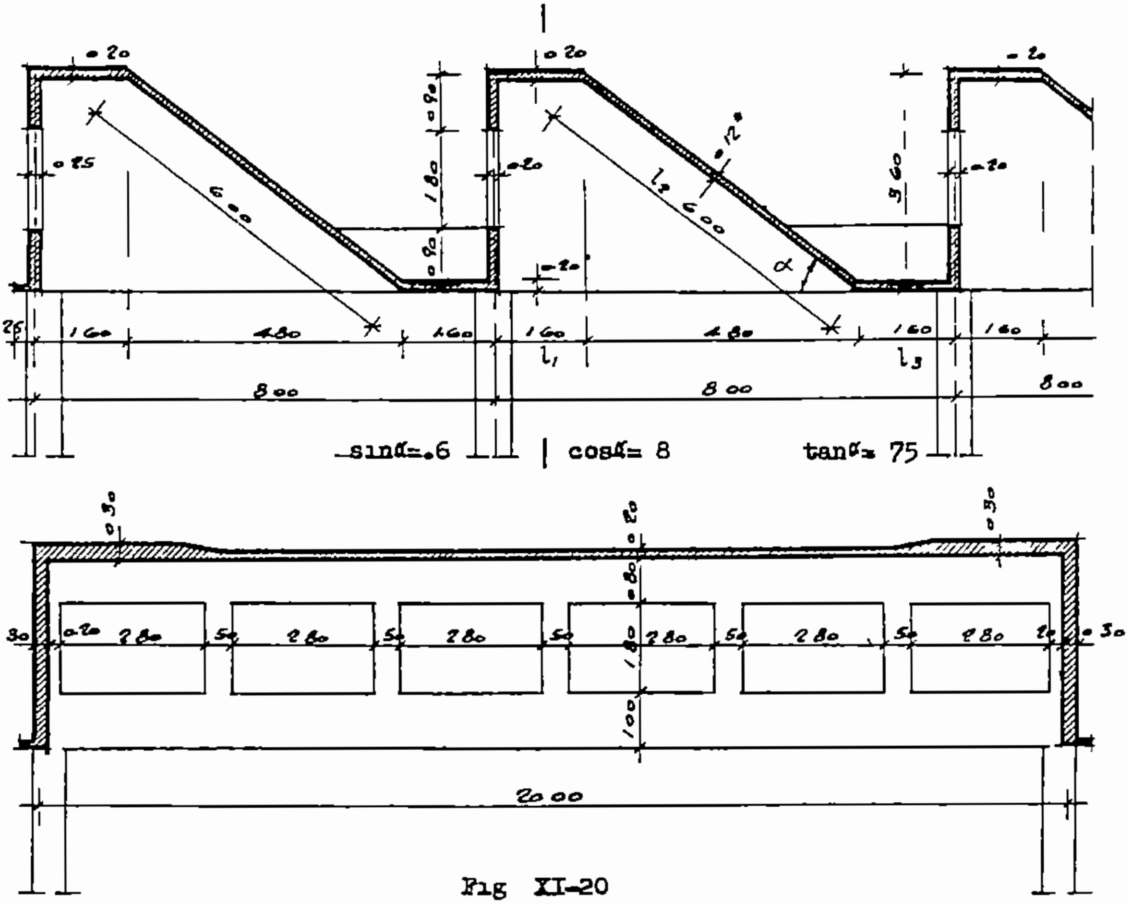


Fig XI-20

<u>Design of interior panel of roof in transverse direction (slab action)</u>

The bending moments of the slab in the transverse direction shall be determined by the equation of three moments in which $M_2 = M_3$ due to symmetry Fig XI-21

Further, the plates 2 and 4 are short relative to plate 3, then their elastic reactions at corners 2 and 3 due to loading of these plates may be neglected Hence

$$2\ M_2\ (\frac{l_2}{I_2} + \frac{l_3}{I_3}) + M_3\ \frac{l_3}{I_3} = -\ 6\ \frac{w'\ l_3^{\ 3}}{24\ I_3}$$

Multiplying both sides by $\frac{I_2}{l_2}$ and assuming $\chi = \frac{I_2}{I_3}\ \ \frac{l_3}{l_2}$, we get

$$2\ M_2\ (1 + \chi) + M_3\chi = -\ \frac{w'\ l_3^{\ 2}}{4}\ \ \chi$$ but

$$\chi = (\frac{0\ 2}{0\ 12})^3\ x\ \frac{6\ 0}{1\ 6} = 17\ 4$$

Assuming w' = 320 kg/m^2 (refer to example 1), then

$$2\ M_2\ (1 + 17\ 4) + 17\ 4\ M_2 = -\ \frac{320\ x\ 6\ 0^2}{4}\ x\ 17\ 4$$

Therefore $$M_2 = M_3 = -\ 920\ \ kg\ m$$ and

$$M_m = \frac{w'\ l_3^2}{8} - 920 = \frac{320\ x\ 6\ 0^2}{8} - 920 = 1440 - 920 = 520\ \ kgm$$

By redistribution, one may assume $M_m = -\ M_2 = -\ M_3 = 720$ kgm so that a thickness of plate 3 of 12 cms and main reinforcements 7 Ø 10 mm/m may be safely accepted The distributers are chosen 5 Ø 6 mm/m

<u>Ridge loads P</u> Fig XI-21

Own weight of plate 1 (and 5) = 0 2 x 0 9 x 2500 = 450 kg/m

Load on plate 2 (and 4), 20 cms thick, + concrete cover + superimposed load = 0 2 x 2500 + 200 + 100 = 800 kg/m

Load on plate 3, 12 cms thick, = 0 12 x 2500 + 100 = 400 kg/m

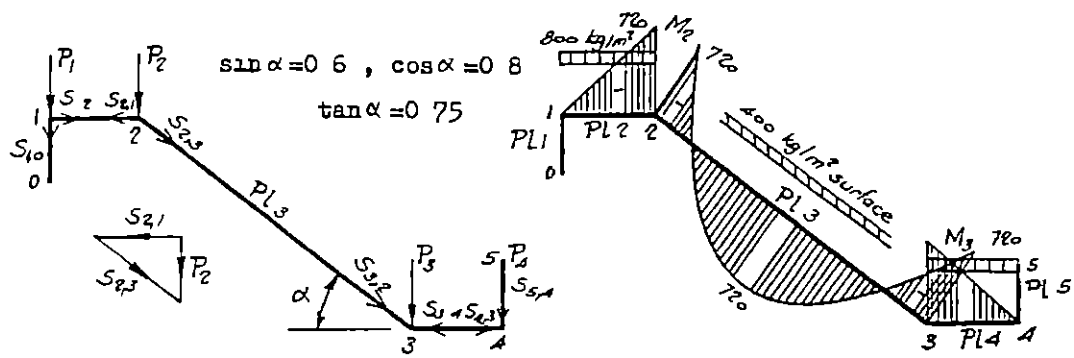


Fig XI-21

Therefore

$$P_1 = P_4 = \text{own wt of plate 1} + \tfrac{1}{2}\ \text{load on plate 2} + \text{glazing} - \frac{M_2}{l_2}$$

$$= 450 + \tfrac{1}{2} \times 800 \times 1\ 6 + 100 - \frac{720}{1\ 6} = 740 \quad kg/m'$$

$$P_2 = P_3 = \text{half loads on plates 2 and 3} + \frac{M_2}{l_2} + \frac{M_3 - M_2}{l_3}$$

$$\tfrac{1}{2}\ (800 \times 1\ 6 + 400 \times 6\ 0) + \frac{720}{1\ 6} + 0 = 2290 \quad kg/m'$$

Resolution of ridge loads Fig XI-21

$S_{1,0} = S_{5,4} = P_1 = P_4 = 740$ kg/m'

$S_{2,1} = S_{3,4} = P_2 / \tan\alpha = 2290 / 0\ 75 = 3053$ kg/m'

$S_{2,3} = S_{3,2} = P_2 / \sin\alpha = 2290 / 0\ 60 = 3817$ kg/m'

Total force on plate 3 $S = 2 \times 3817 = 7634$ kg/m'

Free moments and properties of individual plates

Plate	b m	h m	$A = b\ h$ m^2	$Z = \frac{b\ h^2}{6}$ m^3	S kg/m	$M_o = S\ \frac{20\ 0^2}{8}$ kgm
1	0 20	0 90	A_1 = 0 18	Z_1 = 0 0270	740	M_{o1} = 37 00x10^3
2	0 20	1 60	A_2 = 0 32	Z_2 = 0 0853	3053	M_{o2} = 152 65x10^3
3	0 12	6 00	A_3 = 0 72	Z_3 = 0 7200	7634	M_{o3} = 381 70x10^3
4	0 20	1 60	A_4 = 0 32	Z_4 = 0 0853	3053	M_{o4} = 152 65x10^3
5	0 20	0 90	A_5 = 0 18	Z_5 = 0 0270	740	M_{o5} = 37 00x10^3

Edge shears

The application of equation XI-5 to joints 1 and 2 respectively gives

Joint 1 $$0 + 2\ \left(\frac{T_1}{A_1} + \frac{T_1}{A_2}\right) + \frac{T_2}{A_2} = \frac{1}{2}\ \left(\frac{M_{o1}}{Z_1} + \frac{M_{o2}}{Z_2}\right) \quad \text{or}$$

$$0 + 2\ \left(\frac{T_1}{0\ 18} + \frac{T_1}{0\ 32}\right) + \frac{T_2}{0\ 32} = \frac{1}{2}\ \left(\frac{37000}{0\ 027} + \frac{152650}{0\ 0853}\right) \quad \text{or}$$

$$17\ 36\ T_1 + 3\ 125\ T_2 = 1579968 \quad \text{and}$$

Joint 2 $\frac{T_1}{A_2} + 2\left(\frac{T_2}{A_2} + \frac{T_2}{A_3}\right) + \frac{T_3}{A_3} = \frac{1}{2}\left(\frac{M_{o2}}{Z_2} - \frac{M_{o3}}{Z_3}\right)$ but $T_2 = T_3$

then $\frac{T_1}{0\ 32} + 2\left(\frac{T_2}{0\ 32} + \frac{T_2}{0\ 72}\right) + \frac{T_2}{0\ 72} = \frac{1}{2}\left(\frac{152650}{0\ 0853} - \frac{381700}{0\ 72}\right)$ or

$$3\ 125\ T_1 + 10\ 40\ T_2 = 629713$$

Therefore $\underline{T_1 = 84699 \text{ kgs}}$ and $\underline{T_2 = 35071 \text{ kgs}}$

The final normal stresses in the different plates will be determined by superposition as follows

Normal stresses in the different plates due to beam action

Plate 1 Fig XI-22

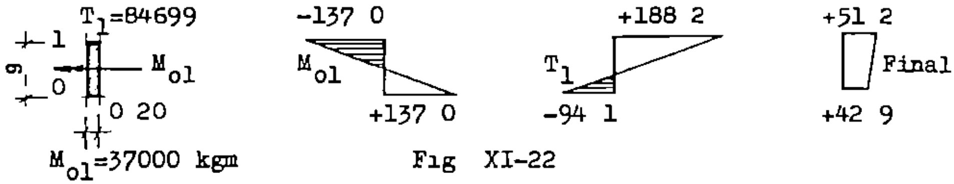


Fig XI-22

Stresses due to M_{o1} = 37000 kgm $\sigma_{\frac{1}{0}} = \mp \frac{M_{o1}}{Z_1} = \mp \frac{3700000}{27000} = \mp 137$ kg/cm^2

T_1 = 84699 kgs $\sigma_1 = + \frac{T_1}{A_1} = + \frac{4x84699}{1800} = + 188\ 22$

$$\sigma_0 = - \frac{2T_1}{A_1} = - \frac{2x84699}{1800} = - 94\ 11$$

Final stresses $\sigma_1 = - 137\ 00 + 188\ 22 = \underline{+ 51\ 20}$

$$\sigma_0 = + 137\ 00 - 94\ 11 = \underline{+ 42\ 90}$$

Plate 2 Fig XI-23

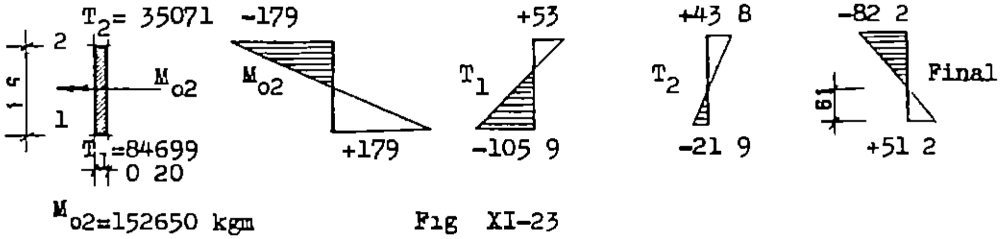


Fig XI-23

Stresses due to M_{o2} = 152650 kgm $\sigma_{\frac{2}{1}} = \mp \frac{M_{o2}}{Z_2} = \mp \frac{15265000}{85300} = \mp 179$ kg/cm^2

" T_1 = 84699 kgs: $\sigma_1 = - \frac{4T_1}{A_2} = - \frac{4x\ 84699}{3200} = - 105\ 9$ "

$\sigma_2 = + \frac{2T_1}{A_2} = + \frac{2x\ 84699}{3200} = + 53\ 0$

" T_2 = 35071 kgs $\sigma_2 = + \frac{4T_2}{A_2} = + \frac{4x\ 35071}{3200} = + 43\ 8$

$\sigma_1 = - \frac{2T_2}{A_2} = - \frac{2x\ 35071}{3200} = - 21\ 9$ "

Final stresses $\sigma_1 = + 179 - 105\ 9 - 21\ 9 = \underline{+51\ 2}$ "

$\sigma_2 = - 179 + 53 + 43\ 8 = \underline{-\ 82\ 2}$

Height of tension zone $= 1\ 6 \times \frac{51\ 20}{82\ 2 + 51\ 2} = 0\ 61$ ms

<u>Plate 3</u> Fig XI-24

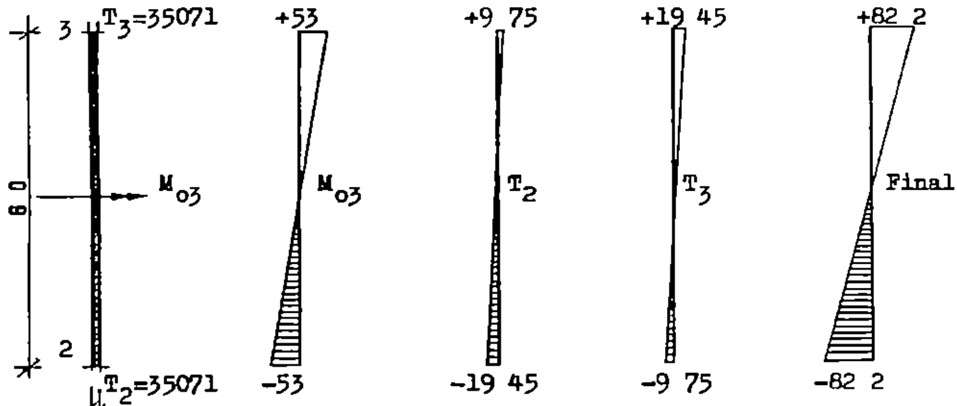


Fig XI-24

Stresses due to M_{o3} = 381700 kgm: $\sigma_{\frac{2}{3}} = \mp \frac{M_{o3}}{Z_3} = \mp \frac{38170000}{720000} = \mp 53$ kg/cm^2

" T_2 = 35071 kgs $\sigma_2 = - \frac{4T_2}{A_3} = - \frac{4x\ 35071}{7200} = -19\ 45$ "

$\sigma_3 = + \frac{2T_2}{A_3} = + \frac{2\ x35071}{7200} = + 9\ 75$ "

" " " T_3 = 35071 kgs $\sigma_3 = + \frac{4T_2}{A_3} = + 19\ 45,\ \sigma_2 = - \frac{2T_2}{A_3} = -9\ 75$

Final stresses $\sigma_2 = - 53 - 19\ 45 - 9\ 75 = \underline{- 82\ 2\ kg/cm^2}$

$\sigma_3 = + 53 + 19\ 45 + 9\ 75 = \underline{+ 82\ 2}$

The final normal stresses calculated above can be determined using the stress-distribution method in the following manner

Areas $A_1 = 1800\ cm^2$ $A_2 \equiv 3200\ cm^2$ $A_3 = 7200\ cm^2$

D F $D_1 = \frac{1800}{1800 + 3200} = 0\ 36$ $D_2 = \frac{3200}{1800 + 3200} = 0\ 64$

$D_3 = \frac{3200}{3200 + 7200} = 0\ 308$ $D_4 = \frac{7200}{3200 + 7200} = 0\ 692$

Plate	1		2		3		4		5	
Joint		1		2		3		4		5
Dist factor		64	36	692	308	308	692	36	64	
Stress	+137	−137	+179	−179	−53	+53	+179	−179	+137	−137
Distribution		+202 2	−113 8	+87 2	−38 8	+38 8	−87 2	+113 8	−202 2	
Carry-over	−101 1	0	−43 6	+56 9	−19 4	+19 4	−56 9	+43 6	0	+101 1
Distribution		−27 9	+15 7	−52 8	+23 5	−23 5	+52 8	−15 7	+27 9	
Carry-over	+14 0	0	+26 4	−7 9	+11 8	−11 8	+7 9	−26 4	0	−14
Distribution		+16 9	−9 5	+13 6	−6	+6	−13 6	+9 5	−16 9	
Carry-over	−8 5	0	−6 8	+4 8	−3	+3	−4 8	+6 8	0	+8 5
Distribution		−4 4	+2 4	−5 4	+2 4	−2 4	+5 4	−2 4	+4 4	
Carry-over	+2 2	0	+2 7	−1 2	+1 2	−1 2	+1 2	−2 7	0	−2 2
Distribution		+1 7	−1 0	+1 7	−0 7	+0 7	−1 7	+1 0	−1 7	
Carry-over	−0 9	0	−0 9	+0 5	−0 4	+0 4	−0 5	+0 9	0	+0 9
Distribution		−0 6	+0 3	−0 6	+0 3	−0 3	+0 6	−0 3	+0 6	
Carry-over	+0 3	0	+0 3	−0 2	+0 2	−0 2	+0 2	−0 3	0	−0 3
Distribution		+0 1	−0 2	+0 3	−0 1	+0 1	−0 3	+0 2	−0 1	
Final	+43 0	+51 0	+51 0	−82 1	−82 1	+82 1	+82 1	−51 0	−51 0	−43 0

The results are approximately the same as in the previous solution

The final normal-stress distribution in the folded plate is shown in Fig XI-25

Longitudinal main reinforcements

The longitudinal tension reinforcements are chosen such that they resist all the tensile forces T in the section given by the area of the tension zones Accordingly, the tension at joint 3 is given by (refer to Fig XI-25)

$$T = \frac{82\ 1 \times 300}{2} \times 12 + \frac{82\ 1 \times 99}{2} \times 20 = 229059 \quad \text{kgs}$$

Using high grade steel with $f_y = 3600$ kg/cm^2 and $\sigma_s = 1800$ kg/cm^2, we get

$$A_s = \frac{229059}{1800} = 127 \quad \text{cm}^2$$

The total tension at joint 1 is given by

$$T = \left[\frac{51 + 43}{2} \times 90 + \frac{51 \times 61}{2}\right] \times 20 = 115710 \quad \text{kgs}$$

$$A_s = \frac{115710}{1800} = 64\ 3 \quad \text{cm}^2$$

The chosen reinforcements are shown in Fig XI-26

Shear stresses in the different plates

Plate 1 $\qquad Q_{max} = \frac{P_1\ l}{2} = \frac{740 \times 20}{2} = 7400 \quad \text{kgs}$

The shear stresses are calculated as follows

Due to Q_{max} at mid-height $\qquad \tau_{omax} = \frac{3}{2}\frac{Q_{max}}{A} = \frac{3}{2}\frac{7400}{20 \times 90} = 6\ 17$ kg/cm^2

Due to T_{max} = 84699 kgs

at top edge $\qquad \tau = 4\frac{T_{max}}{b\ l} = \frac{4 \times 84699}{20 \times 2000} = 8\ 47$

at mid-height $\qquad \frac{\tau}{4} = 8\ 47/4 = 2\ 12$

Total shear at mid-height $\qquad = \tau_{omax} - \frac{\tau}{4} = 6\ 17 - 2\ 12 = 4\ 05$

Plate 2 $\qquad Q_{max} = \frac{S_{2,1}\ l}{2} = \frac{3053 \times 20}{2} = 30530 \quad \text{kgs}$

The thickness of the horizontal plates 2 and 4 is increased to 30 cms at the diaphragms for a length of @ 3 ms hence the shear stresses in plates 2 and 4 are calculated as follows

Due to Q_{max} at mid-height $\tau_{omax} = \frac{3}{2}\frac{Q_{max}}{A} = \frac{3}{2}\frac{30530}{30 \times 160} = 9\ 54$ kg/cm^2

Due to $T_{1\ max}$ = 84699 kgs

at joint 1 $\tau_1 = \frac{4 \times 84699}{30 \times 2000} = 5\ 65$ "

Due to $T_{2\ max}$ = 35071 kgs

at joint 2 $\tau_2 = \frac{4 \times 35071}{30 \times 2000} = 2\ 34$ "

Total shear at middle of plate 2 $= 9\ 54 - \frac{5\ 65}{4} - \frac{2\ 34}{4} = 7\ 55$ "

Plate 3 $Q_{max} = \frac{S\ l}{2} = \frac{7634 \times 20}{2} = 76340$ kgs

The thickness of plate 3 is also increased to 20 cms for a length of @ 3 ms at the diaphragms, hence, the shear stresses are given by

Shear stress due to Q_{max} $\tau_{o\ max} = \frac{3}{2}\frac{76340}{20 \times 600} = 9\ 54$ kgs/cm^2

τ_2 at top edge (jt 2) due to $T_{2\ max}$ = 35071 kgs is

$\tau_2 = \frac{4 \times 35071}{20 \times 2000} = 3\ 51$ "

τ_3 at bot edge (jt 3) due to $T_{3\ max} \doteq 35071$ kgs is

$\tau_3 = \frac{4 \times 35071}{20 \times 2000} = 3\ 51$ "

Total shear at middle of plate 3 $= 9\ 54 - 2 \times \frac{3\ 51}{4} = 7\ 79$ "

Web reinforcements

Arranging 7 Ø 10 mm/m at the top and bottom fibers of the slab in the 3 ms of increased thickness at the diaphragms, the shear stresses resisted by these bars only (i e neglecting the resistance of the longitudinal reinforcements and the concrete) is given by

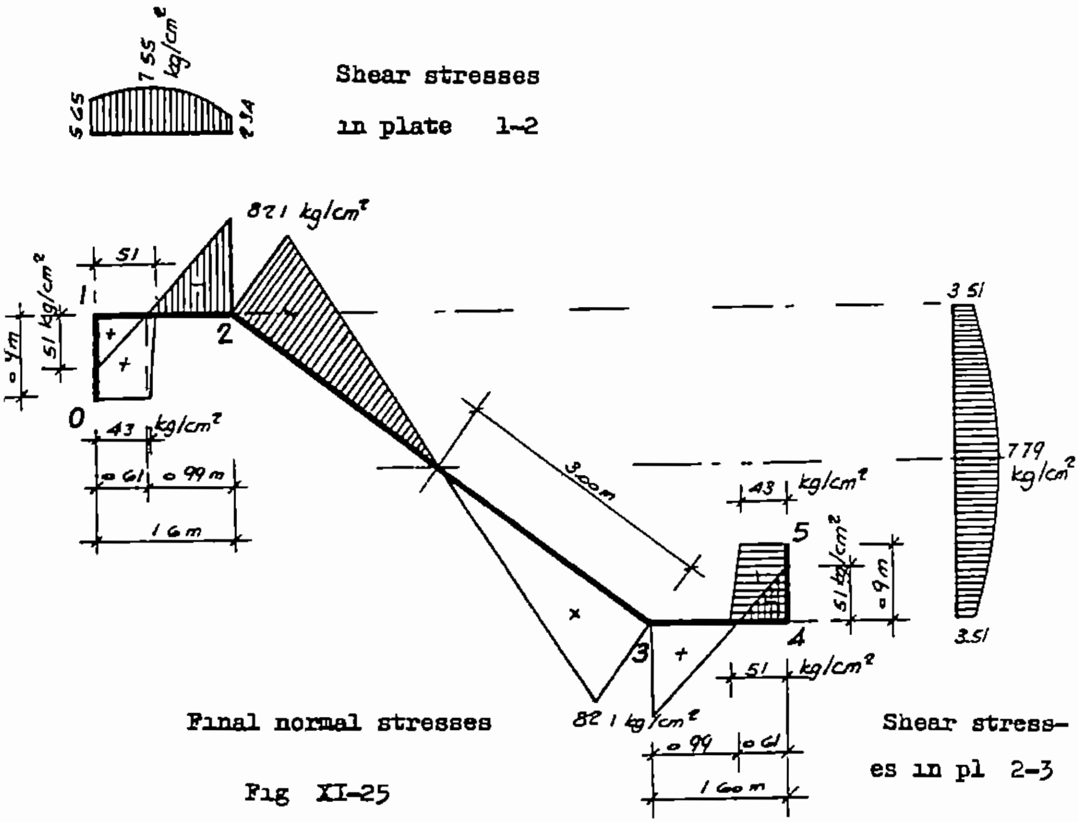
$\tau_{st} = \frac{n\ A_s\ \sigma_s}{b\ s}$ where s is the spacing of the cross bars (14 3cm)

Hence, for plate 2 $\tau_{st} = \frac{2 \times 0\ 79 \times 1400}{30 \times 14\ 3} = 5\ 16$ kg/cm^2

and " " 3 $\tau_{st} = \frac{2 \times 0\ 79 \times 1400}{20 \times 14\ 3} = 7\ 73$ "

Assuming that the concrete can resist a part τ_c of the shear stresses (@ 0 25 $\sqrt{f_{cp}} \simeq 3\ 0$ kg/cm^2), one can dispense with any bent bars in the folded plates of the structure

Fig XI-25 shows the normal and shear stresses in the folded plate



Shear stresses in plate 1-2

Final normal stresses

Fig XI-25

Shear stresses in pl 2-3

The details of reinforcements in the cross section of one intermediate panel, both at middle of span and at supports, is shown in Fig XI-26

The shown solution is only approximate, because the displacements of the joints are neglected while they do affect the internal stresses For the effect of these displacements one may refer to text books on the subject*

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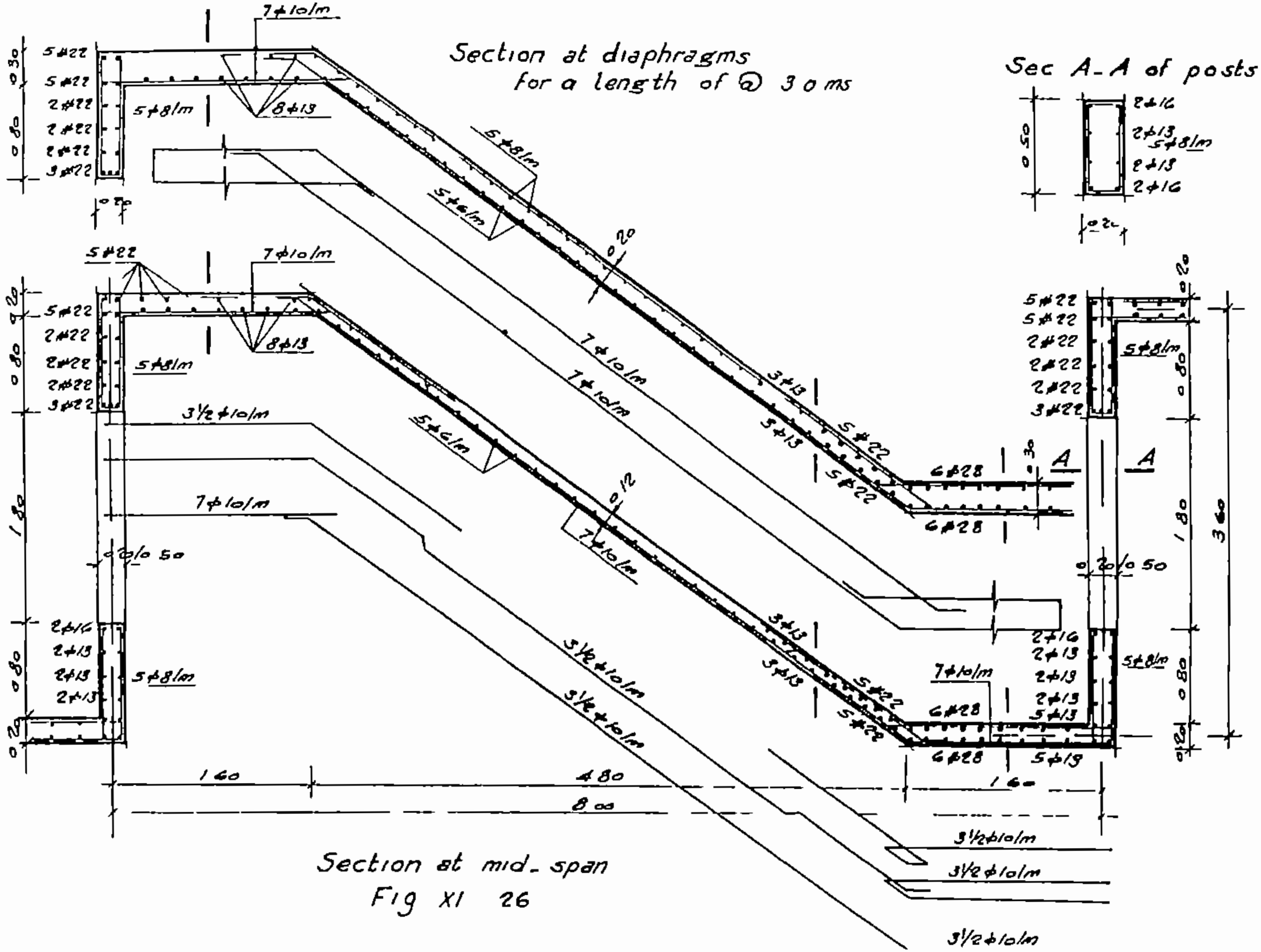
* 'Design and Construction of Concrete Shell Roofs' By Ramaswamy Published by Mc Graw-Hill Book Company New York and London



Section at mid_span
Fig XI 26

XI–7 DESIGN OF DIAPHRAGMS

The stresses arising in the diaphragms supporting folded-plates of symmetrical shape and symmetrically loaded, such as the folded-plate structure shown in Fig XI-27, may be calculated in the following manner

The ridge loads P_1 and P_2 acting on the nodes of the folded-plate, Fig XI-27 a, cause plate forces S_1 and S_2 / m and these exert shearing forces Q_1 and Q_2 on the end diaphragms The shearing forces Q can be assumed to have a parabolic distribution, Fig XI-27 b The shearing force Q_2 can be resolved into the components V_2 and H_2 which are the forces that effectively act upon the end diaphragm (the force Q_1 is transmitted directly to the support) As is apparent from Fig XI-27 c, the two smmetrcal components V_2 produce bending stresses in the end diphragm these can be determined by well known methods The horizontal components H_2 produce tension in the portion of the diaphragm situated between the two sloping plates of the structure These tensile stresses can, likewise, be assumed to have a parabolic distribution, Fig XI-27 c

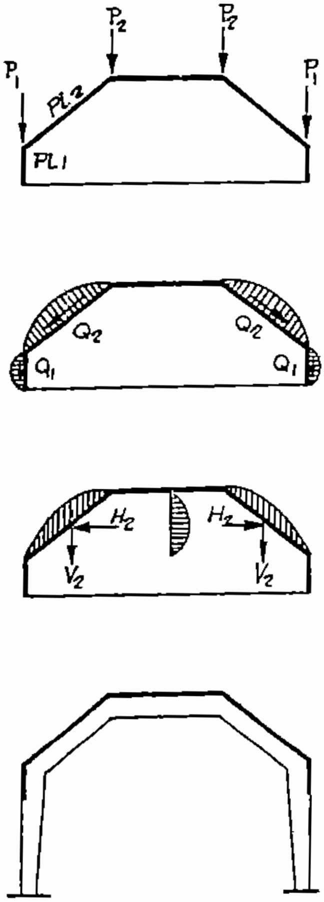


Fig XI-27

As presented in Fig XI-27 d, the end diaphragms may be designed as two hinged frames (or the like), which are loaded by the shearing forces shown in (b) or their components shown in (c) On account of its far greater flexibility in comparison with a solid diaphragm, the frame is more conductive to the development of deformations of the structure at the nodes

Example

It is required to design an intermediate diaphragm for a two-span folded-plate roof (Fig XI-28) Assume the cross-section and dimensions are the same as those of the intermediate panel shown in Fig XI-12 The spans are 2 x 18 4 ms The loads are the same as those of example 1 shown in Fig XI-11

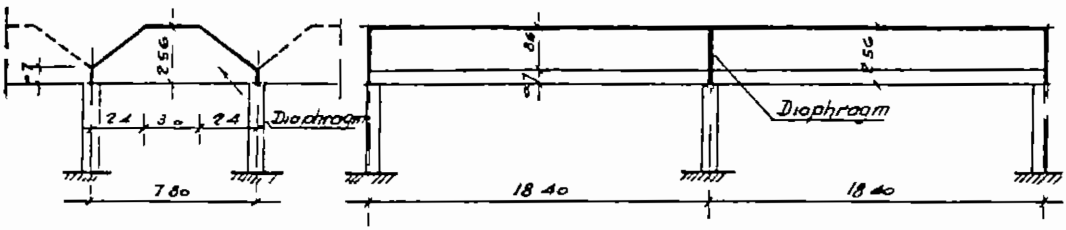


Fig XI-28

The diaphragm shall be assumed as if it were simply supported at both ends although it is continuous with the side panels, because the depth of 0 7 ms at the supports is small relative to the depth of 2 56 ms at the middle of the spans The bending moment that shall take place at the intermediate supports can be resisted by an adequate arrangement of reinforcements as shown in Fig XI-31

Dimensions of diaphragm

Span $l = 7\ 8$ ms Breadth $b = 0\ 30$ ms

Deapths $t_{min} = 0\ 70$ ms $t_{max} = 2\ 56$ ms

Loads (Fig XI-29)

The own weight is composed of three parts, namely

Over the whole span $w_1 = 0\ 3 \times 0\ 7 \times 2\ 5 = 0\ 525$ t/m

Triangular load over outside 2 4 ms with

$$\max w_2 = 0\ 3\ (2\ 56 - 0\ 7) \times 2\ 5 = 1\ 40 \text{ t/m}$$

Uniform load over middle 3 0 ms $w_3 = 1\ 40$ t/m

Having computed the loads, the internal forces can be determined as follows

External and internal forces (Fig XI-29)

Total shearing force Q_{omax} of inclined plate (refer to example1)

$$Q_{omax} = 2 \times 18\ 40 = 36\ 80 \quad t$$

Its vertical component $V = 0\ 6 \times 36\ 8 = 22\ 80 \quad t$

Its horizontal component $H = 0\ 8 \times 36\ 8 = 29\ 44 \quad t$

L o a d s

B M D

S F D

Thrust diag

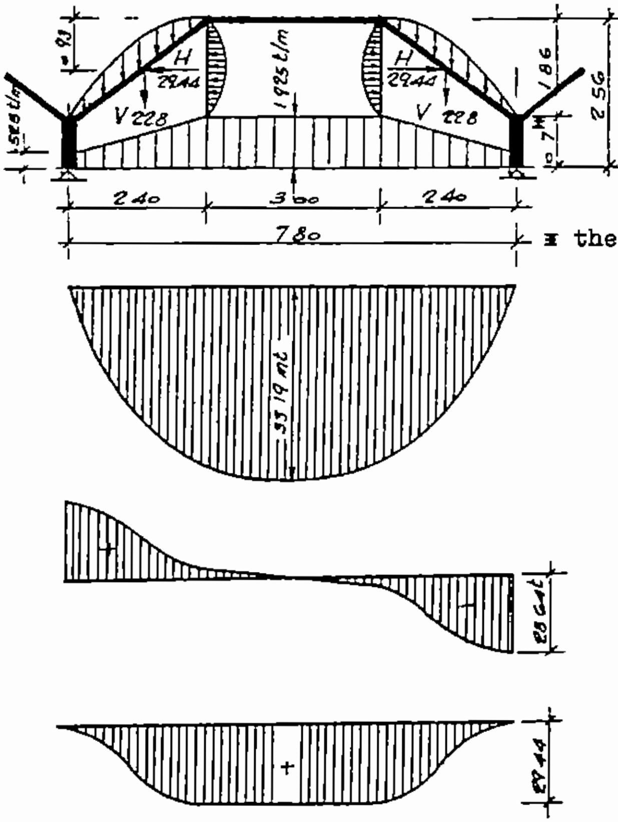


Fig XI-29

$$\text{Reaction } R = \frac{0\ 525 \times 7\ 80}{2} + \frac{1\ 4 \times 2\ 4}{2} + 1\ 4 \times 1\ 5 + 22\ 80 = \underline{28\ 64} \quad t$$

$$\text{Moment } M_{max} = 28\ 64 \times \frac{7\ 8}{2} - \frac{0\ 525 \times 3\ 9^2}{2} - \frac{1\ 4 \times 2\ 4}{2} \times 2\ 3 -$$

$$\frac{1\ 4 \times 1\ 5^2}{2} - 22\ 8 \times 2\ 7 = \underline{33\ 19} \quad t$$

$Q_{max} = R = \underline{28\ 64}$ t $\qquad$ $T = H = \underline{29\ 44}$ t

The bending moment, shearing force and thrust diag^s are in Fig XI-29

Reinforcements (Fig XI-30)

$b = 30$ cms and $d = 251$ cms using normal mild steel, then $\sigma_s = 1400 \text{ kg/cm}^2$

Eccentricity e of thrust T is given by

$e = M/T = 33\ 19/29\ 44 = 1\ 13$ ms ,

i e the resultant T lies between the top and bottom reinforcements of the diaphragm In this case, A_s and A_s' are inversily proportional to x' and x,

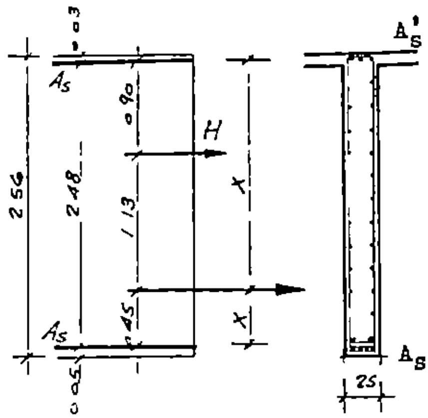


Fig XI-30

where

$$x' = 90^{*} + 113 = 203 \text{ cms} \quad \text{and} \quad x = 248 - 203 = 45 \text{ cms}$$

then

$$A_s = \frac{29\ 44}{1\ 4} \times \frac{203}{248} = 17\ 21 \text{ cm}^2 \quad 6\ \text{Ø}\ 19$$

$$A_s' = \frac{29\ 44}{1\ 4} \times \frac{45}{248} = 3\ 82 \text{ cm}^2 \quad 3\ \text{Ø}\ 16$$

Shear and principal stresses

Shear stress $\tau = \dfrac{28640}{0\ 87 \times 30 \times 80} = 13.7 \text{ kg/cm}^2$

Average normal stress $\sigma = \dfrac{29440}{30 \times 256} = 38\ 5$ tension

Assuming that the inclination of the principal tensile stress with the horizontal is α, then

$$\tan 2\alpha = \frac{2\tau}{\sigma} = \frac{2 \times 13\ 7}{38\ 5} = 0\ 71$$

or $\alpha = 17\ 8$

i e no bent bars are required in spite of that, 3 Ø 19 will be bent and skin vertical and horizontal reinforcements 5 Ø 10 mm/m (> 0 05 % of concrete section) are arranged

The details of reinforcements of the diaphragm are shown in Fig XI-31

* distance between top reinforcement and position of H

Details of Diaphragm

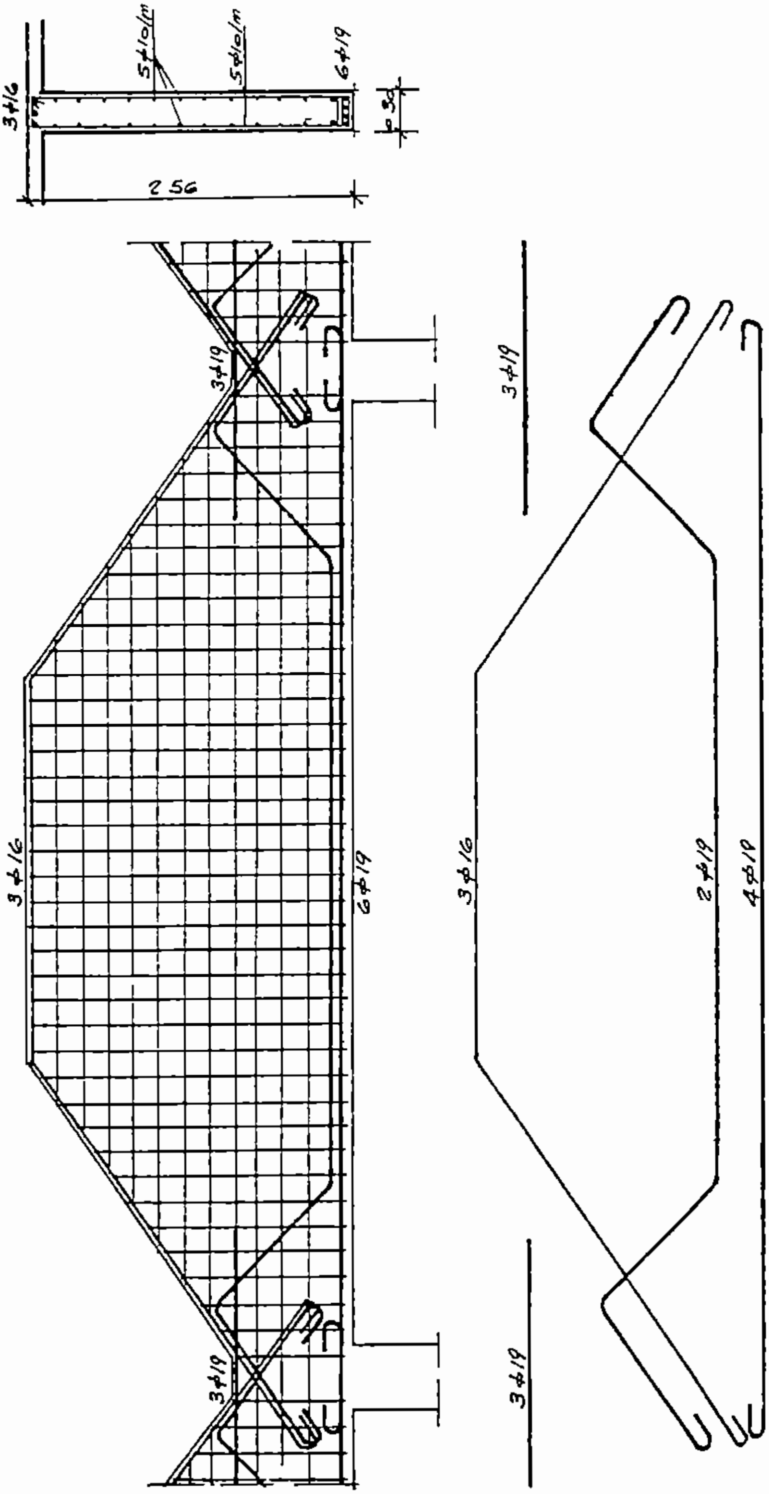


Fig XI 31

XI-8 MULTIPLE FOLDED-PLATE STRUCTURES

If more than two plates of a folded structure intersect at one or more junctions, then it is called a multiple folded plate structure (Fig XI-32)

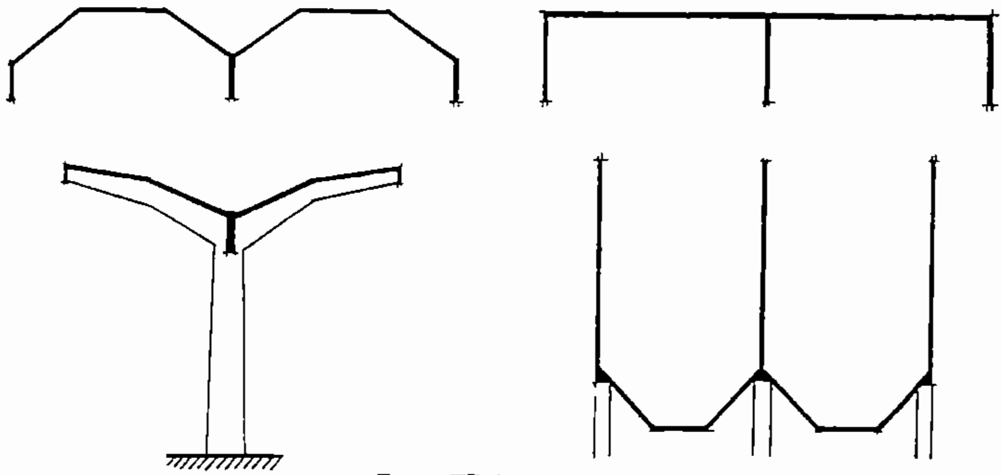


Fig XI-32

In a junction b, where three plates intersect, there now occur two different shear forces T_{b1} and T_{b3} as against only one such force at each junction of a simple folded plate structure Hence, the equation of tree shears will be extended to an equation of four shears

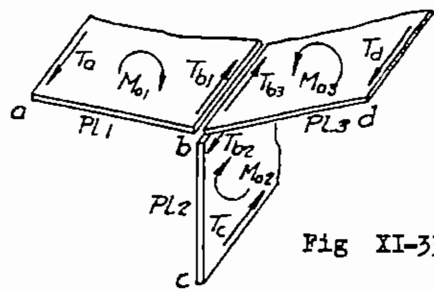


Fig XI-33

From the condition of equal stresses at the junction of intersection, (Fig XI-33) we get

$$\sigma_{b1} = \sigma_{b2} \quad \text{or} \quad \frac{T_a}{A_1} + \frac{2\,T_{b1}}{A_1} + \frac{2\,T_{b2}}{A_2} + \frac{T_c}{A_2} = \frac{1}{2}\left(\frac{M_{o1}}{Z_1} + \frac{M_{o2}}{Z_2}\right) \quad \text{and}$$

$$\sigma_{b2} = \sigma_{b3} \quad \text{or} \quad \frac{T_c}{A_2} + \frac{2\,T_{b2}}{A_2} + \frac{2\,T_{b3}}{A_3} + \frac{T_d}{A_3} = \frac{1}{2}\left(\frac{M_{o2}}{Z_2} + \frac{M_{o3}}{Z_3}\right) \qquad \text{XI-12}$$

These two equations thus determine the shears T_{b1}, T_{b2} and T_{b3} that occur at the junction b

To obtain the right hand side of eqation XI-12 in the form of the equation of three shears, the sense of M_{o3} was changed in relation to that of M_{o2}

In case $T_c = 0$, we have $T_{b2} = T_{b1} + T_{b3}$. XI-13

and equations XI-12 can be given in the form

$$\left.\begin{aligned} \frac{T_a}{A_1} + 2\,T_{b1}\left(\frac{1}{A_1} + \frac{1}{A_2}\right) + 2\,\frac{T_{b3}}{A_2} &= \frac{1}{2}\left(\frac{M_{o1}}{Z_1} + \frac{M_{o2}}{Z_2}\right) \\ \frac{T_d}{A_3} + 2\,T_{b3}\left(\frac{1}{A_3} + \frac{1}{A_2}\right) + 2\,\frac{T_{b1}}{A_2} &= \frac{1}{2}\left(\frac{M_{o3}}{Z_3} + \frac{M_{o2}}{Z_2}\right) \end{aligned}\right\} \quad \text{XI-14}$$

In case of symmetry, $T_{b1} = T_{b3}$, $T_{b2} = 2\,T_{b1} = 2\,T_{b3}$ XI-15

The application of these equations is shown in the following simple example (Fig XI-34)

Illustrative example

It is required to design the shed shown in Fig XI-34 for a vert superimposed dead load of 100 kg/m^2 and a live load of 200 kg/m^2 The shed has a span of 12 0 ms between the columns and two overhanging cantilevers, 3 0 ms each, on both sides The slab may be assumed 10 cms thick and reinforced by 6 Ø 8 mm/m in the transverse direction of the shed

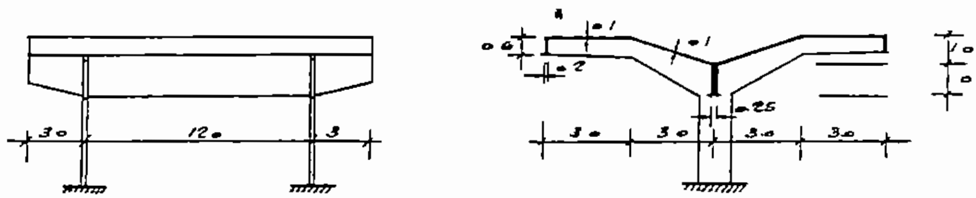


Fig XI-34

The internal stresses in the folded plate shall be determined for the following two cases

A) Symmetrical case of loading with dead, superimposed, and live loads on b d g

B) Unsymmetrical case of loading with dead and superimposed load on b d g and live load on b d only

A) Symmetrical case of loading

In this case, it is sufficient to study one half of the folded-plate only (i e part a b c d e) with half the beam d e (i e 12 5 x 100 cms) and half the load P_d

Ridge and plate loads (Fig XI-35)

$$P_b = 0\ 2 \times 0\ 6 \times 2500 + 1\ 5 \times 550 = 1125 \text{ kg/m'}$$

$$P_c = 3\ 0 \times 550 = 1650$$

$$P_d = 0\ 25 \times 1\ 0 \times 2500 + 3\ 0 \times 550 = 2275$$

Load on pl 1, $S_{ba} = P_b = 1125$ kg/m'

Loads on plate 2, $S_{bc} = 0$ and

$S_{cb} = 3\ P_c = 3 \times 1650 = 4950$ kg/m'

Loads on plate 3, $S_{dc} = 0$ and

$S_{cd} = \sqrt{10}\ P_c = \sqrt{10} \times 1650$

$= 5220$ kg/m'

Loads on ½ plate 4, $S_{de} = ½\ P_d$

$= \frac{2275}{2}$ kg/m'

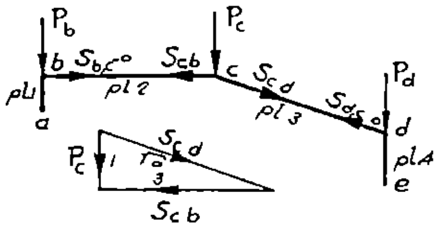


Fig XI-35

Bending moments in longitudinal direction (Fig XI-36)

The bending moments in the longitudinal direction due to a load S is as shown in Fig XI-36

M_o -ve $= S\ 3^2 / 2 = 4\ 5\ S$

max M_o +ve $= S\ \frac{12^2}{8} - 4\ 5\ S = 13\ 5\ S$

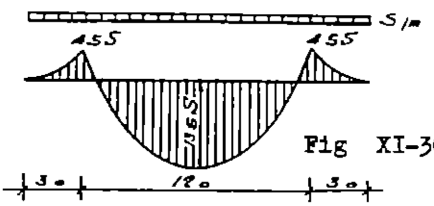


Fig XI-36

Properties of individual plates, Moments M_o and stresses σ_o

Plate	b cm	h cm	A=bh cm^2	Z=$bh^2/6$ cm^3	S/m' kgs	M_o=13 5S 10^3 kgcm	$\sigma_o = \pm M_o/Z$ kg/cm^2
1	20	60	1200	0 12 x10^5	1125	15 19 x 10^5	σ_{oa}=+126 60, σ_{ob}=-126 60
2	10	300	3000	1 50 x10^5	4950	66 83 x 10^5	σ_{ob}=+ 44 60, σ_{oc}=- 44 60
3	10	315	3150	1 65 x10^5	5220	70 47 x 10^5	σ_{oc}=- 42 70, σ_{od}=+ 42 70
½ 4	12 5	100	1250	208 x10^5	$\frac{2275}{2}$	15 33 x 10^5	σ_{od}=- 73 60, σ_{oe}=+ 73 60

Edge shears

Applying equations XI-14 on the different joints, we get

Joint b
$$0 + 2\,T_b\left(\frac{1}{A_1} + \frac{1}{A_2}\right) + \frac{T_c}{A_2} = \frac{1}{2}\left(\frac{M_{o1}}{Z_1} + \frac{M_{o2}}{Z_2}\right)$$
$$0 + 2\,T_b\left(\frac{1}{1200} + \frac{1}{3000}\right) + \frac{T_c}{3150} = \frac{1}{2}\left(126\ 6 \quad 44\ 6\right)$$

or
$$7\,T_b + T_c = 256\ 8 \times 10^3 \qquad (a)$$

Joint c
$$\frac{T_b}{A_2} + 2\,T_c\left(\frac{1}{A_2} + \frac{1}{A_3}\right) + \frac{T_{d3}}{A_3} = \frac{1}{2}\left(\frac{M_{o2}}{Z_2} + \frac{M_{o3}}{Z_3}\right)$$
$$\frac{T_b}{3000} + 2\,T_c\left(\frac{1}{3000} + \frac{1}{3150}\right) + \frac{T_{d3}}{3150} = \frac{1}{2}\left(44\ 6 - 42\ 7\right)$$

or
$$0\ 330\,T_b + 1\ 3\,T_c + 0\ 317\,T_{d3} = 0\ 925 \times 10^3 \qquad (b)$$

Joint d Because of symmetry, T_{d3} acting at joint d of plate 3 is equal to T_{d4} acting at joint d of half plate 4, so that the equation of three shears in its normal form can be applied Hence

$$\frac{T_c}{A_3} + 2\,T_{d3}\left(\frac{1}{A_3} + \frac{1}{A_4}\right) + 0 = \frac{1}{2}\left(\frac{M_{o3}}{Z_3} + \frac{M_{o4}}{Z_4}\right)$$
$$\frac{T_c}{3150} + 2\,T_{d3}\left(\frac{1}{3150} + \frac{1}{1250}\right) + 0 = \frac{1}{2}\left(-42\ 7 - 73\ 6\right)$$

or
$$0\ 317\,T_c + 2\ 235\,T_{d3} = -\,58\ 15 \times 10^3 \qquad (c)$$

Solving the three equations a, b and c, the values of T_a, T_b and T_c can be determined Accordingly

$T_b = 37\ 01 \times 10^3$, $T_c = -\ 2\ 47 \times 10^3$ and $T_{d3} = -\ 25\ 67 \times 10^3$

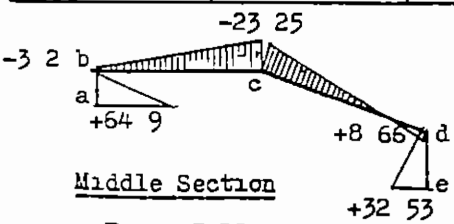
The stresses in the different plates due to M_o and T are shown in Fig XI-37, whereas the final stresses in the middle section of the folded-plate due to this symmetrical case of loading is shown in Fig XI-38

The longitudinal bending moments at the diaphragms of the folded-plate are - 4 5/13 5 = - 1/3 those at the middle section (refer to Fig XI-36), hence, the final stresses at the diaphragms shall have the same ratio both in magnitude and sense and are as shown in Fig XI-39

Symmetrical loading Stresses in the different plates (in kg/cm^2)

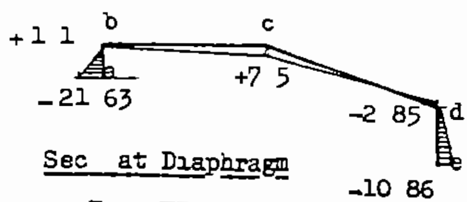
Plate No	M_o kg cm	T kgs		Final stresses kg/cm^2
1 (6; 20; a, b)	M_{o1}=15 19x10^5 -126 6 +126 6	T_a=0 0 0	T_b=+37 01x10^3 +123 4 -61 7	$\sigma_{M_{o1}}+\sigma_{T_a}+\sigma_{T_b}$ -3 2 +64 9
2 (30; 10; c, b)	M_{o2}=66 83x10^5 -44 6 +44 6	T_b=-37 01x10^3 +24 67 -49 35	T_c=-2 47x10^3 -3 29 +1 65	$\sigma_{M_{o2}}+\sigma_{T_b}+\sigma_{T_c}$ -23.22 -3 1
3 (31 5; 10; d, c)	M_{o3}=70 47x10^5 +42 7 -42 7	T_c=+2 47x10^3 -1 57 +3 14	T_d=25 67x10^3 -32 6 +16 30	$\sigma_{M_{o3}}+\sigma_{T_c}+\sigma_{T_d}$ +8 53 -23 26
4 (10; 125; d, e)	M_{o4}=52 98x10^5 -73 6 +73 6	T_d=+25 67x10^3 +82 4 -41 67	T_e=0 0 0	$\sigma_{M_{o4}}+\sigma_{T_d}+\sigma_{T_e}$ +8 8 +32 53

Fig XI-37



Middle Section

Fig _XI-38



Sec at Diaphragm

Fig XI-39

B) <u>Unsymmetrical case of loading</u> (Fig XI-40)

Live loads of 200 kg/m^2 are assumed acting on b c d only

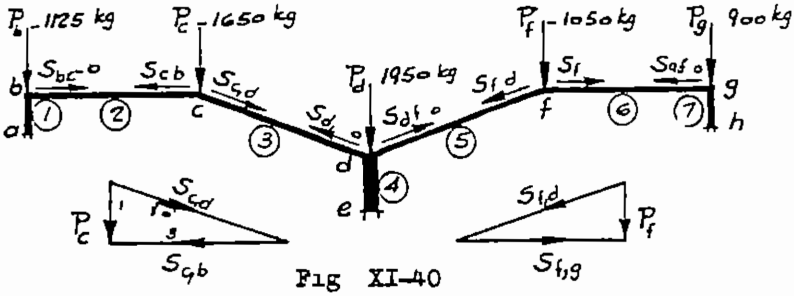


Fig XI-40

<u>Ridge and plate loads</u>

P_b = 1125 kg/m', P_c = 1650 kg/m' (Refer to case A)

P_d = 0 25 x 1 0 x 2500 + 1 5 x 350 = 1950 kg/m'

P_f = 3 0 x 350 = 150 kg/m', P_g = 0 20 x 0 60 x 2500 + 1 5 x 350 =900 kg/m'

<u>Resolution of ridge loads</u>

Loads on plate 1 $S_{b,a} = P_b$ = 1125 kg/m'

" " " 2 $S_{b,c}$ = 0, $S_{c,b} = 3\ P_c$ = 3 x 1650 = 4950 kg/m'

3 $S_{c,d}$ = 10 P_c = 5220 kg/m', $S_{d,c}$ = 0

" 4 $S_{d,e}$ = 1975 kg/m'

" " " 5 $S_{d,f}$ = 0, $S_{f,d}$ = 10 P_f = 3320 kg/m'

" 6 $S_{f,g} = 3\ P_f$ = 3 x 1050 = 3150 kg/m', $S_{g,f}$ = 0

" " " 7 $S_{g,h} = P_g$ = 900 kg/m'

<u>Properties of individual plates, Moments M_o and Stresses</u> $_o$

The properties, moments and stresses of plates 1, 2 and 3 are the same as in the previous case

Plate	b cm	h cm	A=bh cm^2	Z=bh^2/6 cm^3	S/m' kgs	M_o=13 5S 10^3 kg cm	$\sigma_o = \pm M_o/Z$ kg/cm^2
4	25	100	2500	417 x10^5	1975	26 66 x 10^5	σ_{od} = - 63 9 σ_{oe} = + 63 9
5	10	315	3150	1 65 x10^5	3320	44 82 x 10^5	σ_{od} = + 27 16 σ_{of} = - 27 16
6	10	300	3000	1 50 x10^5	3150	42 50 x 10^5	σ_{of} = - 28 33 σ_{og} = + 28.33
7	20	60	1200	0 12 x10^5	900	12 15 x 10^5	σ_{og} = -104 17 σ_{oh} = +104 17

Edge shears

The equations of edge shears applied to joints b and c will be the same as in previous case Hence

Joint b $$7\ T_b + T_c = 256\ 8 \times 10^3 \qquad \text{(a)}$$

Joint c $$0\ 330\ T_b + 1\ 3\ T_c + 0\ 317\ T_{d3} = 0\ 925 \times 10^3 \qquad \text{(b)}$$

Joint d_3 $$\frac{T_c}{A_3} + 2\ T_{d3}\left(\frac{1}{A_3} + \frac{1}{A_4}\right) + 2\frac{T_{d5}}{A_4} = \frac{1}{2}\left(\frac{M_{o3}}{Z_3} + \frac{M_{o4}}{Z_4}\right)$$

$$\frac{T_c}{3150} + 2\ T_{d3}\left(\frac{1}{3150} + \frac{1}{2500}\right) + 2\frac{T_{d5}}{2500} = \frac{1}{2}\left(-42\ 7-63\ 9\right)$$

or $$0\ 315\ T_c + 1\ 43\ T_{d3} + 0\ 80\ T_{d5} = 53\ 3 \times 10^3 \qquad \text{(c)}$$

Joint d_5 $$\frac{2\ T_{d3}}{A_4} + 2\ T_{d5}\left(\frac{1}{A_4} + \frac{1}{A_5}\right) + \frac{T_f}{A_5} = \frac{1}{2}\left(\frac{M_{o4}}{Z_4} + \frac{M_{o5}}{Z_5}\right)$$

$$\frac{2\ T_{d3}}{2500} + 2\ T_{d5}\left(\frac{1}{2500} + \frac{1}{3150}\right) + \frac{T_f}{3150} = \frac{1}{2}\left(-63\ 9-27\ 16\right)$$

or $$0\ 80\ T_{d3} + 1\ 43\ T_{d5} + 0\ 313\ T_f = -\ 45\ 53 \times 10^3 \qquad \text{(d)}$$

Joint f $$\frac{T_{d5}}{A_5} + 2\ T_f\left(\frac{1}{A_5} + \frac{1}{A_6}\right) + \frac{T_g}{A_6} = \frac{1}{2}\left(\frac{M_{o5}}{Z_5} + \frac{M_{o6}}{Z_6}\right)$$

$$\frac{T_{d5}}{3150} + 2\ T_f\left(\frac{1}{3150} + \frac{1}{3000}\right) + \frac{T_g}{3000} = \frac{1}{2}\left(-22\ 16+28\ 33\right)$$

or $$0\ 313\ T_{d5} + 1\ 30\ T_f + 0\ 333\ T_g = +\ 0\ 585 \times 10^3 \qquad \text{(e)}$$

Joint g $$\frac{T_f}{A_6} + 2\ T_g\left(\frac{1}{A_6} + \frac{1}{A_7}\right) + 0 = \frac{1}{2}\left(\frac{M_{o6}}{Z_6} + \frac{M_{o7}}{Z_7}\right)$$

$$\frac{T_f}{3000} + 2\ T_g\left(\frac{1}{3000} + \frac{1}{1200}\right) + 0 = \frac{1}{2}\left(28\ 33+104\ 17\right)$$

or $$0\ 330\ T_f + 2\ 33\ T_g + 0 = -\ 66\ 25 \times 10^3 \qquad \text{(f)}$$

The solution of these six equatins gives the required edge shears

$T_b = +\ 36\ 93 \times 10^3$ kgs $\qquad T_c = -\ 1\ 74 \times 10^3$ kgs

$T_{d3} = -\ 28\ 36 \times 10^3$ kgs $\qquad T_{d5} = -\ 15\ 25 \times 10^3$ kgs

$T_f = -\ 3\ 243 \times 10^3$ kgs $\qquad T_g = +\ 28\ 86 \times 10^3$ kgs

The stresses in the different plates are shown in Fig XI-41

Unsymmetrical loading Stresses in the different plates (in kg/cm²)

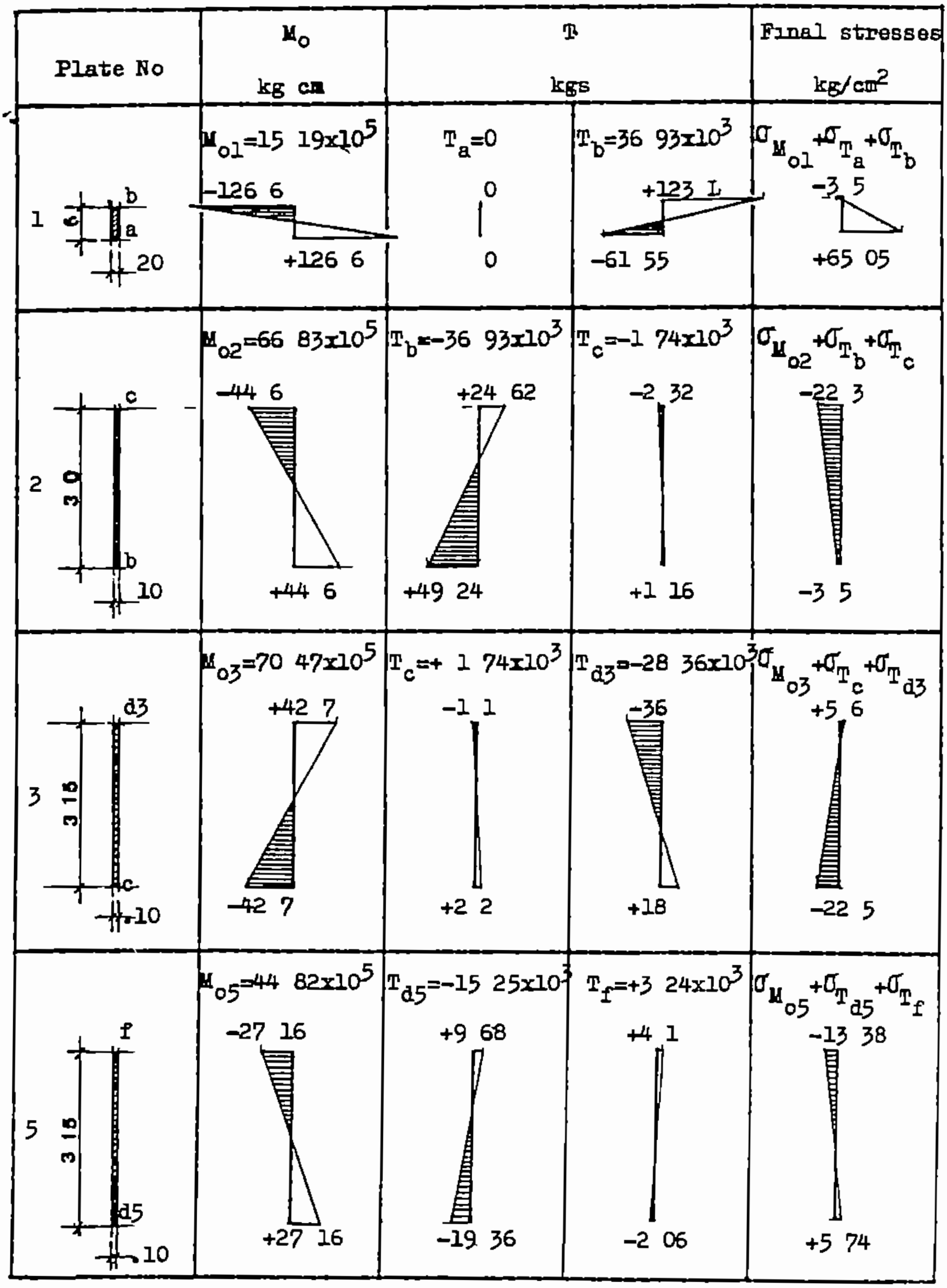


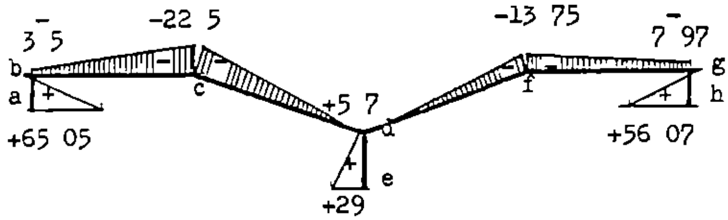
Plate No	M_o kg cm	T kgs		Final stresses kg/cm²
1 (b, a, 6, 20)	M_{o1}=15 19x10^5 -126 6 +126 6	T_a=0 0 0	T_b=36 93x10^3 +123 1 -61 55	$\sigma_{M_{o1}}+\sigma_{T_a}+\sigma_{T_b}$ -3 5 +65 05
2 (c, b, 3 0, 10)	M_{o2}=66 83x10^5 -44 6 +44 6	T_b=-36 93x10^3 +24 62 +49 24	T_c=-1 74x10^3 -2 32 +1 16	$\sigma_{M_{o2}}+\sigma_{T_b}+\sigma_{T_c}$ -22 3 -3 5
3 (d3, c, 3 15, 10)	M_{o3}=70 47x10^5 +42 7 -42 7	T_c=+ 1 74x10^3 -1 1 +2 2	T_{d3}=-28 36x10^3 -36 +18	$\sigma_{M_{o3}}+\sigma_{T_c}+\sigma_{T_{d3}}$ +5 6 -22 5
5 (f, d5, 3 15, 10)	M_{o5}=44 82x10^5 -27 16 +27 16	T_{d5}=-15 25x10^3 +9 68 -19 36	T_f=+3 24x10^3 +4 1 -2 06	$\sigma_{M_{o5}}+\sigma_{T_{d5}}+\sigma_{T_f}$ -13 38 +5 74

Fig XI-41

Unsymmetrical loading Stresses (in kg/cm²) continued

Plate No	M_o kg cm	T kgs		Final stresses kg/cm²
6 (g, f; 30; 10)	M_{o6}=42 5x10⁵ +28.33 -28 33	T_f=-3 243x10³ +2 16 -4 32	T_g=-28 30x10³ -37 8 +18 9	$\sigma_{M_{o6}}+\sigma_{T_f}+\sigma_{T_g}$ -7 31 -13 75
7 (g, h; 6; 20)	M_{o7}=12 15x10⁵ -104 17 +104 17	T_g=+28 36x10³ +96 20 -48 1	T_h=0 0 0	$\sigma_{M_{o7}}+\sigma_{T_g}$ -7 97 +56 07
4 (d4, e; 10; 25)	M_{o4}=26 66x10⁵ -63 9 +63 9	T_{d4}=+43 61x10³ +69 77 -34 9	T_e=0 0 0	$\sigma_{M_{o4}}+\sigma_{T_{d4}}$ +5 7 +29
Note	$T_{d4} = T_{d3} + T_{d5}$ = - (- 28 36 - 15 25) = + 43 61 t			

Fig XI-41 (contd)



Stresses in middle section of folded-plate for case of unsymmetrical loading

Fig XI-42

Longitudinal main reinforcement

It is clear from figures XI-38 and XI-42 that the main tension takes place at the middle of the vertical plates a b, d e and g h It is bigger in the case of symmetrical loading

Assuming that the longitudinal tension reinforcements resist all the tension forces T in the section (area of tension zone), then

For plates a b and g h, we get

Height of tension zone = $\frac{60 \times 64\ 9}{64\ 9 + 3\ 2} = 57\ 2$ cms and

Total tension T $= \frac{57\ 2 \times 64\ 9}{2} \times 20 = 37123$ kgs

Choosing high grade steel with an average $_s$ = 1800 kg/cm^2 for the main reinforcement, we get

Total tension steel A_s = 37123 / 1800 = 20 62 cm^2 chosen 7 # 19

For plate d e, we have

Total tension T $= \frac{32\ 57 + 8\ 54}{2} \times 100 \times 25 = 51400$ kgs

Total tension steel A_s = 51400 / 1800 = 28 50 cm^2 chosen 8 # 22

At the diaphragms, the tension takes place in the upper plates, with a maximum value at joints c and f Hence

Max tension T/m = 7 5 x 100 x 10 = 7500 kgs

Using normal mild steel with allowable stress σ_s = 1400 kg/cm^2, we get

Total steel per m A_s = 7500 / 1400 = 5 30 cm^2 chosen 6 ϕ 8/m top and bottom

The details of reinforcements are shown in Fig XI-46

Shear stresses and web reinforcements

Plates 1 and 7 (a b and g h)

Max shearing force at diaphragms $Q_{max} = \frac{P_b\ l}{2} = \frac{1125 \times 12}{2} = 6750$ kgs

Shear stress at midheight of pl^s $\tau_{omax} = \frac{3}{2}\frac{Q_{max}}{A} = \frac{3}{2}\frac{6750}{1200} = 8\ 44$ kg/cm^2

Shear stress at diaphragms due to $T_{b\ max}$ = 37010 kgs is

at top edge $\tau = \frac{4\ T_{max}}{b\ l} = \frac{4 \times 37010}{20 \times 1200} = 8\ 44\ kg/cm^2$

at mid-height $\tau/4 = 8\ 44 / 4 = 1\ 54$

Total shear at mid-height $= 8\ 44 - 1\ 54 = 6\ 90$

Plates 2 and 6 (b c and f g)

Max shearing force at diaphragms $Q_{max} = \frac{S_{c,b}l}{2} = \frac{4950 \times 12}{2} = 29700$ kgs

Shear stress at midheight of plate $\tau_{omax} = \frac{3}{2}\ \frac{Q_{max}}{A} = \frac{3}{2}\ \frac{29700}{3000} = 14\ 85\ kg/cm^2$

The shear stress being high, it may be advisable to increase the thick- of these plates to 16 cms gradually towards the diaphragms over a length of 3 0 ms, in which case $\tau_{o\ max}$ will be reduced to $\tau'_{o\ max}$, where

$$\tau'_{o\ max} = \frac{3}{2}\ \frac{29700}{16 \times 300} = 9\ 2\ kg/cm^2$$

Shear stress at b (and g) due to $T_{b\ max}$ = 37010 kgs is

$$\tau_b = \frac{4 \times 37010}{16 \times 1200} = 7\ 71\ kg/cm^2$$

Shear stress at c (and f) due to $T_{c\ max}$ = 2470 kgs is

$$\tau_c = \frac{4 \times 2470}{16 \times 1200} = 0\ 51\ kg/cm^2$$

Shear at middle of b c (and f g) $= 9\ 2 - \frac{7\ 71}{4} - \frac{0\ 51}{4} = 7\ 14\ kg/cm^2$

Plates 3 and 5 (c d and d f)

The thickness of these plates is also increased gradually to 16 cms over 3 0 ms at the diaphragms Hence

$$Q_{o\ max} = \frac{5220 \times 12}{2} = 31320\ kgs$$

$$\tau_{o\ max} = \frac{3}{2}\ \frac{31320}{16 \times 315} = 9\ 32\ kg/cm^2$$

Shear stresses at c due to $T_{c\ max}$ = 2470 kgs and at d due to

T_d = 25670 kgs are respectively

$$\tau_c = \frac{4 \times 2470}{16 \times 1200} = 0\ 51 \quad kg/cm^2$$

and

$$\tau_d = \frac{4 \times 25670}{16 \times 1200} = 5\ 35$$

Shear at middle of c d (and d f) = $9\ 32 - \frac{0\ 51}{4} - \frac{5\ 35}{4} = 7\ 86 \quad kg/cm^2$

<u>**Plate 4**</u> (d e)

$$Q_{o\ max} \leq \frac{2275 \times 12}{2} = 13650 \quad kgs$$

$$\tau_{o\ max} = \frac{3}{2} \frac{13650}{25 \times 100} = 8\ 2 \quad kg/cm^2$$

Due to T_d = 25670 kgs

$$\tau_d = \frac{4 \times 25670}{25 \times 1200} = 3\ 42 \quad kg/cm^2$$

Shear at middle of d e = $8\ 20 - \frac{3\ 42}{4} = 7\ 35\ kg/cm^2$

The distribution of the shear stresses on the different plates is shown in Fig XI-43

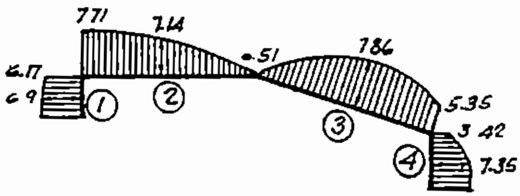


Fig XI-43

<u>Diagonal tension reinforcements</u>

If 6 Ø 10 mm/m are arranged at top and bottom fibers of plates 2, 3, 5 and 6 in the 3 0 ms of increased thickness at the diaphragms, the diagonal tensile stresses that can be resisted by these bars only(i e neglecting the longitudinal reinforcements), are given by

$$\tau = \frac{n\ A_s\ \sigma_s}{b\ s} = \frac{2 \times 0\ 79 \times 1400}{16 \times 16\ 67} = 8\ 27 \quad kg/cm^2$$

This value being bigger than the shear stresses in the mentioned plates, no bent bars are required Further, if the allowable diagonal tensile stress is 6 kg/cm^2, then the required area of steel to be bent in the vert plates shall also be small and may be chosen by designer

Design of diaphragm

The main dimensions of the diaphragms in longitudinal and cross-section are shown in Fig XI-44

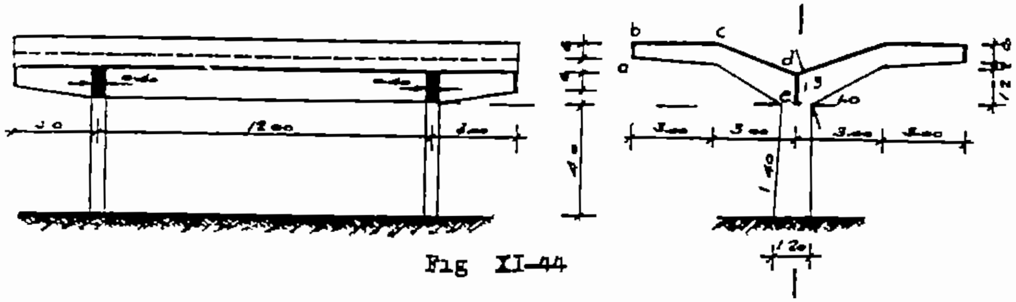


Fig XI-44

To be able to design the diaphragms, one must determine the maximum internal forces (bending moments, shearing forces and thrust) The bending moments and shearing forces shown in Figs XI-45 b and c can be determined from the vertical ridge loads directly The thrust diagram Fig XI-45 e) is to be determined from the parabolic shearing force distribution on the diaphragms (Fig XI-45 d)

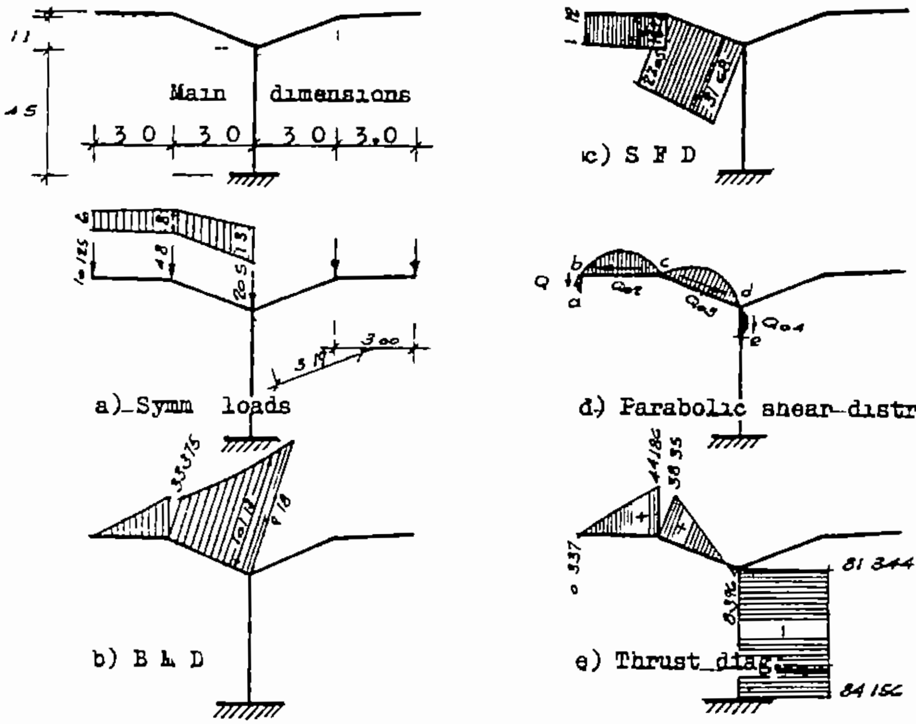


Fig XI-45

Loads

Assume breadth of diaphragm = 40 cms, depths at b, c and d = 60, 80 & 130 cms respectively, i e weight/m depth = 0 4 x 2 5 = 1 0 ton Thus

Own weight at b = 0 6 t/m , at c = 0 8 t/m and at d= 1 3 t/m

Ridge loads

P_b = 1 125 x 9 = 10 125, P_c = 1 650 x 9 = 14 85, P_d = 2 275 x 9 = 20 5^t

Maximum shearing forces of different plates

Plate 1 (a b)	Q_{o1} = 1 125 x 9 = 10 125 t	)	distributed
Plate 2 (b c)	Q_{o2} = 4 950 x 9 = 44 550 t	)	parabolically
Plate 3 (c d)	Q_{o3} = 5 220 x 9 = 46 980 t	)	on plates 1-4
Plate 4 (d e)	Q_{o4} = 2 275 x 9 = 20 475 t	)	Fig XI-45 d

Thrust (along center-line of diaphragm)

At point b N_b = - 10 125 x 0 0333 = - 0 337 t

At middle of b c $= 0\ 337 + \frac{44\ 55}{2} \times 0\ 9994 = 22\ 60$ t

At point c (left) N_c = -0 337 + 44 55 x 0 9994= + 44 186 t

" (right) N_c' = -10 125 x 0 344 + 44 55 x 0 939 = 38 35 t

" d N_d = -10 125 x 0 344 + 44 55 x 0 939 - 46 98 x

x 0 995 = -8 396 t

The corresponding bending moment, shearing force and thrust are shown in Fig XI-45

Design of different sections

The sections shall be designed by the U S D-method assuming an average load factor of 1 6, concrete with f_{cp} = 165 kg/cm^2 and high grade steel 36/50

Section I-I at c 40 x 80 cms

$$M = 10\ 125 \times 3 + 0\ 6 \times \frac{3^2}{2} + 0\ 2 \times \frac{3^2}{6} = 33\ 375 \text{ mt} \qquad M_u = 1\ 6\ M = 53\ 4 \text{ mt}$$

N = + 44 186 t (tension), N_u = 44 186 x 1 6 = 70 698 t

$$e = \frac{M_u}{N_u} = \frac{53\ 4}{70\ 698} = 0\ 755 \text{ ms} \qquad e_s = 0\ 755 - \frac{0\ 8}{2} + 0\ 05 = 0\ 405 \text{ ms}$$

$M_{su} = N_u\ e_s$ or $M_{su'}$ = 70 698 x 0 405 = 28 33 mt

As $d = c\sqrt{\frac{M_{su}}{f_{cp}\, b}}$ then $75 = c\sqrt{\frac{28\ 63 \times 10^5}{165 \times 40}}$ giving c = 3 6

The value of c being bigger than 2, then the failure is ductile and no compression reinforcement is required Table 4-8 gives $\eta = 0\ 85$

$$A_s = \frac{M_{su}}{f_y\, \eta\, d} + \frac{N_u}{\Omega f_y} = \frac{28\ 63 \times 10^5}{3600 \times 0\ 85 \times 75} + \frac{70\ 698 \times 10^3}{0\ 9 \times 3600}$$

$$= 12\ 47 + 21\ 82 = 34\ 29\ cm^2$$ chosen 7 # 25

Section I - I at d 40 x 130 cms

$$M = 10\ 125 \times 6 + 0\ 6 \times 3 \times 4\ 5 + \frac{0\ 2 \times 3}{2} \times 4 + 14\ 85 \times 3 + 0\ 8 \times 3\ 19 \times \frac{3}{2} + 0\ 5 \times \frac{3\ 19}{2} \times \frac{3}{3} = 119\ 182 \text{ mt}$$ at center line of column

Bending moment at face of column is given by

$$M = 10\ 125 \times 5\ 35 + 0\ 6 \times 3 \times 4\ 15 + \frac{0\ 2 \times 3}{2} \times 3\ 35 + 14\ 85 \times 2\ 35 + 0\ 8 \times 3\ 19 \times \frac{2\ 35}{2}$$
$$+ 0\ 5 \times \frac{3\ 19}{2} \times \frac{2\ 35}{3} = 101\ 13 \text{ mt}$$ (design value for section II-II)

$M_u = 101 \times 1\ 6 = 161\ 6$ mt, $N = -\ 8\ 4$ t, $N_u = -8\ 4 \times 1\ 6 = -13\ 44$ t

$e = \frac{161\ 6}{13\ 44} = 12\ 03$ ms, $e_s = 12\ 03 + \frac{1\ 3}{2} - 0\ 07 = 12\ 61$ m So that

$M_{su} = N_u\ e_s$ or $M_{su} = 13\ 44 \times 12\ 61 = 169\ 4$ mt

$123 = c\sqrt{\frac{169\ 4 \times 10^5}{165 \times 40}}$ giving $c = 2\ 43 > 2$ and $\eta = 0\ 875$

$$A_s = \frac{169\ 4 \times 10^5}{3600 \times 0\ 785 \times 123} - \frac{13\ 44 \times 10^3}{0\ 9 \times 3600} = 48\ 65 - 4\ 15 = 44\ 5\ cm^2$$

chosen 9 # 25

Section III-III at top of column 40 x 100 cms

Bending moment at d on loaded side = 119 18 mt (refer to sec II-II)

Bending moment at d on unloaded side d h is given by

$$M = 8\ 1 \times 6 + 0\ 6 \times \frac{3 \times 4\ 5}{2} + 0\ 2 \times \frac{3}{2} \times 4 + 9\ 45 \times 3 + 0\ 8 \times 3\ 19 \times \frac{3}{2}$$
$$+ 0\ 5 \times \tfrac{1}{2}\ 3\ 19 \times 3/3 = 86\ 83 \text{ mt}$$

Bending moment on column = 119 18 - 86 83 = 32 35 mt

Corresponding normal force $N = 70\ 974$ t

Assuming an accidental eccentricity $= t/10 = 0\ 1$ m then

$M_{total} = 32\ 35 + 0\ 1 \times 70\ 974 = 39\ 45$ mt therefore

$M_u = 1\ 6 \times 39\ 45 = 63\ 12$ mt and $N_u = 1\ 6 \times 70\ 974 = 113\ 56$ t

$e = \frac{63\ 12}{113\ 56}\ 0\ 556$ ms, $e_s = 0\ 556 + \frac{1\ 0}{2} - 0\ 05 = 1\ 005$ ms so that

$M_{su} = N_u\ e_s$ or $M_{su} = 113\ 56 \times 1\ 006 = 114\ 24$ mt, and

$95 = c\sqrt{\frac{114\ 24 \times 10^5}{165 \times 40}}$ giving $c = 2\ 28 > 2$ and $\eta = 0\ 76$ so that

$$A_s = \frac{114\ 24 \times 10^5}{0\ 76 \times 3600 \times 95} - \frac{113\ 56 \times 10^3}{0\ 9 \times 3600} = 43\ 95 - 35\ 05 = 8\ 9 \quad cm^2$$

Choosing $A_s = A_s' = 0\ 5\%$, then $A_s = A_s' = \frac{0\ 5}{100} \times 40 \times 100 = 20$ cm^2 or

4 # 25 on each face

Shear stresses and diagonal tension

$Q_b = 10\ 12$ t $\qquad \tau_b = \frac{10120}{0\ 87 \times 40 \times 55} = 5\ 28$ kg/cm^2

$Q_c = 27\ 02$ t $\qquad M_c = 33\ 37$ mt $\qquad \tan\alpha = \frac{0\ 5}{3\ 19} = 0\ 158$

$$\text{Reduced } Q_c = Q_c - \frac{M_c \tan}{0\ 87\ d} = 27\ 02 - \frac{33\ 37 \times 0\ 158}{0\ 87 \times 0\ 75} = 19 \quad t$$

Neglecting the effect of N, $\tau_c = \frac{19000}{0\ 87 \times 40 \times 75} = 7\ 25$ kg/cm^2

$Q_d = 31\ 68$ t $\qquad M_d = 119\ 18$ mt $\qquad \tan\alpha = 0\ 158$

Reduced $Q_d = 31\ 68 - \frac{119\ 18 \times 0\ 158}{0\ 87 \times 1\ 23} = 14\ 18$ t so that

$$\tau_d = \frac{14180}{0\ 87 \times 40 \times 123} = 3\ 3 \quad kg/cm^2$$

The given study shows that the shear stresses in the diaphragms are relatively low in spite of that bent bars are arranged as shown in Fig XI-46 which gives the details of reinforcements in both the diphragms and the folded-plate nearby

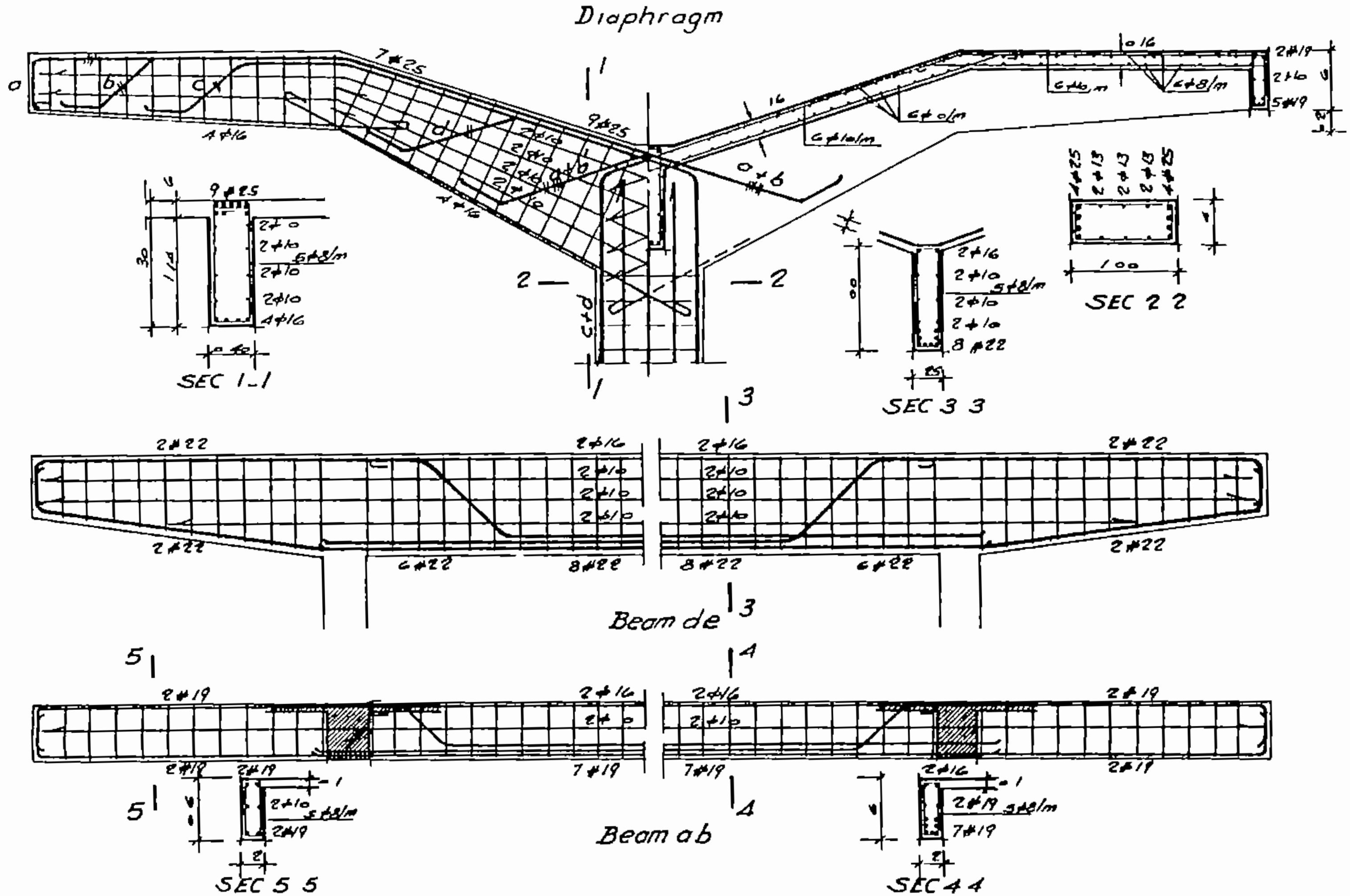


Fig XI 46

XII T H I N S H E L L S T R U C T U R E S

XII-1 INTRODUCTION

The roofing of large unobstructed areas, with a minimum amount of material is a goal that has claimed the attension of engineers and architects for years Of the various types of roof construction, the concrete shell by its strength, beauty, simplicity and economy of construction, offers the best possibility of attaining this desirable end

The famous big monumental domes that have been constructed in the last centuries (16th to 19th) depended in their strength on the big masses of the building materials that were used, as can be seen in the following three examples

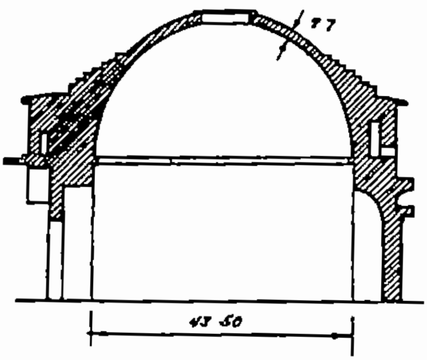


Fig XII-1 Pantneon of Rome

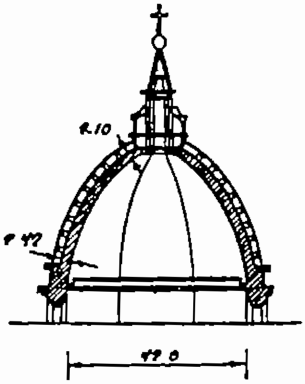


Fig XII-2 Cathedral of Florence

1) The pantheon of Rome shown in figure XII-1 with a span of 43 5 ms and a minimum dome thickness of 2 70 ms

2) The Cathedral of Florence with a span of 42 ms for the inner dome and a thickness varying between 2 1 ms at the crown and 2 42 ms at the foot-ring as shown in figure XII-2

3) The double-walled dome of Saint-Peters Cathedral in Rome, having a span of 42 0 ms The two slabs of the dome are joined together at their shoulders giving a total thickness of 2 8 ms the inner shell is 1 6 ms thick while the outer one is 1 2 ms only as shown in figure XII-3

Due to the continual efforts of engineers and sientists and the available technical tools, it has been possible to analyse the deformations and internal forces in different forms of shells and as a result, it has been found that it is possible to construct very thin

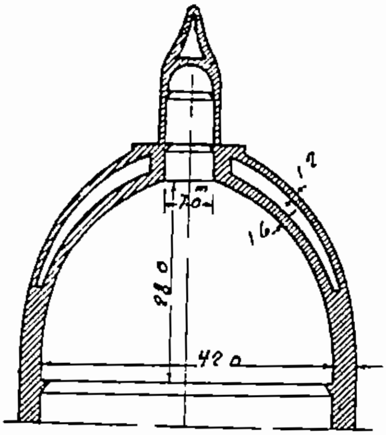


Fig XII-3 Saint-Peters Cathedral

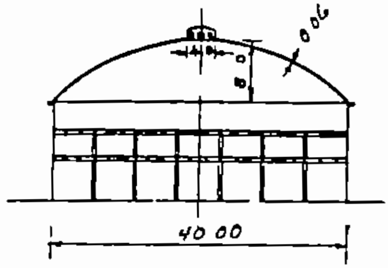


Fig XII-4 Scott Factory at Jena

shells covering relatively big spans and having very big strength by giving them convenient structural forms

The 6cms shell used as a roof for the 40 ms span Scott factory in Jena, shown in figure XII-4 gives an example of the tremendous saving in the building materials that can be achieved by the use of modern shells

The relative roof thickness of the old and new domes shown in figures XII-1 to XII-4 is illustrated in figure XII-5 by proportionally thick rectangles

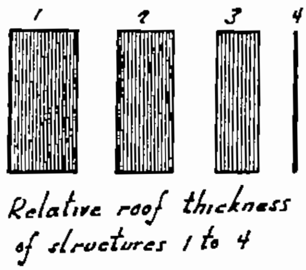


Fig XII-5

Many different shell-forms of single and double curvature proved to be of very high resistance, economic, easy to construct and architecturally very impressive Samples of such exiting shell strctures are shown in the following figures XII-23 and 24 for surfaces of revolution XII-73, 74, 81, 82 and 87 for cylindrical shells and XII-106, 110, 114 & 115 for double curved shells

It has however been possible to construct reinforced concrete shells having a ratio of thickness to span almost equal to that provided by nature in its protective covering of an egg

Except for some simple forms subjected to few simple symmetrical cases of loading, an exact analysis of concrete shells is a difficult, if not impossible, task because the design must take in consideration not only all the variables in loading and physical make-up of the structure but also the physical properties of the materials used and their composite action under load

However, a qualitative understanding of the fundamental nature of the behavior of some forms of shells can be gained without detailed mathematical analysis and in many cases suggests a reasonable safe method of solution

Such solutions are generally accepted because of the remarkable reserve strength of shell construction, which makes it practically impossible for a shell structure to collapse

XII-2 LOADING

<u>Dead weight</u> We shall now discuss the different loading conditions that are of importance in the design of dome structures We are principally concerned with the load components in the direction of the meridian, in the direction of the rings and in the direction normal to the shell surface The most important load is the dead load g composed of the weight of the structure and the roofing Using the notations

g = dead weight per unit area of surface

p_φ = component of g in direction of the tangent to the meridian

p_θ = " " " " ring

p_r = " " " normal to the surface of the shell

we get (Fig XII-6)

$$p_\varphi = g \sin\varphi$$

$$p_\theta = 0$$

$$p_r = g \cos\varphi$$

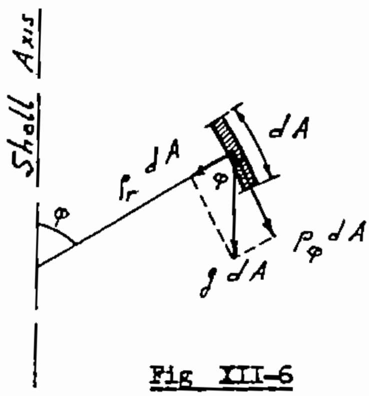


Fig XII-6

<u>Live load</u>

The live load p (e g the snow load) is generally assumed per m^2 horizontal Referring to figure XII-7

we get

$$p_\varphi = p \sin\varphi \cos\varphi$$

$$p_\theta = 0$$

$$p_r = p \cos^2\varphi$$

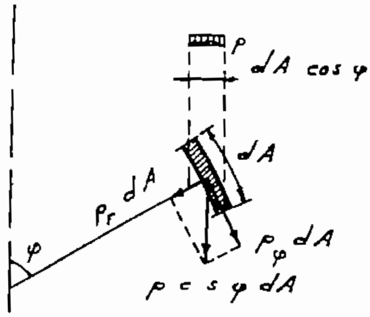


FIG XII-7

Wind load

The wind load of shells is composed of pressure on the wind side and suction on the leeward side Only the load component acting normal to the shell surface is of importance, since the other components are due to friction and are almost equal to zero In order to calculate the wind pressure one can use the following hypothesis, which has the merit of great simplicity (Fig XII-8)

$p_\varphi = 0$ $p_\theta = 0$ $p_r = w \sin\varphi \cos\theta$

in which w = the wind load / m^2 surface at $\theta = 0$, $\varphi = \pi / 2$

This distribution can be used for cylindrical and spherical shells the introduction of more exact laws will unduly complicate the calculations One may assume

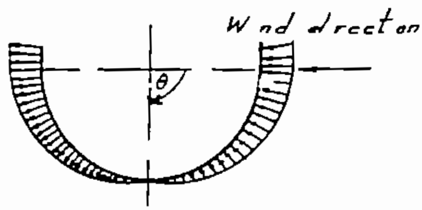


FIG XII-8

$w = 0\ 26\ q$ for the sphere

and

$w = 0\ 45\ q$ for the cylinder

where q = the specified wind pressure

For all other shells of revolution values will have to be assigned according to their shape between these two limits

XII-3 SURFACES OF REVOLUTION

Roofs and floors of circular big span areas may be flat and supported on any convenient system of girders as shown in figure XII-9a or radial frames as shown in figure XII-9b.

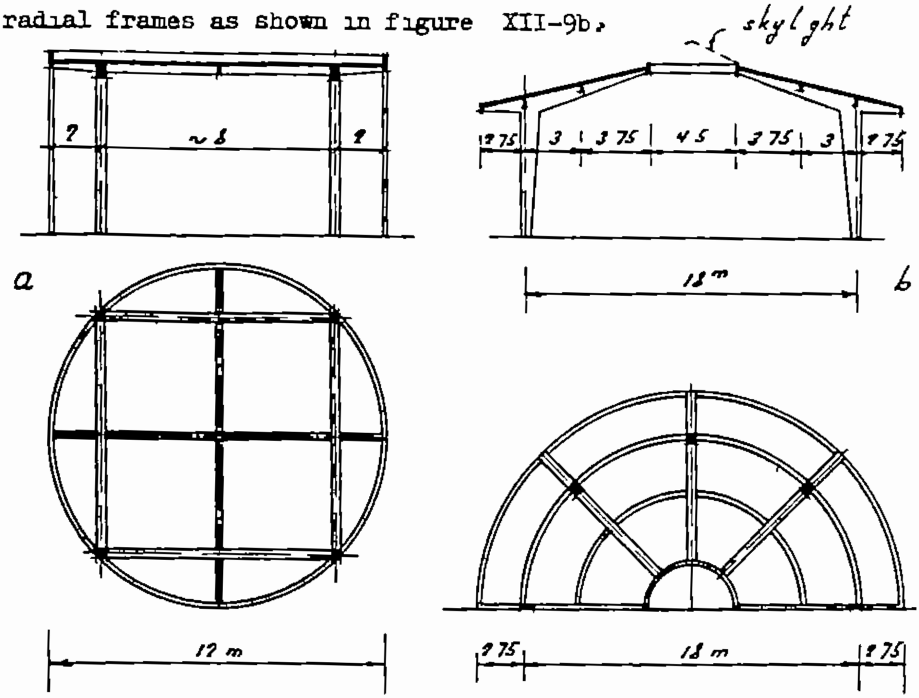


FIG XII-9

In many cases, the choice of a reinforced concrete surface of revolution e g a cone or a dome (Fig XII-4) results in an ultimate saving in materials and cost even when the greater cost of the shuttering is taken into consideration

In the following we give the general principles involved in the design of simple popular forms of surfaces of revolution according to the membrane theory

1 - Membrane Theory of Surfaces of Revolution

In this theory it is assumed that the thickness of the shell is so small that it may be considered as a membrane which can resist meridian and ring forces only i e the bending moments due to the fixation at the supports, unsymmetrical loading and similar effects are neglected

a) Analytical Method (Refer to figure XII-10)

It will be assumed that

r = radius, normal to axis of rotation, of any circular ring at a horizontal plane z z

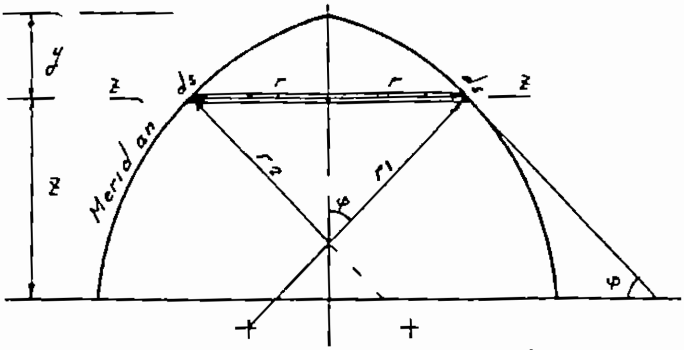


Fig XII-10

r_1 = radius of curvature of meridian

r_2 = cross radius of curvature along the normal to axis of rotation

N_φ = resultant meridian force per unit length of circumference

= σ_φ t where σ_φ = meridian stress and t = thickness of shell

N_θ = resultant ring force per unit length of meridian

σ_θ = σ_θ t where σ_θ = ring stress

H = horizontal thrust of shell per unit length of circumference

W_φ = sum of vertical forces above zz (expressed through the angle φ)

In order to have equilibrium at any horizontal section z z , the vertical component of the meridian forces N_φ must be equal to the vertical load above z z per meter run circumference Hence, we get (fig XII-11)

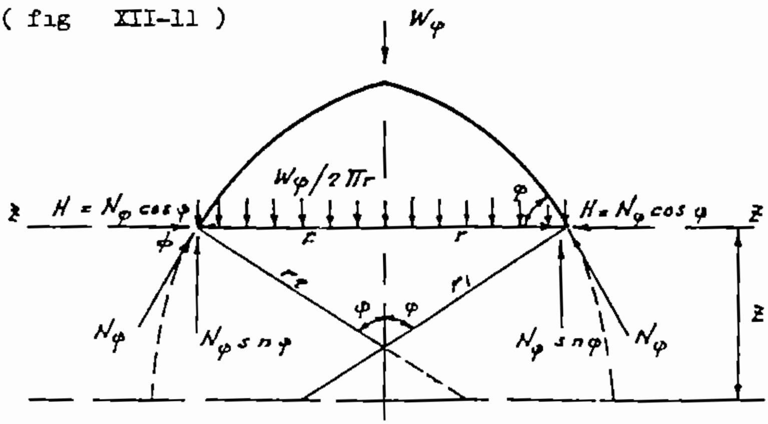


Fig XII-11

$W_\varphi / 2\pi r = N_\varphi \sin\varphi$ or

$$N_\varphi = W_\varphi / 2\pi r \sin_\varphi \qquad (a)$$

But $r = r_2 \sin\varphi$, so that N_φ can also be given in the form

$$N_\varphi = W_\varphi / 2\pi r_2 \sin^2\varphi \qquad (a')$$

The horizontal thrust H per unit length of circumference is

$$H = W_\varphi / 2\pi r \tan\varphi = N_\varphi \cos\varphi$$

considering the equilibrium of the element ds under the forces shown in figure XII-12 , we find that .

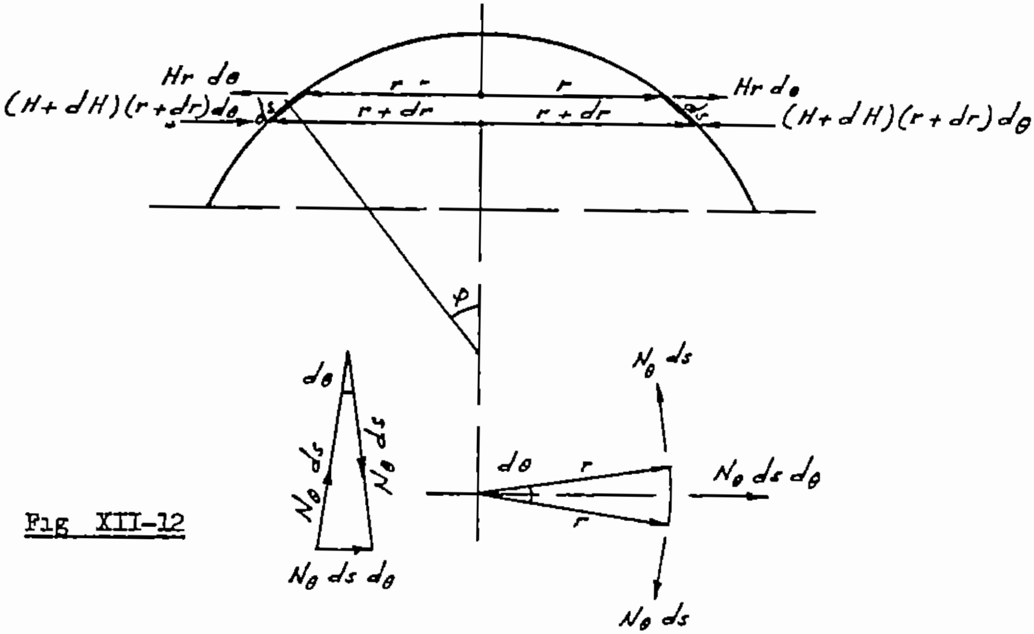


Fig XII-12

$$N_\theta \, ds \, d\theta = (H + dH)(r + dr)\, d\theta - Hr\, d\theta$$

Reducing by $d\theta$ and neglecting $dH\,dr$ being a small value of the second degree, we get

$N_\theta \, ds = Hr + dH\, r + H\, dr - Hr$ or $N_\theta \, ds = d(Hr)$ i e

$$N_\theta = d(Hr) / ds \qquad (b)$$

If the dome were simply supported, the maximum ring force at the lowest strip would be (Fig XII-13)

$$\max N_\theta = H\, r_{max}$$

This force must be resisted by tension ring reinforcement given by

$$A_s = \max N_\theta / \sigma_s$$

The relation between the external forces and the internal stresses can be determined if we consider the equilibrium of all the forces acting on an element ds_1 ds_2, in a radial direction normal to the surface of the shell (fig XII-14) as follows

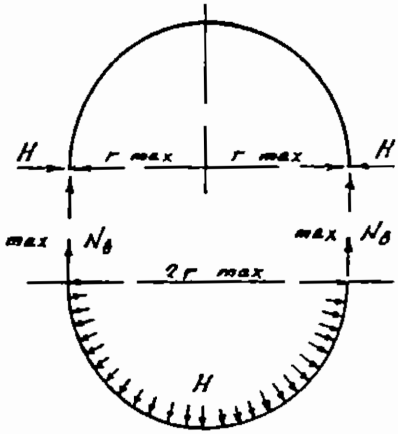


Fig XII-13

Assume radial component of external forces on the element

$$p_r \; ds_1 \; ds_2$$

Radial component of the meridian force $N_\varphi \; ds_2$ due to change of its direction by an angle $d\varphi$ is

$$N_\varphi \; ds_2 \; d\varphi$$

Horizontal component of ring force $N_\theta \; ds_1$ due to change of its direction by an angle $d\theta$ is

$$N_\theta \; ds_1 \; d\theta$$

Its radial component is

$$N_\theta \; ds_1 \; d\theta \; \sin\varphi$$

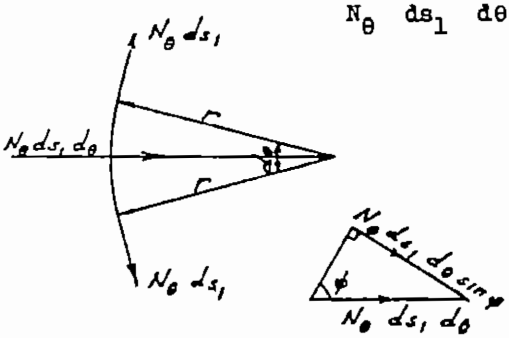
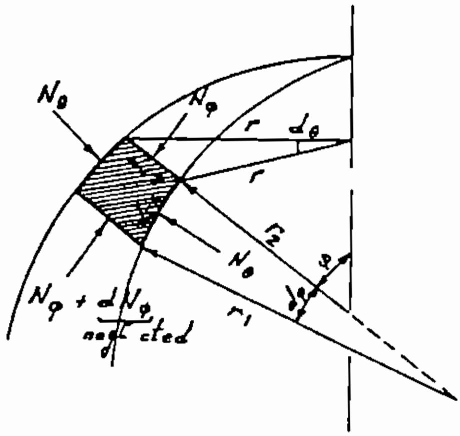


Fig XII-14



The equilibrium between these forces is given by the relation

$$N_\varphi \; ds_2 \; d\varphi \; + \; N_\theta \; ds_1 \; d\theta \; \sin\varphi \; = p_r \; ds_1 \; ds_2$$

But $\quad ds_1 = r_1 \, d\varphi \quad , \quad ds_2 = r \, d\theta = r_2 \sin\varphi \; d\theta$

Therefore, we get

$$N_\varphi\, r_2 \sin\varphi\, d\theta\, d\varphi + N_\theta\, r_1\, d\varphi\, d\theta \sin\varphi = p_r\, r_1\, d\varphi\, r_2 \sin\varphi\, d\theta \quad \text{or}$$

$$N_\varphi\, r_2 + N_\theta\, r_1 = p_r\, r_1\, r_2 \quad \text{i e}$$

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} = p_r \qquad (c)$$

For a spherical surface $r_1 = r_2 = a$ and

$$N_\varphi + N_\theta = p_r\, a \qquad (c')$$

For a conical surface $r_1 = \infty$ and

$$N_\theta = p_r\, r_2 \qquad (c')$$

Accordingly, the meridian force can be determined from equation (a) while the ring force from equations b or c

b) Graphical Method

The meridian and ring forces in surfaces of revolution whose meridian does not follow a simple mathematical equation can be determined using the following graphical method (fig XII-15) which is based on equations a and b

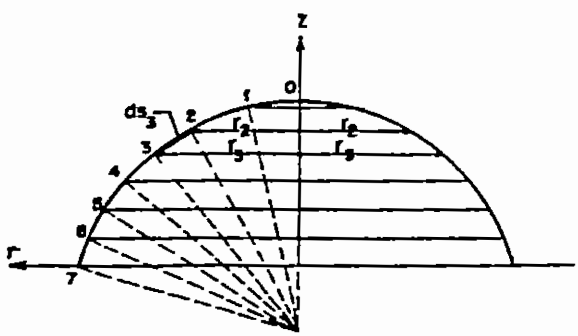
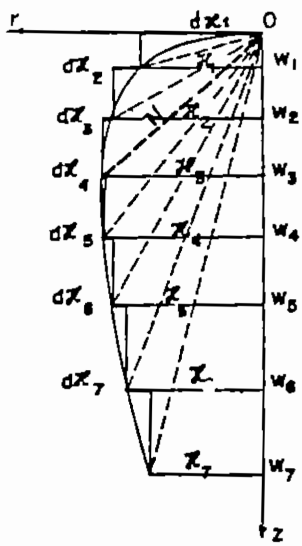


Fig XII-15



Assume W_1 , W_2 , W_3 etc are the loads on the zones 0-1 0-2, 0-3 etc and that they consist of the dead weight g plus the live load p If the surface area of any strip is d A so that

e g $d\,A_4 = ds_4 \quad 2\pi \quad \frac{r_3 + r_4}{2}$ then $d\,W_3 = d\,A_3\,(g + p)$ and

$W_1 = d\,W_1$, $W_2 = W_1 + d\,W_2$, $W_3 = W_2 + d\,W_3$ etc

If we draw through 0 parallels to the tangents of the meridian curve at 1 2, 3 etc , the curve passing through the points of intersection of these parallels with the horizontals through W gives the values of the total horizontal thrust $\mathcal{H}$, where $\mathcal{H} = 2\pi rH$ in the different sections of the shell

This curve can be directly used for determining the meridian and ring forces in the following manner

The meridian forces can be determined according to equation a from the relation

$$N_\varphi = \frac{W}{2\pi r \sin\varphi} = \frac{N}{2\pi r}$$

and the ring forces, according to equation b , from the relation

$$N_\theta = \frac{d\,(H r)}{ds} = \frac{1}{2\pi} \frac{\Delta \mathcal{H}}{\Delta s}$$

It has to be noted that N_φ is always compression while N_θ is compression so long as $\mathcal{H}$ increases (i e $\Delta \mathcal{H}$ is outside the curve) and tension when $\mathcal{H}$ decreases (i e $\Delta \mathcal{H}$ is inside the curve)

In domes with vertical tangent at their foot the horizontal thrust there is equal to zero and vertical reactions only are created at the supports so that no tension ring is required

2 - Application to Popular Reinforced Concrete Surfaces of Revolution

a) Spherical Shells

The relation between a, r and y is given by (Fig XII-16)

$$a = \frac{r^2 + y^2}{2y}$$

The surface area of a spherical shell is

$$A = 2\pi a y$$

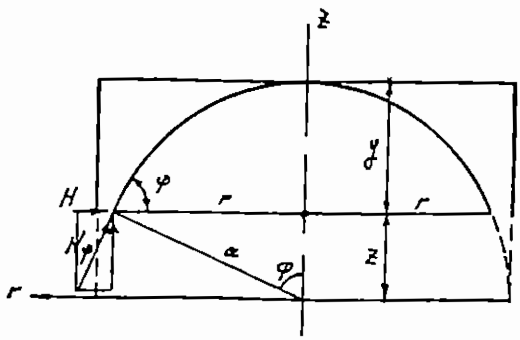


Fig XII-16

i e it is equal to the surface area of a cylinder having the same radius a and height y

<u>Internal forces and Reactions due to Dead Load g/m^2 Surface</u>

The dead weight of a shell height y, and included in a central angle φ is given by

$$W_\varphi = g A = g \; 2\pi \; a \; y$$

but

$$r = a \sin\varphi \qquad \text{and} \qquad y = a (1 - \cos\varphi)$$

then

$$W_\varphi = g \; 2\pi \; a^2 (1 - \cos\varphi)$$

The horizontal thrust H is given by

$$H = \frac{W_\varphi}{2 \pi r \tan\varphi} = g a \frac{\cos\varphi}{1 + \cos\varphi} = g a \frac{z}{a + z}$$

The meridian force N_φ can be calculated from the relation ,

$$N_\varphi = \frac{H}{\cos\varphi} = g \frac{a}{1 + \cos\varphi} = g \frac{a^2}{a + z}$$

at crown $\varphi = 0$, $\cos\varphi = 1$ and $z = a$, then

$$N_\varphi = H = g \frac{a}{2} \qquad \text{compression}$$

at the foot of half spherical shells, where

$\varphi = 90$, $\cos\varphi = 0$ and $z = 0$, we get

$N_\varphi = g a$ (comp) and $H = 0$

The ring force N_θ is given by

$$N_\theta = \frac{d (H r)}{ds} = g a (\cos\varphi - \frac{1}{1 + \cos\varphi}) = g (z - \frac{a^2}{a + z})$$

at crown $\varphi = 0$, $\cos\varphi = 1$ and $z = a$, then

$$N_\theta = H = g \frac{a}{2} \qquad \text{compression}$$

at foot of half spherical shells we have

$N_\theta = - g a$ tension

$N_\theta = 0$ at $\varphi = 51° 49'$ and $z = 0 618a$

or at $y = 0 382 a$ and $r = 0 787a$

Introducing $\varphi = 51° 49'$ or $z = 0 618a$ in the equation of H we get the magnitude of the maximum horizontal thrust, thus

$$H_{max} = 0 382 \; g \; a$$

i e the maximum total horizontal thrust $\mathcal{H}_{max} = 2\pi \; r \; H_{max}$ is therefore given by

$$\mathcal{H}_{max} = 2\pi \times 0 787 \; a \times 0 382 \; g \; a = 0 3 (2\pi \; a^2 \; g)$$

i e the maximum total horizontal thrust of a spherical shell is equal to 0 3 the total dead weights on half the sphere

The meridian and ring forces of spherical shells subject to dead loads g/m^2 surface can accordingly be expressed in the form shown in figure XII-17)

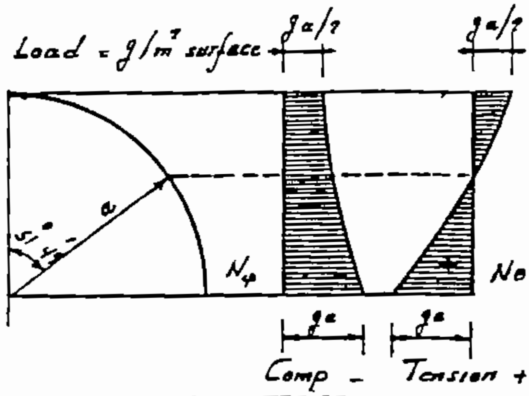


Fig XII-17

Internal Forces and Reactions due to Live Load p/m^2 Horizontal

Total load $W_\varphi = p \pi r^2 = p \pi a^2 \sin^2 \varphi$

The horizontal thrust $H = \dfrac{W_\varphi}{2 \pi r \tan\varphi} = pa \dfrac{\cos\varphi}{2} = p\ z/2$

The meridian force $N_\varphi = \dfrac{H}{\cos\varphi} = pa/2 = \text{constant}$

i e the meridian forces due to p are constant in the shell This striking result can also be proved in the following manner

The general equation a gives $W_\varphi = N_\varphi\ 2 \pi r \sin\varphi$

But

$W_\varphi = p \pi r^2$ and $r = a \sin\varphi$

So that

$$N_\varphi = \frac{p \pi r^2}{2 \pi r \sin\varphi} = p\ a/2 = \text{constant}$$

The ring force $N_\theta = \dfrac{d(H r)}{ds}$ or

$$N_\theta = \frac{p\ a \cos 2\varphi}{2} = \frac{p}{2a}(2 z^2 - a^2)$$

at crown $z = a$ and $N_\varphi = N_\theta = pa/2$ compression

at foot of half spherical shells = $z = 0$

and $N_\theta = -\ pa/2$ tension

The ring force $N_\theta = 0$ where $2 z^2 = a^2$ or $z = 0\ 707a$

This result corresponds to $\varphi = 45°$ i e $r = z = 0\ 707\ a$

Introducing this value in the equation of H, we get the magnitude of

H_{max} , thus

$$H_{max} = p\,z/2 = 0.3535\,p\,a$$

and the maximum total horizontal thrust $\mathcal{H}_{max} = 2\pi\, r\, H_{max}$ is therefore given by

$$\mathcal{H}_{max} = 2\pi \times 0.707\,a \times 0.3535\,p\,a$$

Or

$$\mathcal{H}_{max} = 0.5\,\pi\,a^2\,p$$

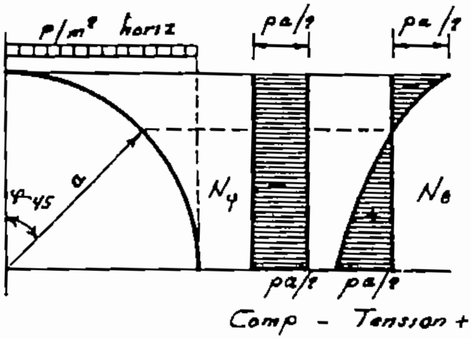


Fig. XII-18

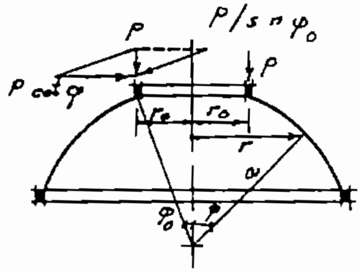


Fig. XII-19

which means that the maximum total horizontal thrust of a spherical shell is equal to half the total live loads on half the sphere

The meridian and ring forces can accordingly be illustrated as shown in figure XII-18

<u>Lantern Load</u>

Most domes are not closed at the vertex but have a skylight or a ventilation opening, covered by a superstructure the lantern Assume its weight is P t/m acting on the upper edge of the shell as a vertical line load Since the shell can resist only tangential forces this edge also needs a stiffening that resists the other compoment - P cotφ_o - and gets a compressive force from it (Fig XII-19) The internal forces are for this case given by

$N_\varphi = - W_\varphi / 2\pi\, r \sin\varphi$ in which

$W_\varphi = 2\pi\, r_o\, P = 2\pi\, a \sin\varphi_o\, P$ and $r = a \sin\varphi$

So that

$N_\varphi = - 2\pi\, a \sin\varphi_o\, P / 2\pi\, a \sin^2\varphi = - P \sin\varphi_o / \sin^2\varphi$ compression

N_θ can be directly calculated from equation 0' , thus

$N_\varphi + N_\theta = p_r\, a = 0$ or $N_\theta = - N_\varphi$ i e

$$N_\theta = + P \sin\varphi_o / \sin^2\varphi$$

b) Conical Shells

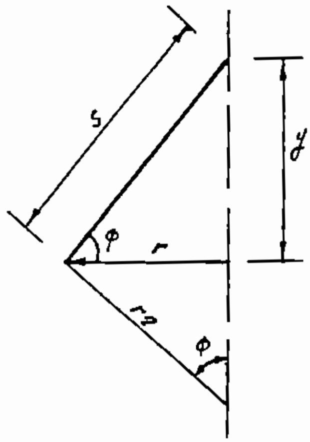


Fig XII-20

The surface area of a conical shell fig XII-20 is given by

$A = 2\pi r \; s/2$ but

$r = y \cot\varphi$ and $s = y / \sin\varphi$

then

$A = \pi y^2 \cos\varphi / \sin^2\varphi$

The meridian force N_s is given according to equation a by

$$N_s = \frac{W_\varphi}{2\pi \; r \sin\varphi} = \frac{W_\varphi}{2\pi y \cos\varphi}$$

The ring force N_θ is given according to equation c by :

$N_\theta = p_r \; r_2$ in which $r_2 = r \dfrac{s}{y} = y \dfrac{\cos\varphi}{\sin^2\varphi}$ or

$$N_\theta = p_r \, y \, \frac{\cos\varphi}{\sin^2\varphi}$$

Internal Forces due to Dead Load g/m² Surface

The dead weight of a cone height y , and included in a central angle φ is given by

$$W_\varphi = g\,A = g\pi y^2 \cos\varphi / \sin^2\varphi$$

The meridian force N_s is therefore given by

$$N_s = \frac{W_\varphi}{2\pi \; y \cos\varphi} = \frac{g\pi y^2 \cos\varphi / \sin^2\varphi}{2\pi \; y\cos\varphi} \quad \text{or}$$

$$N_s = \frac{g\,y}{2\sin^2\varphi} \quad \text{compression}$$

The ring force N_θ can be determined from the given general equation if we replace p_r by $g \cos\varphi$ hence

$$N_\theta = g\cos\varphi \; y \; \frac{\cos\varphi}{\sin^2\varphi} = g\,y\cot^2\varphi \quad \text{or}$$

$$N_\theta = g\,r^2/y \quad \text{always compression}$$

Internal Forces due to Live Load p/m² Horizontal

The live load p/m² horizontal corresponds to p cos φ per meter square surface, so that we can determine N_s and N_θ if we replace g by p cos φ in the previous equations, so that

$$N_s = \frac{p\,y}{2} \frac{\cos\varphi}{\sin^2\varphi} \qquad \text{and}$$

$$N_\theta = p\,y \frac{\cos^3\varphi}{\sin^2\varphi}$$

3 - Tables of Membrane Forces in Popular Shells of Revolution

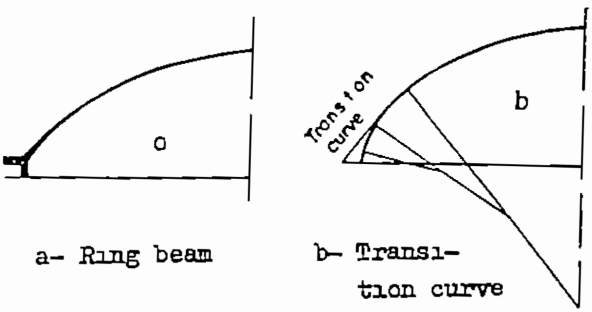
Alf Pfluger gives in his book Elementary Statics of Shells the membrane forces in some popular forms of shells under the effect of different cases of loading We give in the following tables a choice of these cases

4 - Edge Forces and Transition Curves

It has been shown that the upper zones of domes are subject to compressive ring forces while the lower zones are subject to tensile ring forces In case the dome, or cone does not end with a vertical tangent, the horizontal thrust H must be resisted by a tension ring (Fig XII-21a)

On the other hand, the meridian forces in domes and conical roofs due to vertical dead and live loads are always compressive giving relatively low stresses

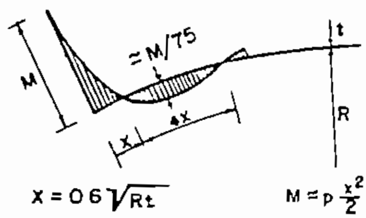
In conical shells and flat spherical domes bending moments will be developed due to the big difference between the high tensile stresses in the foot ring and the compressive stresses or low tensile stresses in the adjacent zones of the shell* The bigger the difference in the strains between the ring and the adjacent zone, the higher will be the bending moments The shape and magnitude of the bending moments at



a- Ring beam

b- Transition curve

Fig XII-21



Edge moments

Fig XII-22

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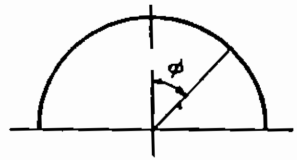
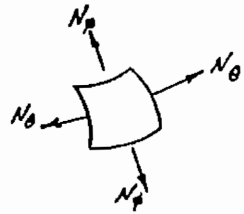
* Refer to text books on shells such as
Flugge 'Stresses in shells published by Springer - Verlag New-York
Ramaswamy Design and Construction of Concrete Shell Roofs Published by Mc Graw-Hill book company
Markus "Theorie und Berechnung Rotationssymmetrischer Bauwerke Published by Werner- Verlag- Dusseldorf

Spherical Shell

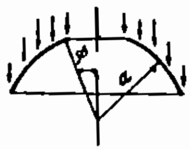
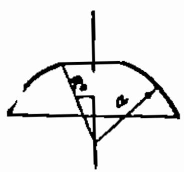
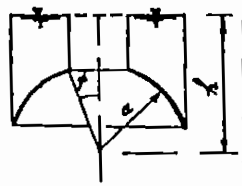


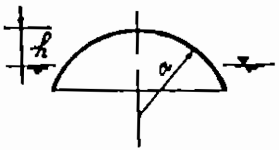
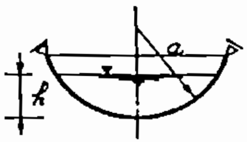
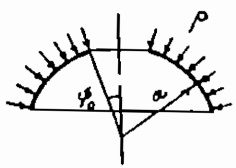


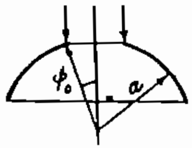
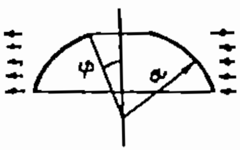
N_φ = Meridian force / m

N_θ = Ring force / m

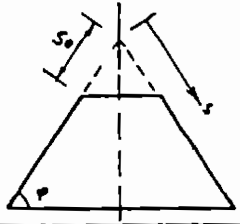
T = Unit central shear

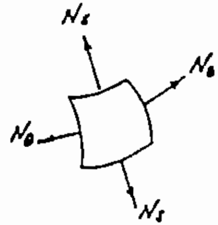
System	Loading	N_φ	N_θ
	Dead Load g/m^2 surface $p_r = g \cos\varphi$	$-g\,a\,\dfrac{\cos\varphi_o - \cos\varphi}{\sin^2\varphi}$ For $\varphi_o = 0$ (no vertex opening) $-g\,a\,\dfrac{1}{1+\cos\varphi}$	$g\,a\left(\dfrac{\cos\varphi_o - \cos\varphi}{\sin^2\varphi} - \cos\varphi\right)$ $g\,a\left(\dfrac{1}{1+\cos\varphi} - \cos\varphi\right)$
	Live load p/m^2 horizontal $p_r = p\cos^2\varphi$	$-p\,\dfrac{a}{2}\left(1 - \dfrac{\sin^2\varphi_o}{\sin^2\varphi}\right)$ For $\varphi_o = 0$ (no vertex opening) $-p\,\dfrac{a}{2}$	$p\,\dfrac{a}{2}\left(1 - \dfrac{\sin^2\varphi_o}{\sin^2\varphi} - 2\cos^2\varphi\right)$ $-p\,\dfrac{a}{2}\cos 2\varphi$
	Liquid pressure $p_r = \gamma\,(h - a\cos\varphi)$ γ = wt/m^3 liquid	$-\gamma a^2\left[\dfrac{h}{2a}\left(1 - \dfrac{\sin^2\varphi_o}{\sin^2\varphi}\right) - \dfrac{1}{3}\,\dfrac{\cos^3\varphi_o - \cos^3\varphi}{\sin^2\varphi}\right]$ For $\varphi_o = 0$ (no vertex opening) $-\gamma a^2\left[\dfrac{h}{2a} - \dfrac{1}{3}\left(1 + \dfrac{\cos^2\varphi}{1+\cos\varphi}\right)\right]$	$-\gamma a\left[\dfrac{h}{2}\left(1 + \dfrac{\sin^2\varphi_o}{\sin^2\varphi}\right) + \dfrac{a}{3}\left(\dfrac{\cos^3\varphi_o - \cos^3\varphi}{\sin^2\varphi} - 3\cos\varphi\right)\right]$ $-\gamma a^2\left[\dfrac{h}{2a} - \cos\varphi + \dfrac{1}{3}\left(1 + \dfrac{\cos^2\varphi}{1+\cos\varphi}\right)\right]$

System	Loading	N_φ	N_θ
		Points above the liquid level	
		0	0
		Points below the liquid level	
	Liquid pressure $p_r = \gamma(a - a\cos\varphi - h)$	$-\gamma \frac{a^2}{6}\left\{\frac{h}{a}\left[\frac{1}{\sin^2\varphi}\ \frac{h}{a}\left(3 - \frac{h}{a}\right) - 3\right] + 1 - \frac{2\cos^2\varphi}{1 + \cos\varphi}\right\}$	$-\gamma a^2\left(1 - \cos\varphi - \frac{h}{a}\right) - N_\varphi$
		Points above the liquid level	
		$\gamma \frac{h^2}{6}\left(3 - \frac{h}{a}\right)\frac{1}{\sin\varphi}$	$-\gamma \frac{h^2}{6}\left(3 - \frac{h}{a}\right)\frac{1}{\sin^2\varphi}$,
		Points below the liquid level	
	liquid pressure $p_r = -\gamma(a - a\cos\varphi - h)$	$\gamma \frac{a^2}{6}\left[3\frac{h}{a} - 1 + \frac{2\cos^2\varphi}{1 + \cos\varphi}\right]$	$\gamma a^2\left[\frac{h}{a} - 1 + \cos\varphi\right] - N_\varphi$
	Normal pressure $p_r = p$	$-p\frac{a}{2}\left(1 - \frac{\sin^2\varphi_0}{\sin^2\varphi}\right)$	$-p\frac{a}{2}\left(1 + \frac{\sin^2\varphi_0}{\sin^2\varphi}\right)$
		For $\varphi = 0$ (no vertex opening)	
		$-p\frac{a}{2}$	$-p\frac{a}{2}$

System	Loading	N_φ	N_θ
	Edge load p/m Vertex load P	$- p \frac{\sin \varphi_0}{\sin^2 \varphi}$	$p \frac{\sin \varphi_0}{\sin^2 \varphi}$
		For $\varphi_0 = 0$ (no vertex opening, single load P a the vertex)	
		$- \frac{P}{2 \pi a \sin^2 \varphi}$	$\frac{P}{2 \pi a \sin^2 \varphi}$
	Wind load w $p_r = w \sin \varphi \cos\theta$	$- w \frac{a}{3} \frac{\cos \theta \cos \varphi}{\sin^3 \varphi} \times$ $\times \left[3 (\cos \varphi_0 - \cos \varphi) - - (\cos^3 \varphi_0 - \cos^3 \varphi) \right]$	$w \frac{a}{3} \frac{\cos \theta}{\sin^3 \varphi} \times$ $\times \left[\cos \varphi (3 \cos \varphi_0 - \cos^3 \varphi_0) - - 3 \sin^2 \varphi - 2 \cos^4 \varphi \right]$
		The unit central shear $T = N_{\varphi\theta}$ is given by :	
		$T = w \frac{a}{3} \frac{\sin \theta}{\sin^3 \varphi} \times \left[3 (\cos \varphi_0 - \cos \varphi) - \cos^3 \varphi_0 - \cos^3 \varphi \right]$	
		For $\varphi_0 = 0$ (no vertex opening)	
		$- w \frac{a}{3} \frac{\cos \theta \cos \varphi}{\sin^3 \varphi} \times$ $\times (2 - 3 \cos \varphi + \cos^3 \varphi)$	$w \frac{a}{3} \frac{\cos \varphi}{\sin^3 \varphi} (2 \cos \varphi -$ $- 3 \sin^2 \varphi - 2 \cos^4 \varphi)$
		$T = w \frac{a}{3} \frac{\sin \theta}{\sin^3 \varphi} \times (2 - 3 \cos \varphi + \cos^3 \varphi)$	

Conical Shell

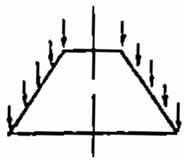
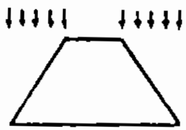
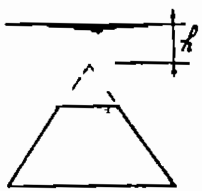


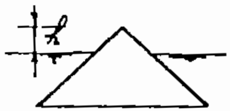
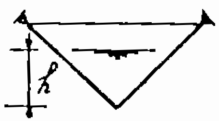
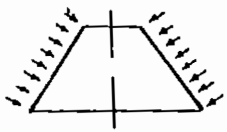
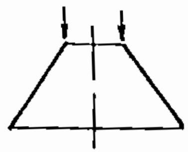
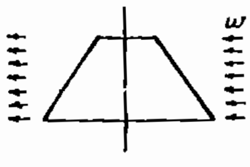


N_s = Meridian force / m

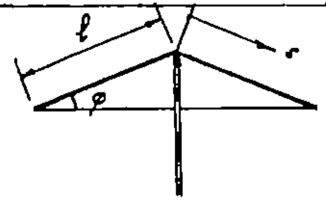
N_θ = Ring force / m

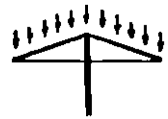
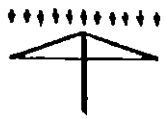
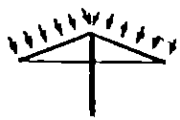
T = Unit central shear

System	Loading	N_s	N_θ
	Dead load g / m^2 surface $p_r = g \cos\varphi$	$- g \dfrac{s^2 - s_o^2}{2 s} \dfrac{1}{\sin\varphi}$ For $s_o = 0$ $- g \dfrac{s}{2 \sin\varphi}$	$- g s \dfrac{\cos^2\varphi}{\sin\varphi}$ (complete cone) $- g s \dfrac{\cos^2\varphi}{\sin\varphi}$
	Live load p / m^2 horizontal $p_r = p \cos^2\varphi$	$- p \dfrac{s^2 - s_o^2}{2 s} \cot\varphi$ For $s_o = 0$ $- p \dfrac{s}{2} \cot\varphi$	$- p s \dfrac{\cos^3\varphi}{\sin\varphi}$ (complete cone) $- p s \dfrac{\cos^3\varphi}{\sin\varphi}$
	Liquid pressure $p_r = \gamma (h + s \sin\varphi)$ γ = wt/m^3 liquid	$- \dfrac{\gamma}{s} \left(h \dfrac{s^2 - s_o^2}{2} \cot\varphi + \dfrac{s^3 - s_o^3}{3} \cos\varphi\right)$ For $s_o = 0$ $- \gamma s \left(\dfrac{h}{2} \cot\varphi + \dfrac{s}{3} \cos\varphi \right)$	$- \gamma s (h \cot\varphi + s \cos\varphi)$ (complete cone) $- \gamma s (h \cot\varphi + s \cos\varphi)$

System	Loading	N_s	N_θ	T
	Liquid pressure $p_r = \gamma\,(s \sin\varphi - h)$	Points above the liquid level 0 Points below the liquid level $-\frac{\gamma}{6\,s}\left[\frac{\cos\varphi}{\sin^3\varphi}\,h^3 + s^2\,(2s\cos\varphi - 3h\cot\varphi)\right]$	 0 $-\gamma\,s\,(s\cos\varphi - h\cot\varphi)$	
	Liquid pressure $p_r = \gamma\,(h - s\sin\varphi)$	Points above the liquid level $\gamma\,\frac{h^3}{6\,s}\,\frac{\cos\varphi}{\sin^3\varphi}$ Points below the liquid level $\gamma\,\frac{s}{2}\,(3\,h\cot\varphi - 2\,s\cos\varphi)$	 0 $\gamma\,s\,(h\cot\varphi - s\cos\varphi)$	
	Normal pressure $p_r = p$	$-p\,\frac{s^2 - s_o^2}{2\,s}\,\cot\varphi$ For $s_o = 0$ (complete cone) $-p\,\frac{s}{2}\,\cot\varphi$	$-p\,s\cot\varphi$ $-p\,s\cot\varphi$	
	Edge load p/m Axial load P	$-p\,\frac{s_o}{s}\,\frac{1}{\sin\varphi}$ For $s_o - 0$ (and axial single load) $-P\,\frac{1}{2\,\pi\,s\sin\varphi\cos\varphi}$	— —	
	Wind load $w\,/\,m^2$ $p_r = w\sin\varphi\cos\theta$	$-w\,\frac{s}{2}\left[\cos\varphi - \frac{1}{3\cos\varphi} - \frac{s_o^2}{s^2}\,x\right.$ $\left.x\left(\cos\varphi - \frac{1}{\cos\varphi}\right) - \left(\frac{s_o}{s}\right)^3\frac{2}{3\cos\varphi}\right]\cos\theta$ For $s_o = 0$ (complete cone) $-w\,\frac{s}{2}\left(\cos\varphi - \frac{1}{3\cos\varphi}\right)\cos\theta$	$-ws\cos\varphi\cos\theta$ $-w\,s\cos\varphi\cos\theta$	$-w\,\frac{s^3 - s_o^3}{3s^2}\,\sin\theta$ $-w\,\frac{s}{3}\,\sin\theta$

<u>Conical Shell</u>
Supported at Vertex
and with Free Edge



System	Loading	N_s	N_θ	T
	Dead load g / m² surface $p_r = g \cos\varphi$	$g \frac{\ell^2 - s^2}{2\,s} \frac{1}{\sin\varphi}$	$- g\, s \frac{\cos^2\varphi}{\sin\varphi}$	0
	Live load p / m² horizontal $p_r = p \cos^2\varphi$	$p \frac{\ell^2 - s^2}{2\,s} \cot\varphi$	$- p\, s \frac{\cos^3\varphi}{\sin\varphi}$	0
	Normal load $p_r = p$	"	$p\, s \cot\varphi$	0
	Wind load w / m² $p_r = w \sin\varphi \cos\theta$	$w\left[\frac{\ell^3 - s^3}{3\,s^2} - \frac{\ell^2 - s^2}{2\,s}\sin^2\varphi\right]$ $x \frac{\cos\theta}{\cos\varphi}$	$- w\, s \cos\varphi \cos\theta$	$w \frac{\ell^3 - s^3}{3\,s^2} \sin\theta$

the foot-ring can be estimated according to the values given in Fig XII-22

As the bending moments are due to the sudden change from high tensile stresses in the foot-ring to low tensile stresses or even compressive stresses in the shell, they can be avoided if the shape of the meridian is changed in a convenient manner This change can be done by a transition curve (Fig XII-21b) which, when well chosen gives a relief to the stresses at the foot-ring It is recommended to make the change of the stresses gradual from foot-ring to shell

In order to decrease the stresses due to the forces at the foot ring, it is recommended to increase the thickness of the shell in the region of the transition curve

5 - Examples

Figure XII-23 shows a project of a covered circular gymnasium, 50 ms diameter The upper part of the roof is covered by a flat dome 30 ms diameter 3 4 ms high and 10 cms thick It is provided with a central lantern 1 5 ms diameter and supported at its lower edge on 96 posts 1 0 m center to center The dome is bounded by two circular rings, one compression ring at the upper edge below the lantern, and the other tension ring at the lower edge over the posts The upper dome is supported by a truncated cone, 4 5 ms high, 10 cms thick and having a diameter of 30 ms at its upper edge and of 50 ms at its lower edge This cone is again provided with two ring beams one at the upper edge to resist the compression forces created from the loads of the upper dome, and one at the lower edge to resist the tensile forces created from the horizontal thrust of the cone The lower ring beam is again here supported on 72 vertical posts 2 0 ms center to center

The three main parts of the roof, the lantern, the upper dome , and the lower cone, are supported on the posts shown in order to give the required architectural effect, to support the windows necessary for lighting and to allow for the free lateral movement of their lower edge

The meridian and ring forces in both upper dome and lower cone are compressive and relatively low so that a thickness of 10 cms is ample The thickness of the shell roof is gradually increased to 15 cms over a length of 1 20 ms in order to resist the edge moments due to the local shear

The shell is reinforced by one mesh 5 Φ 8 mm in each direction except at the edges where two meshes are essential

The reinforcements required in the foot-rings of the upper dome and the lower cone are high because both dome and cone are relatively flat

Figure XII-24a gives an alternative solution for the same project in which the upper truncated cone of the roof with its lantern are supported on the free edge of the upper cantilever of the main stands The details of the cone and the main frames of the stand are shown in Fig XII-24b

6 - Circular beams

Circular beams loaded and supported normal to their plane (fig XII-25) are dealt with in detail in textbooks on theory of elasticity* We give in the following the fundamental equations required to determine the internal forces in a circular beam over n supports and subjected to a uniform load p/m

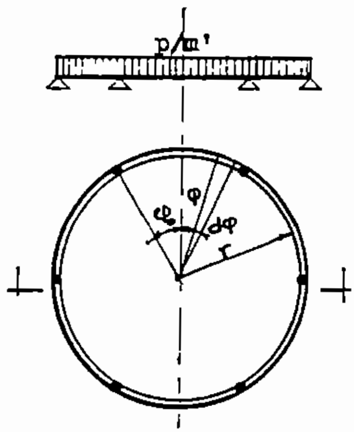


Fig XII-25

The equations of equilibrium governing the relation between the load p, the shearing force Q, the bending moment M and the torsional moment M_t acting on an element ds enclosing a central angle $d\varphi$ of a circular beam having a radius r are as follows

---

* Refer for example to
Kurt Beyer "Statik in Stahlbetonbau", published by Springer-Verlag, Berlin

$$\frac{dQ}{ds} = -p \qquad \text{but} \quad ds = r\,d\varphi \qquad \text{so that}$$

$$\frac{dQ}{d\varphi} = -p\,r \qquad (a) \qquad \text{further}$$

$$\frac{dM}{ds} + \frac{dM_t}{dr} - Q = o \quad \text{as} \quad r = \text{const then} \quad \frac{dM_t}{dr} = \frac{M_t}{r} \quad \&$$

$$\frac{dM}{ds} + \frac{M_t}{r} - Q = o \quad \text{substituting} \quad ds = r\,d\varphi \qquad \text{we get}$$

$$\frac{dM}{d\varphi} + M_t = Q\,r \qquad (b)$$

The component of the moment M along ds in the radial direction must be equal to the difference of the torsional moments along the same element, thus

$$dM_t = M\,d\varphi = \frac{M\,ds}{r} \qquad \text{or}$$

$$\frac{dM_t}{d\varphi} = M \qquad (c)$$

which means that the torsional moment is maximum at point of zero bending moments

Differentiating equation (b) with respect to φ, we get

$$\frac{d^2 M}{d\varphi^2} + \frac{dM_t}{d\varphi} = \frac{dQ\,r}{d\varphi} = r^2\frac{dQ}{ds} \qquad \text{or}$$

$$\frac{d^2 M}{d\varphi^2} + M = -p\,r^2 \qquad (d)$$

The solution of this differential equation is biven by

$$M = A\sin\varphi + B\cos\varphi - p\,r^2 \qquad (e)$$

The integration constants A and B can be determined from the conditions at the supports

The torsional moment M_t can be determined according to equation (c) from the relation

$$M_t = \int M\,d\varphi \qquad (f)$$

The internal forces are in this manner statically indeterminate In many cases, they can be determined from the conditions of equilibrium alone bacause the statically indeterminate values are either known or equal to zero

We give in the following the internal forces for some cases of circular beams

a) <u>Circular Beam Subject to Uniform Load p and Supported Symmetrically on n Supports</u>

The solution of equation (e) is given in the following relations

$$2\varphi_o = \frac{2\pi}{n} \qquad \text{and} \qquad \varphi_o = \frac{\pi}{n}$$

Reaction $\qquad R = \frac{2\pi r p}{n}$

Max shearing force to the right or left of any support

$$Q_{max} = \pm \frac{\pi r p}{n}$$

The bending moment M, the torsional moment M_t and the shearing force Q in any section at an angle φ from the center line between two successive supports, are given by

$$M = r^2 p \left(\frac{\pi \cos\varphi}{n \sin\varphi_o} - 1\right)$$

$$M_t = -r^2 p \left(\frac{\pi \sin\varphi}{n \sin\varphi_o} - \varphi\right)$$

$$Q = -r p \varphi$$

We give in the following table, the reactions, the maximum shearing forces, bending moments and torsional moments in a circular beam of radius r supported symmetrically on n supports and subject to a total, uniformly distributed, load P

$$P = 2\pi r p$$

Number of Cols	Load on each Column	Max Shearing Force	Max Bending Moment		Max Torsional Moment	Angle bet axis of Col & Sec of max M_t
			at C L of span	over C L of columns		
n	V	Q_{max}	M (+)	M (-)	M_t	Degree
4	P/4	P/8	0176 P r	0053 P r	0053 P r	19 21'
6	P/6	P/12	0075 P r	0148 P r	0015 P r	12 44'
8	P/8	P/16	0042 P r	0083 P r	0006 P r	9 33'
12	P/12	P/24	0019 P r	0037 P r	0002 P r	6 21'

b) <u>Cantilever Circular Beam Symmetrically Loaded</u> Fig XII-2

Due to symmetry, the shearing force Q and the torsional moment M_t

at c are equal to zero thus

$$Q_c = o \qquad \text{and} \qquad M_{tc} = o$$

For a single concentrated load

P acting at c

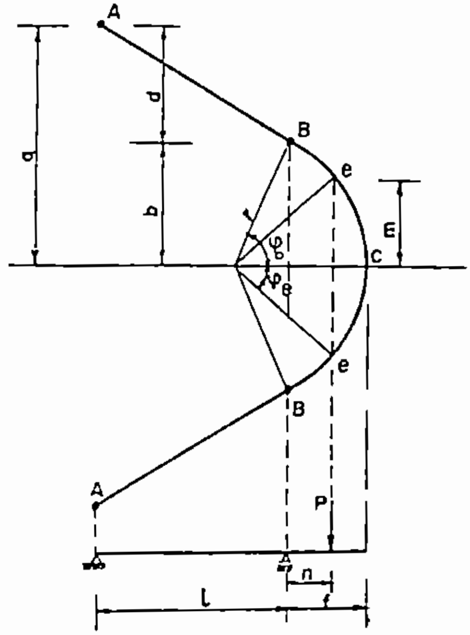


Fig XII-26

Reactions

$$R_A = - \frac{P\,f}{2\,l} \text{ and } R_B = \frac{P}{2}\,(1 + \frac{f}{l})$$

Bending moment at c

$$M_c = \frac{P\,b}{2}\,(1 - \frac{d\,f}{b\,l})$$

For two equal concentrated loads

P acting at e and e'

$$R_A = - \frac{P\,n}{l} \text{ and } R_B = P\,(1 + \frac{n}{l})$$

$$M_c = P\,b\,(1 - \frac{m}{b} - \frac{n\,d}{b\,l})$$

For a uniform load p/m on circular part B B'

$$R_A = - p\,r\,\varphi_o\,\frac{s}{l} \text{ and } R_B = p\,r\,\varphi_o\,(1 + \frac{s}{l})$$

$$M_c = p\,b\,r\left[\varphi_o\,(1 - \frac{s\,d}{l\,b}) - \frac{f}{b}\right]$$

in which s = distance of center of gravity of arch from B B'

and $s = (\sin\varphi_o - \varphi_o\cos\varphi_o)\,\frac{r}{\varphi_o}$

c) Totally Fixed Cantilever Circular Beam Symmetrically Loaded

Fig XII-27

Due to symmetry $Q_c = o$ and $M_{tc} = o$

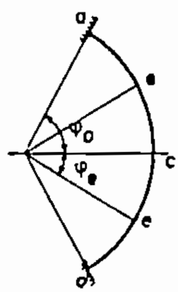


Fig XII-27

This beam is once statically indeterminate, the statically indeterminate value M_c can be determined as follows

$$M_c = - \frac{a_o}{a_1} \text{ in which}$$

$$a_o = 2\,r\int_o^{\varphi_o}(M_o\cos\varphi - M_{to}\,\chi\,\sin\varphi)\,d\varphi$$

and

$$\alpha_1 = 2\, r \int_0^{\varphi_0} (\cos^2\varphi + \varkappa \sin^2\varphi)\, d\varphi =$$

$$\frac{r}{2}\left[2\,(\varkappa + 1)\,\varphi_0 - (\varkappa - 1)\sin 2\varphi_0\right]$$

for half a circle $\varphi_0 = \frac{\pi}{2}$ and $\alpha_1 = \frac{r\pi}{2}(\varkappa + 1)$ in which

$\varkappa = \frac{E}{G}\ \frac{I_y}{I'}$ and

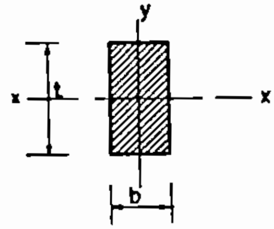


Fig XII-28

E = modulus of elasticity

G = modulus of rigidity $= \frac{E}{2\,(1 + \frac{1}{m})}$

$\frac{1}{m}$ = poisson ratio $= \frac{1}{6} \div \frac{1}{5}$ i e

$\frac{E}{G} =$ 2 4

$I_y =$ moment of inertia about y - y $= \frac{t\, b^3}{12}$ (fig XII-28)

$I' =$ torsion modulus

For a rectangular section with $\frac{t}{b} = n > 1$ $\qquad I' = n\,\psi\, b^4$

In the following table, the values of ψ are given as a factor of $n = \frac{t}{b}$

n	1	1 5	2	3	4	6	8	10	∞
ψ	0 140	0 196	0 229	0 263	0 281	0 298	0 307	0 312	0 333

Therefore $\varkappa = 2\ 4\ \frac{t\, b^3}{12}\ \frac{b}{t\,\psi\, b^4} = \frac{1}{5\,\psi}$ and can be extracted from the following table

n	1	1 5	2	3	4	6	8	10	∞
ϰ	1 425	1 020	0 875	0 760	0 711	0 670	0 651	0 640	0 600

<u>For a single concentrated load P at c we get</u> :

$$M_c = -P\, r\ \frac{2\,\frac{\varkappa}{\varkappa - 1}(\cos\varphi_0 - 1) + \sin^2\varphi_0}{2\,\frac{\varkappa + 1}{\varkappa - 1}\,\varphi_0 - \sin 2\varphi_0}$$

$$M_a = - \frac{P}{2} r \sin \varphi_o - M_c \cos \varphi_o$$

$$M_t = \frac{P}{2} r (1 - \cos \varphi_o) - M_c \sin \varphi_o$$

<u>For a uniform load p/m'</u>

$$M_c = + p r^2 \frac{\frac{\varkappa + 1}{\gamma - 1} (4 \sin \varphi_o - 2 \varphi_o) - \frac{4 \gamma}{\varkappa - 1} \varphi_o \cos \varphi_o + \sin 2 \varphi_o}{2 \frac{\varkappa + 1}{\varkappa - 1} \varphi_o - \sin 2 \varphi_o}$$

$$M_a = - p r^2 (1 - \cos \varphi_o) + M_c \cos \varphi_o$$

$$M_t = p r^2 (\varphi_o - \sin \varphi_o) - M_c \sin \varphi_o$$

If the beam is a fixed cantilever half circle, we get

<u>For a single concentrated load P at c</u>

$$M_c = \frac{P r}{\pi} \qquad M_a = - \frac{P r}{2} \qquad M_{ta} = \frac{P r}{\pi} (\frac{\pi}{2} - 1)$$

<u>For two eoual concentrated loads P at e and e'</u>

$$M_c = \frac{2 P r}{\pi} \left[\cos \varphi_e - (\frac{r}{2} - \varphi_e) \sin \varphi_e \right]$$

$$M_a = - P r \sin (\frac{\pi}{2} - \varphi_e)$$

$$M_{ta} = \frac{2 P r}{\pi} (\frac{\pi}{2} - \cos \varphi_e - \varphi_e \sin \varphi_e)$$

<u>For a uniform load p/m'</u>

$$M_c = p \frac{r^2}{\pi} (4 - \pi) = 0.274 \; p r^2$$

$$M_a = - p r^2$$

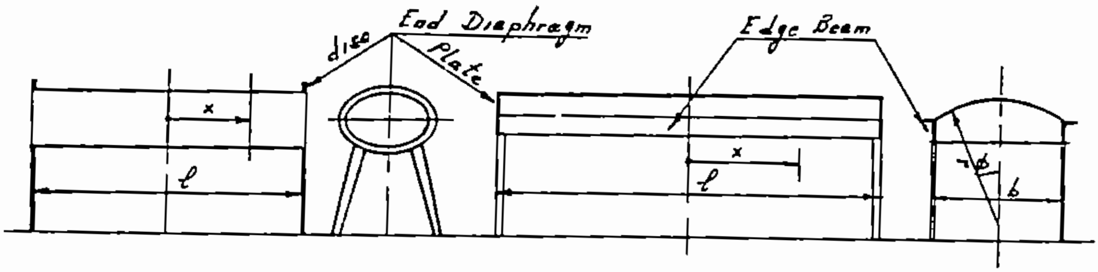
$$M_{ta} = p r^2 (\frac{r}{2} - \frac{4}{\pi}) = 0.3 \, p r^2$$

XII-4 CYLINDRICAL SHELLS

1- Introduction

A cylindrical shell is generated by moving a straight line generator' along a curve profile or basic curve while maintaining it parallel to its original direction

Cylindrical shells of horizontal axis are used in engineering practice either as tube-like closed structures (Fig XII-29a), as barrel-like open structures (Fig XII-29b) or of the saw-tooth form (Fig XII-30) They may be long if their span (distance between



a) Tube-like shell b) Barrel-vault shell

Fig XII-29

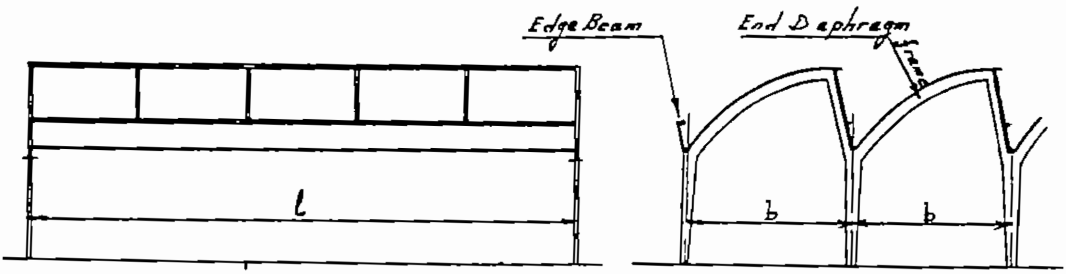


Fig XII-30 Saw-tooth shell

diaphragms) is big relative to its breadth b (distance between edge beams) , Fig XII-29 and 30, or short if their span ℓ is small relative to their breadth b (Fig XII-31)

In the plane of the supports, normal to the generators, such shells must be provided with end diaphragms in the form of discs, plates, arches, trusses, or frames to resist the central shear of the shell

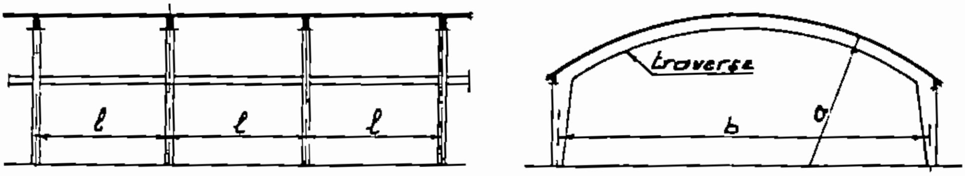


Fig XII-31 a short shell

2- The Membrane Theory

The membrane theory may be used for determining the internal forces in cylindrical shells In the analysis according to this theory , moments, torsion, and transverse shear forces are neglected while considering the equilibrium with normal forces and shear forces in the plane of the shell only

In cylindrical shells, generators and profiles suggest themselves as a natural net of coordinate lines We choose an arbitrary profile as the datum line and from this measure the coordinate x along the generators, positive in one direction and negative in the other The second coordinate must vary from generator to generator In analogy to the angle Φ used on surfaces of revolution, we introduce here the angle φ which a tangent to the profile makes with a horizontal plane (Fig XII-32)

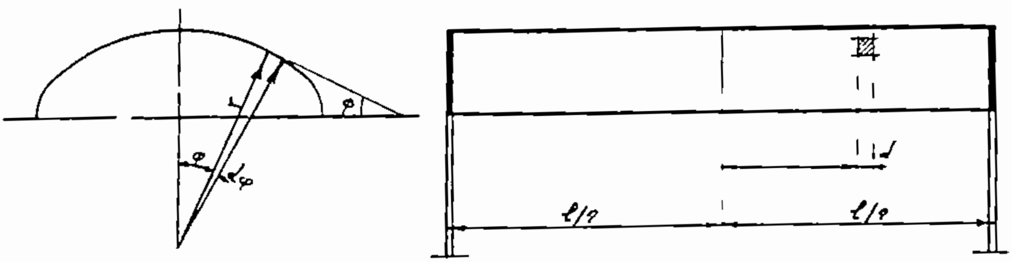


Fig XII-32

We consider now the conditions of equilibrium of a cylindrical shell For this purpose we cut from it an element bounded by two adjacent generators Φ and $\Phi + d\Phi$ and by two adjacent profiles x and x+dx (Fig XII-32) The membrane forces which act on the four edges must all lie in tangential planes to the middle surface and may be resolved into normal and shear components as shown

The internal forces per unit length of section are N_x, N_φ (normal forces) and $N_{x\varphi} = N_{\varphi x}$ (shearing forces). The load per unit area of shell element has the components p_x, p_φ in the directions of increasing x and φ respectively and a radial (normal) component p_r positive upward.

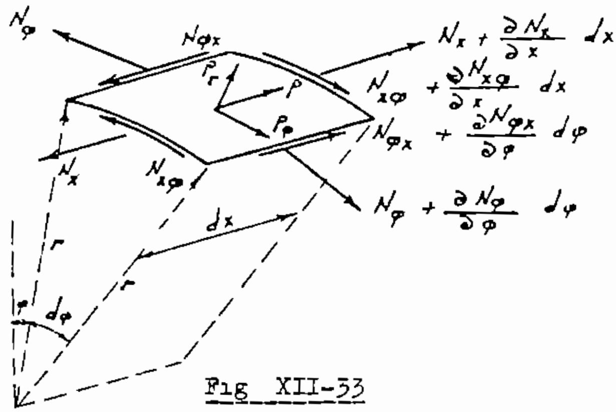


Fig XII-33

The stress resultants N_x, N_φ, $N_{x\varphi}$ can be determined from three conditions of equilibrium. These conditions may easily be read from figure XII-33. For the equilibrium normal to the middle surface we have, besides the external forces $p_r \; dx \; r\,d\varphi$, only the resultant of the two forces $N_\varphi\,dx$, pointing inwards. Thus

$$N_\varphi \; dx \; d\varphi - p_r \; dx \; r\,d\varphi = 0 \tag{1}$$

For the forces parallel to a tangent to the profile, we have

$$\frac{\partial N_\varphi}{\partial \varphi} d\varphi \; dx + \frac{\partial N_{x\varphi}}{\partial x} \; dx \; r\,d\varphi + p_\varphi \; dx \; r\,d\varphi = 0 \tag{2}$$

The equilibrium in the x-direction yields the equation

$$\frac{\partial N_x}{\partial x} \; dx \; r\,d\varphi + \frac{\partial N_{\varphi x}}{\partial \varphi} d\varphi \; dx + p_x \; dx \; r\,d\varphi = 0 \tag{3}$$

Dividing by the two differentials, we get the following differential equations for the membrane forces in the shell

$$N_\varphi = p_r \; r \tag{4}$$

$$\frac{\partial N_{x\varphi}}{\partial x} = - p_\varphi - \frac{1}{r} \; \frac{\partial N_\varphi}{\partial \varphi} \tag{5}$$

$$\frac{\partial N_x}{\partial x} = - p_x - \frac{1}{r} \; \frac{\partial N_{x\varphi}}{\partial \varphi} \tag{6}$$

In order to calculate N_φ, $N_{x\varphi} = N_{\varphi x}$, N_x, we proceed as follows

1) First, we determine N_φ according to equation 4 This hoop force depends only on the local intensity of the normal load p_r and cannot be influenced by the boundary conditions This is not of great importance for shells whose profiles are closed curves and which have two profiles as boundaries, e g , the shell shown in figure XII-29a But for shells as that shown in figure XII-29b, the impossibility of prescribing arbitrary values of N at the straight edges leads to the crucial point in the membrane theory of cylindrical shells

2) For the determination of the shearing forces $N_{x\varphi} = N_{\varphi x}$, we integrate equation 5 with respect to x and get an integration constant C_1

3) In order to determine the normal force N_x, we integrate equation 6 with respect to x and get a new integration function C_2

The integration functions C_1 and C_2 do not permit any boundary conditions so that the problem of cylindrical shells can only be solved under some special supporting conditions as shown in the following special cases

Special Cases

In the special case $p_x = 0$ the internal forces can be given in the form

1) $$N_\varphi = p_r r \qquad (4)$$

2) $$N_{x\varphi} = N_{\varphi x} = -\left(p_\varphi + \frac{1}{r} \frac{dN_\varphi}{d\varphi}\right) x + C_1(\varphi)$$

The value $p_\varphi + \frac{1}{r}\frac{dN_\varphi}{d\varphi}$ is a function of φ and if we call it $F(\varphi)$ we shall have

$$N_{x\varphi} = N_{\varphi x} = - x F(\varphi) + C_1(\varphi) \qquad (7)$$

3) $$N_x = \frac{x^2}{2r} \frac{dF(\varphi)}{d\varphi} - \frac{x}{r} \frac{dC_1(\varphi)}{d\varphi} + C_2(\varphi) \qquad (8)$$

In the following we are going to limit our discussion to shells stiffened at both ends by diaphragms which can resist forces acting in their plane only, and symmetrically loaded with respect to the middle cross-section In such cases, we have

at $x = 0$ $\quad N_{x\varphi} = 0 \quad$ and $\quad C_1(\varphi) \equiv 0$

at $x = \frac{l}{2}$ $\quad N_x = 0 \quad$ and $\quad C_2(\varphi) = \frac{-l^2}{8r} \frac{dF(\varphi)}{d\varphi}$

Introducing these values in the previous equations 4, 7 & 8 , we get finally $N_\varphi = p_r r$

$$N_{x\varphi} = N_{\varphi x} = - x F(\varphi)$$

$$N_x = - \frac{1}{8r}\left(l^2 - 4x^2 \right) \frac{dF(\varphi)}{d\varphi} \qquad (9)$$

It is clear from these formulae that the shearing force $N_{x\varphi}$ and the longitudinal normal force N_x vary in the direction of the generators according to the same pattern as in the case of a simple beam of span l loaded by a uniformly distributed load as shown in fig XII-34

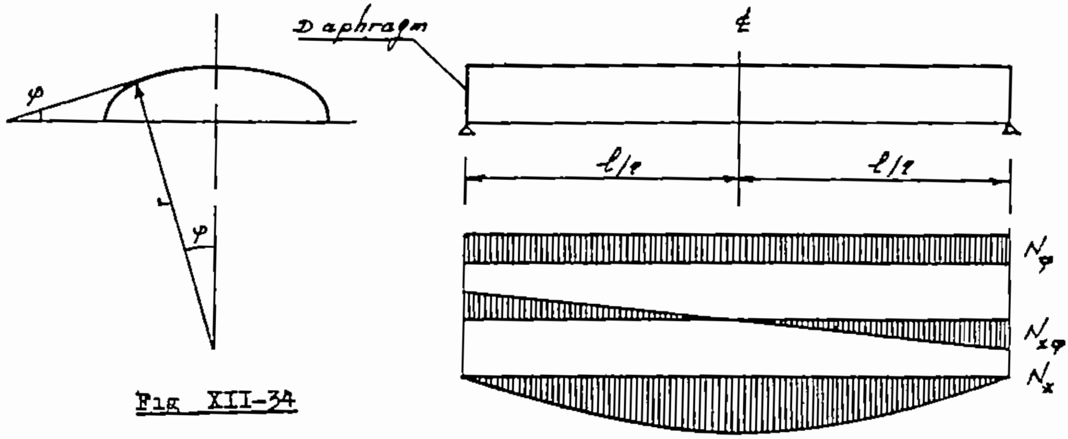


Fig XII-34

Of coarse the distribution of the forces $N_{x\varphi}$ and N_x over the cross section cannot be derived from the beam formulas but is governed by equation 1 for the equilibrium of the shell element

Another case of boundary conditions is represented by fig XII-35

Here one end, $x = l$, of the shell is completely built in, i e the support at this side can resist not only shearing forces $N_{x\varphi}$ but also normal forces N_x The other end, $x = 0$, may then be left without any support at all and we have the boundry conditions

$x = 0$ $N_{x\varphi} = 0$ and $N_x = 0$

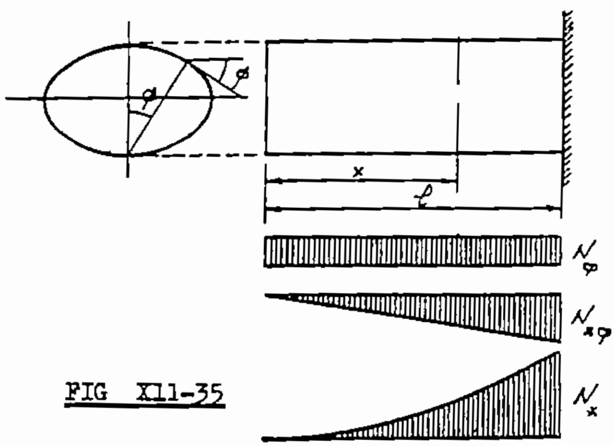


FIG XI1-35

Equations 7 and 8 show that in this case $C_1(\varphi)$ and $C_2(\varphi)$ must be zero and hence we have

$$N_{x\varphi} = - x F(\varphi) \qquad N_x = \frac{x^2}{2r} \frac{dF(\varphi)}{d\varphi} \tag{10}$$

This shell is supported like a cantilever beam, and, again the span wise distribution of $N_{x\varphi}$ and N_x are those of the shear and the bending moment of the beam analogue

The three-dimensional support of such a cantilever shell will scarcely be accomplished by a solid wall as shown in figure XII-35, but rather by an adjoining span of the same shell (fig XII-36)

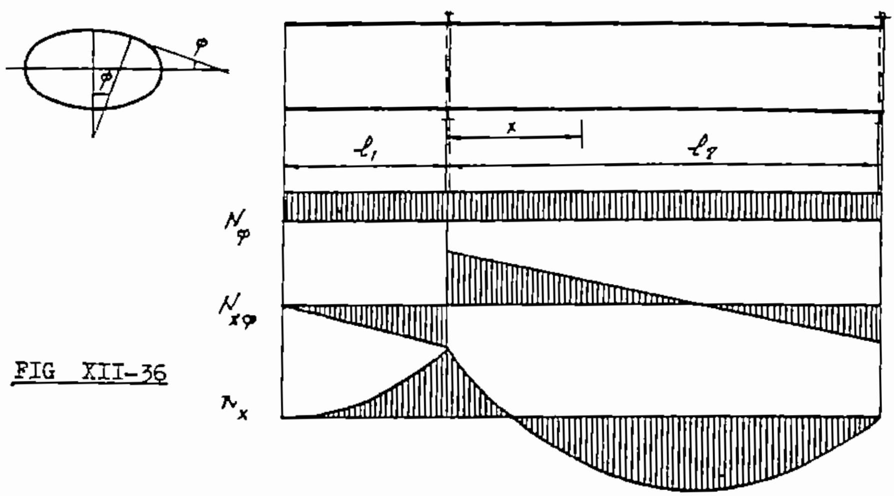


FIG XII-36

In such a structure we have again two diaphragms of the usual type which resist only shearing forces but do not accept forces N_x from the shell The forces N_x coming from the cantilever section must therefore be transmitted across the diaphragm to the adjoining bay of the shell which therefore has the boundary conditions

$$x = 0 \qquad N_x = \frac{l_1^2}{2r} \frac{dF(\varphi)}{d\varphi}$$

$$x = l_2 \qquad N_x = 0$$

Then we determine $C_1(\varphi)$ and $C_2(\varphi)$ from these conditions, we get

$$\left.\begin{aligned} N_{x\varphi} &= \left(\frac{l_1^2 + l_2^2}{2 l_2} - x \right) F(\varphi) \\ N_x &= \frac{1}{2r} \left(x^2 - x \frac{l_1^2 + l_2^2}{l_2} + l_1^2 \right) \frac{dF(\varphi)}{d\varphi} \end{aligned}\right\} \tag{11}$$

Again here $N_{x\varphi}$ and N_x have the same distribution as the shearing force and the bending moment of a beam with overhanging cantilever This coincidence will also be found if another cantilever shell is added at the other end of the main span but the analogy cannot be extended to statically indeterminate cases as for example that of a two span continuous cylindrical shell between three diaphragms Here the result will be influenced by the deformation of the shell which is different from that of a simple beam

In all the preceeding cases N_φ is found from equation 1 which is not affected by the choice of the boundary conditions

The Circular Cylindrical Shell

A) Tube-Like Shells

It will be assumed here that the radius of the middle surface is equal to a i e $r = a$ and that the circular edges of the shell are stiffened by diaphragms capable of resisting forces in their plane only The loads are symmetrical with respect to the middle axis

The internal forces can be calculated according to equations 9 from the relations

$$\left.\begin{aligned} N_\varphi &= p_r\, a \\ N_{x\varphi} = N_{\varphi x} &= - x\, F(\varphi) \\ N_x &= \frac{1}{8a}\left(l^2 - 4x^2\right) \frac{dF(\varphi)}{d\varphi} \end{aligned}\right\} \qquad (12)$$

Examples

1) Circular Cylindrical Shell Loaded by its own Weight

Assuming that the dead weight per square meter surface = g , then the surface load components are

$$p_\varphi = g \sin\varphi \qquad \text{and} \qquad p_r = - g \cos\varphi$$

The first of equations 12 is therefore given by

$$N_\varphi = - g\, a \cos\varphi \qquad (13a)$$

The function $F(\varphi)$ and its derivative are

$$F(\varphi) = p_\varphi + \frac{1}{r}\frac{dN_\varphi}{d\varphi} = g \sin\varphi + \frac{1}{a}\, g\, a \sin\varphi = 2\, g \sin\varphi \qquad \text{and}$$

$$\frac{dF(\varphi)}{d\varphi} = 2\, g \cos\varphi$$

accordingly we get

$$N_{x\varphi} = N_{\varphi x} = -\,2\, g\, x \sin\varphi \qquad (13b)$$

$$N_x = \frac{g}{4a}\left(l^2 - 4\, x^2 \right) \cos\varphi \qquad (13c)$$

The distribution of the stresses on the cross section is shown in figure XII-37. One has to notice that N_x is linear as is generally assumed in elementary statics. However, this statement cannot be generalized for other shells and for other cases of loading.

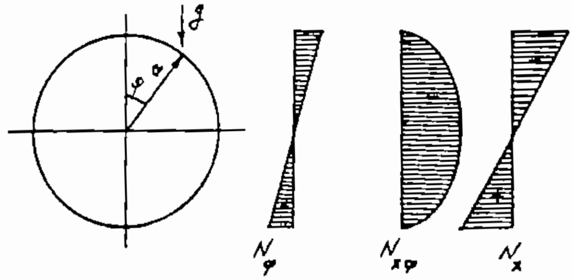


FIG XII-37

2) Circular Cylinder Filled with Liquid

For a pipe filled with liquid of specific weight γ the external forces (the water pressure) are

$$p_x = p_\varphi = 0 \qquad\qquad p_r = p_0 - \gamma\, a \cos\varphi$$

Here p_0 is the pressure at the level of the axis of the pipe and may be anything $\geqslant \gamma a$ but not less

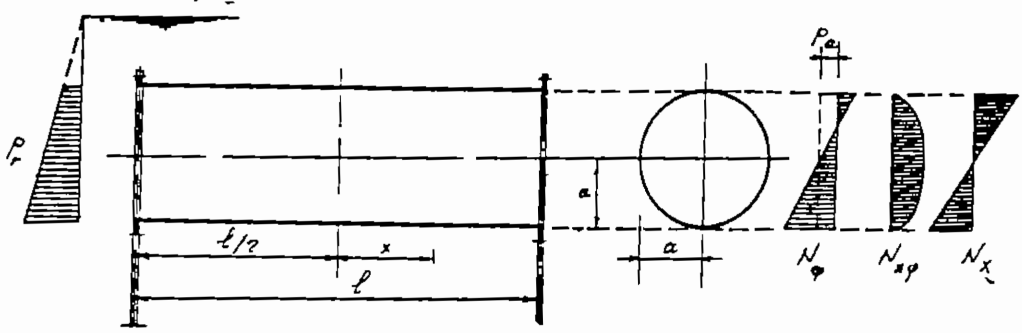


FIG XII-38

If we choose the boundary conditions of figure (XII-34) we find at once from equations 12 the stress resultants (Fig XII-38)

$$N_\varphi = p_0\, a - \gamma\, a^2 \cos\varphi$$

$$N_{x\varphi} = N_{\varphi x} = -\,\gamma\, a\, x \sin\varphi \qquad (14)$$

$$N_x = -\,\frac{1}{8}\gamma\left(l^2 - 4\, x^2 \right) \cos\varphi$$

The average pressure p_o produces only hoop stresses The load term with γ represents the weight of the liquid and produces a kind of overall bending of the pipe which acts as a beam carrying this weight between the supports We have already seen that therefore the shear $N_{x\varphi}$ and the normal force N_x have the same span wise distribution as the shear and the bending moment of a beam The distribution of N_x over the profile is incidentially the same linear distribution as that of bending stresses in common beam theory This result is a peculiarity of the circular cylinder and even there, is restricted to certain simple loads

B) Barrel Vaults

1) Semicircular Simple Barrel Vault Loaded by its own Weight

As an introduction to the theory of barrel vaults we are going to discuss the stress distribution of the internal forces, given by equations 13, in the circular cylindrical shell subjected to its own weight treated in example 1 and shown in figure XII-37

The most remarkable feature of this force system is that on the generators at midheight $\varphi = \pm \pi/2$ we have $N_\varphi = 0$ If we cut away the lower half of the shell, the upper half need not be supported at the straight edges and may carry its weight freely between the diaphragms, just as the tubular shells do Such barrel vaults have been used as roof structures

However the straight edges of a barrel vault are not completely free of external forces There is a shear $N_{x\varphi} = \pm 2\, g x$, and a structural element must be provided to which it can be transmitted This so called " edge beam is a straight member and if properly placed, it is stressed only in tension (fig XII-39) Its axial tensile force T is of coarse variable along the span It can easily be found by integrating the shear $N_{x\varphi_1}$, beginning at the end $x = \frac{l}{2}$ where $T = 0$ For the edge $\varphi = + \pi/2$ the integration gives

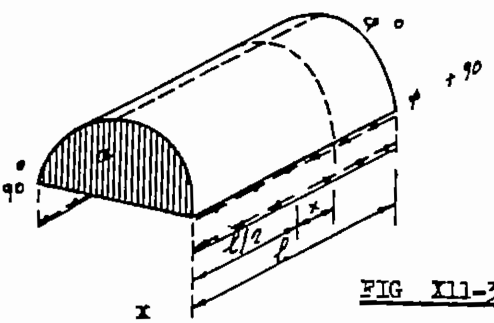


FIG XII-39

$$T = \int_{-l/2}^{x} N_{x\varphi}\, dx = -2\, g \int_{l/2}^{x} x \sin\varphi\, dx = -2\, g \int_{-l/2}^{x} x\, dx = \frac{1}{4}\, g\, (l^2 - 4 x^2)$$

and at the other end, we get a similar result

The statical necessity of this force may be understood from a look at the N_x diagram in fig XII-40 which shows the stress resultants of a barrel vault with semicircular profile

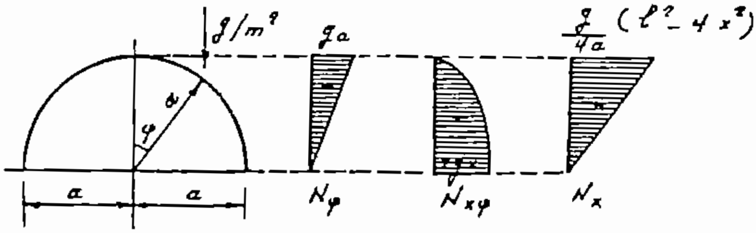


FIG XII-40

The section has only longitudinal compressive forces N_x and if we cut the shell apart in a plane x = constant the horizontal equilibrium of each half requires that tensile forces of the same amount also appear Now the integral of the forces N_x in the cross-section

$$\int_{-\pi/2}^{+\pi/2} N_x \, a \, d\varphi = -\frac{1}{4} g \, (l^2 - 4x^2) \int_{-\pi/2}^{+\pi/2} \cos\varphi \, d\varphi = -\frac{1}{2} g \, (l^2 - 4x^2)$$

This is exactly the same compressive force as the two tensile forces T in the edge beams so that they maintain the horizontal equilibrium

The resultant of the compressive forces lies somewhere in the semi-circular profile and higher than the tensile force 2T and both combine to form a couple equal to the external moment When we consider the barrel vault as a beam of span l Since the load of the beam per unit length is $\pi a g$, its bending moment is

$$M = \pi a g \, (l^2 - 4x^2)/8$$

The moment of the stress resultants N_x and T in the cross-section can be determined by taking moments about the lower horizontal diameter which is at the same time the line of action of the tension T Hence

$$\int_{-\pi/2}^{+\pi/2} N_x \, a \cos\varphi \, a \, d\varphi = \int_{-\pi/2}^{+\pi/2} \frac{g}{4a} (l^2 - 4x^2) \cos^2\varphi \, a^2 d\varphi$$
$$= \pi a g \, (l^2 - 4x^2)/8$$

which is exactly equal to M In the same way check that the vertical resultant of the shearing forces $N_{x\varphi}$ in the cross-section is equal to the transverse shearing force $-\pi a g x$ of the beam analogue

This comparison between the barrel vault and its beam analogue gives a good general idea of the stress system in the shell and yields a useful check for the computations. It cannot disclose details of the membrane stress distribution, since they depend essentially on the shape of the shell.

2) Semicircular Simple Barrel Vault Loaded by a Vertical Live Load

$p\ t/m^2$ horizontal

The load components are in this case given by

$$p_\varphi = p \sin\varphi \cos\varphi \qquad p_r = -p\cos^2\varphi$$

Substituting these values in equations 12, we get (fig XII-41)

$$N_\varphi = -p\,a\cos^2\varphi$$

$$N_{x\varphi} = N_{\varphi x} = -\frac{3}{2}\,p\,x\,\sin 2\varphi$$

$$N_x = -\frac{3}{8}\frac{p}{a}\,(l^2 - 4x^2)\cos 2\varphi$$

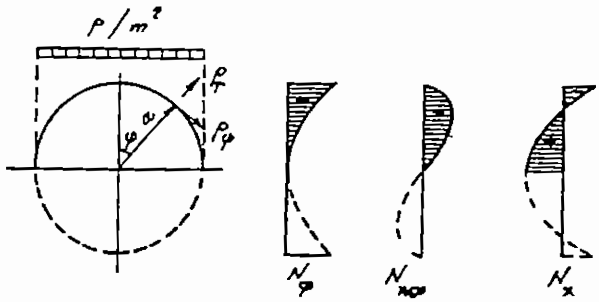


FIG XII-41

It is interesting to observe that at the shell edges where $\varphi = \pm\pi/2$, both N_φ and $N_{x\varphi} = N_{\varphi x}$ equal zero. Consequently, in this loading case, the shell need no support at its longitudinal edges.

In order to make a shell suitable for the construction of free-spanning barrel vaults, the force N_φ must be zero at the straight edges. This is always possible for shell profiles subject to vertical loads and terminating with vertical tangents. Accordingly, the semi-circular, elliptical, cycloidal and similar profiles are considered as convenient forms for freespanning barrel vaults.

As an example, we give in the following, the stress resultants in a cycloidal barrel shell loaded by dead and superimposed loads acting per square meter surface.

The Cycloidal Barrel Shell

The equation of the cycloid (fig XII-42) is given by the formulae

$y = a\,(2\varphi + \sin 2\varphi)$, $z = a(1+\cos 2\varphi)$

The radius of curvature is therefore

$$r = 4\,a \cos\varphi$$

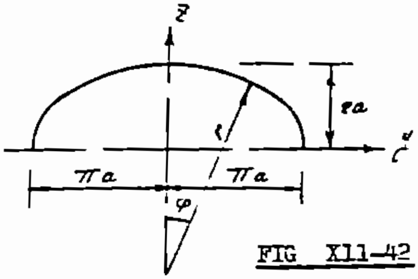


FIG XII-42

Thus

In case of own weight (fig XII-43)

$p_\varphi = g \sin\varphi \quad p_r = -\,g \cos\varphi$

$N_\varphi = -\,4\,g\,a \cos^2\varphi$

$N_{x\varphi} = N_{\varphi x} = -\,3\,g\,x \sin\varphi$

$N_x = -\,\frac{3\,g}{32\,a}\,(l^2 - 4\,x^2)$

= constant

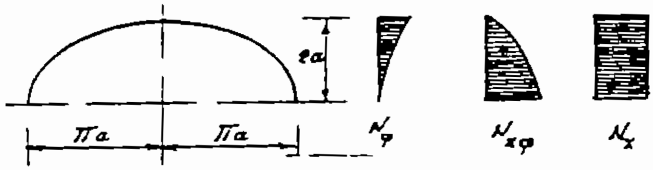


FIG XII-43

Here the force N_x is constant over the whole profile This may sometimes be desirable since it indicates that all material is efficiently used But in most cases we should rather like to have a tensile force N_x along the edge to avoid large differences between the length wise strain in the shell and in its edge member

The Wiedemann's Barrel Shell

It is however possible to improve the properties of the cycloidal shell by using the Wiedemann's profile which is given by the equation

$$r = a_o + a_1 \cos\varphi$$

This curve can be originated by rolling a circle on the outside of a cycloid as shown in figure XII-44 The Wiedemann's curve is the locus of the center point of the circle We find the ordinates y and z in the plane of a profile by simple integration of the projections of the line element $r\,d\varphi$ thus

$$y = \int_0^{\varphi} r \cos\varphi \; d\varphi = a_o \sin\varphi + \frac{1}{4}\,a_1\,(\,2\varphi + \sin 2\varphi\,)$$

$$z = -\int_{\pi/2}^{\varphi} r \sin\varphi \, d\varphi = a_o \cos\varphi + \frac{1}{4} a_1 (1 + \cos 2\varphi)$$

These relations show that the curve lie somewhere between the circle ($a_1 = o$) and a cycloid ($a_o = o$) But this does not mean that the stresses lie halfway between the two cases The stress resultants are accordingly

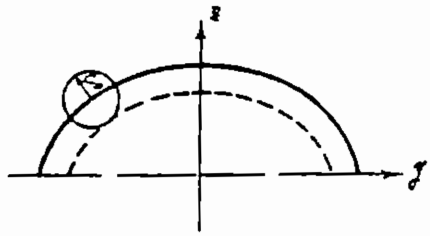


FIG XII-44

For the own weight g/m^2 surface

$$p_\varphi = g \sin\varphi \qquad p_r = - g \cos\varphi$$

$$N_\varphi = - g (a_o + a_1 \cos\varphi) \cos\varphi$$

$$N_{x\varphi} = N_{\varphi x} = - g \, x \sin\varphi \frac{2 a_o + 3 a_1 \cos\varphi}{a_o + a_1 \cos}$$

$$N_x = - \frac{g}{8} (l^2 - 4 x^2) \frac{(2 a_o + 3 a_1 \cos\varphi)(a_o + a_1 \cos\varphi)\cos\varphi - a_o a_1 \sin^2\varphi}{(a_o + a_1 \cos\varphi)^3}$$

at the lower edge where $\varphi = \pm \pi/2$

$$N_\varphi = o , \qquad N_{x\varphi} = N_{\varphi x} = - 2 g x \qquad \text{and} \qquad N_x = + \frac{g a_1}{8 a_o^2} (l^2 - 4 x^2)$$

For a live load p/m^2 horizontal

$$p_\varphi = p \sin\varphi \cos\varphi \qquad p_r = - p \cos^2\varphi$$

$$N_\varphi = - p (a_o + a_1 \cos\varphi) \cos^2\varphi$$

$$N_{x\varphi} = N_{\varphi x} = - p \, x \sin\varphi \cos\varphi \frac{3 a_o + 4 a_1 \cos\varphi}{a_o + a_1 \cos\varphi}$$

$$N_x = - \frac{p}{8}(l^2 - 4 x^2) \frac{(3 a_o^2 + 4 a_1^2 \cos^2\varphi) \cos 2\varphi - a_o a_1 \cos\varphi (8 - 15 \cos^2\varphi)}{(a_o + a_1 \cos\varphi)^3}$$

at the lower edge where $\varphi = \pm \pi/2$

$$N_\varphi = o \qquad N_{x\varphi} = N_{\varphi x} = o \qquad \text{and} \qquad N_x = + \frac{3 p}{8 a_o} (l^2 - 4 x^2)$$

This means that at the edges $\varphi = \pm \pi/2$, tensile N_x forces of reasonable values are developed both for dead weight and vertical live loads Since tensile forces are also acting on the upper edge of the straight edge beams, we have the possibility to eliminate or to decrease the difference of elongations by proper choice of the dimensions of the shell

3- <u>The Beam Theory</u>

We have seen in figures XII-34, 35 & 36, that the shearing force $N_{x\varphi}$ and the longitudinal force N_x vary in the direction of the generators according to the same pattern as in the case of a simple beam of span l loaded by a uniformly distributed load We will discuss now whether it is possible to consider the shell as a beam

It has been found that the analysis of a shell as a three dimensional problem in a strict sense , is very complicated but its structural behavior can be best understood by comparing its action to that of an ordinary bean, whose cross-section is a thin curved slab spaning between the transverse stiffeners* -the diaphragms-

The application of load on the shell induces longitudinal fiber stresses similar to those induced in any homogenious beam under load, and if the span l is large compared to the transverse chord width b, the variation of the fiber stresses across any section is the same as that created in a beam, that is, it is linear from top to bottom

As the ratio of b to l increases the variation of the longitudinal force per unit width, N_x , changes from straight line to curvilinear The extent of this change is shown in figure XII-45, in which the ordinate represents vertical distance measured from the bottom of a cylindrical shell in terms z'/f and the absessa indicates magnitud of N_x - forces For convenience, the curves are drawn for various values of a/l , the ratio of shell radius to span length , rather than the previosly discussed ratio of b to l Although the two ratios are independent, a/l is a more suitable parameter since it includes the effect of shell rise as well as chord width When a/l is less than 0 2, the variation is almost linear, while for larger values of a/l the force at the lower edge is greater than that given by ordinary beam theory

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* Refer to H Lundgren Cylindrical Shells " Published by the Danish Technical Press The institution of Danish Civil Engineers.

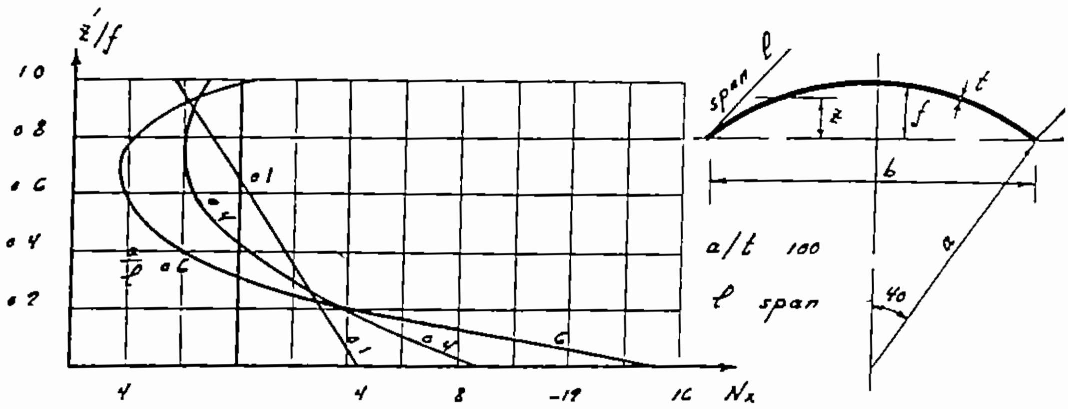


FIG X11-45

Although the magnitude of the tensile stresses at the lower edge of the shell is sometimes quite senstive to variations of a/l the total tensile area under these curves does not vary to the same degree For example the ratio of bottom fiber stresses for values of a/l of 0 6 and 0 1 is 3 2 while the ratio of tensile areas below the neutral axis for these cases is only 1 4 Since the reinforcement required depends essentially on the total tensile area rather than the maximum fiber stress, the design of a shell is not so sensitive a problem Moreover, the curves are prepared for a single shell unrestrained at the lower edges When these edges are restrained by adjoining shells or edge beams longitudinal stresses conform more closely to the straight line variation Studies indicate that the beam method is applicable to the following classes of shells provided they are uniformly loaded

1) Single shells without edge beams if a/l < 0 2
2) Long single shells with not too deep edge beams if a/l < 0 3
3) Interior shells of a group of multiple shells with feather edge beams if a/l < 0 6
4) Interior shells of a multiple group of shells with edge beams if a/l < 0 4

However, close agreement with the results of the analytical theory have been obtained for ratios of a/l ≈ 0 9

The basic assumptions used in the beam theory are

1) Plane sections before deformation remain plane after deformation
2) Form of cross section is not changed i e points A and B shown in figure XII-46 deflect the same amount

Accordingly

1) Distribution of N_x is similar to σ

2) Distribution of $N_{x\varphi}$ is similar to τ

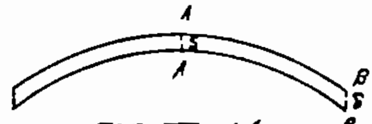


FIG XII-46

as shown in figure XII-47

The analysis of the normal and shearing forces in a long barrel shell is similar to that of a beam, span l, and having a curved cross section as shown in following simple example

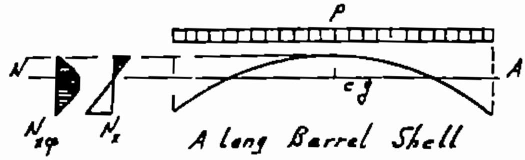


FIG XII-47

Analysis of Beam Action of Symmetrical Circular Cylindrical Shells

1 Properties of Shell section

a) Shell without edge beams Fig XII-48

The area of cross section of a symmetrical circular shell of radius a and subtending a central angle $2\,\varphi_o$ is given by

$$A = 2\,a\,\varphi_o\,t$$

The position of the c g of the section can be determined from the relation

$$z_o = a \sin\varphi_o / \varphi_o$$

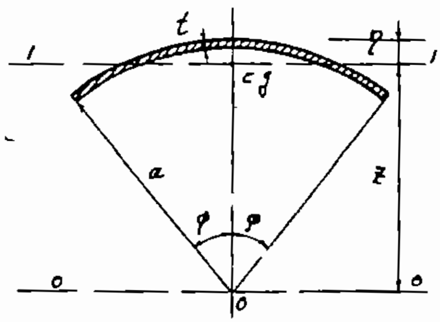


FIG. XII-48

So that

$$\eta = a\left(1 - \frac{\sin\varphi_o}{\varphi_o}\right)$$

The moment of inertia of the section about axis 0-0 passing through the center 0 is

$$I_{o-o} = t\,a^3\,(\varphi_o + 1/2 \sin 2\varphi_o)$$

Hence, the moment of inertia of the section about axis 1-1 passing through the c g is therefore

$$I_{1-1} = t\,a^3\left(\varphi_o + 1/2 \sin 2\varphi_o - \frac{2\sin^2\varphi_o}{\varphi_o}\right)$$

The statical moment of the section above or below the neutral axis(c g) axis can be calculated from the relation

$$S_{1-1} = \frac{4}{3}\ t\ \eta\ \sqrt{2\ \eta\ \ a}\ (\ 1 + \frac{\eta}{20\ a}\ )$$

b) <u>Shell with edge beams</u> (Fig XII-49)

The area of cross-section of the shell is given by

$$A = 2\ (\ a\ \varphi_o\ t + b_o\ d\ )$$

The statical moment of the section of the shell about axis o–o is

$$S_{o-o} = 2\ (t\ a^2\ \sin\ \varphi_o + b_o d\ \bar{z}_b)$$

where

$$\bar{z}_b = a\ \cos\ \varphi_o\ -\ \frac{d}{2}$$

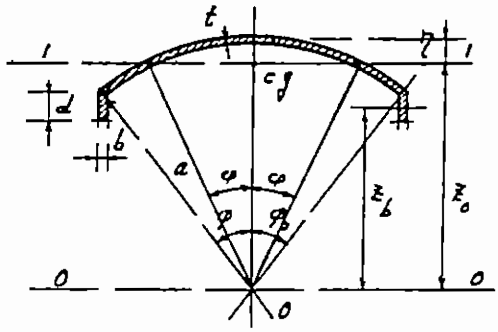


FIG XII-49

The position of the c g can be determined from the relation

$$z_o = S_{o-o}\ /\ A \qquad \text{i e} \qquad \eta = a - z_o$$

The angle φ is therefore given by $\cos\ \varphi = z_o\ /\ a$ and the statical moment S_{1-1} about axis passing through the c g will be the same as in previous case

The moments of inertia about axes o–o and 1–1 are given by

$$I_{o-o} = a^3\ t\ (\ \varphi_o + 1/2\ \sin\ 2\ \varphi_o\ ) + (\ \frac{b_o d^3}{6} + 2\ b_o\ d\ \ \bar{z}_b^2\ )$$

$$I_{1-1} = I_{o-o} - A\ z_o^2$$

2 <u>Internal Forces in the Shell</u>

<u>Load</u>

Total load per meter run shell is given by

$\bar{p}$ = $2\ a\ \varphi_o\ p$ for a circular shell without edge beams,

= $2\ (\ a\ \varphi_o\ p + p_b\ )$ " ' " with "

in which

p = total vertical load, including own weight, per square meter shell

p_b = load of edge beam per meter

The dead weight of the shell slab may practically always be considered uniformly distributed even in cases where light stiffening ribs are provided

Since the greater part of the shell has only light reinforcement the weight per cubic meter slab may be assumed = 2 40 t

On the other hand, for long shells, the edge beams are so heavily reinforced that in computing the weight due regard has to be given to the reinforcement

As regards the live loads, one may follow the following rules
1) For ordinary long shells where the exact distribution of load is less important an average live load of 65 kg/m^2 is assumed for the entire shell surface provided that the maximum slope does not differ much from 40

2) For other shells where the discrease of the load by increasing inclination is of importance a load of $\sim$ 100 $\sin^2\varphi$ kg/m^2 of shell surface may be assumed

3) For parallel shells the valleys between the individual shells are to be considered half filled with snow having a unit weight of 150 kg/m^3 or an equivalent weight of rain water

4) For the composite shell structure of the ordinary long type, the additional load from one valley, due to snow, rain water or gutter plain concrete may be assumed as a line load on the edge beam Thereby, the snow - or rain water - load on the edge beam will be

$25\, b \left(f + \frac{1\,5}{\sqrt{f}} - 2\,5 \right)$ kg/m' where b = width of shell and
f = rise in meters

When using this formula a uniformly distributed snow or live load of 65 kg/m^2 is to be applied to the entire shell surface

5) The wind load on a cylindrical roof with an inclination of less than 40 may be assumed as a uniformly distributed suction of $\sim$ 100 kg/m^2 It reduces the stresses due to dead and snow or live loads

<u>Stresses and Stress Resultants</u>

For a symmetrically loaded symetrical section as that shown in fig XII-50, a long shell may be considered as a beam subject to bending in the vertical plane

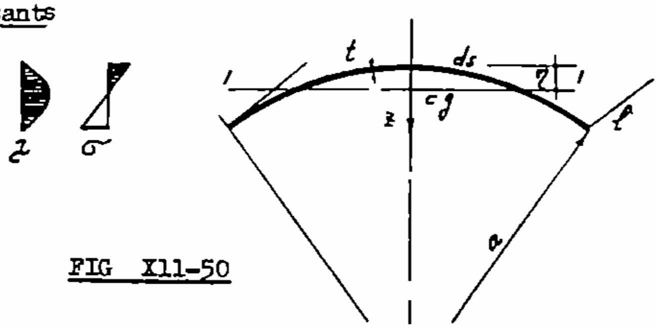


FIG XII-50

only hence, the normal stress σ in the longitudinal direction is given by

$$\underline{\sigma = M z / I_{1-1}}$$

The maximum compressive stress $\sigma_{c\ max}$ at the upper fiber must be smaller than the allowable buckling stress σ_{buckl} Hence

$$\sigma_{c\ max} = M \eta / I_{1-1} < \text{allow } \sigma_{buckl}$$ where

$$\text{allow } \sigma_{buckl} = \frac{\sigma_{cb}}{1 + \frac{5\ \sigma_{c28}}{E} \frac{a}{t}} \quad \frac{1}{s}$$ s = factor of safety

σ_{cb} = The bending compressive strength of concrete = $\frac{4}{3}$ σ_{c28}

σ_{c28} = The cube compressive strength of concrete

E = Modulus of elasticity of concrete ≈ 200 000 kg/cm² for normal cases

s = Factor of safety = 4 in buckling problems

Assuming σ_{c28} = 200 kg/cm², we should have

$$\sigma_{c\ max} < 65 / \left(1 + \frac{a}{200\ t} \right)$$

Thus, for normal cases, the maximum allowed concrete compressive stres $\sigma_{c\ max}$ can be extracted from the curve shown in figure XII-51

FIG XII-51

The longitudinal stress resultant N_x is therefore given by

$$N_x = M\ z\ t\ /\ I_{1-1}$$

The total tension T below the neutral axis being equal to the total compression C above the same axis, we get

$$C = T = \int_{z=o}^{z=\eta} N_x\ ds = \frac{M}{I_{1-1}} \int_{z=o}^{z=\eta} t\ z\ ds = \frac{M\ S_{1-1}}{I_{1-1}}$$

Therefore, the total tension T in the section is given by

$$\underline{T = M\ S_{1-1}\ /\ I_{1-1}}$$

Knowing further that $I_{1-1}\ /\ S_{1-1} = y_{CT}$ = the lever arm between the center of tension and the center of compression, then

$$\underline{T = M\ /\ y_{C-T}}$$

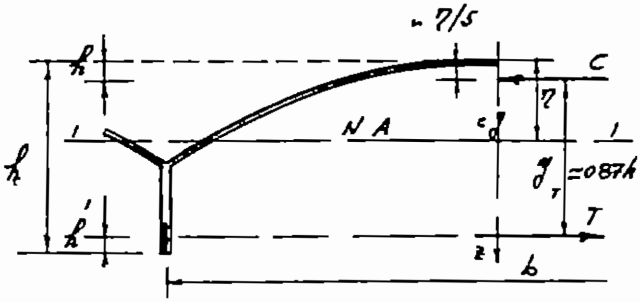


FIG XII-52

In reinforced concrete shells (Fig XII-52), the concrete in tension is generally neglected and all the tensile stresses are resisted by tension steel so that the real y_{CT} may be assumed equal to the distance between the center of the tension reinforcement and the center of compression which is generally bigger than $I_{1-1}\ /\ S_{1-1}$

For normal dimensions of symmetrical circular cylindrical shells where b = 8 - 12 ms and l = 16 - 25 ms, h may be chosen equal to l/10 In a preliminary estimate h' may be assumed 10-20 cms and h' = /5 may be assumed 15-25 cms , so that y_{CT} can be assumed - equal to h = /10 minus 25 to 40 cms y_{CT} may however be assumed equal to 0 87 h

The required tension steel in the shell can therefore be given by

$$A_s = T\ /\ \sigma_s$$

This reinforcement is generally arranged in a narrow edge beam as shown in figure XII-53 , the free space between the bars is equal to their diameter so that if the tension rein forcement is for example 13 Φ 25 mm then h'= 6 Φ = 15^cm The maximum stress in the tension steel σ_s max takes place in the lowest row the average stress σ_s at the center of gravity of the tension steel is generally smal - ler If no exact calculation is done σ_s may be assumed equal to 0 9 the maximum allowed values

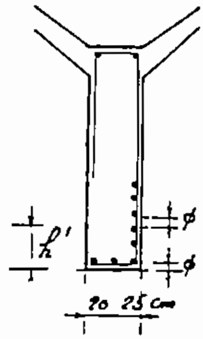


FIG X11-53

Assuming that $N_{x\varphi}$ denotes the total internal shearing force per unit length of the section then

$$N_{x\varphi} = \tau t = Q S / I_{1-1}$$

in which Q and τ are the shearing force and the shearing stress respectively, S is the statical moment of the section above any plane about the c g axis Its distribution is as stated before parabolic and attains its maximum values at the diaphragms where Q is maximum

So that

$$\max N_{x\varphi} = Q_{max} S / I_{1-1}$$

The maximum ordinate of the shear diagram lies at the axis passing through the center of gravity of the section, hence

$$\max \max N_{x\varphi} = Q_{max} S_{1-1} / I_{1-1} = Q_{max} / y_{CT}$$

As in ordinary beams this shearing force N_x will cause a diagonal tension T_x , resisted by both sides of the shell, and of equal magnitude, so that each side resists a diagonal force of the magnitude

$$T_{x\varphi} = \frac{1}{2} Q / y_{CT}$$

All shear stresses τ bigger than the allowed values (6-8 kg/cm^2)must be resisted either by stirrups or diagonal bars inclined 45° with the axis The latter are prefered in cases when τ is bigger than 10 kg/ cm^2

The shear stress τ is given by

$$\tau = T_{x\varphi} / A$$

where A is the area of 1 m strip of the shell = 100 x t cm^2

Example

It is required to analyse the beam action and to determine the longitudinal and diagonal reinforcements for the intermediate long cylindrical shell roof with edge beams shown in figure XII-54

The shell is simply supported on two edge diaphragms 25 ms apart

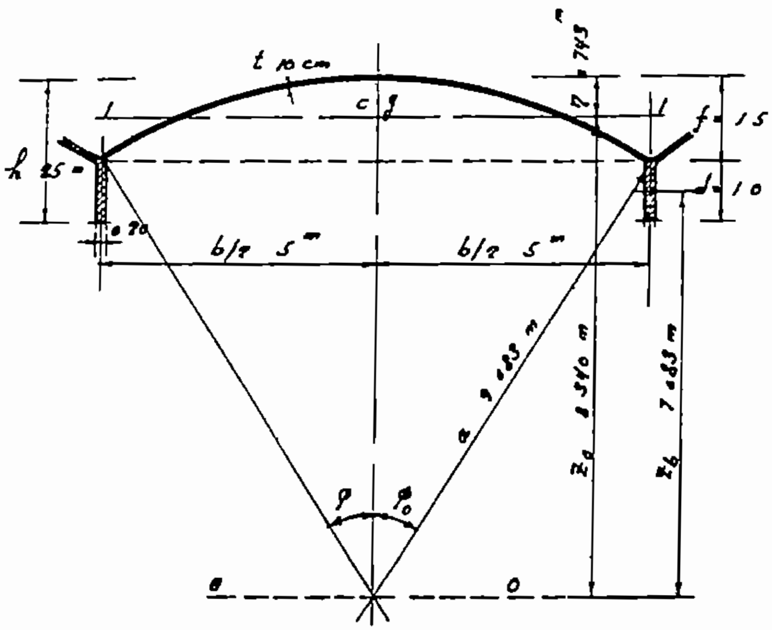


FIG. XII-54

Data

Span l = 25 m, width b = 10 m, total rise h = 2 5 m, rise of arc f=1 5m
Depth of edge beam d = 1 00 m, breadth b_o^{*} = 0 10 m Thickness of shell = 0 10 m

Radius a can be calculated from the relation :

$$a^2 = 5^2 + (a - 1\ 5)^2 \qquad \text{or} \qquad a = 9\ 083 \text{ ms}$$

Loads

Own weight of 10 cms shell-slab	0 1 x 2400	=	240	kg/m^2
Weight of 3 cms fine concrete	0 03 x 2200	=	66	"
Weight of plaster + isolation		=	34	"
Live load		=	60	
	total p	=	400	"

* Since each of the 20 cm wide edge beams is common for two shells, the beams belonging to one shell are to be introduced with a breadth b_o of 10 cms

Own weight of edge beam $= 0.2 \times 0.9 \times 2500 = 450$ kg/m'
Concrete slopes + rain water $= 300$

total $p_b = 750$ kg/m'

$\sin \varphi_o = b/2a = 10/2 \times 9.083 = 0.5505 \quad \varphi_o = 33° 24' = 33.4^o$

i e $\varphi_o = \frac{33.4}{180} \times 3.1416 = 0.583$ radians

Total load $\bar{p} = 2 \varphi_o \; a \; p + p_b$

$= 2 \times 0.583 \times 9.083 \times 400 + 750 = 5000$ kg/m'

Bending Moments and Shearing Forces

Maximum external bending moment at mid-span

$$\max M = 5 \times 25^2 / 8 = 390 \text{ m t}$$

Maximum shearing force at diaphragms

$$\max Q = 5 \times 25 / 2 = 62.5 \text{ tons}$$

Determination of Center of Gravity, Statical Moment and Moment of Inertia of Section of Shell

Referring to figure XII-49 , we get

$A = 2 (a \varphi_o t + b_o d) = 2 (9.083 \times 0.583 \times 0.1 + 0.1 \times 1) = 1.259 \text{ m}^2$

$\bar{z}_b = a \cos \varphi_o - \frac{d}{2} = 9.083 \times 0.835 - 0.50 = 7.583 - 0.5 = 7.083$ ms

$S_{o-o} = 2 (t a^2 \sin \varphi_o + b_o d \bar{z}_b) = 2(0.1 \times 9.083^2 \times 0.5505 + 0.1 \times 1.0 \times 7.083) = 10.500 \text{ m}^3$

$z_o = S_{o-o} / A = 10.500 / 1.259 = 8.340$ m

Therefore, the distance of the c g - axis (N A) from the top fiber is given by

$\eta = a - z_o = 9.083 - 8.340 = \underline{0.743 \text{ m}}$

$I_{o-o} = a^3 t (\varphi_o + \frac{1}{2} \sin 2 \varphi_o) + (1/6 \, b_o d^3 + 2 \, b_o d \, \bar{z}_b^2)$

$= 9.083^3 \times 0.1 (0.583 + 0.5 \times 0.919) + (1/6 \times 0.1 \times 1.0^3 + 2 \times 0. \times 1.0 \times 7.083^2)$

$= 88.180 \text{ m}^4$

The moment of inertia of the section about the c g axis is

$I_{1-1} = I_{o-o} - A z_o^2 = 88.180 - 1.259 \times 8.340^2 = \underline{0.61 \text{ m}^4}$

The statical moment of the cross-sectional area above or below the c g axis is given by

$$S_{1-1} = \frac{4}{3}\, t\, \eta \sqrt{2\, \eta\, a}\,\left(1 + \frac{\eta}{20\, a}\right)$$

$$= \frac{4}{3} \times 0\ 1 \times 0\ 743 \sqrt{2 \times 0\ 743 \times 9\ 083}\,\left(1 + \frac{0\ 743}{20 \times 9\ 083}\right) = 0\ 365\ m^3$$

The theoretical lever arm is therefore

Theoretical $y_{CT} = I_{1-1} / S_{1-1} = 0\ 610 / 0\ 365$ $= 1\ 670$ ms

Normal stresses and longitudinal reinforcements

Maximum concrete stress at top fiber

max σ_c = max $M\,\eta / I_{1-1} = 390 \times 0\ 743/0\ 61 = 475\ t/m^2 = \underline{47\ 5}\ kg/cm^2$

For $a/t = 9\ 083/0\ 1 = 90\ 83$, the maximum allowed compressive stress in concrete according to relations given in Fig XIII-51 is

max allowed $\sigma_c = 45\ kg/cm^2$ only!

The existing value of $47\ 5\ kg/cm^2$ may be accepted, it means that the factor of safety 's' against buckling is 3 86 which is sufficient

The center of compression is assumed at $h'' = \eta/5$ from the upper fiber Thus $h' = \eta/5 = 0\ 743/5 \simeq 15$ cms Whereas, the center of tension is assumed at the center of gravity of the tension steel which lies at a distance $h' = 20$ cms from the lower fiber of the edge beam The actual lever arm is therefore given by

Actual $y_{CT} = h - (h' + h') = 2\ 5 - (0\ 2 + 0\ 15) = 2\ 5 - 0\ 35 = \underline{2\ 15}$ ms

Assuming $y_{CT} = 0\ 87\ h$, we get $y_{CT} = 0\ 87 \times 2\ 5$ $= \underline{2\ 175}$ ms

The two values are approximately the same and are about 1 3 times the theoretical value given before Hence

The maximum total tension in the section at midspan is

max T = max M / act y_{CT} = 390 / 2 15 = $\underline{180}$ tons

Using high grade steel with an allowable average stress $\sigma_s = 0\ 9 \times 2 = 1\ 8\ ton/cm^2$, then the required tension reinforcement at mid-span is

max A_s = max $T / \sigma_s = 180 / 1.8 = \underline{100}\ cm^2$ chosen 20 # 25 mm

Shear stresses and diagonal reinforcements

The maximum total internal shearing force per unit length of section is given by

max $N_{x\varphi}$ = max $\tau\, t$ = max $Q / y_{CT} = 62\ 5 / 2\ 15 = 29$ t/m

The maximum diagonal tension resisted by each side of the shell is therefore

max $T_{x\varphi} = 1/2$ max $N_{x\varphi} = 29 / 2 = \underline{14\ 5}$ t/m

Hence the maximum shear stress is

$$\max \tau = \max T_{\lambda\varphi} / A = 14500/100 \times 10 = \underline{14\ 5}\ \text{kg/cm}^2$$

All shear stresses bigger than the allowed value of 8 kg/cm^2 (giving 8 t/m diagonal tension) will be resisted by diagonal reinforcements in the manner shown in figure XII-55

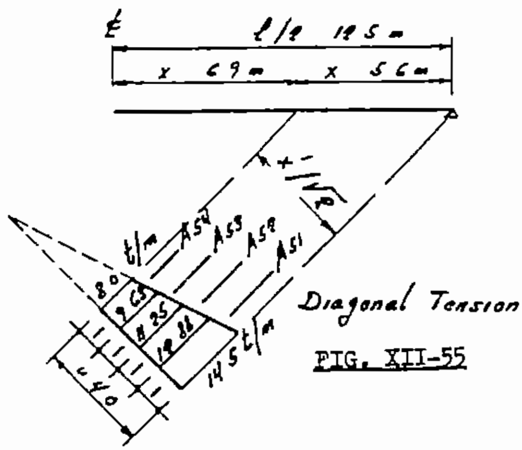


FIG. XII-55

The shear stress is equal to zero at mid-span it will be equal to 8 kg/cm^2 at a distance x from the center line where x = 8 x 12 5/14 5=6 9 m Dividing the distance $x' = l/2 - x$ = 12 5 - 6 9 = 5 6 ms into four equal strips, then the area of the diagonal tension reinforcement required in each strip is given by

$$A_{s1} = \frac{14\ 5 + 12\ 88}{2} \times \frac{1\ 0}{\sigma_s} = \frac{13\ 69 \times 1}{1\ 4} = 9\ 75\ \text{cm}^2 \qquad 8\ \Phi\ 13$$

$$A_{s2} = \frac{12\ 88 + 11\ 25}{2} \times \frac{1\ 0}{1\ 4} = 8\ 60\ \text{cm}^2 \qquad 7\ \Phi\ 13$$

$$A_{s3} = \frac{11\ 25 + 9\ 63}{2} \times \frac{1\ 0}{1\ 4} = 7\ 45\ \text{cm}^2 \qquad 6\ \Phi\ 13$$

$$A_{s4} = \frac{9\ 63 + 8}{2} \times \frac{1\ 0}{1\ 4} = 6\ 30\ \text{cm}^2 \qquad 5\ \Phi\ 13$$

The lengths x and x are to be measured at the neutral plane

Analysis of Arch Action of Symmetrical Circular Cylindrical Shells

The characteristic feature of shell action is the transmission of loads primarily by direct stresses with relatively small bending stresses This unique property of cylindrical shells stems from the behavior of the shell in the transverse direction To form a clear picture of the manner in which the shell operates and to contrast it with the behaviour of a plate typical strips cut from a plate and a shell are shown in figure XII-56

Considering the strip of fig a as a free body, it is evident that shearing forces and bending moments are required in order to maintain the external load in equilibrium On the other hand in the

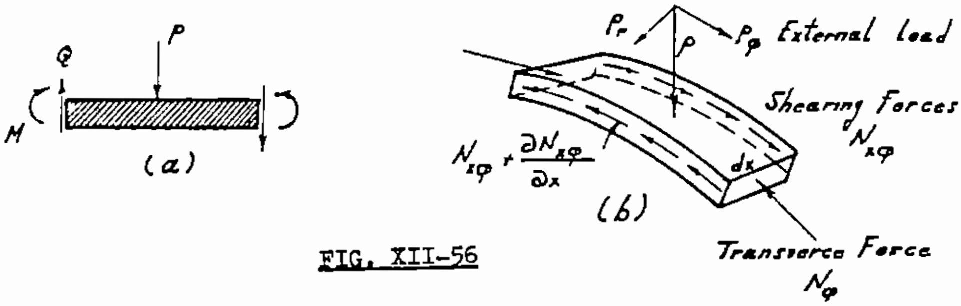


FIG. XII-56

strip of fig b the radial component p_r of the external load p is resisted by transverse force N_φ causing normal stresses on the radial sections, whereas the tangential component P_φ is resisted by shearing forces on the transverse sections The presence of these shearing forces distinguishes shell action from arch action Because of them the shell regardless of its shape, can support any type of loading by direct stresses, whereas the arch can carry, by direct stresses only one type of loading

In order to determine the transverse bending moments $_\varphi$ we consider a free body in the form of an elementary arch included between two adjacent cross-sections of the shell which are dx apart The equilibrium of the arch is maintained by two sets of forces, namely the load p acting on the element and the forces $\partial N_{x\varphi}/\partial x$ (Fig XII-57) the latter is known as the specific shear The specific shear at any point, acting in the direction of the tangent to the shell arch, may be resolved into horizontal and vertical components It is clear that the sum of the vertical components of the specific shear would balance the load on the shell arch the horizontal components of the specific shear which are symmetrically disposed about the crown balance themselves

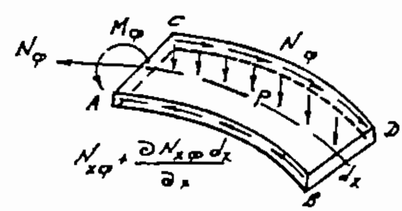


FIG. XII-57

We consider first a single shell with or without edge beams It is clear that the elementary shell arch cut out from such a shell will not develop any restraining forces or moments at its ends Hence we have a statically determinate arch

Next, we consider the elementary arch cut out from an interior shell of a symmetrically loaded group of multiple shells As the shell arch is restrained at the ends it would behave as a fixed arch. If the loading on the shell arch is symmetrically distributed across the cross-section, one would expect the degree of indeterminancy to be three. But the degree of indeterminancy involved is only two as no vertical reactions develop at the ends, the vertical loading on the shell being fully balanced by the sum of the vertical components of the specific shears. The elementary shell arch fixed at the ends and acted upon by the load and the specific shear may be analysed by any method applicable to fixed arches to determine the transverse moments M_φ.

Applications

1) Single circular shell without edge beams

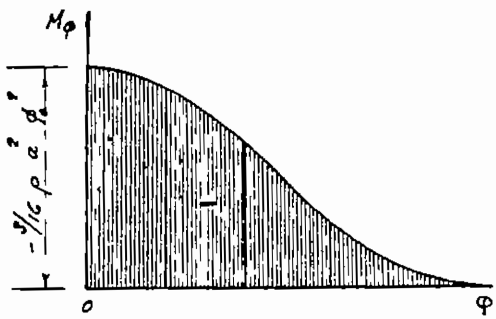
Denoting the angular distance from the crown to an arbitrary point by φ and half the central angle of the shell by φ_0, and assuming the unit load p to be uniformly distributed over the surface, Lundgren in his book on cylindrical shells gives the following relation

$$M_\varphi = -\frac{p\,a^2}{16\,\varphi_0^4}\,(\varphi_0^2 - \varphi^2)^2\,(3\,\varphi_0^2 - \varphi^2)$$

The moment is negative along the whole width of the shell and has the form shown in figure XII-58.

FIG. XII-58

Transverse Moments in a Circular Shell without Edge Beams



The maximum moment at the crown is therefore

$$M_0 = -\,3/16\;p\,a^2\,\varphi_0^2$$

For a vertical concentrated load P, acting on the middle of a stiffening rib, the moment in the rib for $\varphi > 0$, is given by

$$M_\varphi = \frac{P\,a}{32\,\varphi_0^5}\,(\varphi_0 - \varphi)^4\,(5\,\varphi_0^2 + 4\,\varphi_0\,\varphi + \varphi^2)$$

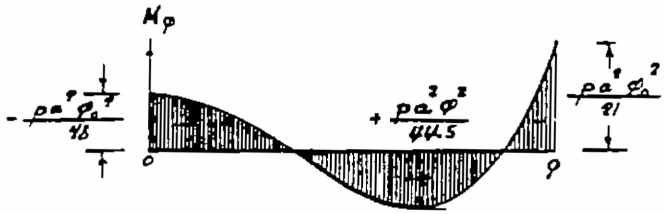
2) Circular inner shells without edge beams

The transverse bending moment of an inner circular shell , according to Lundgren, is given by

$$M_\varphi = -\frac{p\,a^2}{336\,\varphi_0^4}\left(7\,\varphi_0^6 - 75\,\varphi_0^4\,\varphi^2 + 105\,\varphi_0^2\,\varphi^4 - 21\,\varphi^6\right)$$

The moment is negative at the springing and at the crown but positive at the quarter point (Fig XII-59) Numerically, the statically indet-

Fig XII-59
Transverse moments in an inner shell without edge beams



erminate negative moment at the springing is the greatest and equal to

$$M\varphi_0 = -p\,.\,a^2\ \ \varphi_0^2 / 21$$

i e only ¼ maximum moment in a single shell The moment at the crown is

$$M_0 = -p\ \ a^2\ \ \varphi_0^2 / 48$$

a value that is only 1/9 of maximum moment in a single shell

The maximum positive bending moment takes place at an angle φ from the crown that can be determined from the relation

$$dM\varphi / d\varphi = 0$$

which gives $$\varphi = 0\ 638\ \varphi_0$$

Introducing this value in the general equation of $M\varphi$, we get

$$\max M\varphi + = p\ \ a^2\ \ \varphi_0^2 / 44\ 5$$

The normal force in long shells with a $\leqslant$ 10 ms may be estimated from the following relation

$$N\varphi = k\ \ p\ \ a$$

The values of k are shown in Fig XII-60

The bending moments and normal forces in the critical sections of an inner long cylindrical shell subject to uniform load p can accordingly be summarized as follows

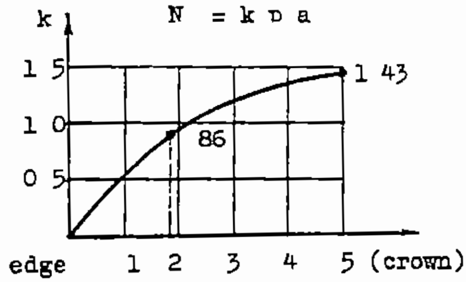


Fig XII-60

Section	Springing	max $M_{\varphi+}$	Crown
φ	$\varphi = \varphi_o$	$\varphi = 0\ 638\ \varphi_o$	$\varphi = 0$
Bending moment M_φ	$-\ p\ a^2 \varphi_o^2 \ /\ 21$	$+\ p\ a^2 \varphi_o^2 \ /\ 44\ 5$	$-\ p\ a^2 \varphi_o^2 \ /\ 48$
Normal force N_φ	—	0 86 p a	1 43 p a

The transverse bending moments in circular symmetrical inner shells with edge beams may be assumed of the same values shown in Fig XII-59 Accordingly, the transverse bending moments and normal forces of the example given in Fig XII-54 are

Section	Springing	max $M_{\varphi+}$	Crown
φ	$\varphi = \varphi_o$	$\varphi = 0\ 638\ \varphi_o$	$\varphi = 0$
$M_\varphi = \dfrac{p\ a^2\ \varphi^2}{k}$	$-\ \dfrac{11220}{21} = -530$	$+\ \dfrac{11220}{44\ 5} = 250$	$-\ \dfrac{11220}{48} = -\ 235$
$N_\varphi = k\ p\ a$	—	0 86 x 3633=3124	1 43x3633=5200

in which

$p\ a^2\ \varphi_o{}^2 = 400 \times 9\ 083^2 \times 0\ 583^2 = 11220$ kg m and

$p\ a = 400 \times 9\ 083 = 3633$ kg

It is clear that the transverse bending moments are relatively low so that a shell thickness of 10 cms reinforced by two meshes 6 Φ 8 mm/m and 5 Φ 6 mm/m circular and longitudinal bars respectively is generally ample It is however recommended to increase the thickness of the shell gradually from 10 cms to 15 cms at the springing and at the diaphragms along a distance equal to 1/10 length of arch

If the span of the shell is small (<~15ms), a slab thickness of 8 cms, increased to 12 cms at the springing and diaphragms, is generally sufficient Such shells are reinforced by one mesh, 6 ϕ 8 mm/m circular and 5 ϕ 6 longitudinal, except at the edges where we need two meshes

At the end diaphragms, local bending moments are induced in the longitudinal direction of the form shown in Fig XII-61 The maximum ordinate can be estimated by

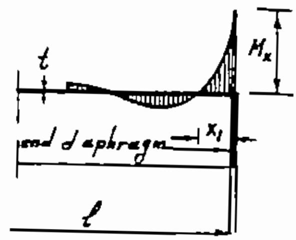


Fig XII-61

$$\max M_x = - p \quad x_1^2 / 2$$

in which

$$x_1 = 0\ 76 \sqrt{a\ t}$$

a = radius and t = thickness of shell , p = total load / m^2

The determination of the transverse bending moments at different points of a simple circular shell without edge beams can be treated in the following manner

a slice of the shell one meter long , is considered as an arch which is in equilibrium under the action of the loads p and the specific shear $\partial N_{x\varphi} / \partial x$ (Fig XII-62)

In a statically determinate shell subject to uniform load p per square meter, $\partial N_{x\varphi} / \partial x$ must be constant along any generator in direction x because N_x varies linearly in this direction (refer to Figs XII-34, 35 and 36) so that in a simple shell, we have

$$N_{x\varphi} = \frac{\partial N_x}{\partial x} \quad \frac{l}{2}$$

It has further been shown that

$$\max N_{x\varphi} = \frac{\max Q}{y_{CT}} = \frac{\bar{p}}{2\, y_{CT}}$$

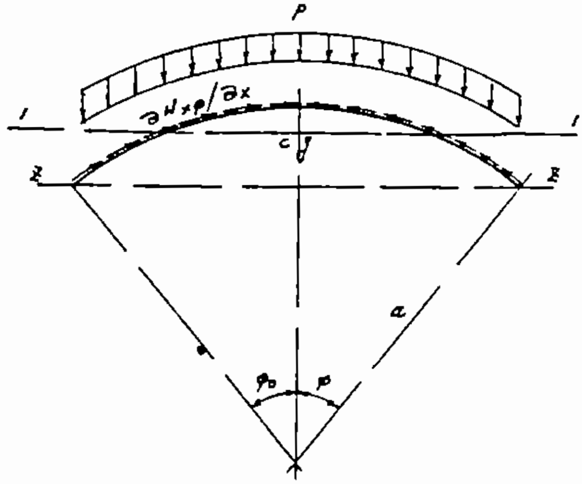
in which

$\bar{p}$ = the total load on the shell per meter run So that

$$\max \frac{\partial N_x}{\partial x} = \frac{\bar{p}}{y_{CT}}$$

Its distribution along the section of the shell is similar to that of

$N_{x\varphi}$, i e , with zero ordinates at crown and foot of arc and maximum at the c g axis It may be assumed triangular with a maximum ordinate equal to $\frac{4}{3} \frac{\bar{p}}{y_{CT}}$ as shown in Fig XII-63

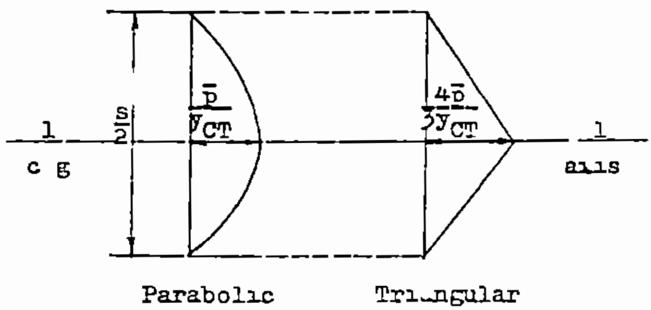


Length of arc $s = 2 a \varphi_0$

$$\max \frac{\partial N_{x\varphi}}{\partial x} = \frac{\bar{p}}{y_{CT}}$$

Fig XII-62

$$\frac{\bar{p}}{y_{CT}} = \max \frac{\partial N_{x\varphi}}{\partial x}$$



Distribution of $\partial N_{x\varphi}/\partial x$

Fig XII-63

All the statical values required for the design can be determined [illegible] the strip method as follows

1) Determine the arc length from the relation $s = 2 a \varphi_o$, where φ_o is measured in radians

2) Divide half the arc into a convenient number of strips, each of length Δs (normally 2 to 3 meters) Determine the coordinate z' from

a horizontal axis z-z and y from the middle vertical axis both for the middle and the edges of each strip so that Δ z and Δ y are known (Fig XII-64)

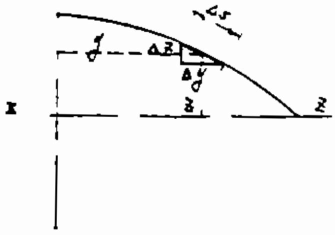


FIG XII-64

3) The c g axis can be determined by dividing the statical moment of the elemental areas about any axis (e g z-z) by their total area The statical moments S S_{1-1} and the moment of inertia I_{1-1} can then be easily determined

4) Determine the shear forces $N_{x\varphi}$ and the specific shear $\partial N_{x\varphi} / \partial x$ from the given relations either the real parabolic distribution or the equivalent triangular distribution (fig XII-63) may be used

5) The statically determinate transverse moments for p and $\partial N_{x\varphi} / \partial x$ at the middle of the different increments may be determined directly assuming the specific shear in every increment parallel to the tangent at its c g in which case if we assume that the specific shear on an element (e g 2-3) is δ ,

then its moment at points 3 4 and 5 are $\delta\, n_3$ $\delta\, n_4$ and $\delta\, n_5$ where n_3 n_4 and n_5 are the normals from points 3, 4 and 5 on the line of action of δ (Fig XII-65)

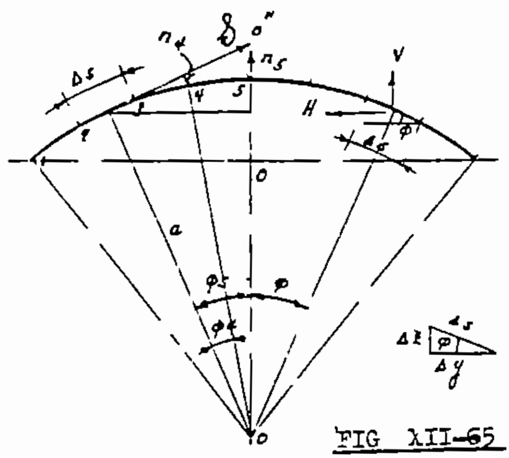


FIG XII-65

In order to determine n_5 we have

$\cos \varphi_5 = a / \overline{o - o}$,

hence $\overline{o - o} = a / \cos \varphi_5$ and

$\cos \varphi_5 = n_5 / \overline{o'' - 5} =$

$= n_5 / \overline{o - o''} - a$

therefore

$$n_5 = (\overline{o - o''} - a) \cos \varphi_5 = (\frac{a}{\cos \varphi_5} - a) \cos \varphi_5 \qquad \text{or}$$

$$n_5 = a (1 - \cos \varphi_5)$$

Similarly
$$n_4 = a (1 - \cos \varphi_4) \qquad \text{and}$$
$$n_3 = a (1 - \cos \varphi_3) \qquad \text{etc}$$

The transverse moments due to specific shear can also be determined by resolving δ to its vertical and horizontal components and multiplying them by their distances from the different points on the arc It has to be noted that

The vertical component V of the specific shear δ acting on the element Δs is

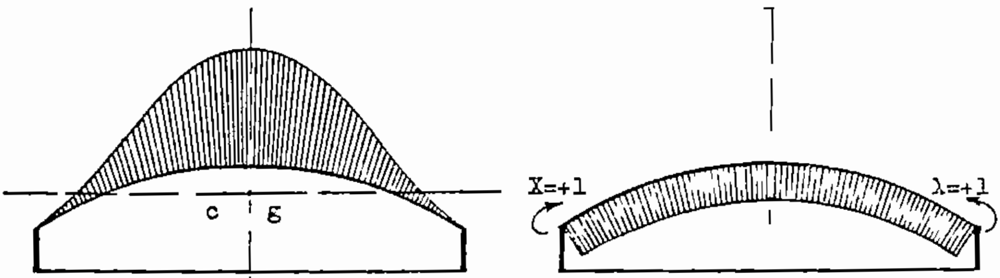
$$V = \delta \sin\varphi = \frac{\partial N_x}{\partial x} \Delta s \quad \frac{z}{s} = \frac{\partial N_x}{\partial x} \Delta z$$

The horizontal component H of the specific shear acting on the element Δs, is

$$H = \delta \cos\varphi = \frac{\partial N_x}{\partial x} \Delta s \quad \frac{y}{s} = \frac{\partial N_x}{\partial x} \Delta y$$

The transverse bending moment is negative althrough as shown in Fig XII-66a

<u>Transverse Bending Moments in a Simple Circular Shell with Edge Beams</u>



(a) M_o-diag (b) M_1-diag

FIG XII-66

A single shell with edge beams may be assumed for symmetrical loading, as once statically indeterminate The statically indeterminate value is the connecting moment X between the shell slab and the edge beam (Fig XII-66) If the edge beams are sufficiently deep and the necessary constructional provisions are taken to prevent them from rotation, we may assume that the angle of rotation at the springing is equal to zero and the statically indeterminate moment X can be determined from the equation of elasticity

$$0 = \delta_o + X \delta_1$$

So that

$$X = - \delta_o / \delta_1 = - \sum \frac{M_o M_1}{E I} \Delta s / \sum \frac{M_1^2}{E I} \Delta s$$

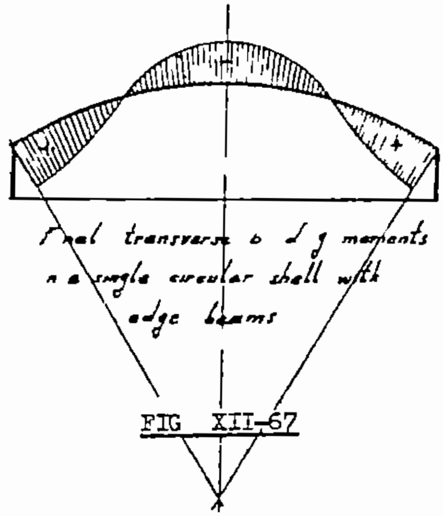


FIG XII-67

The M_o- diagram can be determined as shown in the previous case (Fig XII-66 a), the M_1 diagram is rectangular (Fig XII-66b) and the final transverse moments are therefore determined from the relation (Fig XII-67)

$$M = M_o + X M_1$$

Transverse Bending Moments in an Inner Circular Shell with Edge Beams

This case is twice statically indeterminate, the unknowns are the connecting moments X_1 and the horizontal thrust X_2 at the joint between the shell slab and the edge beams The equations of elasticity are

$$\delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} = 0$$
$$\delta_{20} + X_1 \delta_{21} + X_2 \delta_{22} = 0$$

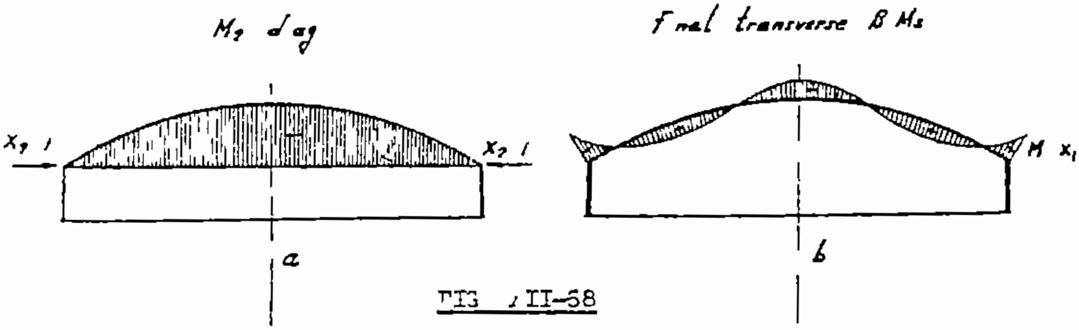


FIG XII-68

in which

$\delta_{10} = \sum M_o M_1 \Delta s / E I$, $\delta_{11} = \sum M_1^2 \Delta s / E I$ are the same as in previous case

$\delta_{20} = \sum M_o M_2 \Delta s / E I$, $\delta_{22} = \sum M_2^2 \Delta s / E I$ and $\delta_{12} \sum M_1 M_2 \Delta s/EI$

The M_o and the M_1 – diagrams are the same as those of the previous case (fig XII–66 a and b) The M_2– diagram is shown in fig XII–68a Generally X_1 and X_2 are negative which means that the bending moment at the springing is negative and that λ_2 acts outwards reducing the statically determinate axial force. The final transverse moments are given by

$$M = M_o + \lambda_1 M_1 + \lambda_2 M_2$$

The axial force in the shell is equal to the resultant of the specific shear and the components of the load p and X_2 parallel to the axis of the shell

External Loads and Internal Forces Acting on End Diaphragms

The end diaphragm of a shell may be an arch with a tie, a frame with curved girder , avertical plate , a truss etc as shown in figure XII–69 a, b and c)

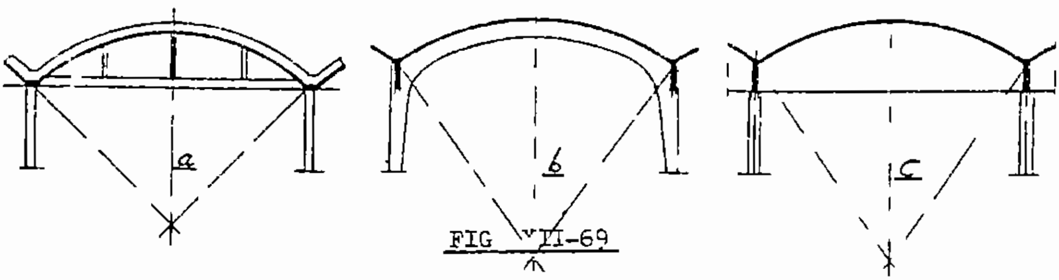


FIG XII–69

The loads of the shell slab are transmitted to the diaphragms through the direct shearing forces $N_{x\varphi}$ acting along the axis of the shell at the end cross-section

Their magnitude can be determined from the relation

$$N_{x\varphi} = Q_{max} \quad S / I_{1-1}$$

Their distribution is parabolic with a maximum ordinate ($N_{x\varphi}$ max) at the neutral axis of the shell given by

$$N_{x\varphi} \max = Q_{max} / y_{CT}$$

They may also be assumed triangular with a maximum ordinate equal to

$$N_{x\varphi} \max = 4\, Q_{max} / 3\, y_{CT}$$

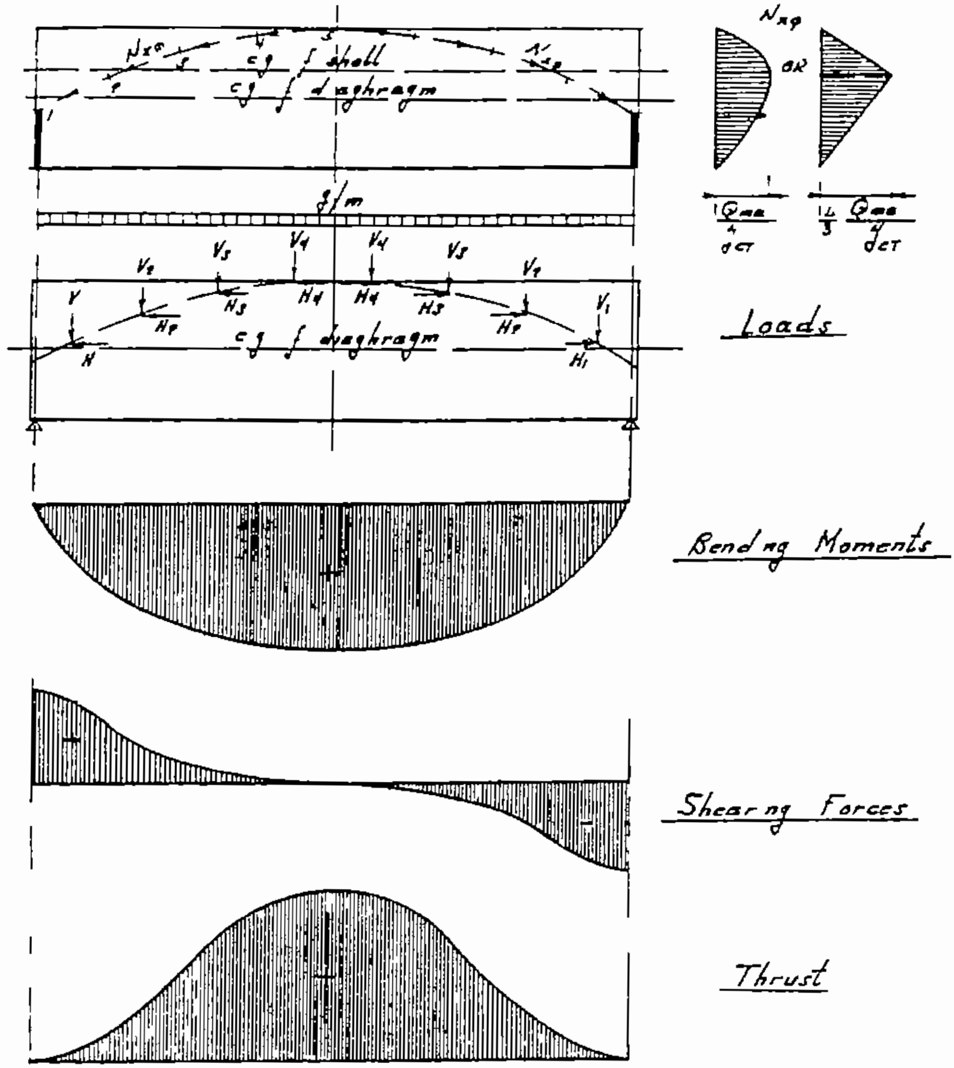


Fig XII-70 External load and internal forces acting on end diaphragm

which is more convenient for the numerical calculations Fig XII-70 shows a simple diaphragm in which the forces $N_{x\varphi}$ acting on the different elements Δs of the shell are resolved to their vertical and horizontal components V and H where

$$V = N_{x\varphi} \Delta s \sin\varphi = N_{x\varphi} \Delta s \frac{\Delta z}{\Delta s} = N_{x\varphi} \Delta z$$ and

$$H = N_{x\varphi} \quad \Delta s \cos\varphi = N_{x\varphi} \quad \Delta s \quad \frac{\Delta y}{\Delta s} = N_{x\varphi} \quad \Delta y$$

in which Δz and Δy have the same meaning shown in figure XII-65

As a check for the calculations $\sum V$ must be equal to Q_{max}

Having determined V and H, the corresponding bending moment shearing force and thrust with respect to the axis of the diaphragm can be easily determined

Prof Dr A Shaker* in an ' Introduction to Three Dimensional Analysis of Structures " has given three examples showing the numerical analysis of the internal forces in long shell roofs by the beam method namely

1) Single long cylindrical shell roof without edge beams
2) " " " " " with vertical edge beams
3) Intermediate long cylindrical shell roof with vertical edge beams

Constructional Details

For normal barrel vaults the maximum practical span l is 25 ms It can be increased to 50 ms if prestressing is used

Figure XII-71 shows cylindrical parallel barrel vaults with and without edge valley beams

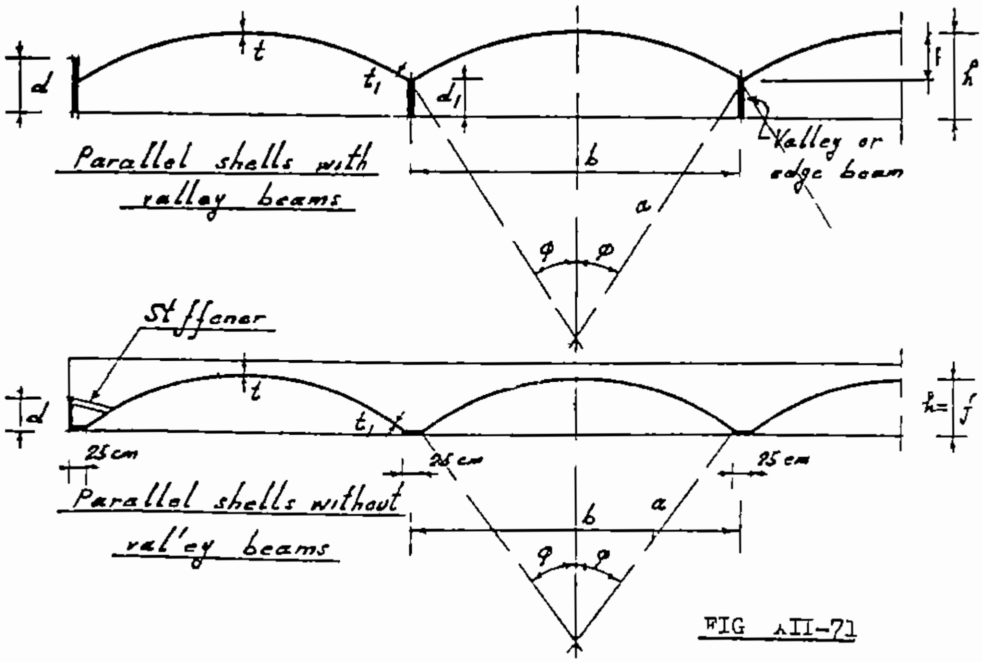


FIG XII-71

* Professor of Theory of Structures at the Faculty of engineering Ain Shams University

The following gives guide lines for the conveniient proportions of cylindrical barrel vaults

$$l / b \approx 2 \qquad \text{and} \qquad h / l \approx 1/10$$

These proportions give the optimum value for economy consistent with strength and deflection requirements l/b may however be increased to 3 or even more and h/l may be decreased to 1/12
d/l may be chosen 1/15 If edge beams are not supported by intermediate columns, stiffeners with a cross-section varying between 15 x 15 cm and 20 x 20 cms at the third points of the spans are usually required to prevent lateral instability of the compression zone of the edge beam

If the edge beams are supported by intermediate columns and $d \geq 50$ cms, then stiffeners need not be used

For cast in situ shells the thickness t should not be less than 8 cms to be increased to 10 cms for $16 m < l < 25$ ms and 12 cms for bigger spans

The thickness t_1 at the springing may be chosen equal to $l/120$ usually between 12 and 20 cms It is a common practice to increase the thickness by ~ 4 cms for a distance of $l/10$ from each end

The angle φ_o should always be kept less than 45 , otherwise double shuttering will be needed near the springing. If the above proportions are used φ_o will automatically be less than 45

Sachnors..1 in his text book Stahlbetonkonstruk+ionen recommends the following proportions (fig XII-72)

$b < 20$ ms $\qquad h \geq l/10$ and $\geq b/c$

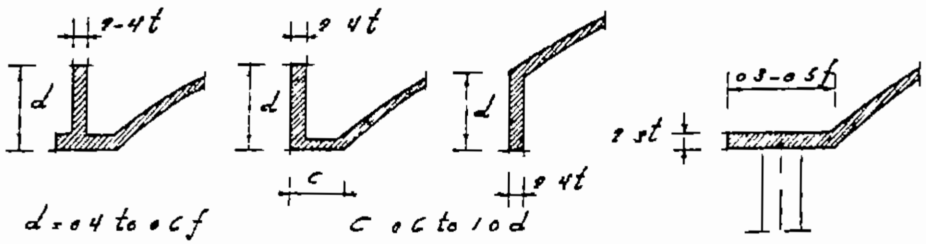


FIG XII-72

Fig XII-73 shows the details of reinforcements of an intermediate shell 9 meters wide and 18 ms span The shell is 7 cms thick, increased to 13 cms over a length of 1 00 m at the springing and end diaphragms It is reinforced with one mesh 6φ 8 mm/m circular and 5φ 6 mm/m longitudinal except at the edges where we have two meshes The diagonal reinforcement at the corners is arranged at the middle of the shell slab they must be well anchored with the bent bars of the valley edge beams which are reinforced by 12 ϕ 25 normal mild stee. It is however recommended to replace these longitudinal reinforcements by equivalent deformed high grade or cold twisted steel for bigger strength and higher bond

Figures XII-74 a and b show the general layout, main dimensions and details of reinforcements of the inspection shed constructed at Gisr-El-Suez garage area They show one of the wide possible applications of circular cylindrical shells The shed in this structure , covering an area of 22 x 49 ms, is supported on six columns only Its thickness is 10 cms increased to 14 cms on a small distance at the edge beams and end diaphragms The shell, in its cross-section , is composed of a circular part 12 4 ms wide and two overhanging circular cantilevers 4 8 ms each It has four edge beams two outside ones 15 x 100 cms each and two intermediate ones 25 x 75 cms each a the joint between the cantilever arms and the central part In the longitudinal direction, the shell is continuous over two spans 24 6 ms each It is supported on three diaphragms one intermediate and two edge ones The reinforcement of the shell is composed of two meshes each 6 ϕ 10 mm/m circular and 6 ϕ 8 mm/m longitudinal The longitudinal reinforcement in the intermediate edge beams is 18 ϕ 25 mm and in the outside edge beams is 5 ϕ 25 + 6 ϕ 22 mm The longitudinal reinforcement resisting the connecting moment between the two spans of the shell over the intermediate diaphragm is 25 ϕ 13 mm top and bottom The diagonal reinforcement shown in plan is arranged to resist the principal diagonal tensile stresses in the shell

The diaphragms are two hinged frames with overhanging cantilevers Their main girders are inverted in order to have a plane bottom surface for the roof Due to the severe variation of the moment of inertia of the diaphragm-girder, it has been possible to make its middle section 60 cms deep only

The soil at the site of the garage is composed of clayey layers that are much affected by water to depths varying between 10 and 14 meters, underlaid by medium sand The foundations for the main columns are single isolated footings composed of rectangular reinforced concrete footings 50 to 60 cms thick resting on rounded plain concrete deep ones reaching the sand and having a depth varying between 7 and 9 meters Their top surface is 3 ms from ground level

The cross-section x-x of the underground path necessary for in-section is 1 70 ms deep and 0 80 ms wide it has been designed as a reinforced concrete U-section on elastic foundation subject to the rolling wheel load of the busses Its walls and floor are chosen 20 cms thick and its main longitudinal reinforcement is 11 ϕ 16 mm at the bottom of the floor slab and 8 ϕ 19 mm at the top of the walls as shown in Fig XII-74a

4- Cross-Supported Cylindrical Shells

If the cross-section of a shell is simply supported at the edges e and e' , there is only one statically indeterminate value, consisting of two vertical reactions X , from the longitudinal walls (Figure XII - 75 a) X is to be so determined that the total vertical displacement at the springing is zero For solving the problem, we choose as main system the free arc e e' without supports at the edges e and e' The equation of elasticity is therefore

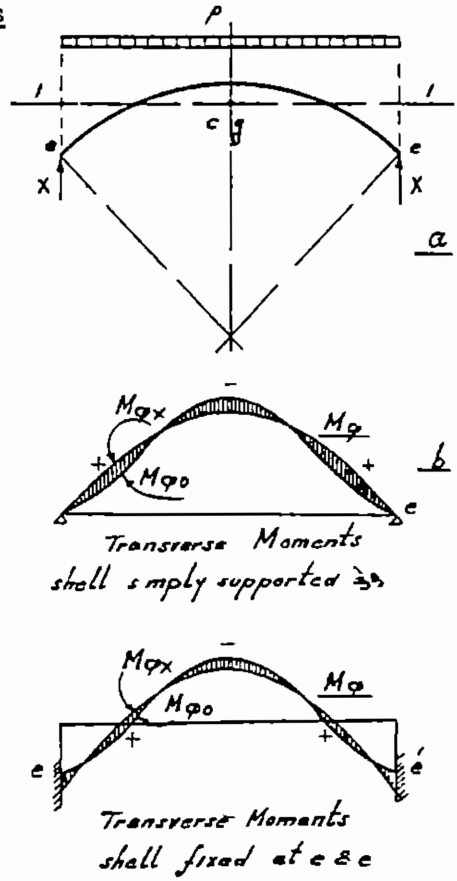


FIG XII-75

$$\delta = 0 = \delta_o + \delta_X$$

where

δ_o = vertical displacement of the middle section of the main system due to loads and

δ_X = vertical displacement of the same section due to the statically indeterminate edge loads X

Both values are to be calculated taking the beam and the arch action in consideration in the following manner

$\delta_o = \delta_{bo} + \delta_{ao}$ = vertical displacement due bending moments in lon- longitudinal direction (beam action) plus vertical displacement due to bending moments in cross direction (arch action)

For a shell simply supported between the diaphragms subjected to a uniform load $\bar{p}$, we have

$$\delta_{bo} = 5\,\bar{p}\,l^4 / 384\ E\,I_{1-1}$$

where, $\bar{p}$ is the total load on the arc per meter run in the longitudinal direction, and I_{1-1} is the moment of inertia of the cross-section of the shell about the c g axis The displacement δ_{ao} can be computed according to the theorem of virtual work from the relation

$$\delta_{ao} = \int \frac{M_{\varphi o}\ M_{\varphi 1}}{E\ I_{\varphi}}\,ds$$

where $I_{\varphi} = t^3 / 12$, $M_{\varphi o}$ is the transverse bending moment due to the loads p and the specific shear $\partial N_{x\varphi}/\partial x$ shown in figure XII -75b- heavy curve - and is to be determined according to methods given in the previous article $M_{\varphi 1}$ is the transverse bending moment due to X=1 It is however convenient to determine the transverse bending moment for $X = -\bar{p} / 2$ because for this load, the corresponding specific shear and transverse moments are the same as for the uniform load $\bar{p}$ but with opposite sign (Fig XII-75b, thin curve) In this manner, we have

$$\delta_{ao} = \frac{2}{\bar{p}} \int \frac{M_{\varphi o}\ M_{\varphi X}}{E\ I_{\varphi}}\,ds$$

Analogously for a load $X = -\bar{p} / 2$ the total edge vertical displacement can be given in the form

$$\delta_X = \delta_{bX} + \delta_{aX}$$

where

$$\delta_{bX} = \delta_{bo}$$

and

$$\delta_{aX} = \frac{2}{\bar{p}} \int \frac{M^2_{\varphi X}}{E\ I}\,ds$$

the value of X required to bring the edge back to the original level can now be computed from the relation

$$X = \frac{2}{\bar{p}}\ \frac{\delta_o}{\delta_X} = \frac{2}{\bar{p}}\ \frac{\delta_{bo} + \delta_{ao}}{\delta_{bX} + \delta_{aX}}$$

The variation of the final transverse moment is shown in figure XII-75 b it is negative in the neighbourhood of the crown and positive near the quarter points Numerically the positive moments are bigger.

Whether the shell is a little longer or shorter is not of much importance to X because the beam deformations are small compared to the arch deformations.

If the cross-section is restrained at the springings, it is twice statically indeterminate. The two redundants are the vertical reactions and the restraining moments. Since the two restraining moments equilibrate each other, they have no influence on the beam action of the shell. The transverse moments for this case are shown in figure XII-75 c. In shells of ordinary lengths, the resulting moments are negative at the springing and in the neighbourhood of the crown, but positive near the quarter points. Numerically, the moments at the springings are the largest. However, the thickness of the shell may be increased here if necessary.

Shells restrained at the edges transfer a greater part of the load in the transverse direction than do shells with a simply supported cross-section, especially in shells of considerable lengths.

Lundgren in his text-book on cylindrical shells gives the following data for cross-supported circular shells.

a) <u>Simply Supported Cross-Section</u>

$$M_\varphi = - \frac{p\, a^2}{\varphi_o^4} \left(0.0234\, \varphi_o^6 - 0.2138\, \varphi_o^4\, \varphi^2 + 0.2379\, \varphi_o^2\, \varphi^4 - 0.0476\, \varphi^6 \right)$$

The maximum moment is obtained for $\varphi = 0.73\, \varphi_o$ and is given by

$$\underline{\max\ M_\varphi = 0.03\, p\, a^2\, \varphi_o^2}$$

The reaction from the longitudinal walls is

$$\underline{X = 0.239\, p\, a\, \varphi_o}$$

i.e. $\sim$ 24% of the load is transmitted to the transverse direction. In these relations, it was assumed that the beam deformation δ_b is negligible compared with the arch deformation δ_a at the edge of the main system. The ratio δ_b / δ_a can however be given in the form

$$\delta_b / \delta_a = 10.7\, l^4\, I_\varphi / t\, a^6\, \varphi_o^8$$

and the beam deflection will increase X with factors $(\delta_a + \delta_b) / \delta_a$; the consequent increase of the moment is somewhat larger. As an example, a 10% increase of X will increase the maximum moment by $\sim$ 20%.

It has been stated before that transverse bending moments in single shells without edge beams (fig XII-58) are relatively high and negative all through with a maximum value at the crown of

$$M_0 = - 0.187\ p\ a^2\ \varphi_o^2$$

Whereas in cross supported single shells the transverse bending moments (fig XII-75 a) are negative at the crown and with a value of

$$M_0 = - 0.023\ p\ a^2\ \varphi_o^2$$

and positive in the outer parts of the arc the maximum values lie at $\varphi = 0.73\ \varphi_o$ and are equal to

$$\max\quad M_\varphi + = 0.03\ \ p\ a^2\ \varphi_o^2$$

i e smaller than 1/6 of M_0 in cross-unsupported shells

Moreover the bending moments and the corresponding concrete compressive stresses and tension steel as well as the shearing forces and the corresponding shear stresses and diagonal steel are reduced by ~ 24 %

Lundgren in his text-book on cylindrical shells has given an example on a simple cross-supported shell We show in the following a summary of the data, the design and a discussion to the final results (Fig XII-76)

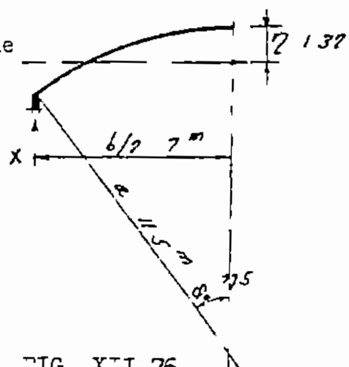


FIG XII-76

1) Data

a shell, 30 ms long and 14 ms wide is simply supported in both directions The slope at the springings will be $\varphi_o = 37.5 = 0.654$ radius The thickness of the shell is 8 cms the sectional area of one edge beam including reinforcement may be put equal to $A_b = 0.30\ m^2$

The total dead and live load is p = 220 dead + 65 live = 285 kg/ m^2

Rise $f = a (1 - \cos 37.5) = 11.5 (1 - 0.793) = 2.37$ ms

Distance of c g from top of arc $\eta = 1.32$ ms

Statical moment S and moment of inertia I about c g axis

$$S_{1-1} = 0.788\ m^3 \qquad I_{1-1} = 1.595\ m^4$$

Theoretical lever arm

$$y_{CT} = 1\ 595 / 0\ 788 = 2\ 02 \text{ ms}$$

Actual lever arm act $y_{CT} = h - (\frac{\eta}{5} + h')$

Assuming $h = 2\ 5$ ms and $h' = 10$ cms, we get

$$\text{actual } y_{CT} = 2\ 5 - (\frac{1\ 32}{5} + 0\ 1) = 2\ 15 \text{ ms}$$

Total load $\bar{p} = 2\ p\ a\ \varphi_o = 2 \times 0\ 285 \times 11\ 5 \times 0\ 654 = 4\ 32$ t/m shell

Internal Forces

a) Shell Free at Edges

max $M = \bar{p}\ l^2 / 8 = 4\ 32 \times 30^2 / 8 = 485$ m t

max $Q = \bar{p}\ l / 2 = 4\ 32 \times 30 / 2 = 64\ 8$ t

max $\sigma_c = M\eta / I_{11} = 485 \times 1\ 32 / 1\ 595 = 400$ t/m² $= 40$ kg/cm²

For $a/t = 11\ 5/\ 08 = 144$ max allow $\sigma_c = 38$ kg/cm²

max $T = M/y_{CT} = 485 / 2\ 15 = 225$ ton

max $A_s = T/2\ \sigma_s = 225 / 2 \times 1\ 8 = 62\ 5$ cm² high grade steel/edge

max $N_{x\varphi} = Q/y_{CT} = 64\ 5/\ 2\ 15 = 30$ t/m on both sides

max $T_{x\varphi} =$ max $N_{x\varphi} / 2 = 30/2 = 15$ t/m on each side

max $\tau =$ max $T_{x\varphi} / A = 15000/100 \times 8 = 18\ 7$ kg/cm²

max transverse bending moment at crown

$$M_O = -0\ 187\ p\ a^2\ \varphi_o^2 = -0\ 187 \times 285 \times 11\ 5^2 \times 0\ 654^2 = -3000 \text{ kgm}$$

Lundgren has computed the cross bending moment by the strip method and the maximum value at the crown was found to be equal to - 2028 kgm The difference is due to the existance of the edge beam However a shell 8 cms thick cannot sustain such big moments by any means

b) Shell Supported at Longitudinal Edges

The statically indeterminate vertical reaction can be estimated from the relation

$$X = 0\ 239\ p\ a\ \varphi_o = 0\ 239 \times 285 \times 11\ 5 \times 0\ 654 = 510 \text{ kg/m}$$

The max positive transverse bending moment can be estimated by max M_φ + = 0 03 p a^2 φ_o^2 = 0 03 x 285 x 11 5^2 x 0 654^2 = 480 kgm

If these values are determined by the strip method, they are found to be X = 436 kg/m and max M_φ + = 321 kgm

These differences show clearly the sensitivity of the calculations But it is clear that the cross supports have reduced the transverse moments to less than 1/6 of M_0 created in free shells that a thickness of 8 - 10 cms is possible

Accordingly, the total load transmitted in the longitudinal direction is given by

$\bar{p}$	= 4320 - 2 X	= 4320 - 2 x 436	=	3442	kg/m' and
max M	$= \bar{p}\, l^2 / 8$	= 3 442 x 30^2 / 8	=	387	mt
max Q	$= \bar{p}\, l / 2$	= 3 442 x 30 / 2	=	52	t
σ_c max	$= M\eta / I_{1-1}$	= 387x1 32/1 595	=	320	$t/m^2 = 32 < 38\ kg/cm^2$
max T	$= M / y_{CT}$	= 387 / 2 15	=	180	t i e 90 ton each side
max A_s	$= T / 2\sigma_s$	= 180 / 2 x 1 8	=	50 cm^2	high grade steel/edge
max $N_{x\varphi}$	$= Q / y_{CT}$	= 52 / 2 15	=	24 2	t/m
max $T_{x\varphi}$	= max $N_{x\varphi}$ /2	= 24 2/ 2	=	12 1	t/m on each side
max τ	= max $T_{x\varphi}$ /A	= 12100/ 100 x 8	=	15 1	kg/cm^2

B) Restrained Cross-Section

$$M_\varphi = - \frac{p\,a^2}{\varphi_o^4} (0\ 0134\ \varphi_o^6 - 0\ 1519\ \varphi_o^4\ \varphi^2 + 0\ 2173\ \varphi_o^2\ \varphi^4 - 0\ 0435\ \varphi^6)$$

Numerically, the largest moment occurs at the springing, where

$$\text{min}\ M_\varphi = - 0\ 0353\ \ p\ a^2\ \varphi_o^2$$

The reactions from the longitudinal walls are

$$X = 0\ 305\ \ p\ a\ \varphi_o$$

i e 30% of the total load is transmitted to the transverse direction The beam deflection gives a correction which is considerably greater than for a simply supported cross-section First, the factor 10 7 in the formula of δ_b / δ_a must be replaced by 42 2 Secondly an increment to X of 10% will increase the transverse moment at the springing by 36 %

5-Saw - Tooth Shells

Northlight cylindrical shell roofs as shown in figure XII - 77 provide ample and uniform daylightning and large column - free spaces for factory buildings located anywhere in the Northern Hemisphere They are especially suitable for factories inside which precision operations are to be carried out The distribution of the light in halls

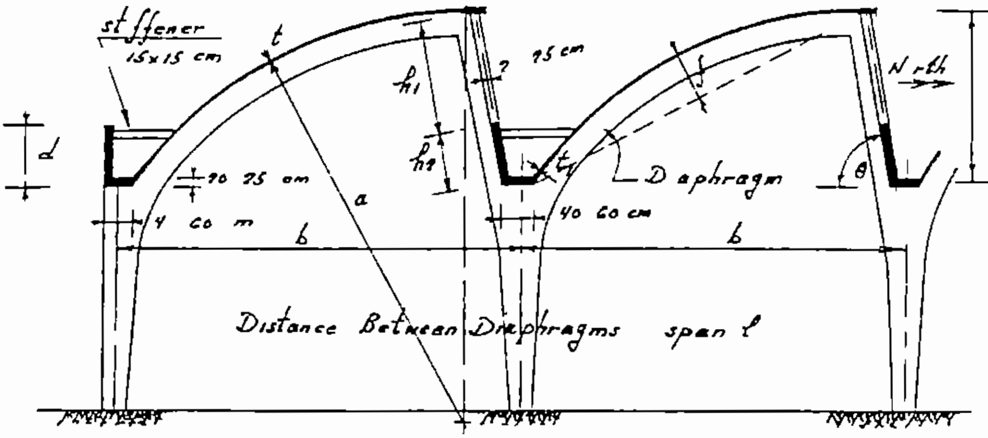


FIG XII-77

covered by saw-tooth shells is much better than in halls covered by plain saw-tooth roofs as the luminance is uniform shadow free and of much higher value

The span of northlight shells if not prestressed is limited to about 30 ms The more usual spans lie in the range of 12 to 25 ms With prestressing spans up to 40 ms are possible Where large un - obstructed column - free spaces are desired, the spacing of columns at right angles to the span may be made as large as 25 ms by supporing a number of shells on a single diaphragm as shown in figure XII-78

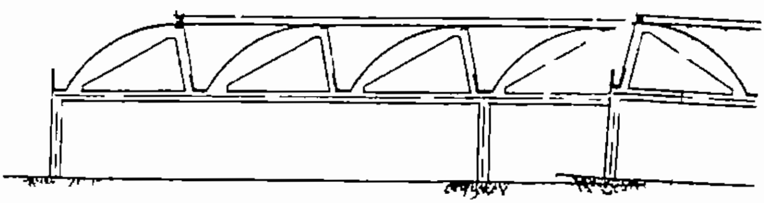


FIG XII-78

The following guiding rules may be used for the proportioning of a saw-tooth shell

The radius a is to be less than 12 ms The width b to span l may be between 1 2 and 1 4 The ratio of 1 2 leads to economic structures The height h shall not be smaller than l/6 the rise f of the circular segment shall as a rule be greater than l/18 the height h_2 of the gutter beam shall not generally be less than l/18 The angle θ lies usually between 60° and 90 The thickness of the shell t for cast in situ roofs should not be less than 8 cms, 10 cms for spans exceeding 15 ms and 12 cms for spans bigger than 20 ms The thickness t_1 at the springing is usually 12 cms (for t = 8 cms) and 20 cms (for t = 12 cms) If d or h_2 are chosen smaller than l/15 stiffeners 15 x 15 cms at the third points of the span are required to prevent lateral instability of the compression zone of the dege beam

Determination of Internal Forces

The analysis of a saw-tooth shell is more time consuming than that of a symmetrical cylindrical shell because it is unsymmetrical However, if the shell is long, i e , $l/a \geqslant 3$, the beam method, described in the previous article, with small modifications, gives satisfactory results Instead of using the $M z/I$ formula we have now to use the formulas relating to unsymmetrical bending because the shell beam cross-section is asymmetric Otherwise the procedure remains the same as before This method has the advantage that it avoids the use of higher mathematics, but it is exactly equivalent to the Lundgren beam method

We give in the following[x] the main steps required for determining the internal forces (Fig XII-79)

1) Determination of the Properties of the Section

The total area of the section of the shell A is equal to the area of the edge beams, in which the reinforcements may be replaced by an equivalent concrete area, plus the area of the arc equal to 2 a φ_0 t

In order to determine the center of gravity, the statical moment and the moment of inertia of the section, choose an origin O, the mid-point of the arc the axes Oz and Oy are chosen along the normal and tengent to the shell arc at O

[x] Ramaswamy Design and Construction of concrete shell roofs Published by Mc Graw-Hill Book Company New York and London

Divide the shell to a convenient number of strips of length Δs_1, Δs_2 Δs_3 etc determine the area of each strip ΔA and the co-ordinates z and y of its center of gravity The co-ordinates z_G and y_G of the centroid G can be calculated from the relations

$$z_G = \sum z \ \Delta A/A \qquad \text{and} \qquad y_G = \sum y \ \Delta A/A$$

When the position of the centroid is thus determined the moments and product of inertia about the axes Gz_1 and Gy_1, passing through the centroid and parallel to $\bar{O}z$ and $\bar{O}y$ respectively, are next calculated That is

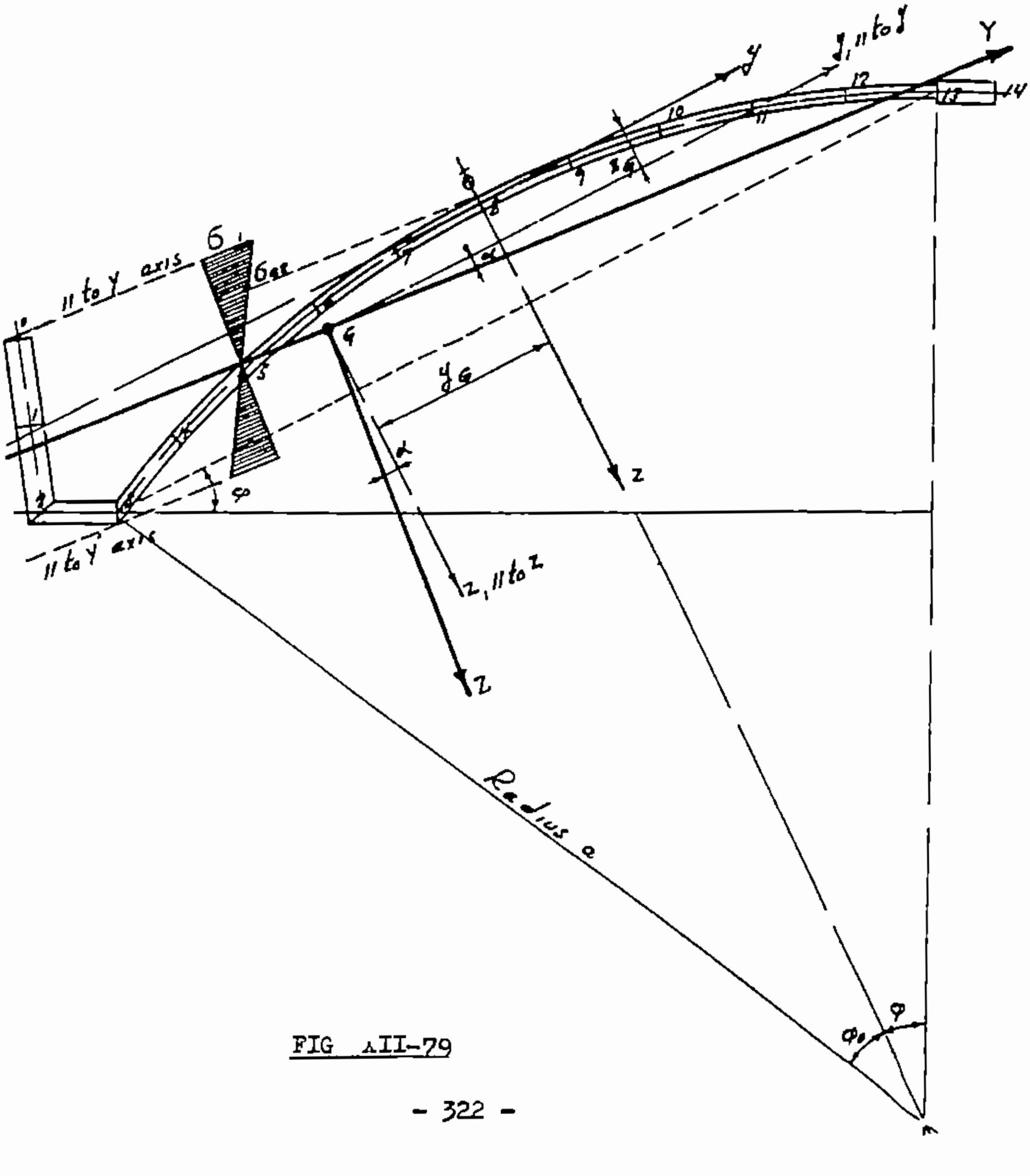


FIG AII-79

$$I_{z_1 z_1} = \sum \Delta A \, y^2 - A \, y_G^2$$

$$I_{y_1 y_1} = \sum \Delta A \, z^2 - A \, z_G^2$$

$$I_{y_1 z_1} = \sum \Delta A \, y \, z - A \, y_G \, z_G$$

From these values, the orientation of the principal axes of inertia G Z and G Y can be fixed from the known relation

$$\tan 2\alpha = 2 I_{y_1 z_1} / (I_{z_1 z_1} - I_{y_1 y_1})$$

(The angle α is to be taken in the clockwise direction from G z_1)

The principal moments of inertia are therefore given by

$$I_{ZZ} = I_{z_1 z_1} \cos^2\alpha + I_{y_1 y_1} \sin^2\alpha - I_{y_1 z_1} \sin 2\alpha$$

$$I_{YY} = I_{y_1 y_1} \cos^2\alpha + I_{z_1 z_1} \sin^2\alpha + I_{y_1 z_1} \sin 2\alpha$$

2) Bending Moments, Longitudinal Stresses σ_x and Longitudinal Forces N_x

The co-ordinates of the points with reference to the principal axes G Z and G Y are next calculated using the formulas

$$Z = (z - z_G) \cos\alpha - (y + y_G) \sin\alpha$$

$$Y = (z - z_G) \sin\alpha + (y + y_G) \cos\alpha$$

The load p is resolved into its components p_Z and p_Y along the two principal axes p_Z causes bending about Y Y axis and p_Y about the Z Z axis The moments caused by p_Z and p_Y are denoted by M_Z and M_Y , respectively

The longitudinal stress $\sigma_x = N_x / t$ at any point in the shell is then given by the formula

$$\sigma_x = N_x / t = \frac{M_Z}{I_{YY}} Z + \frac{M_Y}{I_{ZZ}} Y$$

where

$$M_Z = \frac{p_Z \, l^2}{8} \quad \text{and} \quad M_Y = \frac{p_Y \, l^2}{8}$$

in which

$$p_Z = p \cos(\varphi_0 - \alpha) \quad \text{and} \quad p_Y = p \sin(\varphi_0 - \alpha)$$

p being the vertical load per square meter surface

Assuming $L_Z / I_{YY} = (F_Z)$ and $L_Y / I_{ZZ} = (F_Y)$ we can write

$$\sigma_x = N_x/t = (F_Z)\, Z + (F_Y)\, Y$$

This expression gives the values of σ_x and N_x at all points of the shell they are equal to zero at the neutral axis Hence, the equation of the neutral axis is given by

$$0 = (F_Z)\, Z + (F_Y)\, Y$$

Assuming further $(F_Z) / (F_Y) = (F)$

the equation of the neutral axis can be written in the form

$$(F)\ Z + Y = 0$$

The longitudinal force N_x on each elemental area is determined by multiplying the stress σ_x at the center of the element by its area ΔA

In order to have sufficient safety against buckling the maximum normal stress in the shell σ_{c2} must be smaller than σ_c max shown in figure XII-51, while the stiffeners shown in figure XII-77 prevent the lateral buckling of the edge beam

The correctness of the statical calculation can be verified if the following two conditions are satisfied

a) The sum of the normal forces acting on the section = 0, or

$$\sum N_x\ \Delta s = 0$$

b) The internal resisting moment is equal to the bending moment due to loading, i e ,

$$\sum N_x\ \Delta s\ y = M$$

The check is made at the midspan section and the moments are taken about the Oy-axis which is equal to $p \cos \varphi_0\, l^2 / 8$

3) Shear Stresses

In case of saw-tooth shells, the direction of the shearing force Q does not coincide with the principal axes Z and Y of the section hence

$$\tau = \frac{Q_Z\, S_Y}{I_Y\, t} + \frac{Q_Y\, S_Z}{I_Z\, t}$$

in which

$$Q_Z = Q \cos (\varphi_o - \alpha)$$

$$Q_Y = Q \sin (\varphi_o - \alpha)$$

are the components of the shearing force along the principal Z and Y axes

However, Ramaswamy in his previously mentioned text book on shell structures computes the shear stress using the principle that in a beam subjected to uniform loading, the difference between the specific shears at any two points is equal to the longitudinal force between those two points divided by the bending moment factor which is $l^2 / 8$ for a simply supported beam The proof of this statement is as follows

It has been shown that

$$\int N_x \, ds = M S / I_{1-1} \qquad\qquad N_{x\varphi} = Q S / I_{1-1}$$

and for the section at the diaphragm, we have

$$\max N_{x\varphi} = Q_{max} S / I_{1-1}$$

So that

$$N_{x\varphi} / Q = \max N_{x\varphi} / Q_{max} = S / I_{1-1}$$

But for a simple beam subject to uniformly distributed load $\bar{p}$, we have $Q_{max} = \bar{p}\, l /2$ so that

$$N_{x\varphi} / Q = \max N_{x\varphi} / \bar{p}\, l /2 = S / I_{1-1}$$

at middle section , we have further $M = \bar{p}\, l^2/8$ so that

$$\int N_x \, ds = \bar{p} \frac{l^2}{8} \frac{S}{I_{1-1}} = \bar{p} \frac{l^2}{8} \frac{\max N_{x\varphi}}{\bar{p}\, l /2} = \frac{l^2}{8} \frac{\max N_{x\varphi}}{l /2}$$

It has also been stated before that in a simple beam subject to uniform loads, the relation between the specific shear at the middle and the max shear at the diaphragm is given by

$$\partial N_{x\varphi} / \partial x = \max N_{x\varphi} / l /2 ,$$ so that

$$\frac{\int N_x \, ds}{l^2 / 8} = \frac{\partial N_{x\varphi}}{\partial x} \qquad \text{hence}$$

The difference between specific shears at points A and B = longit force N_x on A B / ($l^2/8$) The specific shears $\partial N_{x\varphi}/\partial x$ at the c g of the strips are obtained by adding the differences of the specific shears already obtained The shear resultant $N_{x\varphi}$ is equal to the specific shear multiplied by $l/2$

4) Arch Calculation

In order to determine the cross bending moments M_φ and axial forces N_φ, a slice of the shell, of unit length, is regarded as an arch subjected to the action of the external loads p and the specific shears $\partial N_{x\varphi}/\partial x$ Calculate first their components q_z and q_y in the directions of the axes $\bar{O}_z$ and $\bar{O}_y$ according to the relations

$$q_z = p_z + \frac{\partial N_{x\varphi}}{\partial x} \Delta z$$

$$q_y = p_y + \frac{\partial N_{x\varphi}}{\partial x} \Delta y$$

Next, add these loads starting from point 0 and ending with point 14 to get N_z and N_y according to the relations

$$N_z = \sum q_z \quad \text{and} \quad N_y = \sum q_y$$

Now starting from point 0 (and assuming that the shell is not connected to the adjacent shells), the increments of the bending moments from point to point of the cross-section are calculated as

$$\Delta M_{\varphi_0} = N_y \Delta z - N_z \Delta y$$

in which Δz and Δy are the projected lengths of the elements The summation of these increments gives the transverse bending moments M_{φ_0} at all the points of the cross-section

This procedure does not include the force in the window posts and hence the bending moments at point 14 will not be zero The cross bending moments can however be corrected due to this force in the following manner

The correcting moment $M_{\varphi 1}$ must satisfy the following conditions

1) $M_{\varphi 1} = 0$ at points o and 14

2) $M_{\varphi 1}$ at point 8 (y = o, z = o) = component of force in window

points perpendicular to the line joining 8 and 14 multiplied by the perpendicular distance between the two points

M_{φ_1} is however a linear function of y and z and may be represented as

$$M_{\varphi_1} = A\,z + B\,y + C$$

The constants A, B and C can be determined from the three conditions given under 1 and 2

The final transverse moments are thus given by

$$M_{\varphi} = M_{\varphi o} + M_1$$

According to the above mentioned principles, Ramaswamy, in his text book on shell structures has given the numerical analysis of a saw-tooth shell as that shown in Fig XII-79 The internal forces in the shell were as shown in Fig XII-80

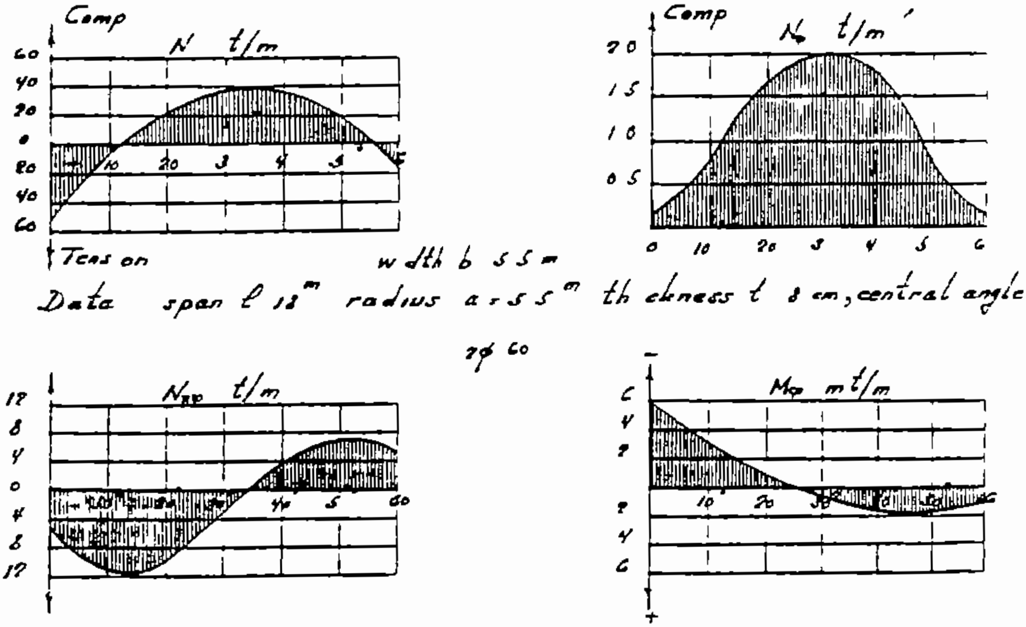
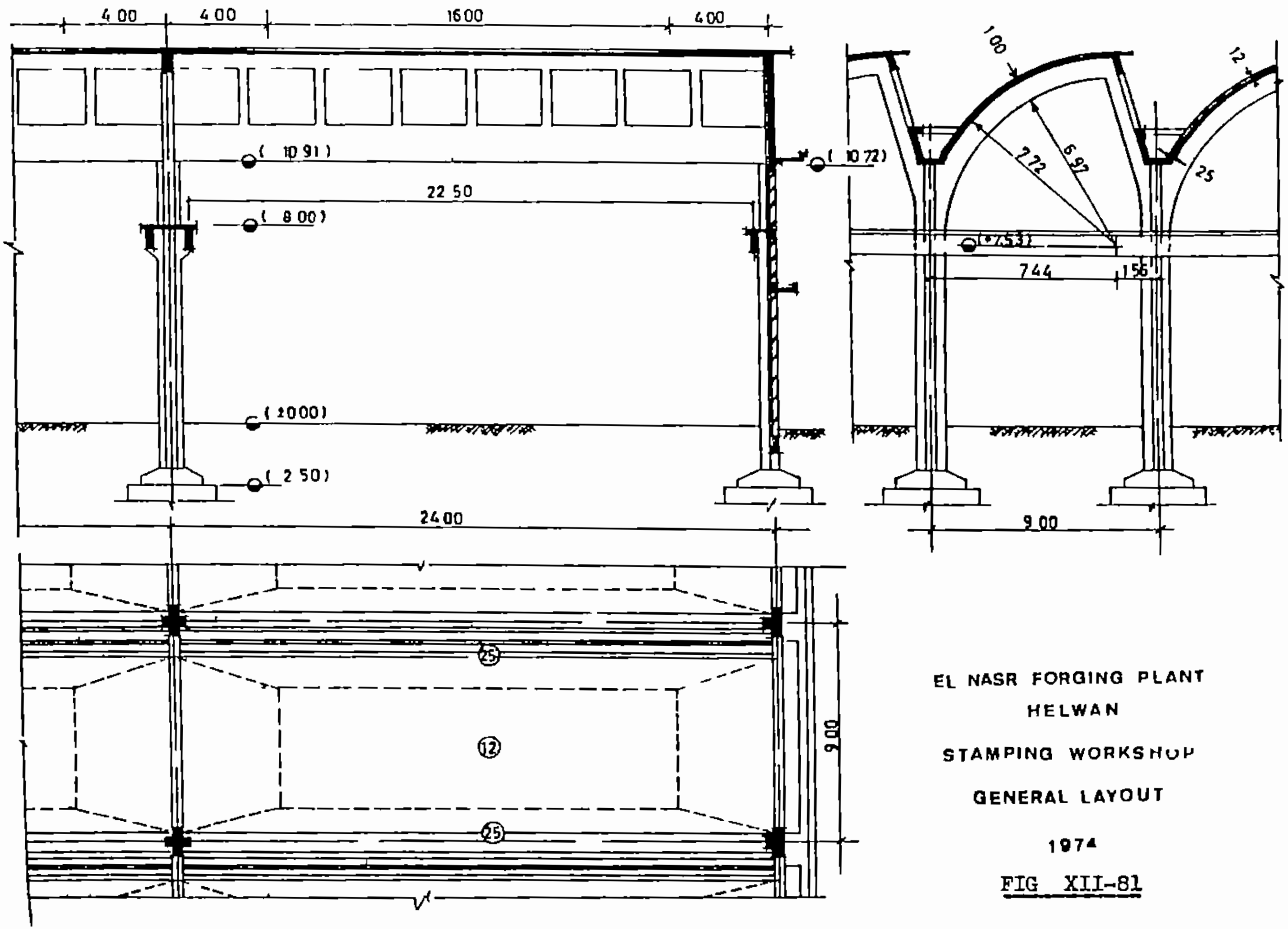


Fig XII-80

Fig XII-81 shows the general layout and main dimensions of a part of a saw-tooth shell covering one of the main halls of El Nasr Forging Plant at Helwan 48 ms wide and 144 ms long The north is parallel to the longer side of the hall and the saw-tooth form of the shell is chosen such that the windows are facing the north The distance between the windows (breadth b of saw-tooth shell) is 9 ms The bigger length of the hall, 144 ms, is divided into 5 blocks one block



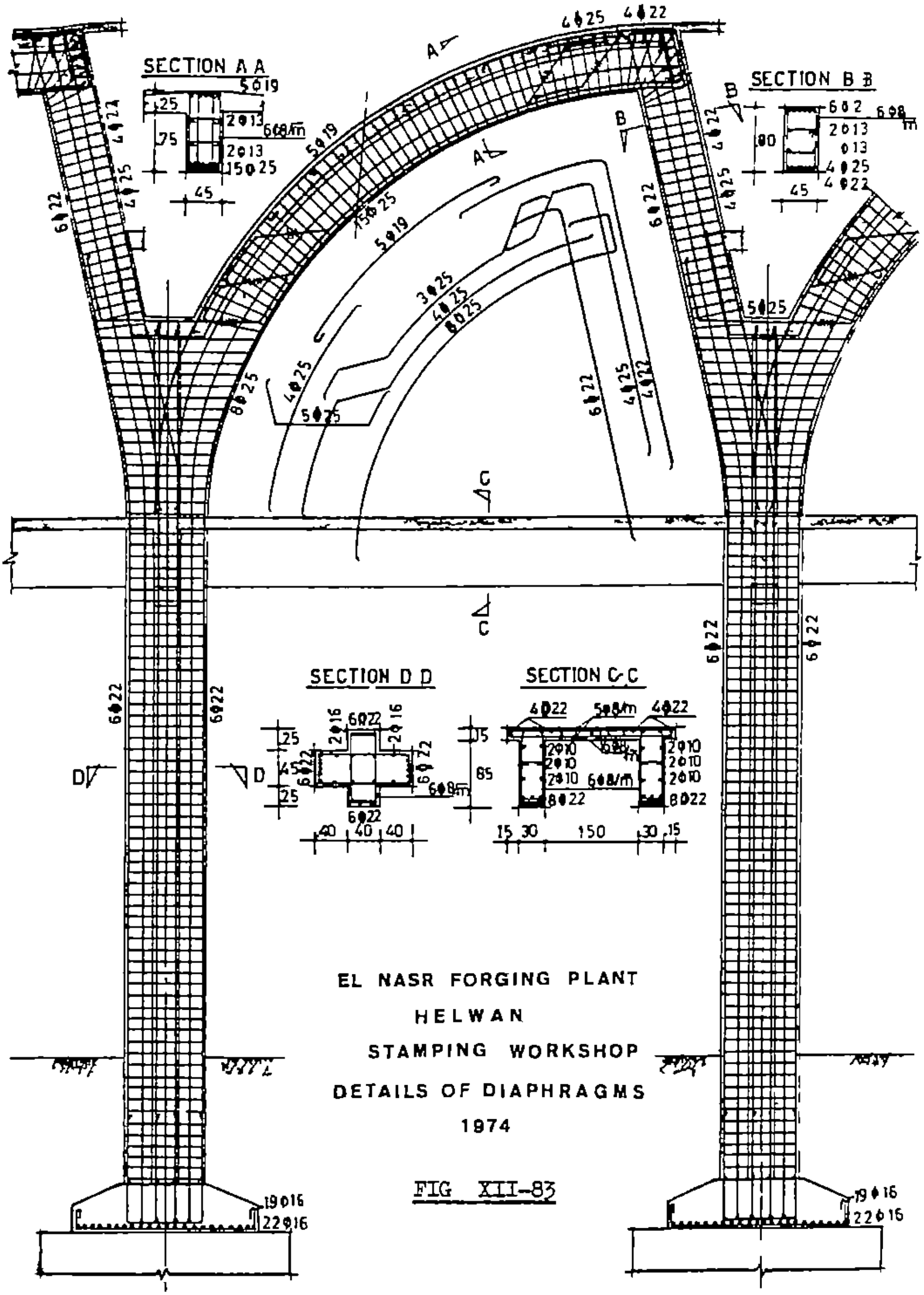
EL NASR FORGING PLANT
HELWAN

STAMPING WORKSHOP

GENERAL LAYOUT

1974

FIG XII-81



EL NASR FORGING PLANT
HELWAN
STAMPING WORKSHOP
DETAILS OF DIAPHRAGMS
1974

FIG XII-83

4 x 9 ms = 36 ms and four blocks of 3 x 9 ms = 27 ms each In the cross direction, the shell is continuous over two spans 24 ms each The radius 'a' of the shell is 7 78 ms, its height h is 4 74 ms i e $l/5$ of which, the depth h_1 of the lower edge beam is 1 50 ms i e $l/16$ The shell slab is 12 cms thick increased to 25 cms at the edge beams and diphragms

Fig XII-82 shows the details of reinforcements of the shell and its edge beams The shell slab is reinforced by two meshes, each has 6 ϕ 10 mm/m circular and 7 ϕ 8 mm/m longitudinal bars The diagonal reinforcements are laid between the two meshes in the four corners of each panel (9 x 24 ms) of the shell being 7 ϕ 13 mm/2 5 m at the lower corners and 11 ϕ 13 mm/1 5 m at the upper corners The main longitudinal reinforcements are 19 ϕ 25 + 8 ϕ 22 + 7 ϕ 19 (146 cm^2) at the lower edge of the shell, 6 ϕ 25 (30 cm^2) in the upper edge beam and 7 ϕ 16 /m top and bottom extending over a breadth of 2 5 ms in the shell slab and have a length varying between 8 and 12 ms symmetrically placed with respect to the intermediate diaphragm in order to resist the connecting moment between the two spans of the shell These reinforcements are however continued by 7 ϕ 13 mm/m top and bottom over both sides Top longitudinal reinforcements 7 ϕ 13/m are also arranged at the outside diaphragms for a length of 4 ms to resist the local bending moments that are liable to take place at the edges

Fig XII-83 shows the details of reinforcements of the intermediate diaphragm which has to resist in addition to its own weight, the central shear N_x from the two spans of the shell acting along their curved axis, the crane loads, the wind loads and any other loads or actions that are liable to affect its design

6- Short Shells

In the calculation of short shells, the membrane theory is applied This theory, however, represents the actual condition of stresses only if the load varies sufficiently smoothly over the shell surface, and it is not applicable to the immediate vicinity of the edges We shall deal here with the perturbations which originate from the discontinuities of the load and partly with perturbations originating from the edge

Edge perturbations are the most important in short shells If the shell is free, the membrane theory does not satisfy the two edge conditions $N_\varphi = 0$ and $N_{\varphi x} = 0$ When the edge values of N_φ and $N_{\varphi x}$ have been computed by the membrane theory, it is therefore necessary, afterwards, to consider a case where the edge carries the loads - $_{\varphi}$ and - $N_{\varphi x}$

The load - $_{\varphi x}$ does not give any essential perturbation (small) more important than the edge load from $N_{\varphi x}$ is the load originating from $N_{\varphi} = p_r\, r$ For vertical loads p this force is $N_{\varphi} = - p\, r \cos\varphi_0$ at the edge This means that the shell is under the influence of a tangential line load at the edge of $P = p\, r \cos\varphi_0$ (Fig XII-84) This load is transferred to the traverses by the lowermost part of the shell which acts as a slightly curved deep beam with span l and width t

Since the membrane shear forces are small, the stress distribution in a short shell may be imagined in such a way that the shell carries the load as a vault, and at the springings the normal forces in the vault are taken by the edge zones, which transfer their load to the traverses

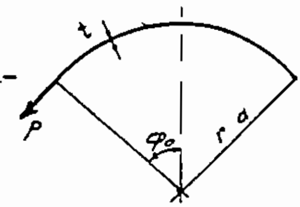


Fig XII-84

According to the theory of deep beams*, we have:

For a simple deep beam of span l subject to a uniform load P

$$\max\ T = 0\ 2\ P\ l \qquad (1)$$

Assuming $y_{CT} = 0\ 5\ l$, we get for continuous beams

At middle of outer spans and over intermediate supports

$$\max\ T = 0\ 2\ P\ l \qquad (2)$$

and at middle of inner spans

$$\max\ T = 0\ 13\ P\ l \qquad (3)$$

where max T is the tension in the critical sectins of the deep beam

Central shear

Since the tensile force in the deep beam varies approx parabolically between the traverses, the maximum central shear at the traverses may be set equal to

$$\max\ N_x = 4\ T/l \qquad (4)$$

Trasverse bending moments

According to Lundgren, the transverse bending moments may be taken

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* Refer to Theory and Design of Reinforced Concrete Tanks by A Hilal Published by J Marcou & Co Cairo

as a function of the two dimensionless parameters ρ and χ which may be found from the relations

$$\rho^2 = 5.85 \frac{a}{l}\sqrt{\frac{a}{t}} \qquad (5) \qquad \text{and} \qquad \chi = 1.688 \frac{\sqrt{t\,a}}{l} \qquad (6)$$

For the transverse bending moment, the shell is assumed simply supported over the traverses because, so far, the analytical method is not practicable in other cases. This is a very important limitation since, in general, short shells are actually continuous over several spans.

For a shell simply supported at the springings, the transverse bending moments are negative, with a maximum value of

$$\max M_\varphi = - \frac{1.4}{(1 + 3\chi)^2} \; \frac{P\,a}{\rho^2} \qquad (7)$$

The corresponding normal force is given by

$$N_\varphi = - \frac{0.84}{(1 + 1.4\chi)^2} \; P \qquad (8)$$

It has to be noted that a free edge at the springing is seldom encountered in practice but even if the edge is strengthened by a light edge beam, the formulea 7 and 8 for M_φ and N_φ may very well be applied.

If the support can take a restraining moment, it is reasonable to base the design on the following case.

For a shell fixed at the springings, the maximum negative fixing moment at the edge is given by

$$\max M_\varphi = - 0.50 \, (1 + 0.15\chi) \, \frac{P\,a}{\rho^2} \qquad (9)$$

The corresponding normal force is P.

<u>Example</u>

Fig. XII-85 represents the cross section of a circular short shell with a radius a = 27.4 ms and a width b = 35 ms. The rise f = 6.32 ms and the corresponding angle at the springing φ_0 = 39.7° = 0.693 radians. The length of the shell l = 10 ms and its thickness t = 10 cms.
Assuming that every second span is covered by a skylight, the shell may be considered as simply supported at

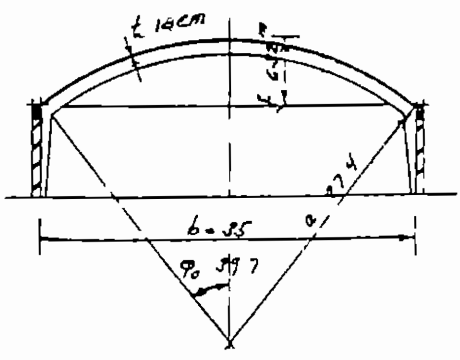

Fig. XII-85

at the traverses At the springings, the shell is provided with edge beams which rest on vertical columns or walls An investigation has shown that if the edge beams are of the order say 25 x 50 cms, their torsional rigidity is sufficient to secure complete clamping of the shell Therefore the condition for applying formula 9 is fairly satisfied

The internal forces in the shell be calculated for a total vertical load equal to p, where $p = 350$ kg/m^2 surface

The shell is first calculated by the membrane theory Hence

Maximum transverse comp at crown $N_\varphi = -\, p\, a = -350 x 27\ 4 = -9590$ kg/m

Corresponding conc comp stress $\sigma_c = \frac{N}{A_c} = \frac{9590}{100 \times 10} = 9\ 59$ kg/cm^2

Maximum allowed buckling stress $\sigma_c = 65/(1 + \frac{2740}{200 \times 10}) = 27\ 4$ kg/cm^2

Max shear at the springing $N_{x\varphi} = p l \sin\varphi_o = 350 x 10 x\ 639 = 2236$ kg/m

Corresponding shear stress $\tau = N_{x\varphi} / A = 2236/100x10 = 2\ 24$ kg/cm^2

Membrane tesile force at middle of span (x = 0)

$$T_o = \int_x^{\frac{l}{2}} N_{x\varphi}\, dx = \int_0^{\frac{l}{2}} p\, x \sin\varphi_o\, dx = p\,\left(\frac{l^2}{4} - x^2\right) \sin\varphi_o \qquad \text{or}$$

$$T_o = 350 \times \frac{10^2}{4} \times 0\ 639 = 5590 \text{ kgs}$$

Perturbational load at edge $P = -N_{\varphi_o} = p\, a \cos\varphi_o$ but

$$\cos\varphi_o = \frac{a - f}{a} = \frac{27\ 4 - 6\ 32}{27\ 4} = 0\ 769 \qquad \text{therefore}$$

$$P = 350 \times 27\ 4 \times 0\ 769 = 7374 \text{ kg/m'}$$

Tension due to P (equ 2) $T_p = 0\ 2\ P\ l = 0\ 2x7374x10 = 14740$ kgs

Total tensile force $T = T_o + T_p = 5590 + 14740 = 20330$ kgs

Required reinforcement $A_s = \frac{T}{\sigma_s} = \frac{20330}{1400} = 14\ 5$ cm^2 chosen 5 Ø 19 mm

Max shear at traverses according to equation 4 is given by

$$\max N_{x\varphi} = \frac{4}{l} T = \frac{4}{10} \times 20\ 33 = 8\ 13 \text{ t/m}$$

If this shear is to be resisted by transverse reinforcements (stirrups) xc usive v a.. .rea of $_s = \frac{o\ 1}{-\ ?} = 6\ 25\ cm^2/m$. e 6 Ø 1c mm/- is sufficient However, a part of the shear may be resisted by bent up bars from the longitudinal reinforcement The design corresponds to that of an ordinary reinforced concrete deeep beam with an arm of internal resistance $v_{CT} = 0\ 5\ l = 5\ 0$ ms

The maximum transverse bending moment at the springings is, according to equ 9, given by

$$\max\ M_\varphi = -\ 0\ 50\ (1 + 0\ 15\,\gamma)\ \frac{P\ a}{\rho^2}$$

in which

$$\gamma = \frac{1\ 688\sqrt{a\ t}}{l} = \frac{1\ 688\sqrt{27\ 4 \times 0\ 1}}{10} = 0\ 28$$

and

$$\rho^2 = 5\ 85\ \frac{a}{l}\sqrt{\frac{a}{t}} = 5\ 85 \times \frac{27\ 4}{10}\sqrt{\frac{27\ 4}{0\ 10}} = 265 \qquad \text{so that}$$

$$\max\ M_\varphi = -\ 0\ 5\ (1 + 0\ 15 \times 0\ 28) \times \frac{7374 \times 27\ 4}{265} = -\ 400\ \text{kgm}$$

The corresponding normal force is $N_\varphi = P = 7374$ kgs (compression) The section of the shell at the springing is to be designed for these internal force.

It is however advisable to increase the thickness of the shell gradually to 14 cms at the springing on a length λ where

$$\lambda = 0\ 76\sqrt{a\ t} = 0\ 76\sqrt{27\ 4 \times 0\ 10} = 1\ 25\ \text{ms}$$

A thickness of 14 cms and top reinforcement of minimum 6 Ø 8 mm/m at the springing are supposed to be ample for the acting forces

Fig II-86

Details of reinf

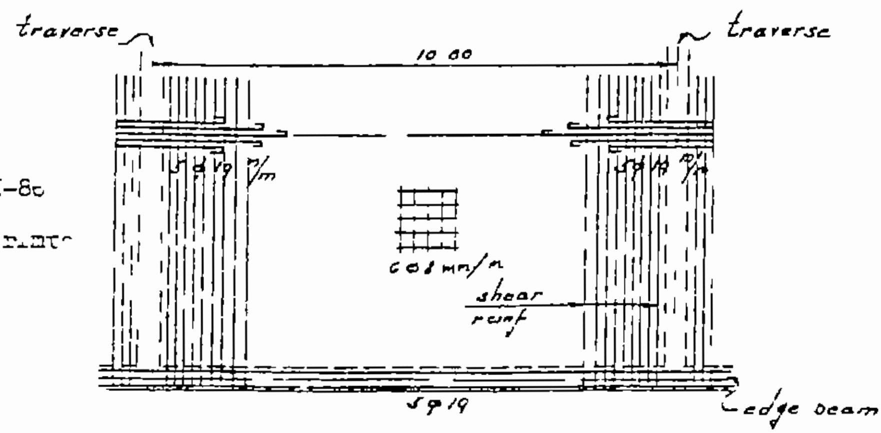

Fig XII-87 shows the general layout, main dimensions and details of reinforcements of the main supporting elements of a part of the garage of "El Nasr Steel Pipes and Fittings Co ' at Helwan The main garage covers an area 38 x 90 ms without intermediate supports Attached to it, there is a shed for cars having a span of 9 ms and a cantilever arm of 3 ms In the longitudinal direction, the hall is divided into three blocks 3 x 9 m + 4 x 9 m + 3 x 9 m = 90 ms

As a cover for the garage, a short shell 10 cms thick, increased to 14 cms over a length of 2 ms at the springing, is used Its radius is 36 5 ms, its width is 38 ms and is supported in the longitudinal direction on arched frame traverses with ties every 9 ms The traverses are 0 4 ms wide and 1 35 ms deep, increased to 2 ms at the corners and decreased to 0 7 ms at the lower hinges The side shed is covered by a 25 cms two-way hollow block slab 9 x 9 ms It is supported on longitudinal beams and inverted cross beams, 9 ms span with 3 ms overhanging cantilevers

The shell is reinforced at its bottom surface by a mesh 6 Ø 8 mm/m except for the first 8 ms from the springing where the circular reinforcement only, is increased to 6 Ø 10 mm/m in a distance of 1.5 ms at each side of the traverses Top reinforcement for the shell is needed over the traverses and normal to them for a distance of 2 5 ms at each side It is also needed at the straight edges and normal to them, for a distance of 2 ms from the springing The top longit reinforcement over the traverses is 6 Ø 13 mm/m for the middle 8 ms and then reduced to 6 Ø 10 mm/m on both sides The top circular reinforcement is 6 Ø 8 mm/m except for the first 8 ms from the springing where it is increased to 6 Ø 10 mm/m in a distance of 1 5 ms at each side of the traverses The top reinforcement at the springing is one mesh 6 Ø 8 mm/m in the space between the top reinforcements resisting the connecting moments at the traverses The details of the intermediate traverses and their ties are also shown in Fig XII-87

XII- 5 MEMBRANE THEORY OF SHELLS OF GENERAL SHAPE

1- Basic Idea

The purpose of the following is to develop a a general membrane theory for shells of arbitrary shape and then to show its application to some simple cases of shells of double curvature and especially those which do not fall in one of the special groups considered before

The basic idea of the solution, developed by Pucher, is to examine the entire system of external and internal forces acting on the shell in the plan projection It will be shown that the determination of the internal forces in many cases can be done by a surprisingly simple way

We choose for this purpose an orthogonal co-ordinate system x, y, z in which the z-axis is vertical If we cut the shell surface under consideration by two pairs of vertical planes x, x+dx and y, y+dy they meet the shell surface in curves which in general are not normat to each other (Fig XII-88) The elemental rectangular area in plan projection dA = dx dy will correspond to a rhombus on the shell surface of area τ dA in which

$$\tau = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2} \qquad (1)$$

This equation can naturally be used only if $\partial z/\partial x \neq \infty$ & $\partial z/\partial y \neq \infty$

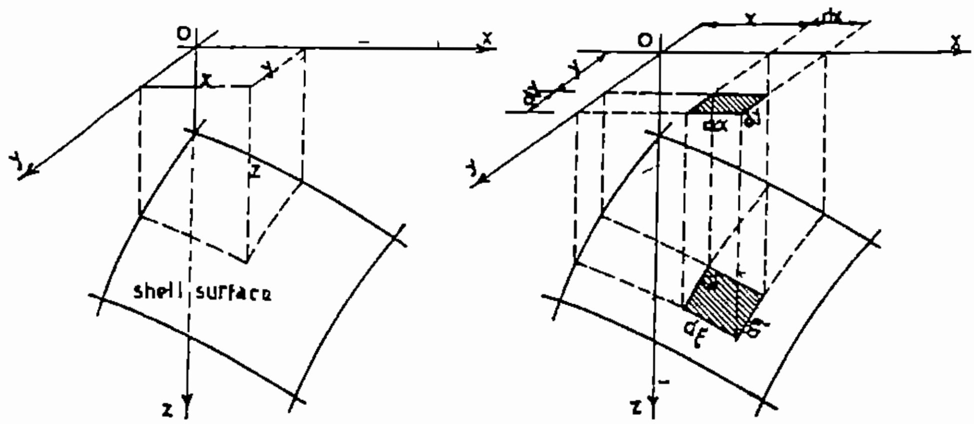

FIG XII-88

assuming that the components of the load on the shell surface are p_x, p_y, p_z and that their corresponding values on the horizontal projection are g_x g_y g_z then we have

$$\underline{g_x = \lambda p_x} \qquad \underline{g_y = \lambda p_y} \qquad \underline{g_z = \lambda p_z} \qquad (2)$$

It will be assumed further that the real normal and shearing forces on the sides of the elemental area of the shell are N_x, N_{xy}, N_{yx} N_y and the corresponding reduced values on the plan projection n_x, n_{xy}, n_{yx}, n_y as shown in figure XII-89

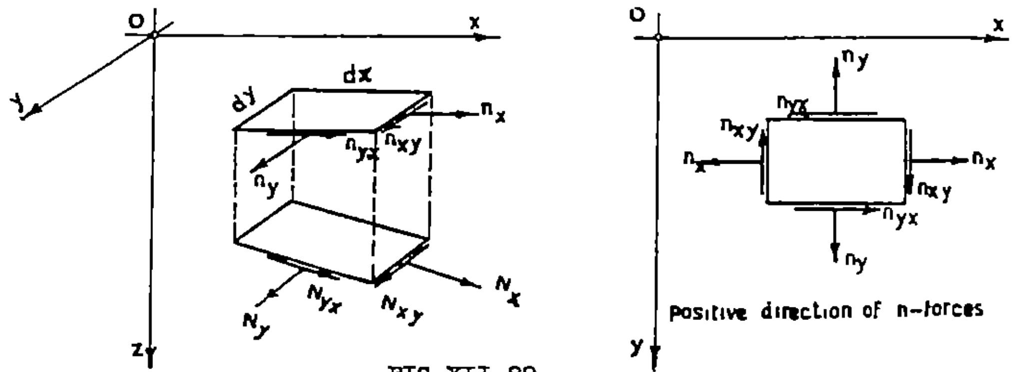

FIG XII-89

The relation between the real and reduced forces can be expressed in the following manner (Fig XII-90)

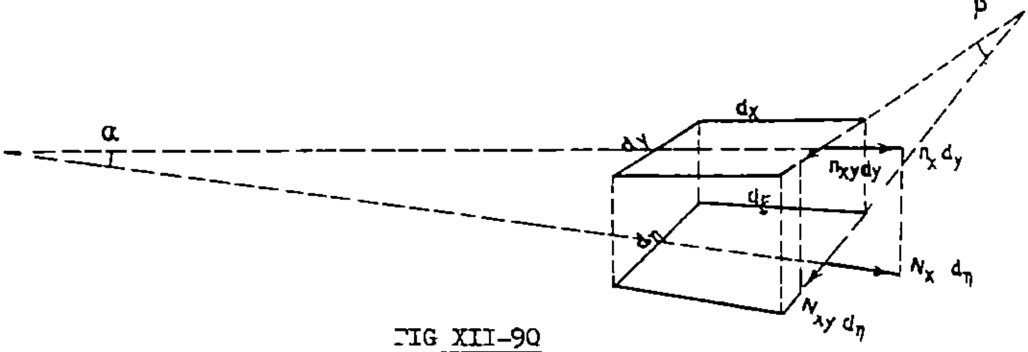

FIG XII-90

$$n_x \; dy = N_x \; d\eta \quad \cos\alpha \, ,$$

$$n_{xy} \; dy = N_{xy} \; d\eta \quad \cos\beta$$

so that

$$N_x = \frac{n_x}{\cos\alpha}\,\frac{dy}{d\eta} = \frac{n_x}{\cos\alpha}\,\cos\beta$$

$$N_{xy} = \frac{n_{xy}}{\cos\beta}\,\frac{dy}{d\eta} = \frac{n_{xy}}{\cos\beta}\,\cos\beta = n_{xy}$$

Similar equations can be extracted for the relations between n_y, n_{yx} and N_y, N_{yx}. Hence

$$\left.\begin{aligned} &\underline{N_x = n_x \frac{\cos\beta}{\cos\alpha}}\ , \quad \underline{N_{xy} = n_{xy}} \\ &\underline{N_y = n_y \frac{\cos\alpha}{\cos\beta}}\ , \quad \underline{N_{yx} = n_{yx}} \\ \text{but } &\underline{N_{xy} = N_{yx}} \ \text{ then } \ \underline{n_{xy} = n_{yx}} \end{aligned}\right\} \qquad (3)$$

This means that the reduced shearing forces are also equal. The values of $\cos\alpha$ and $\cos\beta$ are given by the following relations

$$\tan\alpha = \frac{\partial z}{\partial x} \qquad \text{and} \qquad \tan\beta = \frac{\partial z}{\partial y} \qquad (4)$$

Hence, α and β and the cos-values required for calculating N_x and N_y can be easily determined.

These equations can naturally only be used if

$$\partial z/\partial x \neq \infty \qquad \partial z/\partial y \neq \infty$$

The previous relations give the reduced internal forces in a point in the two normal directions x and y. The forces acting at the same point in any two other normal directions u and v making an angle α with x and y can be determined in the same way known in plane structures, thus (Fig XII-91)

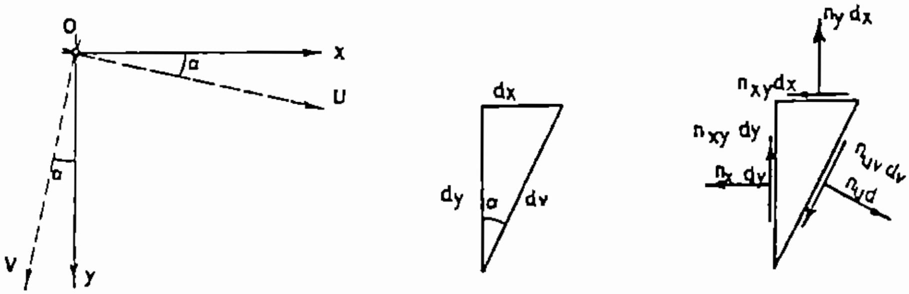

FIG XII-91

$$n_u = n_x \cos^2\alpha + n_y \sin^2\alpha + 2\, n_{xy} \cos\alpha \sin\alpha$$

$$n_v = n_x \sin^2\alpha + n_y \cos^2\alpha - 2\, n_{xy} \cos\alpha \sin\alpha$$

$$n_{uv} = n_{vu} = (n_y - n_x) \cos\alpha \sin\alpha + n_{xy} (\cos^2\alpha - \sin^2\alpha)$$

The magnitude and directions of the principal reduced (or actual forces can therefore be calculated by the known relations

$$n_{\frac{1}{2}} = \frac{n_x + n_y}{2} \pm \sqrt{\left(\frac{n_x - n_y}{2}\right)^2 + n_{xy}^2} \qquad (5)$$

and

$$\tan 2\alpha = \frac{2\, n_{xy}}{n_x - n_y} \qquad (6)$$

2- Conditions of Equilibrium

We give in the following the conditions of equilibrium of the reduced forces n_x n_y and $n_{xy} = n_{yx}$ acting on the sides dx and dy of the plan projection of an element of a shell due to the reduced load components g_x , g_y and g_z

2- 1 Condition of Equilibrium in the x - Direction

The reduced forces and loads acting in the x- direction are shown in figure XII-92 One has to notice that

$$dn_x = \frac{\partial n_x}{\partial x} dx \quad \text{and} \quad dn_{xy} = \frac{\partial n_{xy}}{\partial y} dy$$

The condition of equilibrium in the x direction is given by

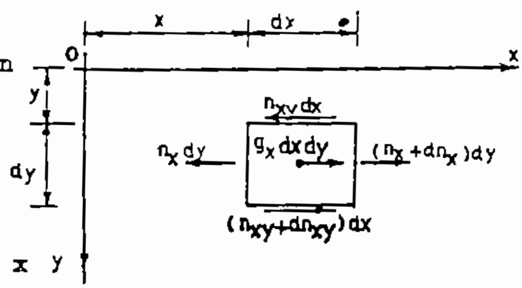

FIG XII-92

$$dn_x\, dy + dn_{xy}\, dx + g_x\, dx\, dy = 0$$

Substituting for dn_x and dn_{xy} , we get

$$\frac{\partial n_x}{\partial x} dx\, dy + \frac{\partial n_{xy}}{\partial y} dy\, dx + g_x\, dx\, dy = 0 \qquad \text{or}$$

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_{xy}}{\partial y} + g_x = 0 \qquad (7)$$

2- 2 Condition of Equilibrium in the y- Direction

The conditon of equilibrium in the y - direction can similarly be expressed by the relation (Fig. XII-93)

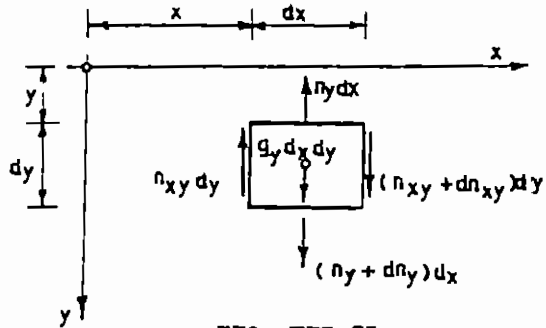

FIG XII-93

$$\frac{\partial n_y}{\partial y} + \frac{\partial n_{xy}}{\partial x} + g_y = 0 \qquad (8)$$

2-3 Condition of Equilibrium in the z - Direction

When writing the condition of equilibrium in the z-direction, the vertical components of the real forces are to be considered in the following manner (Fig XII-94)

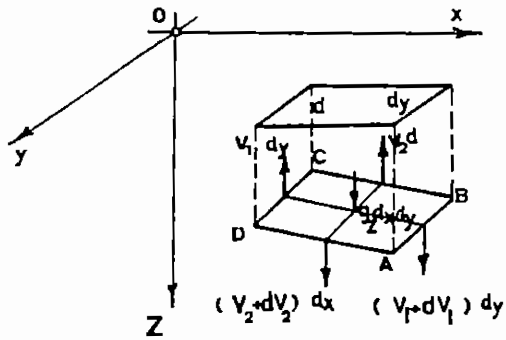

FIG XII-94

The vertical component of N_x along the edge CD is given by

$$n_x \frac{\partial z}{\partial x}$$

whereas that of N_{xy} along the same edge is

$$n_{xy} \frac{\partial z}{\partial y}$$

so that the total vertical component along CD is given by

$$V_1 = n_x \frac{\partial z}{\partial x} + n_{xy} \frac{\partial z}{\partial y}$$

Similarly , along the edge BC ,we have

$$V_2 = n_y \frac{\partial z}{\partial y} + n_{yx} \frac{\partial z}{\partial x}$$

Therefore the condition of equilibrium in the z - direction can be given in the form

$$dV_1 \, dy + dV_2 \, dx + g_z \, dx \, dy = 0$$

in which

$$dV_1 = \frac{\partial V_1}{\partial x}\, dx \quad \text{and} \quad dV_2 = \frac{\partial V_2}{\partial y}\, dy$$

So that

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \varepsilon_z = 0$$

Introducing the values of V_1 and V_2 , we get

$$\frac{\partial}{\partial x}\left(n_x \frac{\partial z}{\partial x} + n_{xy} \frac{\partial z}{\partial y} \right) + \frac{\partial}{\partial y}\left(n_y \frac{\partial z}{\partial y} + n_{yx} \frac{\partial z}{\partial x} \right) + \varepsilon_z = 0$$

Solving this equation we get

$$n_x \frac{\partial^2 z}{\partial x^2} + \frac{\partial n_x}{\partial x} \frac{\partial z}{\partial x} + n_{xy} \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial n_{xy}}{\partial x} \frac{\partial z}{\partial y} + n_y \frac{\partial^2 z}{\partial y^2} + \frac{\partial n_y}{\partial y} \frac{\partial z}{\partial y}$$

$$+ n_{yx} \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial n_{yx}}{\partial y} \frac{\partial z}{\partial x} + \varepsilon_z = 0$$

This equation can also be given in the form

$$n_x \frac{\partial^2 z}{\partial x^2} + (n_{xy} + n_{yx}) \frac{\partial^2 z}{\partial x \partial y} + n_y \frac{\partial^2 z}{\partial y^2} + \left(\frac{\partial n_x}{\partial x} + \frac{\partial n_{yx}}{\partial y} \right) \frac{\partial z}{\partial x} +$$

$$+ \left(\frac{\partial n_y}{\partial y} + \frac{\partial n_{xy}}{\partial x} \right) \frac{\partial z}{\partial y} + \varepsilon_z = 0$$

But we have $\quad n_{xy} = n_{yx} \quad$ and

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_{yx}}{\partial y} = -\varepsilon_x \qquad \frac{\partial n_y}{\partial y} + \frac{\partial n_{xy}}{\partial x} = -\varepsilon_y$$

so that the equilibrium equation in the z - direction can be given in the form

$$n_x \frac{\partial^2 z}{\partial x^2} + 2\, n_{xy} \frac{\partial^2 z}{\partial x \partial y} + n_y \frac{\partial^2 z}{\partial y^2} + \bar{\varepsilon}_z = 0 \qquad (9)$$

in which

$$\bar{\varepsilon}_z = \varepsilon_z - \varepsilon_x \frac{\partial z}{\partial x} - \varepsilon_y \frac{\partial z}{\partial y} \qquad (10)$$

For shells subject to vertical loads only

$$\varepsilon_x = 0 \quad , \quad \varepsilon_y = 0 \quad \text{and} \quad \bar{\varepsilon}_z = \varepsilon_z \qquad (11)$$

The Pucher Stress Differential Equation

In order to simplify the problem, we will limit the following studies to <u>shells subjected to vertical loads only</u>. In this special case the three conditions of equilibrium can be expressed by the relations

$$\left.\begin{aligned} \frac{\partial n_x}{\partial x} + \frac{\partial n_{yx}}{\partial y} &= 0 \\ \frac{\partial n_y}{\partial y} + \frac{\partial n_{xy}}{\partial x} &= 0 \\ n_x \frac{\partial^2 z}{\partial x^2} + 2\, n_{xy} \frac{\partial^2 z}{\partial x\, \partial y} + n_y \frac{\partial^2 z}{\partial y^2} + g_z &= 0 \end{aligned}\right\} \qquad (12)$$

The determination of the reduced internal forces in shells by the use of these three simultaneous differential equations is generally complicated.

In order to simplify the problem Pucher has introduced a stress function F such that

$$\left.\begin{aligned} n_x &= \frac{\partial^2 F}{\partial y^2} \\ n_{xy} = n_{yx} &= - \frac{\partial^2 F}{\partial x\, \partial y} \\ n_y &= \frac{\partial^2 F}{\partial x^2} \end{aligned}\right\} \qquad (13)$$

With these relations, the first two conditions of equilibrium will be satisfied and it is sufficient to satisfy the third condition which can be put in the form

$$\frac{\partial^2 z}{\partial x^2}\ \frac{\partial^2 F}{\partial y^2} - 2\, \frac{\partial^2 z}{\partial x \partial y}\ \frac{\partial^2 F}{\partial x\, \partial y} + \frac{\partial^2 z}{\partial y^2}\ \frac{\partial^2 F}{\partial x^2} + g_z = 0 \qquad (14)$$

giving <u>Pucher differential equation</u>' for shells subjected to vertical loads only.

With respect to the orthogonal axes x, y, z, the stress function F represents a stress surface independent of the origin O and the position of the horizontal plane x y.

When dealing with shell problems , the edge conditions of the shell define the shape of the stress surface at the edges and hence they must be taken in consideration as will be shown in the following illustrative examples

4- Illustrative Examples

4- 1 Paraboloid Shell of Revolution with an Equilateral Triangular Plan

Shells supported on three points and bounded by three arches at the edges as shown in figure XII-95-a may give an architecturally striking solution for the roof covering of relatively big span halls

It will be assumed that the edge arches can resist loads in thier planes only and have no resistance to lateral forces so that the force components normal to them must be equal to zero In order to satisfy this condition , we should have

$$F_{edge} = 0 \qquad (15)$$

A paraboloid of revolution as shown in figure XII-91-b has the equation

$$z = \frac{h}{4\,a^2} (x^2 + y^2) \qquad (16)$$

Our study will be limited to shells subjected to uniformly

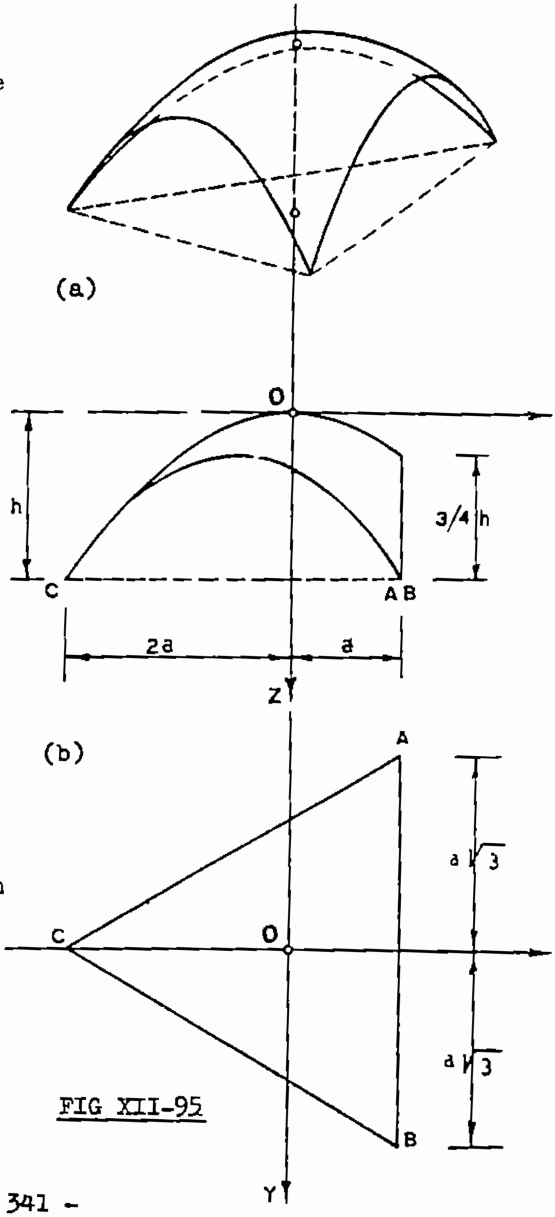

FIG XII-95

distributed load g In this case we have

$\partial^2 z/\partial x^2 = h / 2 a^2$, $\partial^2 z/\partial x \; \partial y = 0$, $\partial^2 z/\partial y^2 = h / 2a^2$

so that Pucher differential equation can be given in the form

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = - \frac{2 a^2}{h} g \qquad (17)$$

In order to satisfy the edge condition stated before we have further

$$F_{edge} = 0$$

A function equal to zero along the three sides of the ground plan can be constructed as follows

a) Write the equations of the three sides of the triangular plan of the shell such that the right hand side of each equation is equal to zero Thus

$$a - x = 0 \quad , \quad y - \frac{2 a}{\sqrt{3}} - \frac{x}{\sqrt{3}} = 0 \quad \text{and} \quad y + \frac{2 a}{\sqrt{3}} + \frac{x}{\sqrt{3}} = 0$$

b) The stress function F will be equal to a constant C multiplied by the left hand side of the equations of the sides Hence

$$F = C (a - x)(y - \frac{2 a}{\sqrt{3}} - \frac{x}{\sqrt{3}})(y + \frac{2 a}{\sqrt{3}} + \frac{x}{\sqrt{3}})$$
$$= C (a x^2 + a y^2 + \frac{x^3}{3} - x y^2 - 4 \frac{a^3}{3})$$

c) It is possible to prove that this function satisfies Pucher differential equation if

$$C = - a g / 2 h$$

So that the stress function* F is given by

$$F = - \frac{ag}{2h} (a x^2 + a y^2 + \frac{x^3}{3} - x y^2 - 4 \frac{a^3}{3}) \qquad (18)$$

and the reduced internal forces* by

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* Refer to Csonka 'Membranschalen' Bauingenieur - Praxis No 16 Published by Wilhelm Ernst & Sohn Berlin Munchen

$$
\left.
\begin{aligned}
n_x &= \frac{\partial^2 F}{\partial y^2} = - \frac{a\,g}{h}(a - x) \\
n_{xy} &= \frac{\partial^2 F}{\partial x\, \partial y} = - \frac{a\,g}{h}\, y \\
n_y &= \frac{\partial^2 F}{\partial x^2} = - \frac{a\,g}{h}(a + x)
\end{aligned}
\right\} \qquad (19)
$$

The internal forces N_x $N_{xy} = N_{yx}$ and N_y acting on the shell can be calculated using equations 3 and 4 as follows

According to equation 4 , we have

$$\cos\alpha = 1 / \sqrt{1 + (\partial z/\partial x)^2} \qquad \text{and} \qquad \cos\beta = 1 / \sqrt{1 + (\partial z/\partial y)^2}$$

but $\partial z/\partial x = h\, x / 2\, a^2$ and $(\partial z/\partial x)^2 = h^2\, x^2 / 4\, a^4$

so that

$$\cos\alpha = 1/\sqrt{1 + (h^2 x^2 / 4a^4)} \text{ , similarly } \cos\beta = 1/\sqrt{1 + (h^2 y^2/4a^4)}$$

and

$$\cos\beta / \cos\alpha = \frac{\sqrt{1 + (h^2\, x^2 / 4\, a^4)}}{\sqrt{1 + (h^2\, y^2 / 4\, a^4)}} = \sqrt{\frac{4\, a^4 + h^2\, x^2}{4\, a^4 + h^2\, y^2}}$$

similarly

$$\cos\alpha / \cos\beta = \sqrt{\frac{4\, a^4 + h^2\, y^2}{4\, a^4 + h^2\, x^2}}$$

according to equation 3 , we have

$$
\left.
\begin{aligned}
N_x &= n_x \frac{\cos\beta}{\cos\alpha} = - \frac{a\,g}{h}(a - x) \sqrt{\frac{4\, a^4 + h^2\, x^2}{4\, a^4 + h^2\, y^2}} \\
N_y &= n_y \frac{\cos\alpha}{\cos\beta} = - \frac{a\,g}{h}(a + x) \sqrt{\frac{4\, a^4 + h^2\, y^2}{4\, a^4 + h^2\, x^2}} \\
N_{xy} &= N_{yx} = n_{xy} = n_{yx} = - \frac{a\,g}{h}\, y
\end{aligned}
\right\} \qquad (20)
$$

Example

To illustrate the application of these equations the internal forces in a triangular shell subject to uniformly distributed load g will be determined for the sepcial case

$$a = h$$

In this case , we have

$$\left.\begin{aligned} N_x &= - g (a - x) \sqrt{\frac{4 a^2 + x^2}{4 a^2 + y^2}} \\ N_y &= - g (a + x) \sqrt{\frac{4 a^2 + y^2}{4 a^2 + x^2}} \\ N_{xy} &= N_{yx} = - g y \end{aligned}\right\} \quad (21)$$

The evaluation of these values gives

1- Along A-B $\qquad x = a$

$$N_x = 0 \qquad N_y = - 2g a \sqrt{\frac{4 a^2 + y^2}{5 a^2}} = - 2g \sqrt{\frac{4 a^2 + y^2}{5}} \text{ , thus}$$

For	$y =$	0	$\pm \frac{a}{2} \sqrt{3}$	$\pm a \sqrt{3}$
	$N_y =$	$-1\ 79ga$	$-1\ 95ga$	$-2\ 36ga$

and

$N_{xy} = - g y$ (linear relation) hence

For $y = 0$, $N_{xy} = 0$ and for $y = \pm a \sqrt{3}$ $\quad N_{xy} = \mp 1\ 735\ ga$

2- Along the x-axis $\qquad y = 0$

$$N_x = - g (a - x) \sqrt{\frac{4 a^2 + x^2}{4 a^2}}$$ hence

For	$x =$	0	$+ a$	$- a$	$- 2a$
	$N_x =$	$-ga$	0	$-2\ 24ga$	$-4\ 25ga$

$$N_y = - g (a + x) \sqrt{\frac{4 a^2}{4 a^2 + x^2}}$$ hence

For	$x =$	0	$+ a$	$- a$	$- 2a$
	$N_y =$	$-ga$	$-1\ 79ga$	0	$+0\ 71ga$

We have further $\quad N_{xy} = 0$ due to symmetry

3- Along n-n $\qquad y = \frac{a}{2} \sqrt{3}$

$$N_x = - g (a - x) \sqrt{\frac{4\ a^2 + x^2}{4\ 75\ a^2}}$$ hence

For	x =	0	+ a/2	a	− a/2
	N_x =	−0 92ga	−0 47ga	0	−1 42ga

$$N_y = - g (a + x) \sqrt{\frac{4\ 75 a^2}{4\ a^2 + x^2}}$$ hence

For	x =	0	+a /2	a	− a/2
	N_y =	−1 09ga	−1 59ga	−1 95ga	−0 53ga

$$N_{xy} = - g \frac{a}{2}\sqrt{3} = - 0\ 867\ g\ a = \text{constant}$$

The calculated internal forces are shown in figure XII-96

It is clear from the given diagrams that the shell is subject to compressive stresses over its whole surface with maximum values at the corners along the bisectors of the angles The maximum compressive force is given by

$$\max\ N_x = - 4\ 25\ g\ a$$

Due to symmetry , the compressive force at point 0 is the same in all directions

$$N_{x0} = N_{y0} = -g\ a$$

Tensile stresses exist at the corners normal to the bisectors of the angles The maximum tensile force is

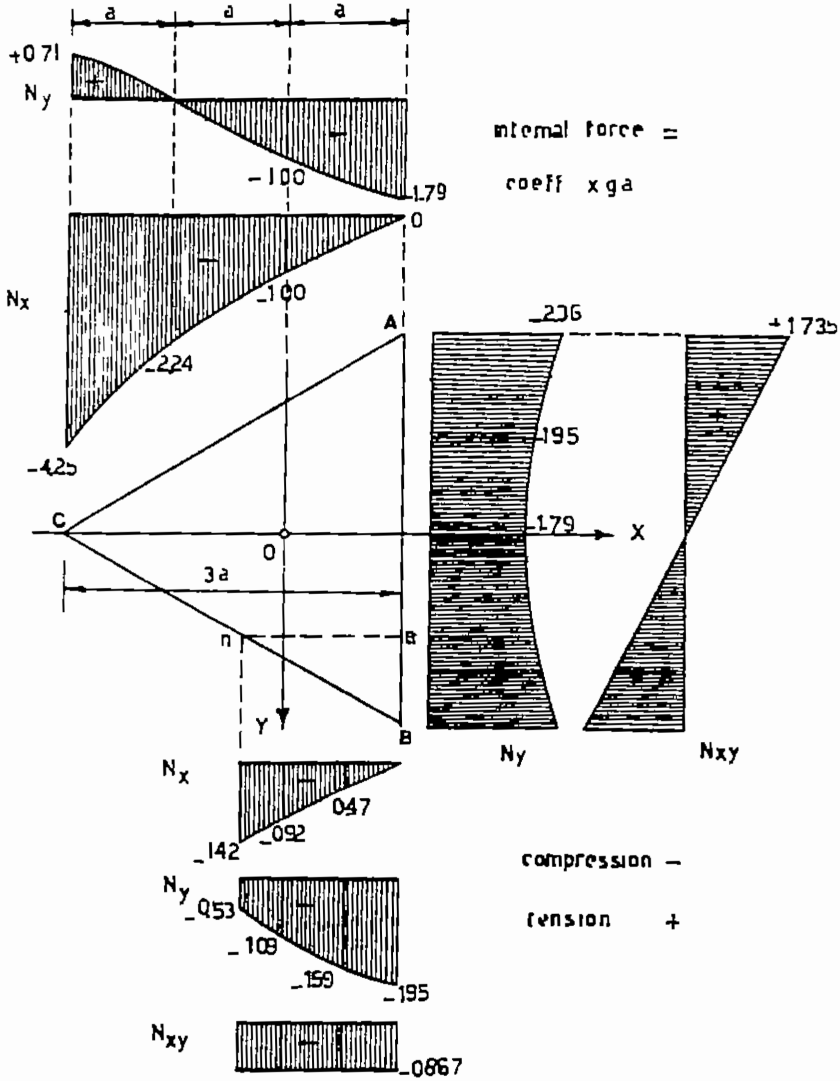
$$\max\ N_y = + 0\ 71\ g\ a$$

The biggest shear stresses exist at the edges , zero at the middle - due to symmetry - and maximum at the corners The maximum value is

$$\max\ N_{xy} = \pm 1\ 735 ga$$

At the corners , a rhomboidal element with edges parallel to those of the the shell is in a state of pure shear Hence the principal compressive and tensile stresses along the bisectors and normal to them are equal to the shear stresses , thus

$$-N_1 = N_2 = N_{xy} = 1\ 735\ g\ a$$



Stress Resultants in a Triangular Shell for a = h

<u>FIG XII-96</u>

Accordingly , the stresses in a 10 cms thick shell with 3a = 30 ms and h = 10 ms subject to a uniform vertical load g = 400 kg / m horizontal are given by

Maximum compressive stress at the corners

$max\ \sigma_x = max\ N_x / A_c = - 4.25 \times 400 \times 10 / 100 \times 10 = -17.0\ kg/cm^2$

Compressive stress at crown

$$\sigma_c = - 1.00 \times 400 \times 10 / 100 \times 10 = - 4.0$$

Principal tensile stress at corners normal to bisectors

$$\sigma_1 = +1.733 \times 400 \times 10 / 100 \times 10 = +6.94$$

Although the stresses are lo it is recomended to increase the thickness of the shell graduall/ to 15 cms at the corners in a lenghh of ca 0.5 a measured along the bisectors of the angles Accordingly, the compressive stress σ_c and the tensile stress σ_1 at the corners will be reduced to 10 / 15 = 2 / 3 the given values hence

$$max\ \sigma_x = - 17 \times 2/3 = - 11.3\ kg/cm^2$$

$$max\ \sigma_1 = +6.94 \times 2/3 = + 4.63$$

The reinforcement is generally one mesh 5ϕ8 /m in each direction except at the corners and the edges where it is recommended to use two meshes as shown in figure XII-97

The supporting arches are subject to the shearing forces of the shell acting parallel to the edge These forces are zero at the crown and increase linearly to their maximum value at the springing The sum of vertical components of these forces acting on any of the arches must be equal to 1/3 total vertical loads acting on the shell

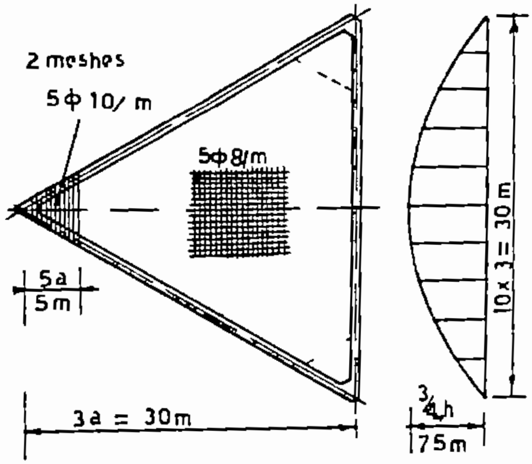


FIG XII-97

4- 2 Membrane Shells with Rectangular Ground Plan

For roof constructions, shells which permit covering rectangular big free areas (fig XII-98 a) are of much interest They are generally supported by four vertical arches that can resist vertical forces and are uncapable to resist horizontal forces

The equation of the middle surface of a paraboloid shell of revolution using an orthogonal system of coordinates O (x y z) shown on figure XII-98-b is given by

$$z = \frac{h}{2\,a^2}\,(x^2 + y^2) \quad (22)$$

Its second derivatives are

$$\partial^2 z / \partial x^2 = h / a^2$$
$$\partial^2 z / \partial x \partial y = 0$$
$$\partial^2 z / \partial y^2 = h / a^2$$

Limiting our study to the case of uniformly distributed load g/m² horizontal the Pucher differential equation can be given in the form

$$\frac{h}{a^2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + g = 0 \quad (23)$$

As the edge arches are unable to resist lateral forces the stress function F has to satisfy the boundary condition

$$F_{edge} = 0 \qquad (24)$$

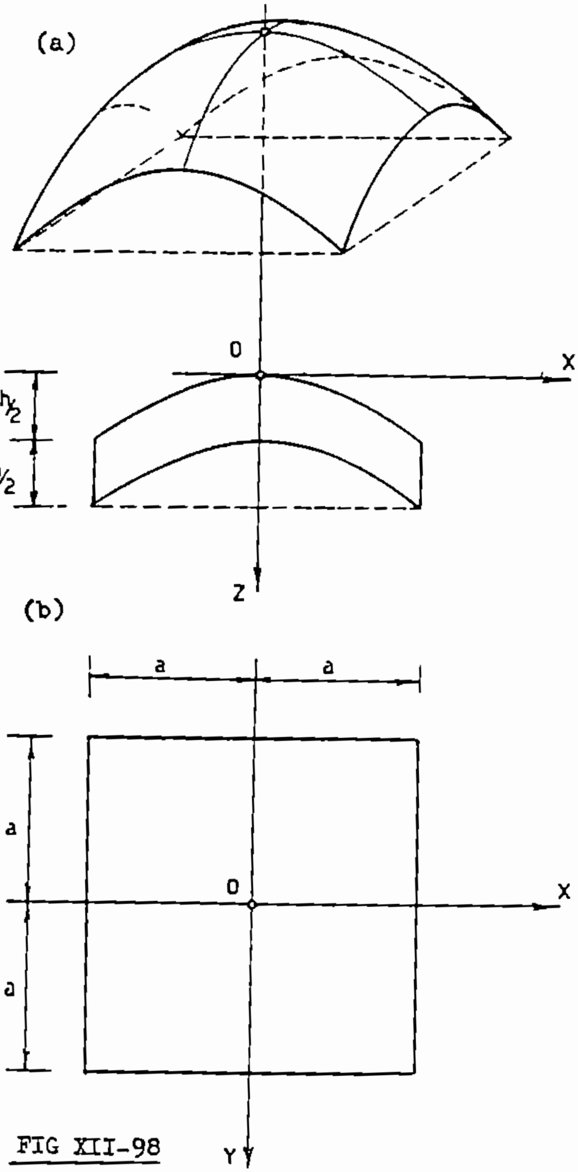


FIG XII-98

The following relation(*) gives

*) Refer to Csonka "Membranschalen Bauingenieur - Praxis No 16 Published by Wilhem Ernst & Sohn Berlin München

an approximate expression for the stress function satisfying equations (23) and (24)

$$F = \frac{a^4}{24\ h} g \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right)\left(7 + \frac{x^2+y^2}{a^2} + \frac{x^2y^2}{a^4} + 4\frac{x^4y^4}{a^8}\right) \quad (25)$$

The reduced internal forces are accordingly given by

$$n_x = \frac{a^2 g}{2h}\left(-1 + \frac{x^2}{a^2} - \frac{y^2}{a^2} + \frac{5x^4y^2}{a^6} - \frac{4x^6y^2}{a^8} - \frac{10x^4y^4}{a^8} + \frac{10x^6y^4}{a^{10}}\right)$$

$$n_{xy} = \frac{a^2 g}{2h}\left(-\frac{2xy}{a^2} - \frac{20x^3y^3}{3a^6} + \frac{8x^3y^5}{a^8} + \frac{8x^5y^3}{a^8} - \frac{12x^5y^5}{a^{10}}\right) \quad (26)$$

$$n_y = \frac{a^2 g}{2h}\left(-1 + \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{5x^2y^4}{a^6} - \frac{4x^2y^6}{a^8} - \frac{10x^4y^4}{a^8} + \frac{10x^4y^6}{a^{10}}\right)$$

However, Murashev, Sigalov and Baikov in their text book "Design of Reinforced Concrete Structures (*) have given the internal forces in a shallow convex shell of constant curvature The shell is rectangular in plan and supported at the edges on four diaphragms absolutely rigid in their own plane and absolutely flexible normal to that plane (Fig XII-99) This determines the edge conditions of the shell, namely

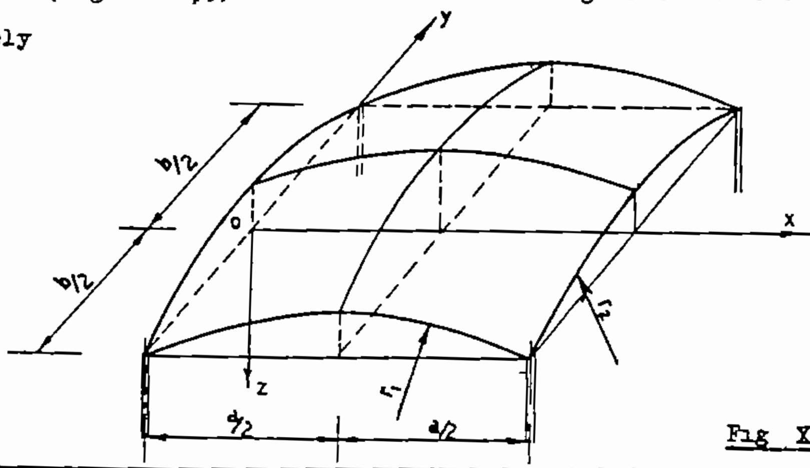


Fig XII-99

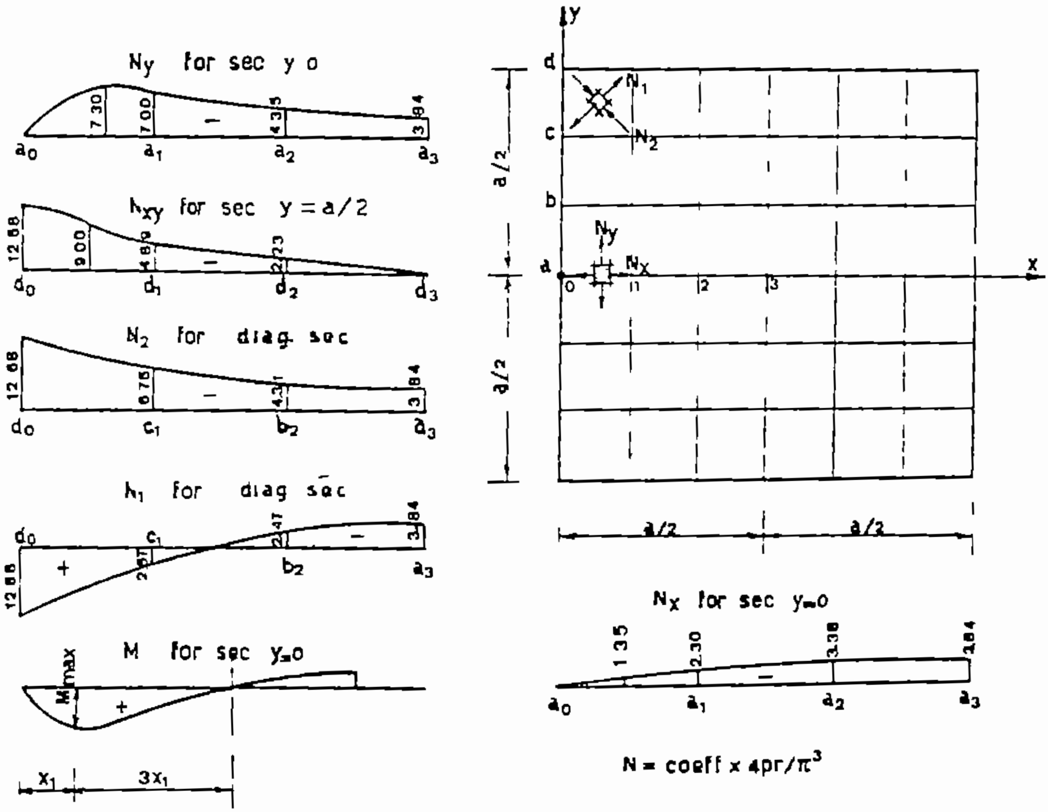
*) Mir - Publishers - Moscow

when $x = 0$ and $x = a$ $N_x = 0$

$y = \pm b / 2$ $N_y = 0$

The shell is subject to compressive normal forces N_x N_y and shear forces N_{xy} At the corners , principal diagonal compressive and tensile forces N_2 and N_1 are created

With $a = b$ and $r_1 = r_2 = r$, the coefficients of the forces can be determined according to figure XII-100 The forces are obtained for the various points of the shell by multiplying the corresponding coefficients by a constant equal to $4\ p\ r/\pi^3$



Internal Forces in Shallow Convex Shells with Square Plan

FIG XII-100

In the zones adjacent to the supports, bending moments are created Their numerical values are not large but however must be considered in the design Their magnitude can be determined from the relations

$$M = 0.3\ p\ r\ t\ e^{-\varphi} \sin\varphi \qquad (27)$$

where e is the base of natural logarithms, $\varphi = x / s$ and t = thickness of the shell

The maximum bending moment is

$$\max M = 0.094\ r\ t\ p \qquad (28)$$

and acts in a section at a distance x_1 from the edge , where

$$x_1 = 0.6 \sqrt{r\ t} \qquad (29)$$

The diaphragms resist the shearing forces transmitted by the shell , they act parallel to its edge

Example

The numerical evaluation of the given method is shown in the following example

It is required to determine the internal forces acting in a square shallow roof shell with $l = 2a = 2b = 40$ ms its rise h at the center is 6 ms , radius r = 68 2 ms and shell thickness t = 10cms if it is subject to a uniformly load g = 500 kg/m^2 horizontal

All the required forces can be determined from the data given in figure XII-100 The constant factor for determining the forces is

$$4\ p\ r / \pi^2 = 4 \times 0.5 \times 68.2 / \pi^2 = 4.35\ \text{t/m}$$

The maximum compressive force at the middle of the shell

$$N_x = N_y = -3.84 \times 4.35 = -16.7\ \text{t/m}$$

The maximum compressive force at the zone adjacent to the edge

$$N_y = -7.30 \times 4.35 = -31.8\ \text{t/m}$$

The maximum principal compressive and principal tensile forces , and the shear forces at the corners of the shell, are

$$-N_2 = N_1 = N_{xy} = 12.68 \times 4.35 = 55.2\ \text{t/m}$$

The maximum bending moment

$$\max M = 094\ r\ t\ p = 094 \times 68\ 2 \times 0\ 1 \times 500 \approx 300 \text{ kgm/m}$$

acting at a distance

$$x_1 = 0\ 6\sqrt{r\ t} = 0\ 6\sqrt{68\ 2 \times 0\ 1} = 1\ 57 \text{ m}$$

The compressive force N_x in this section can be determined by linear interpolation Thus

$$N_x = -1\ 35 \times 4\ 35 \times 1\ 57 \times 12 / 40 = -2\ 95 \text{ t/m}$$

Due to the high principal compressive and tensile stresses at the corners it is recommended to increase the thickness of the shell gradually from 10 to 16 cms along a length of about 0 15 l measured along the diagonal

Accordingly the maximum compressive stresses are

At crown $\sigma_c = -16700 / 100 \times 10 = -16\ 7 \text{ kg/cm}^2$

At edges $\sigma_c = -31800 / 100 \times 10 = -31\ 8$

At corners $\sigma_c = -55200 / 100 \times 16 = -34\ 8$

In spite of that the shell is reinforced over the greater part of its surface by one mesh 5 ϕ 8mm/m

The diagonal tension reinforcement on both sides at the corners can be calculated as follows (Fig XII-101)

$$A_{s1} = \frac{55 + 44}{2\ \sigma_s} = \frac{49\ 5}{1\ 4} \approx 35 \text{ cm}^2/\text{m}$$

$$A_{s2} = \frac{44 + 34}{2\ \sigma_s} = \frac{39}{1\ 4} \approx 28$$

$$A_{s3} = \frac{34 + 26}{2\ \sigma_s} = \frac{30}{1\ 4} \approx 22$$

$$A_{s4} = \frac{26 + 20}{2\ \sigma_s} = \frac{23}{1\ 4} \approx 16$$

$$A_{s5} = \frac{20 + 11\ 6}{2\ \sigma_s} = \frac{15\ 8}{1\ 4} \approx 11\ 2$$

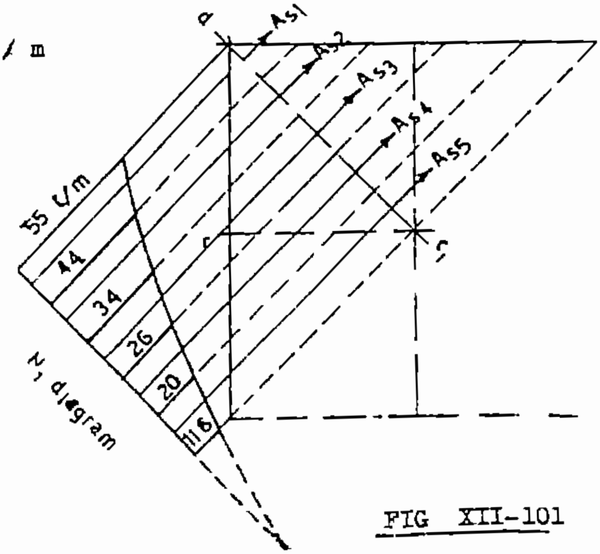


FIG XII-101

In order to resist the bending moments at the edges of the ... additional reinforcement may be required normal to the edge.

The edge diaphragms have to support their own weight plus the shear of the shell $\simeq N_{xy}$

The arrangement of the reinforcement is shown in figure XII -102

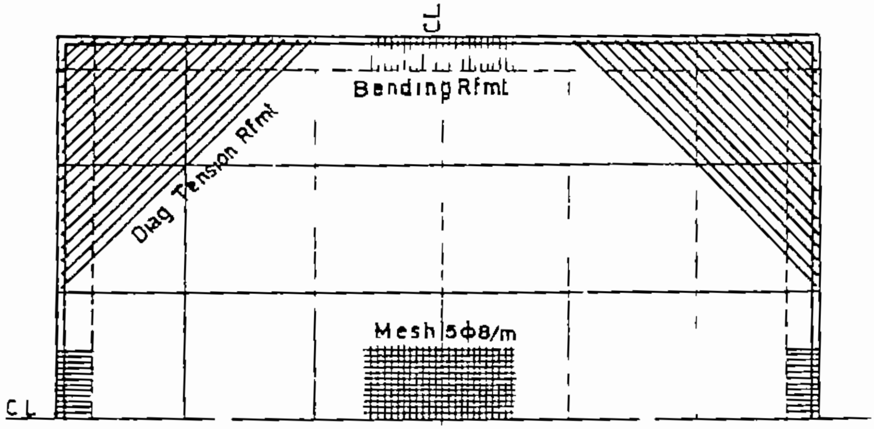


FIG XII-102

For an elliptic paraboloid on rectangular plan with sides 2a and 2b and rise h_1 and h_2 as shown in figure XII-103 the equation of the surface is given by

$$z = \frac{h_1}{a^2} x^2 + \frac{h_2}{b^2} y^2 \qquad (30)$$

The total central rise h of the shell is

$$h = h_1 + h_2$$

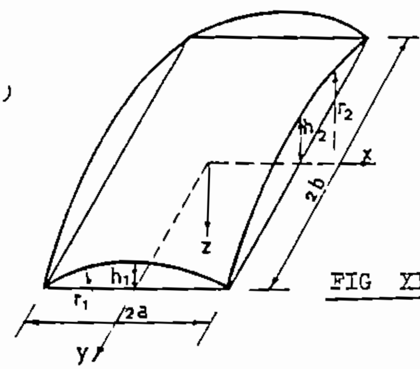


FIG XII-103

The derivatives of the equation are

$$\left.\begin{array}{lll} \partial z/\partial x = 2h_1x/a^2 & & \partial z/\partial y = 2h_2y/b^2 \\ \partial^2 z/\partial x^2 = 2h_1/a^2 & \partial^2 z/\partial x\,\partial y = 0 & \partial^2 z/\partial y^2 = 2h_2/b^2 \end{array}\right\} \quad (31)$$

If r_1 and r_2 are the radii of the generating parabolas then

$$1/r_1 \approx \partial^2 z/\partial x^2 = 2h_1/a^2 \qquad \text{and} \qquad 1/r_2 \approx \partial^2 z/\partial y^2 = 2h_2/b^2 \quad (32)$$

Accordingly the Pucher differential equation for a reduced vertical load g is given by

$$\frac{2h_1}{a^2} \frac{\partial^2 F}{\partial y^2} + \frac{2h_2}{b^2} \frac{\partial^2 F}{\partial x^2} = - g \tag{33}$$

If the two generating parabolas are identical the surface is called a rotational paraboloid and is characterized by the relation

$$h_1/a^2 = h_2/b^2 \tag{34}$$

so that

$$1/r_1 = 1/r_2 = 1/r \tag{35}$$

and $$\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2 = 1/r \ , \quad \partial^2 z/\partial x \, \partial y = 0 \tag{36}$$

The equation of the surface may therefore be given in the form

$$z = \frac{1}{2\,r} (x^2 + y^2) \tag{37}$$

Hence

$$\partial z/\partial x = x/r \qquad \text{and} \qquad \partial z/\partial y = y/r \tag{38}$$

The Pucher differential equation can therefore be given in the form

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = - g\, r \tag{39}$$

For the solution of this equation Ramaswamy proposes to construct a polynomial stress function For a vertical uniform load p/m^2 surface, it takes the form

$$F = (x^2 - a^2) (y^2 - b^2) (Ax^2 + B\, y^2 + C) \tag{40}$$

which automatically satisfies the desired boundary conditions

$$\left.\begin{aligned} n_x &= 0 \quad \text{at} \quad x = \pm a \\ \text{and} \quad n_y &= 0 \quad \text{at} \quad y = \pm b \end{aligned}\right\} \tag{41}$$

Thus

$$\left.\begin{aligned} \partial^2 F/\partial y^2 = n_x &= 2 \Big[A\, x^4 + 6\, B\, x^2 y^2 + (C - Aa^2 - Bb^2) x^2 \\ &\qquad - 6\, B\, a^2 y^2 + a^2 (Bb^2 - C) \Big] \quad (a) \\ \partial^2 F/\partial x^2 = n_y &= 2 \Big[B\, y^4 + 6\, A\, x^2 y^2 + (C - Aa^2 - Bb^2) y^2 \\ &\qquad - 6\, A\, b^2 x^2 + b^2 (Aa^2 - C) \Big] \quad (b) \end{aligned}\right\} \tag{42}$$

$$- \partial^2 F / \partial x \partial y = n_{xy} = - 8 (Ax^3 y + Bx y^3) + 4 x y (Aa^2 + Bb^2 -) \tag{42c}$$

When the load acting is a vertical uniform load p per square meter surface then the Pucher differential equation assumes the form

$$\partial^2 F / \partial x^2 + \partial^2 F / \partial y^2 = - g r = - p r \lambda \tag{43}$$

Substituting for λ the value given in equation 1 and expanding the right-hand side by the binomial theorem and limiting ourselves to the first two terms then

$$\partial^2 F / \partial x^2 + \partial^2 F / \partial y^2 = - p r \left[1 + \frac{1}{2 r^2} (x^2 + y^2) \right] \tag{44}$$

Substituting further for the derivatives of F from 42 a and 42b and equating the coefficients of like terms on the left- and right-hand sides of the equation we arrive at the following three simultaneous equations in the three unknowns A B and C

$$\left. \begin{aligned} 2 (C - Aa^2 - Bb^2 - 6Ab^2) &= - p/2r \\ 2 (C - Aa^2 - Bb^2 - 6Ba^2) &= - p/2r \\ 2 a^2 (Bb^2 - C) + 2b^2 (Aa^2 - C) &= - pr \end{aligned} \right\} \tag{45}$$

From the first two equations of the set, we find that

$$A b^2 = B a^2 \tag{45a}$$

Making use of this relation B may be eliminated to give the two following simultaneous equations

$$C - Aa^2 - \frac{Ab^4}{a^2} - 6 A b^2 = - p/4r \tag{45b}$$

$$b^2 A - C = - p r / 2 (a^2 + b^2) \tag{45c}$$

Knowing A, B and C, the reduced stresses are easily md The method gives satisfactory accuracy if the shell is not too deep

Example

In order to show the application of the given relations the membrane analysis of a paraboloid of revolution will be illustrated for the following data (Fig XII-104)

$2a = 30$ m, $2b = 40$ m

Rise at crown $h = h_1 + h_2 = 5$ ms

Thickness $t = 10$ cms

Total load p/m^2 surface = 400 kgs

Choosing the origin at the crown, the equation of the surface is given by

$$z = \frac{1}{2r} (x^2 + y^2)$$

Noting that $z = h$ for $x = a$ and $y = b$, the radius of curvature of the surface is given by

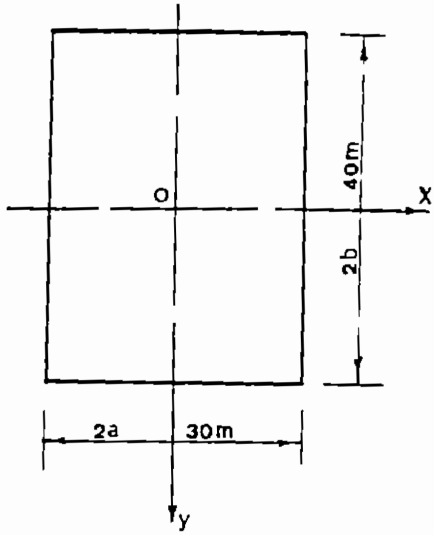


FIG XII-104

$$r = \frac{1}{2h} (a^2 + b^2) = \frac{1}{2 \times 5} (15^2 + 20^2) = 62\ 5 \text{ ms}$$

The constants A, B & C of equations 45a 45b and 45c can be calculated from the relations

(45a) $A b^2 = B a^2$ or $20^2 A = 15^2 B$ i e $B = 1\ 7778\ A$

(45b) $C - A a^2 - \frac{Ab^4}{a^2} - 6 A b^2 = - p/4 r$ or $C - 3336\ 1111A = - 1\ 6$

(45c) $b^2 A - C = - pr/2 (a^2 + b^2)$ or $-C + 400\ A = - 20$

Solving these three equations, we get

$A = 0\ 007356$ $B = 0\ 013077$ $C = 22\ 9?24$

The reduced internal forces are therefore given by

$$n_x = 0\ 014712\ x^4 + 0\ 156924\ x^2 y^2 + 32\ 1130\ x^2 - 35\ 3080\ y^2 - 7970\ 22$$

$$n_y = 0\ 026154\ y^4 + 0\ 088272\ x^2 y^2 + 32\ 1130\ y^2 - 35\ 3080\ x^2 - 17030\ 24$$

$$n_{xy} = - 0\ 058848\ x^3 y - 0\ 104616\ x y^3 - 64\ 226\ x y$$

Their values are as follows

On the x - axis y = 0 Fig XII-105a

x	0	5	10	15	m
$-n_x$	7970	7158	4612	0	kg/m
$-n_y$	17030	17913	20560	24974	"
$-n_{xy}$		0			"

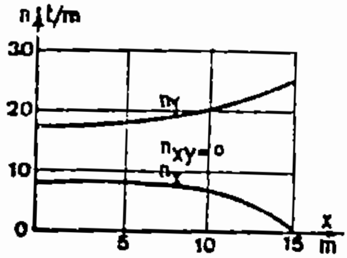


Fig XII-105a

At the edge y=b/2=20m Fig XII-105b

x	0	5	10	15	m
$-n_x$	22093	19712	12458	0	kg/m
$-n_{xy}$	0	12416	22391	35794	"
$-n_y$		0			"

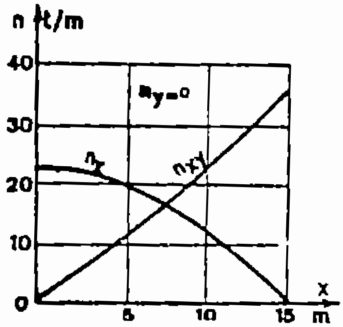


Fig XII-105b

On the y - axis x = 0 Fig XII-105c

y	0	5	10	15	20	m
$-n_x$	7970	8853	11501	15914	22093	kg/m
$-n_y$	17030	16211	13557	8453	0	"
$-n_{xy}$		0				"

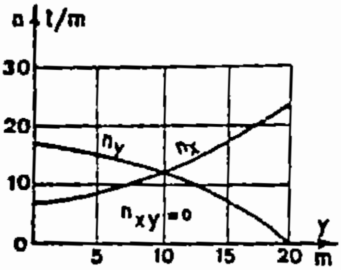


FIG XII-105c

At the edge x=a/2=15m Fig XII-105d

y	0	5	10	15	20	m
$-n_y$	24975	23659	19516	11956	0	kg/m
$-n_{xy}$	0	6006	13189	22726	35794	"
$-n_x$		0				"

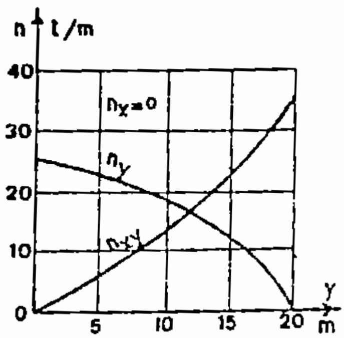


FIG XII-105d

It is clear from the tables that the shell is subject to compressive stresses over its whole surface except at the corners where we have high shearing forces N_{xy} = 35794 kg/m causing principal diagonal tesile forces N_1 (normal to the bisector of the corner angle) and compressive forces N_2 (in the direction of the bisector) The maximum magnitude of both N_1 and N_2 is equal to max N_{xy} at the corner

It is therefore recommended to increase the thickness of the shell at the corners to about 20 cms and to resist the principal tensile forces N_1 by top and bottom corner reinforcements arranged at 45° to the axes x and y normal to the bisector of the corner angle

Fig XIII-106 shows the general layout, main dimensions and details of reinforcements of two seperate units of a series of double curved shell roofs of the elliptic paraboloid type constructed in Egypt near Cairo Every two units are attatched and supported on 6 main columns only Each unit covers an area 23.94 x 24.75 ms The rises h_1 and h_2 of the shorter and longer diaphragms are approximately equal and each is 3 ms The shell thickness within a horizontal radius of 10 ms is equal to 8 cms increased to 12 cms within a radius of 12.4 ms and to 24 cms at the four corners The central part, 8 cms thick, is reinforced by one orthogonal mesh, 6 φ 8 mm/m, the rest is reinforced by two meshes, each is composed of circular bars increasing from 6 φ 10 mm/m, in the zone limited by the radii 10 and 12.4 ms, to 8 φ 13 mm/m at the corners, and radial bars 5 φ 8 mm/m arranged radially in the manner shown in figure Each of the end diaphragms is an arch 35 x 55 cms with a prestressed tie 25 x 25 cms The arch is reinforced by 8 φ 19 mm and the tie by 4 φ 16 mm in addition to four Freyssinet cables 20 tons each

The roof structure constructed according to this system adapts itself in an impressive, simple, easy and economic manner to rectangular areas of relatively big spans

4-3 Conoid Shells

A conoid surface is originated when a straight line moves at one of its ends on a basic curve C (called directrix) and at the other end on a key line K in such a way that the straight line is always parallel to a vertical guide plane G Accordingly, any vertical section between the basic curve C & the key line K has the same form and span as C but with a smaller rise h, it is called an affine curve All sections parallel to the guide plane G are straight lines and are called the generators of the conoid Fig XII-107

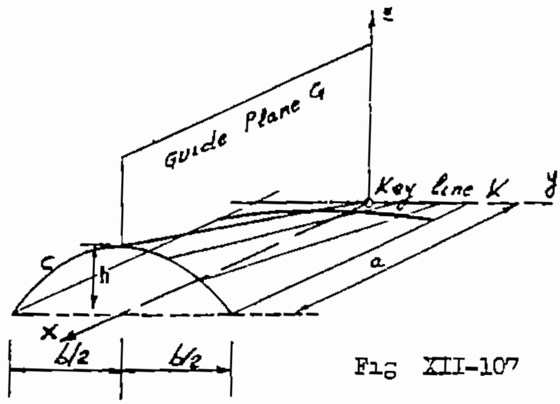


Fig XII-107

A shell roof is generally a truncated conoid, bounded by two arches C and C' and two straight edges parallel to the guide plane G Its formwork is very easy as it is constructed from straight planks

Depending upon the curve used as the directrix, conoids are described as parabolic, cirular etc Of these, the parabolic conoid is by far most common

The generators of the conoid surface being parallel to the vertical symmetry axis of the shell, they appear as parallel straight lines in the ground plan Fig XII-108

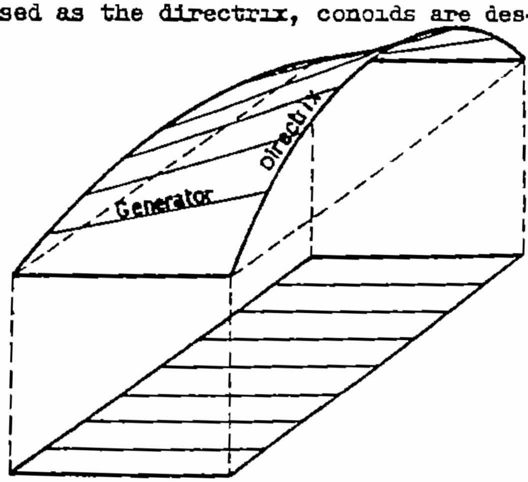


Fig XII-108

The equation of the middle surface of a parabolic conoid shell is given by (Fig XII-109)

$$z = -\frac{h\,x}{a}\left(1 - \frac{4\,y^2}{b^2}\right) \qquad (46)$$

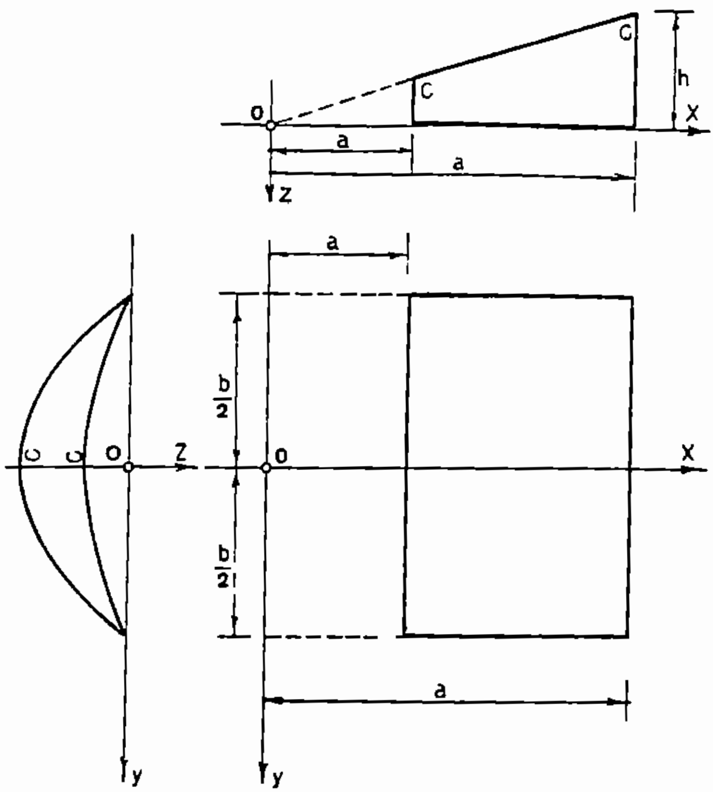


Fig XII-109 A parabolic conoid

The derivatives of this relation are given by

$$\left.\begin{array}{lll} \dfrac{\partial z}{\partial x} = -\dfrac{h}{a}\left(1 - \dfrac{4\,y^2}{b^2}\right) & & \dfrac{\partial z}{\partial y} = \dfrac{8\,h\,x\,y}{a\,b^2} \\ \dfrac{\partial^2 z}{\partial x^2} = 0 & \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{8\,h\,y}{a\,b^2} & \dfrac{\partial^2 z}{\partial y^2} = \dfrac{8\,h\,x}{a\,b^2} \end{array}\right\} \quad (47)$$

In the following, the internal forces due to a vertical load g_z will be given $\partial^2 z/\partial x^2$ being equal to zero, the conditions of equilibrium, expressed by equations 12, can in this case , be given in the form

$$\left.\begin{array}{ll} \dfrac{\partial n_x}{\partial x} + \dfrac{\partial n_{yx}}{\partial y} = 0 & \dfrac{\partial n_y}{\partial y} + \dfrac{\partial n_{xy}}{\partial x} = 0 \\ n_y \dfrac{\partial^2 z}{\partial y^2} + 2\,n_{xy}\dfrac{\partial^2 z}{\partial x\,\partial y} = -g_z & \end{array}\right\} \quad (48)$$

It will be assumed also here that the edge arches can resist loads in their planes only and have no resistance to lateral forces so that the force components normal to them must be equal to zero, i e

$$\text{at } x = a_o \quad \text{and} \quad x = a \qquad n_x = 0 \tag{49a}$$

It is further known that in a conoid shell, symmetrical about the middle x-axis (y = 0) subject to symmetrical vertical load g_z, we should have

$$\text{for} \qquad y = 0 \qquad n_{xy} = 0 \tag{49b}$$

Due to a vertical distributed load g / m^2 surface, we have

$$g_z = g \left[1 + \frac{h^2}{2\,a^2} \left(1 - \frac{4\,y^2}{b^2}\right)^2 + 32\,\frac{h^2\,x^2\,y^2}{a^2\,b^4} \right] \tag{50}$$

Satisfying the conditions of equilibrium (48), the edge conditions (49) and the relation between g and g_z given in (50), the reduced internal forces can be determined, according to Fischer[x] from the following relations

$$
\left.
\begin{aligned}
n_x &= \frac{g\,h}{4\,a\,b^2} \Bigg\{ (b^2 - 6\,y^2)(a - x) - \frac{16}{3}(a^3 - x^3) \\
&\quad - \frac{b^2 (a - a_o) - \frac{16}{3}(a^3 - a_o^3)}{a^5 - a_o^5}(a^5 - x^5) \\
&\quad + 6\,\frac{a - a_o}{a^9 - a_o^9}(a^9 - x^9)\,y^2 \Bigg\} \\
n_y &= -\frac{g\,a\,b^2}{8\,h\,x}\left[1 + \frac{h^2}{2\,a^2} - \frac{32\,h^2\,x^2\,y^2}{a^2\,b^4}\right] + \frac{g\,h\,y^2}{2\,a\,b^2}\Bigg\{\frac{5\,x^3}{a^5 - a_o^5} \\
&\quad \left[b^2 (a - a_o) - \frac{16}{3}(a^3 - x_o^3)\right] - \frac{18\,y^2\,x^7}{a^9 - a_o^9} - (a - a_o)\Bigg\} \\
n_{xy} &= \frac{g\,h\,y}{4\,a\,b^2}\Bigg\{b^2 - 2\,y^2 - 16\,x^2 - \frac{5\,x^4}{a^5 - a_o^5}\Big[b^2 (a - a_o) - \\
&\quad - \frac{16}{3}(a^3 - a_o^3)\Big] + \frac{18\,y^2\,x^8}{a^9 - a_o^9}(a - a_o)\Bigg\}
\end{aligned}
\right\} \tag{51}
$$

For the numerical calculations, it is recommended to replace x and y by ξ and η according to the following relations

$$x = \xi a \qquad (a_o = \xi_o\, a) \qquad \text{and} \qquad y = \frac{b}{2}\,\eta \tag{52}$$

so that the reduced internal forces can be given in the form

$$\left.\begin{aligned}
n_x &= \frac{gh}{4ab^2}\left\{ b^2 a\,(1-\frac{3\eta^2}{2})(1-\xi) - \frac{16\,a^3}{3}(1-\xi^3) - \frac{1-\xi^5}{1-\xi_o^5}\left[b^2 a\,(1-\xi_o) \right.\right.\\
&\qquad \left.\left. - \frac{16\,a^3}{3}(1-\xi_o^{\,3}) \right] + \frac{3b^2 a}{2}\,(1-\xi_o)\,\eta^2\,\frac{1-\xi^9}{1-\xi_o^{\,9}} \right\}\\
n_y &= -\frac{gb^2}{8h\xi}\left[1 + \frac{h^2}{2a^2} - \frac{8h^2}{b^2}\xi^2\eta^2 \right] + \frac{gh\eta^2}{8a}\left\{ \frac{5\,\xi^3}{a(1-\xi_o^{\,5})}\left[b^2\,(1-\xi_o) - \right.\right.\\
&\qquad \left.\left. - \frac{16\,a^2}{3}\,(1-\xi_o^{\,3}) \right] - \frac{9}{2}\,\frac{b^2\eta^2\xi^7\,(1-\xi_o)}{a\,(1-\xi_o^{\,9})} \right\}\\
n_{xy} &= \frac{g\,h\,\eta}{8\,a\,b}\left\{ b^2(1-\frac{\eta^2}{2}) - 16\,a^2\,\xi^2 - \frac{5\,\xi^4}{1-\xi_o^{\,5}}\left[b^2(1-\xi_o) - \right.\right.\\
&\qquad \left.\left. - \frac{16\,a^2}{3}\,(1-\xi_o^{\,3}) \right] + \frac{9}{2}\,\frac{b^2\eta^2\,\xi^8}{(1-\xi_o^{\,9})}\,(1-\xi_o) \right\}
\end{aligned}\right\} \tag{53}$$

Having determined the reduced internal forces n_x, n_y and n_{xy}, the actual internal forces N_x, N_y and $N_{xy} = n_{xy}$ can be calculatd using relations 3 & 4 and the principal forces N_1 and N_2 using conditions 5 & 6

Fig XII-110 shows the general layout, main dimensions and details of a conoid shell solved by Fischer[x] for the following data

Theoretical length $a = 12\ 00$ m , Actual length $= a - a_o = 8\ 00$ m

Breadth $b = 20\ 00$ m , Max rise $h = 4\ 50$ m

Shell thickness $t = 10$ cms , Load/m² surface $g = 300$ kg/m²

$\xi_o = \frac{a_o}{a} = \frac{4}{12} = 0\ 3333 \quad \xi_o^{\,3} = 0\ 03703 \quad \xi_o^{\,5} = 0\ 004115$ and $\xi_o^{9} = 000051$

Introducing these values in equation 53 of the reduced internal forces, it is easy to prove that

---

x Fischer Theorie und Praxis der Schalenkonstruktionen Published by Wilhelm Ernst und Sohn Berlin und Munchen 1967

$$n_x = 0.3375\,(1-\xi)(1-\frac{3\eta^2}{2}) - 0.6480\,(1-\xi^3) + 0.4006\,(1-\xi^5) + 0.3375\,\eta^2\,(1-\xi^{o})$$

$$n_y = \frac{+}{\xi}\,(-3.5675 + 1.3500\,\eta^2\xi^2 - 2.7823\,\eta^2\xi^4 - 1.4063\,\eta^4\xi^8)$$

$$n_{xy} = \eta\left\{0.2813\,(1-\frac{\eta^2}{2}) - 1.6200\,\xi^2 + 1.6694\,\xi^4 + 0.8438\,\eta^2\xi^8\right\}$$

The real internal forces N_x, N_y and N_{xy} are, according to equation 3, given by

$$N_x = n_x \cos\beta / \cos\alpha \;, \quad N_y = n_y \cos\alpha / \cos\beta \quad \text{and} \quad N_{xy} = n_{xy}$$

Knowing further that

$$\tan\alpha = \frac{\partial z}{\partial x} = \frac{h}{a}\,(1 - \frac{4\,y^2}{b^2}) = \frac{h}{a}\,(1-\eta^2)$$

and

$$\tan\beta = \frac{\partial z}{\partial y} = \frac{8\,h\,x\,y}{a\,b^2} = \frac{4\,h}{b}\,\xi\,\eta$$

then α and β, and the corresponding cos-values required for calculating N_x and N_y can be easily determined

The calculated internal forces in the shell are as shown in Fig XII-111

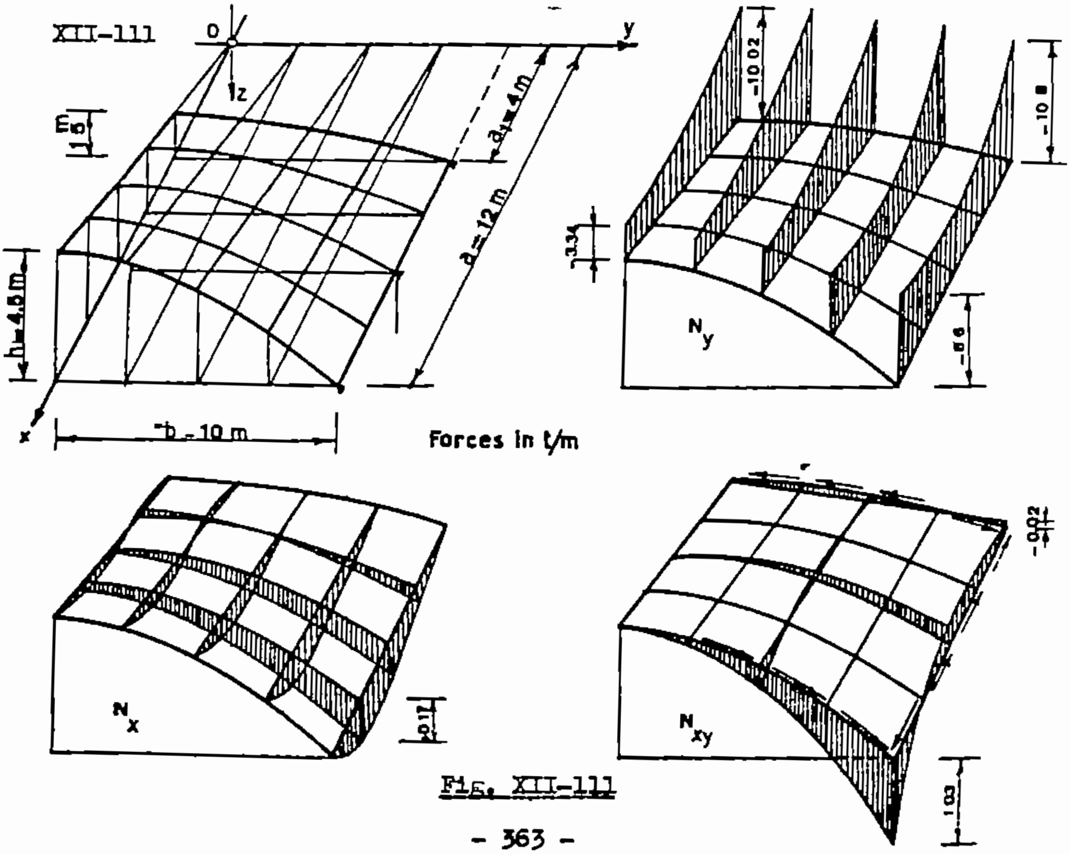


Fig. XII-111

The longitudinal beams at the straight edges of the conoid (Fig XII-112) carry the components of the normal forces N_y. The vertical continuous beam (spans 4 ms) carries the vertical component V of the normal force N_y,

where $V = n_y \tan\beta$

The horizontal beams (spans 8 ms) carry the horizontal component H of the normal force N_y

where $H = n_y$

The shearing force $N_{xy} = n_{xy}$ acting along the straight edge of the shell is small and may be neglected

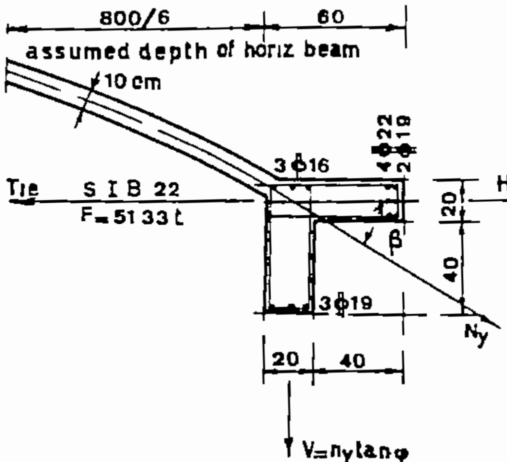


Fig XII-112

The reactions of the horizontal beam will be resisted by the tie of the arches

The diaphragm of the conoid is composed of two arches with a tie. The shearing forces transmitted to the flat arch are very small and hence, it does not need any design. The deep arch with the tie (Fig XII-113) carries its own weight + weight of flat arch and tie G + the shearing forces $N_{xy} = n_{xy}$ transmitted to it from the shell (components H and V)

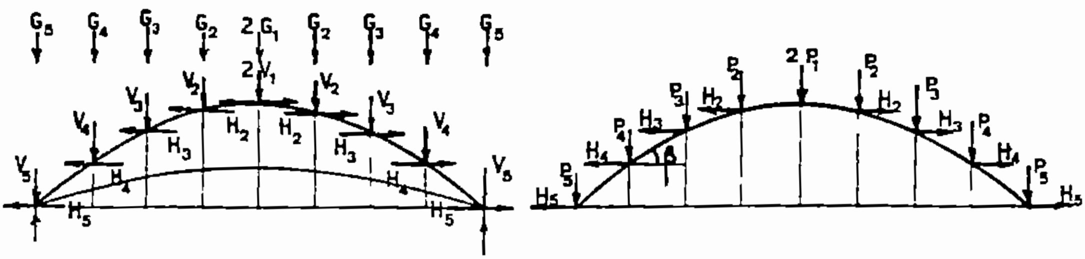


Fig XII-113

The bending moments and normal forces in the deep arch are generally small that light arches are normally sufficient. The statically indeterminate thrust is also very small that the force in the tie is approximately equal to the reaction of the horizontal beam only

Fig XII-114 shows the general layout and main dimensions of a part of the main stores of El-Nasr Pipes and Fittings Company at Helwan The main part of these stores is ~ 92 ms wide and ~ 108 ms long with one longitudinal and three transverse expansion joints, so that the store is divided into six blocks each 46 ms (2 x 23 ms) x 36 ms (4 x 9 ms) It consists of two floors a ground floor ~ 4 5 ms high and a top floor ~ 9 ms high

The ground floor is used as store for the fittings and the small light products, while the top floor is used for storing the main product of the company, the pipes

The live load on the first floor is prescribed to be 6 ton/m^2 For this reason, it has been chosen of the solid flat slab type ~ 4 5 x 4 5 ms The slab thickness is chosen 25 cms and provided with a 10 cms thick drop panel 2 25 x 2 25 ms The typical intermediate columns are 50 x 50 cms and provided with column heads 140 x 140 cms

The top floor of each block is covered by continuous two-bay conoids having a breadth b = 23 ms and a span $(a - a_o)$ = 9 ms The conoid is bounded by two arched diaphragms, one deep having a rise h = 4 5 ms and one very flat having a rise of 1 0 m only No horizontal tie is arranged, and the flat arch acts as a tie for the deep arch

The thickness of the conoid slab is 10 cms increased to 16 cms at the outside edges It is reinforced by one bottom mesh 6ϕ8 mm/m except at the free straight edge where it is increased to 6ϕ10 mm/m All edges are however reinforced by another top mesh 6ϕ 10 mm/m The top arch is 40 x 45 cms reinforced by 10ϕ 16 mm, the lower arch is 40 x 45 cms at the crown, its top surface is chosen horizontal for architectural requirements and reinforced by 20ϕ25 mm high grade steel The details of their reinforcements are shown in Fig XII-115

This interesting structure is still under construction

—4 The Hyperbolic Paraboloid

Figure XII-116 shows a shell formed according to a hyperbolic paraboloid surface Its equation with respect to the orthogonal

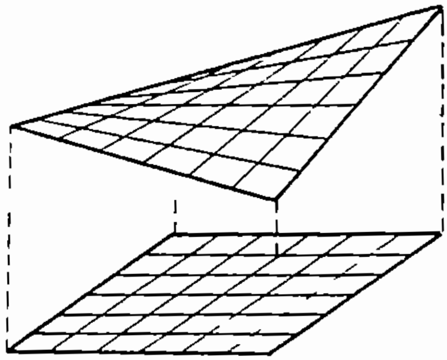


FIG XII-116

system of axes x y, z shown in figure XII - 117 is given by

$$z = x\,y / c \qquad (54)$$

in which

c = constant

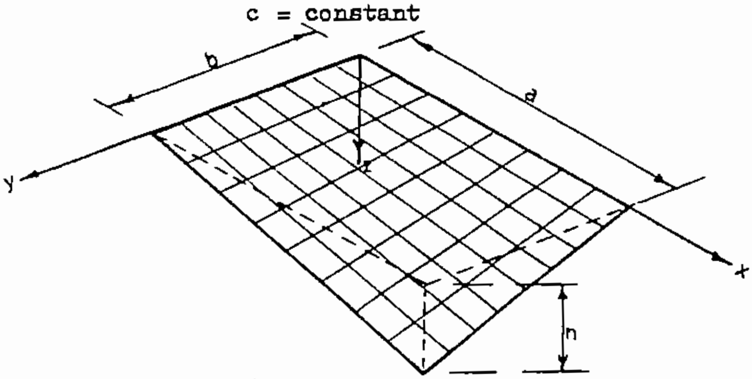


FIG XII-117

It is easy to see that the z - co - ordinate of any point is $h\,x\,y / a\,b$ Hence, the equation of the surface can be given in the form

$$z = \frac{h}{a\,b}\,x\,y \qquad (55)$$

which means that

$$c = a\,b / h \qquad (56)$$

A hyperbolic paraboloid may be regarded as a warped surface formed by elevating or depressing one corner of a rectangle by a certain amount (h) while the other three corners remain in their original level The surface shown in figures XII-116 and 117 is formed by two sets of straight line generators lying entirely on the surface The plan projections on the x - y plane of these generators constitute two families of parallel lines which are the characteristic lines of the surface

This surface being determined by two intersecting systems of straight lines its formwork requires only straight wood joist generators The smooth warped surface may be secured merely by covering these joists with flexible plywood sheathing

The hyperbolic paraboloid shells have been successfully used in many different ways to form roofs giving a striking appearance needed in such diverse structures such as banks, churches, restaurants, exhibition and assembly halls etc Some of these possibilities are given by Ramaswamy* and shown in figure XII-118 a to h

The derivatives of equation (66) are :

$$\left.\begin{array}{lll} & \partial z/\partial x = y/c \quad \text{and} & \partial z/\partial y = x/c \\ \partial^2 z/\partial x^2 = 0 & \partial^2 z/\partial x \partial y = 1/c & \partial^2 z/\partial y^2 = 0 \end{array}\right\} \quad (57)$$

Inserting these values in the membrane condition of equilibrium expressed by equation (12), we get

$$\underline{n_x = 0} \quad , \quad \underline{n_y = 0} \quad \text{and} \quad \underline{n_{xy} = - g\, c/2} \quad (58)$$

In shallow shells the load p/m^2 surface is approximately equal to the projected load g so that, we may write also

$$\underline{n_{xy} = - p\, a\, b \,/\, 2\, h} \qquad (59)$$

Thus we arrive at the important conclusion

" A shallow hyperbolic paraboloid submitted to the action of dead loads develops a state of pure shear, constant over the whole surface and

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* Ramaswamy " Design and Construction of Concrete Shell Roofs "

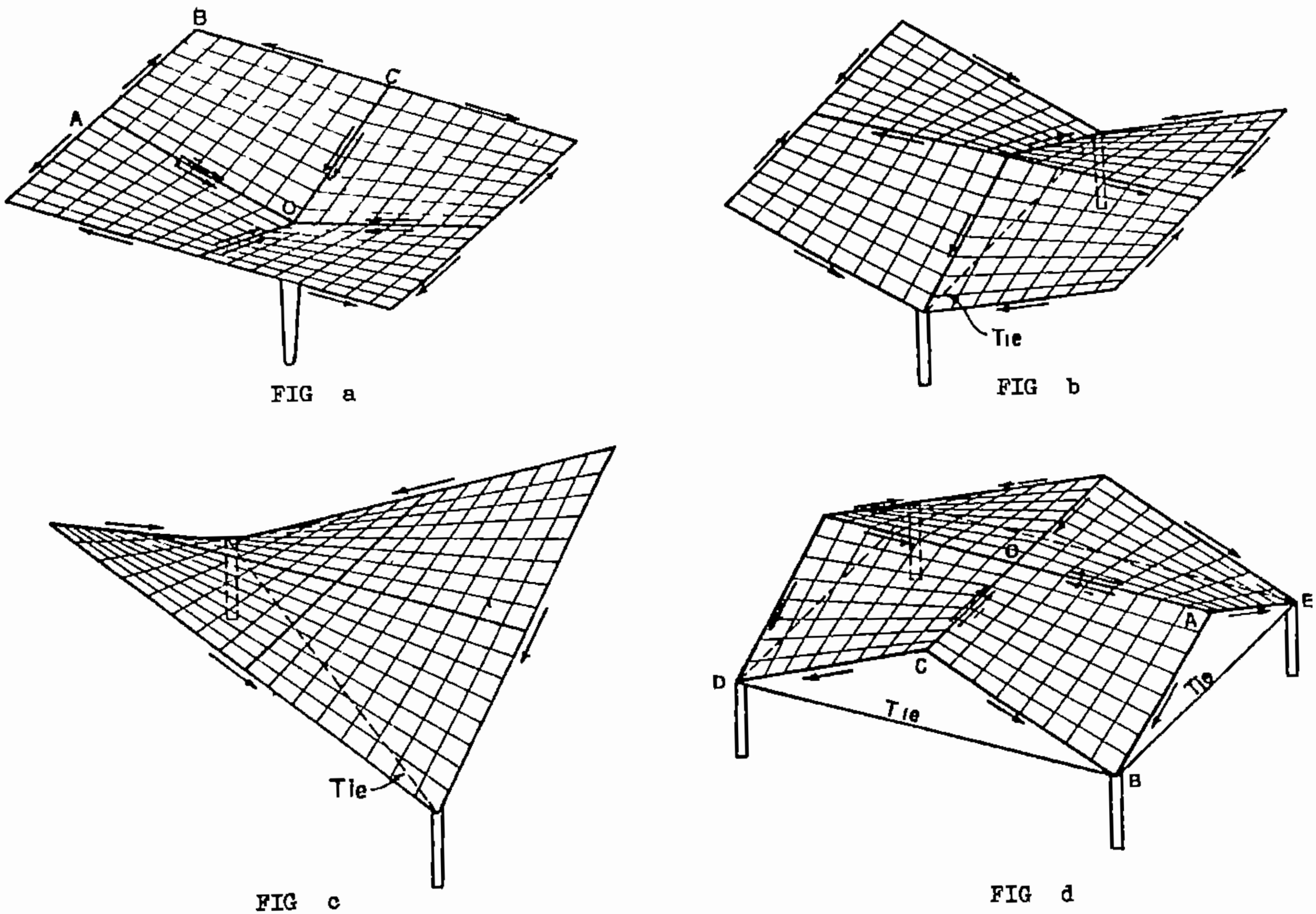


FIG XII-118
a to d

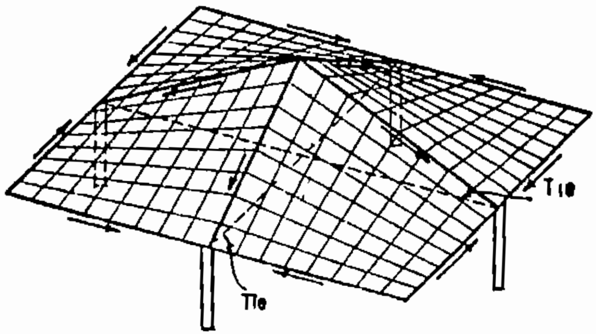


FIG e

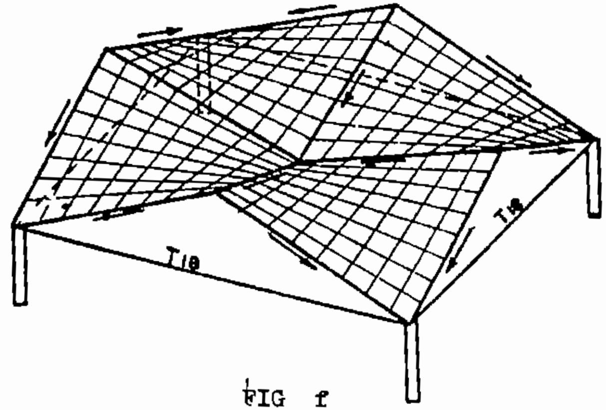


FIG f

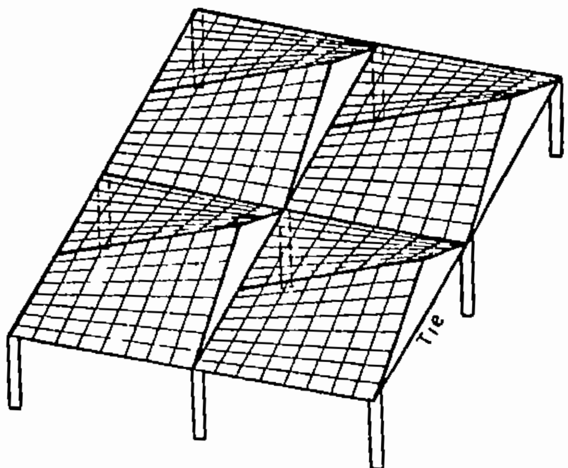


FIG g

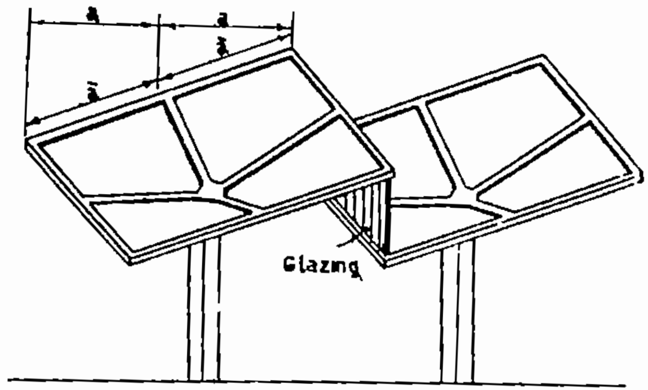


FIG h

FIG XII-118
e to h

unaccompanied by normal stresses. Due to this state of stresses, the shell will be subject to principal tensile and compressive forces along the vertical sections forming an angle of 45 with the directions x and y, in the arch - like fibers there will be compression while in the inverted arches tension will arise Fig XII-119 The absolute value of the principal tensile and compressive forces is equal to the absolute value of the shearing force This means $n_1 = - n_2 = n_{xy}$ and $N_1 = - N_2 = N_{xy}$

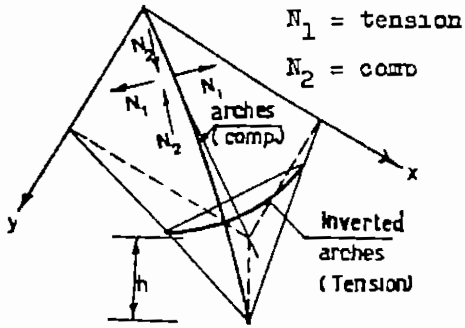


FIG XII-119

We consider now the arrangement of the umbrella roof formed by four hyperbolic paraboloids resting on four triangular frames (fig XII-118d) along their edges Taking any one of the hyperbolic paraboloids say O A C B, it abuts against the adjacent hyperbolic paraboloids along the edges OA and OB Along the two remaining edges AC and BC, it is supported on frames which are stiff in their plane only Each of the edges is subject to axial forces that may be found by summing up the shears transferred to them by the shell, their magnitude varies from zero at the free end and increases gradually to its maximum value H at the supported end So that

$$H = n_{xy} l \qquad (60)$$

in which l (= a or b) is the projected length of the edge beam under consideration

The sense of the forces H depends on the relative height of the corner points of any of the units of the hyperbolic paraboloid roof (Fig XII-120 If one of the four corner points is situated lower (or higher) than the other three, then the resultant of the shearing forces will act from the higher corner points towards the lower ones

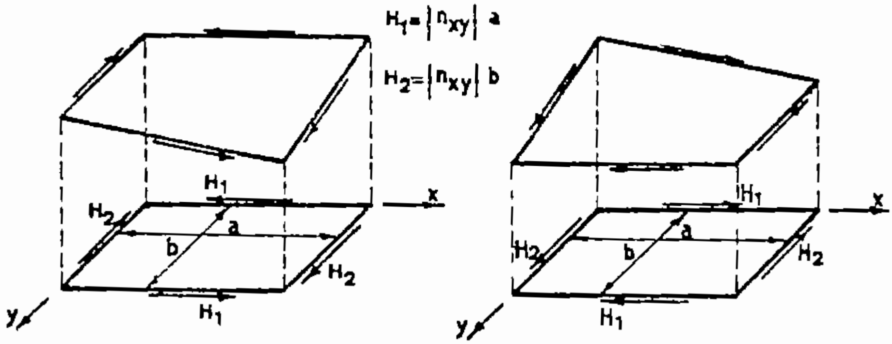


FIG XII-120

Reverting again to the umbrella roof of figure XII-118d and according to the given principles, we find that the corner B is lower than the other three corners C, O and A and hence the forces in the edge beams A B and C B are directed downwards from A and C towards B Since the supporting column lies at B, then
Axial forces in A B and C B are compressive zero at A and C and increase gradually to their maximum values at B Axial forces in C O and A O are opposite in direction to those in A B and C B and since they are free at C and A and supported at O then the forces in the two beams are compressive zero at C and A and increase gradually to their maximum values at O Each of these two edge beams carries the axial forces transmitted to it from two adjacent hyperbolic paraboloids

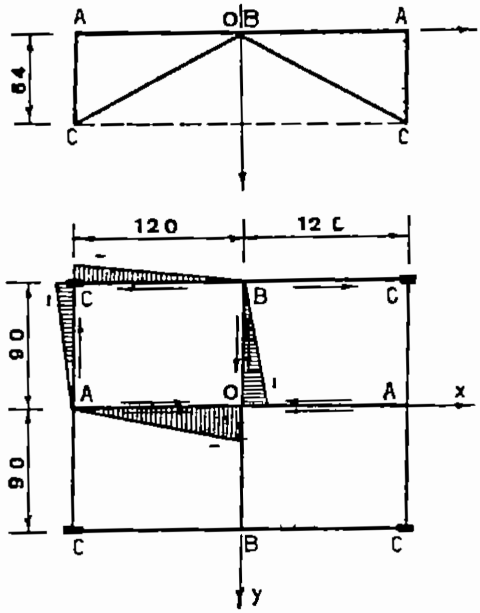
If we examine the forces acting in the edge beams of the inversed umbrella shown in figure XII-118a we find that the forces in A C and C O are compressive zero at A and C and maximum at O Each of the two edge beams carries the forces from two adjacent elements of the roof The forces in A B and B C are tensile zero at B and maximum at A and C

The forces acting in the edge beams tend to distort the frame formed by them In order to prevent this, ties joining the supporting columns as shown in figure XII-118b to g are to be arranged

Since n_x and n_y are equal to zero the own weight of the ridge beams cannot be resisted by the shell Therefore it is necessary that the ridge beams take care of themselves and sustain their own weight as beams supported at the gables

In order to illustrate the calculations, the statical investigation of some simple forms will be shown in the following examples

Example 1



In this example a hyperbolic paraboloid shell supported on four corner columns, of the form shown in figure XII-121 is presented The geometrical data of the shell are the following

a = 12 0 m   b = 9 0 m   h = 6 4 m

The specific value of the load acting on the shell is

$$g = 400 \quad kg/m^2$$

The shearing force arising in the shell due to this loading is

$$n_{xy} = - \frac{a\ b\ g}{2\ h} = - \frac{12 \times 9 \times 400}{2 \times 6\ 4} = -3375 \ kg/m$$

FIG. XII-121

The rib B C is subject to compression, zero at B and maximum at C The horizontal projection of the compressive force at C is given by $H_1 = n_{xy}$ a = 3375 x 12 = 40500 kgs

Similarly the horizontal projection of the compressive force in A C at C is $H_2 = n_{yy}$ b = 3375 x 9 = 30375 kgs

Each of the ribs A O and B O carry two elements of the roof Hence Max horizontal projection of compression in AO at O=2x40500=81000kgs

" " " " " BO at O=2x30375=60750

The tensile forces in the tie rods C B C and C A C are of the same magnitude as H_1 = 40500 kgs and H_2 = 30375 kgs The dimensions of the edge beams must be sufficient to resist the acting compressive forces and the bending moments due to their own weight

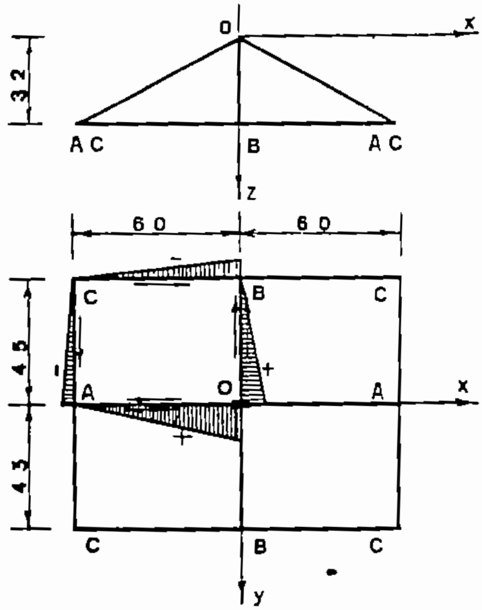


FIG XII-122

Example 2

Umbrella roof supported on a single column at the center (Fig XII-122)

a = 6 0 m b = 4 5 m h= 3 2m

Assume specific value of load

$$g = 320 \text{ kg/m}^2$$

The shearing force in the shell is given by

$$n_{xy} = -\frac{a\ b\ g}{2\ h} = -\frac{6 \times 4\ 5}{2 \times 3\ 2} \times 320$$

$$= -1350 \text{ kg/m}$$

Max horizontal projection of tensile force in AO at O=1350x6x2=16200kgs

" " " " " BO at O=1350x4 5x2=12150 "

" " " comp " " CB at B=1350x6 0 =8100 "

" " " CA at A=1350x4 5 =6075

Example 3

For the inverted umbrella roof with O below A,C and B and having the same specific load and dimensions (Fig XII-123) the forces in the edge beams will be of the same magnitude but opposite in sense Hence

$$n_{xy} = 1350 \text{ kg/m}$$

The reduced forces in A O and

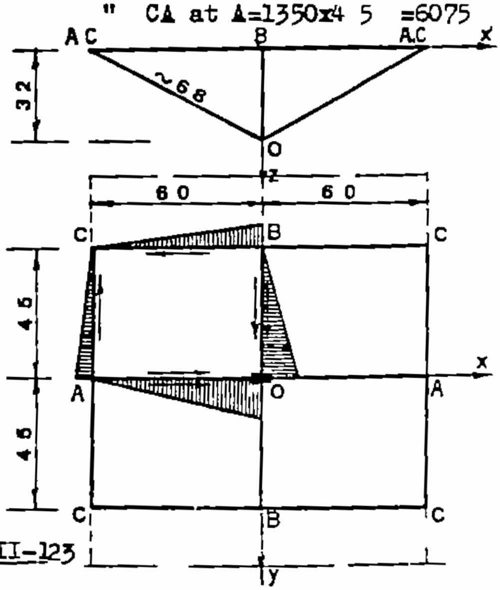


FIG XII-123

B O are compressive, with maximum values of 16200 kg and 12150 kg at O respectively The reduced forces in C B and C A are tensile, with maximum values of 8100 kgs at B and 6075 kgs at A

The actual shearing force in the shell slab N_{xy} being equal to the reduced value n_{xy}, then the actual principal forces N_1 and N_2 are also equal to $\pm n_{xy}$ Hence the max concrete compressive stress is given by

$$\sigma_c = n_{xy} / A_c \qquad (63)$$

σ_c is generally very low and the minimum executable thickness sufficient to protect the steel reinforcement from rusting (6 - 8 cms) is sufficient

The area of the steel reinforcement if it is arranged diagonally is

$$A_s = n_{xy} / \sigma_s \qquad (64)$$

For normal dimensions of hyperbolic paraboloid roofs A_s is low and it is recommended in such cases, to arrange the reinforcements parallel to the edge beams in which case

$$A_s = n_{xy} / \sqrt{2}\ \sigma_s \qquad (64a)$$

Assuming the thickness of the shell slab of example 1 is 6 cms, then

$$\sigma_c = 3375/100 \times 6 = 5\ 63\ kg/cm^2$$

and the area of the required reinforcement if arranged parallel to the sides is

$$A_s = 3375 / \sqrt{2}\ 1400 = 1\ 69\ cm^2/m$$

chosen 5 ϕ 8 mm/m

The actual forces in the edge beams are equal to

$$H / \cos\alpha$$

in which H is the horizontal projection of the force in the beam and α is the angle between the edge beam and its horizontal projection

The ribs must further be checked for bending moments caused by their own weight, unsymmetrical live loads and the eccentricity of the load (the compressive force is applied through the shell slab at the lower edge of the beam)

In order to give an idia about the order of these moments , the internal forces acting on the intermediate rib A O of example 3 will be shown

Actual max compressive force in rib A O at O = 16200 x 6 8 = 18360 kgs

Assuming the cross-section of beam A O at O is as shown in figure XII-124 and decreases to zero at A, then average own weight $= \frac{1}{2} \times 1 \times \frac{0\ 4}{2} \times 2\ 5 = 0\ 25$ t/m acting at $\sim \frac{1}{3}$ a from O

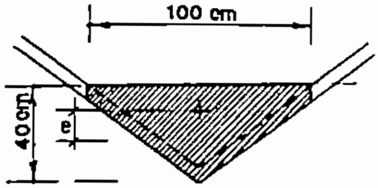


FIG. XII-124

Bending moment due to own weight = $0\ 25 \times 6^2/3 = 3\ 0$ m t

The bending moment due to unsymmetrical live load on C C B B is according to Parme, given by (T h) in which T is the maximum tension in the edge beam C B due to live load and h is the height of the paraboloid

Assuming the live load = 50 kg/m^2 and h being 3 2 ms, then maximum T = 8100 x 50/320 = 1 256 tons

Bending moment due to unsymmetrical load = 1 2o5 x 3 2 = 4 05 m t

If the shell is continuous as shown dotted, one may assume that half the bending moment is resisted by the adjacent shells and the other half is resisted by the beam A O

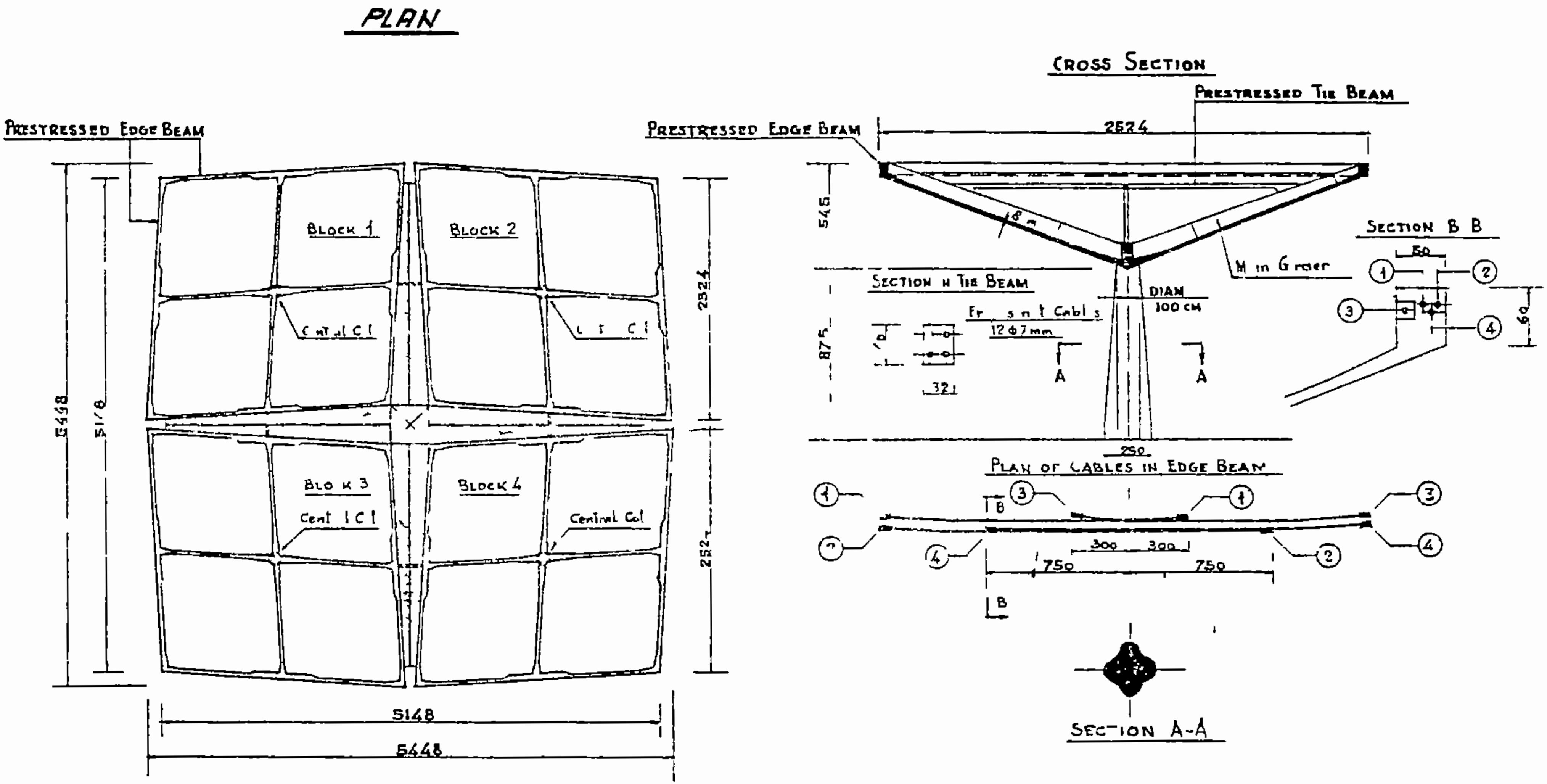
The shearing forces are transmitted to the beam through the shell slab

Assuming the average depth of the beam is 25 cms the bending moment due to eccentricity of force = $18\ 36 \times \frac{0\ 25}{2} = 2\ 42$ m t The total bending moment for one single shell

$$\text{total } M = 3\ 0 + 4\ 05 + 2\ 42 = 9\ 47 \ \underline{m\ t}$$

The edge beams C B are to be calculated for a max tensile force at O = 8100 kgs plus a bending moment equal to this force multiplied by the eccentricity from the center of shell slab to the center of the edge beam

Figs XII-125 and 126 give the general layout, main dimensions and details of reinforcements of the national exhibition halls at El Nasr city, Cairo The roof of the halls is composed of a series of hyperbolic paraboloids of the inverted-umbrella type Each unit of the series is 25 x 25 ms supported on a single column at the middle In order to reduce the bending moments in the intermediate beams, its cantilevering arms were connected together by a tie Prestressing was used both in the ties and in the tension edge beams



CAIRO INTERNATIONAL FAIR
EL NASR CITY
NATIONAL EXHIBITION HALLS 1964

Fig. II-125

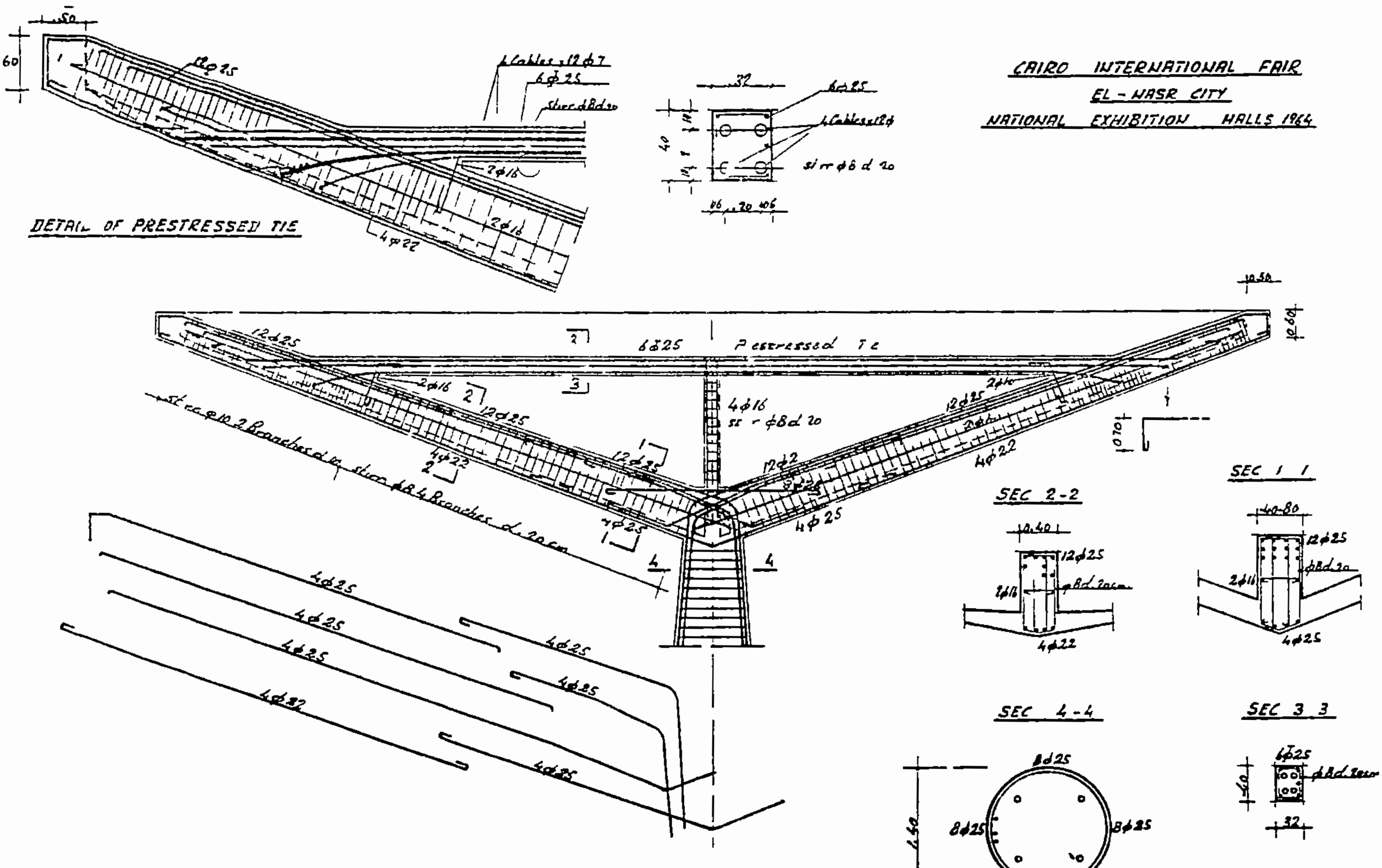

CAIRO INTERNATIONAL FAIR
EL - NASR CITY
NATIONAL EXHIBITION HALLS 1964
DETAIL OF PRESTRESSED TIE
Prestressed Tie
SEC 1 1
SEC 2-2
SEC 3 3
SEC 4-4
4φ25
4φ32
4φ22
12φ25
6φ25
4φ16
8φ25

Fig XII-126

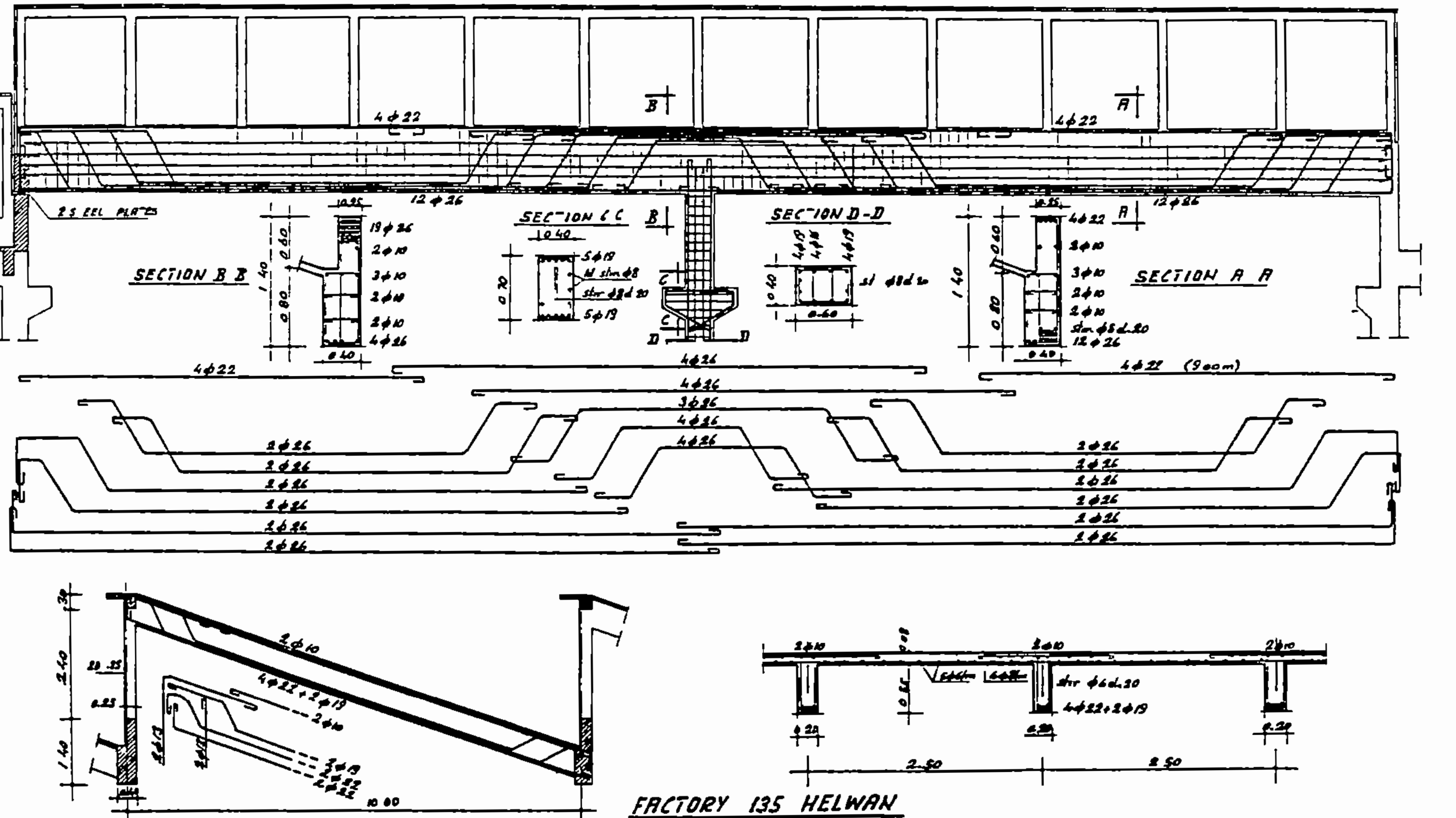

SECTION B B
SEC-ION C C
SEC-ION D-D
SECTION A A
FACTORY 135 HELWAN

Fig III-16

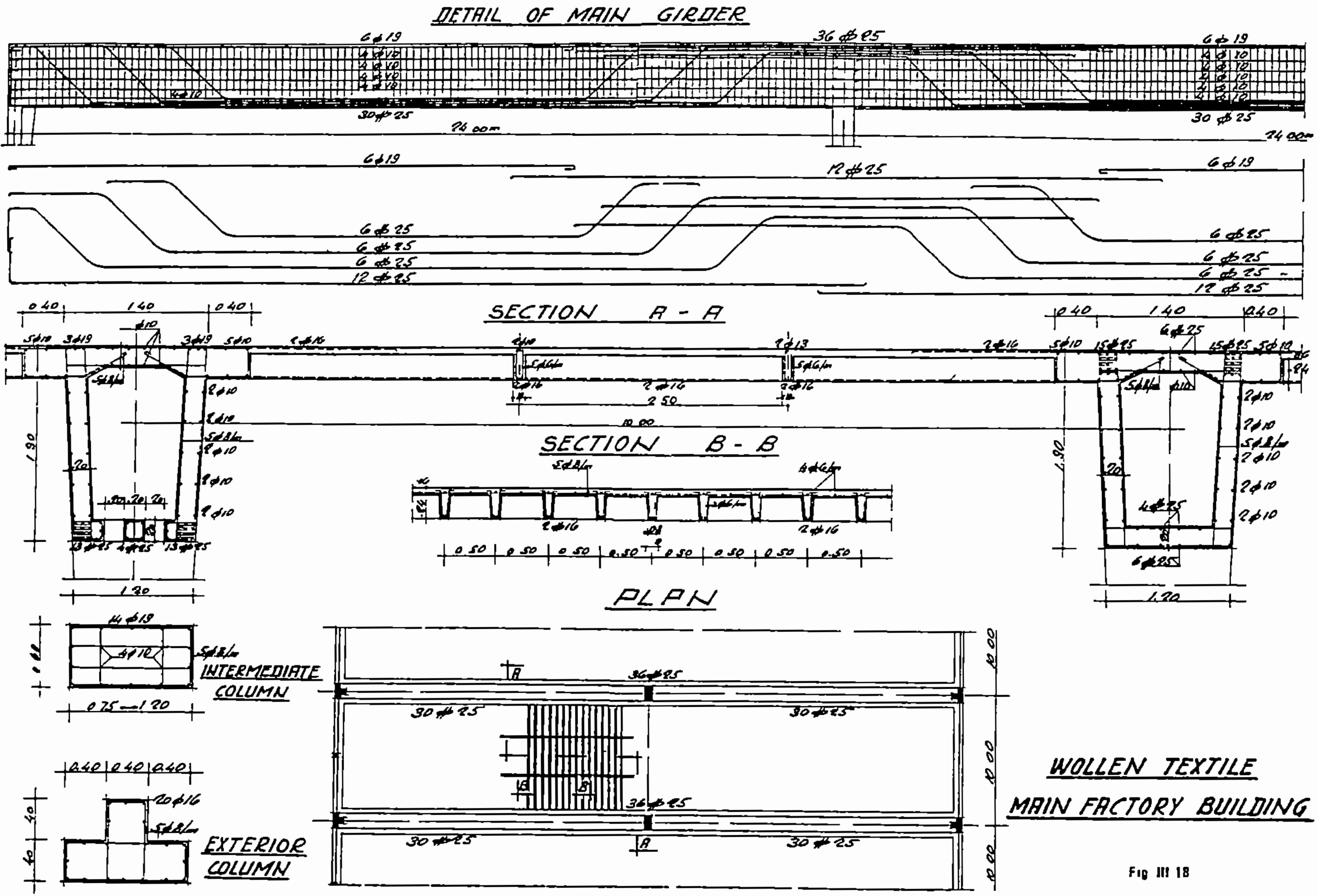

DETAIL OF MAIN GIRDER
SECTION A - A
SECTION B - B
PLAN
INTERMEDIATE COLUMN
EXTERIOR COLUMN
WOLLEN TEXTILE
MAIN FACTORY BUILDING

Fig III 18

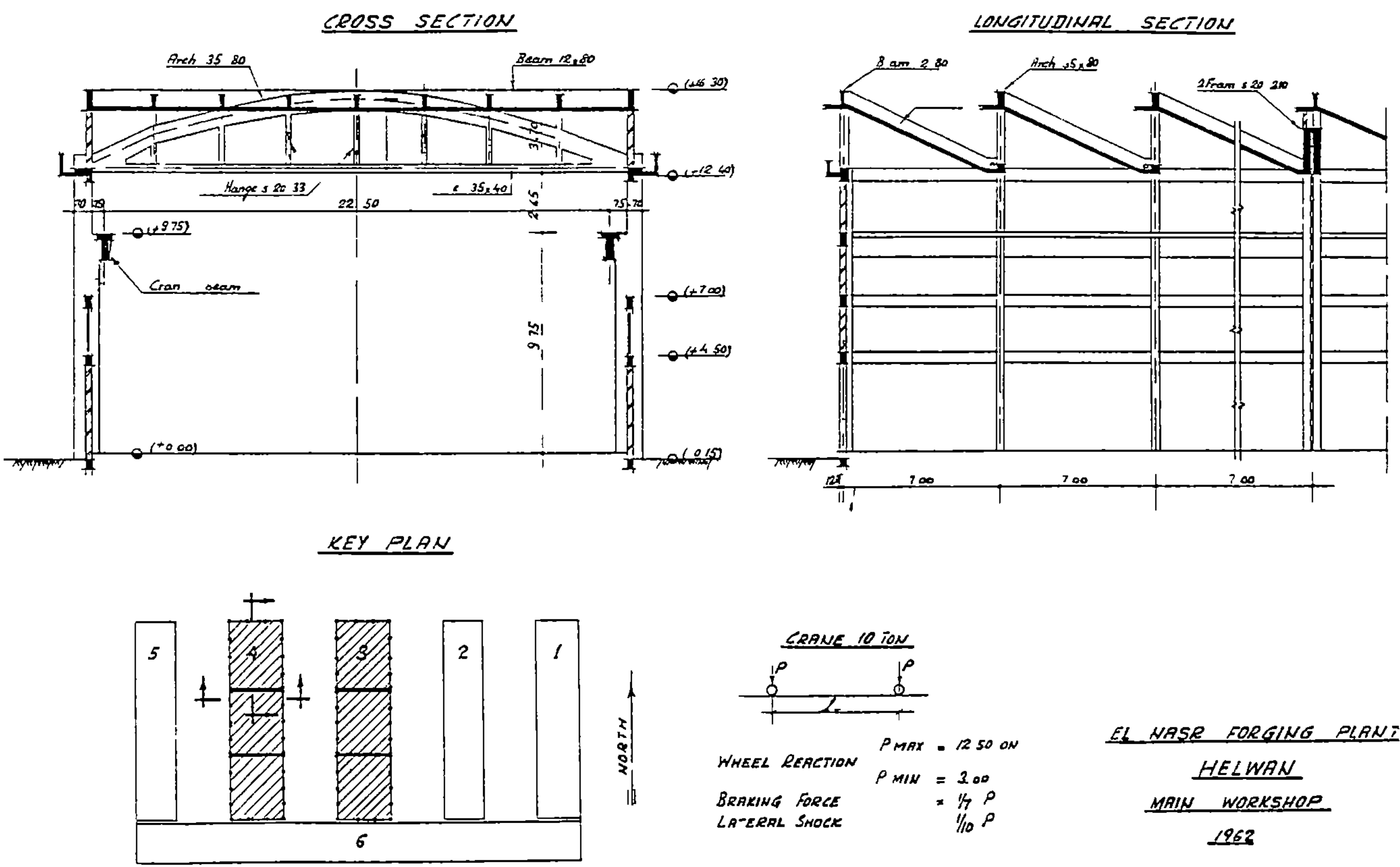
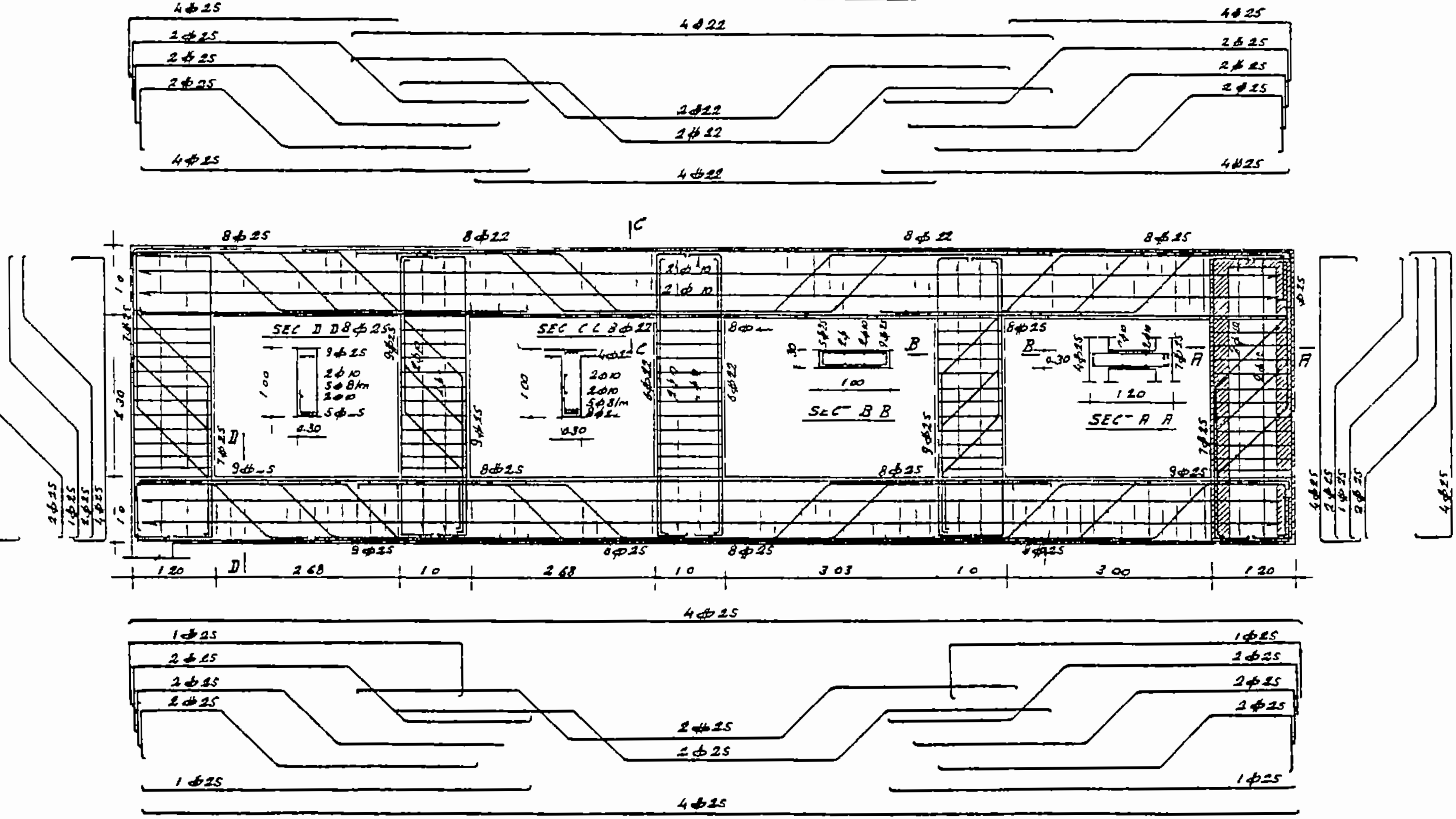

CROSS SECTION
Arch 35 80
Beam 12x80
(+16 30)
(+12 40)
Hange s 20 33
c 35x40
22 50
2 45
3 70
(+9 75)
Cran beam
(+7 00)
(+4 50)
9 75
(+0 00)
(0 15)
LONGITUDINAL SECTION
Beam 2 80
Arch 35x80
2 Fram s 20 210
7 00
7 00
7 00
KEY PLAN
5
4
3
2
1
6
NORTH
CRANE 10 TON
P
P
WHEEL REACTION
P MAX = 12 50 ON
P MIN = 3 00
BRAKING FORCE = 1/7 P
LATERAL SHOCK 1/10 P
EL NASR FORGING PLANT
HELWAN
MAIN WORKSHOP
1962

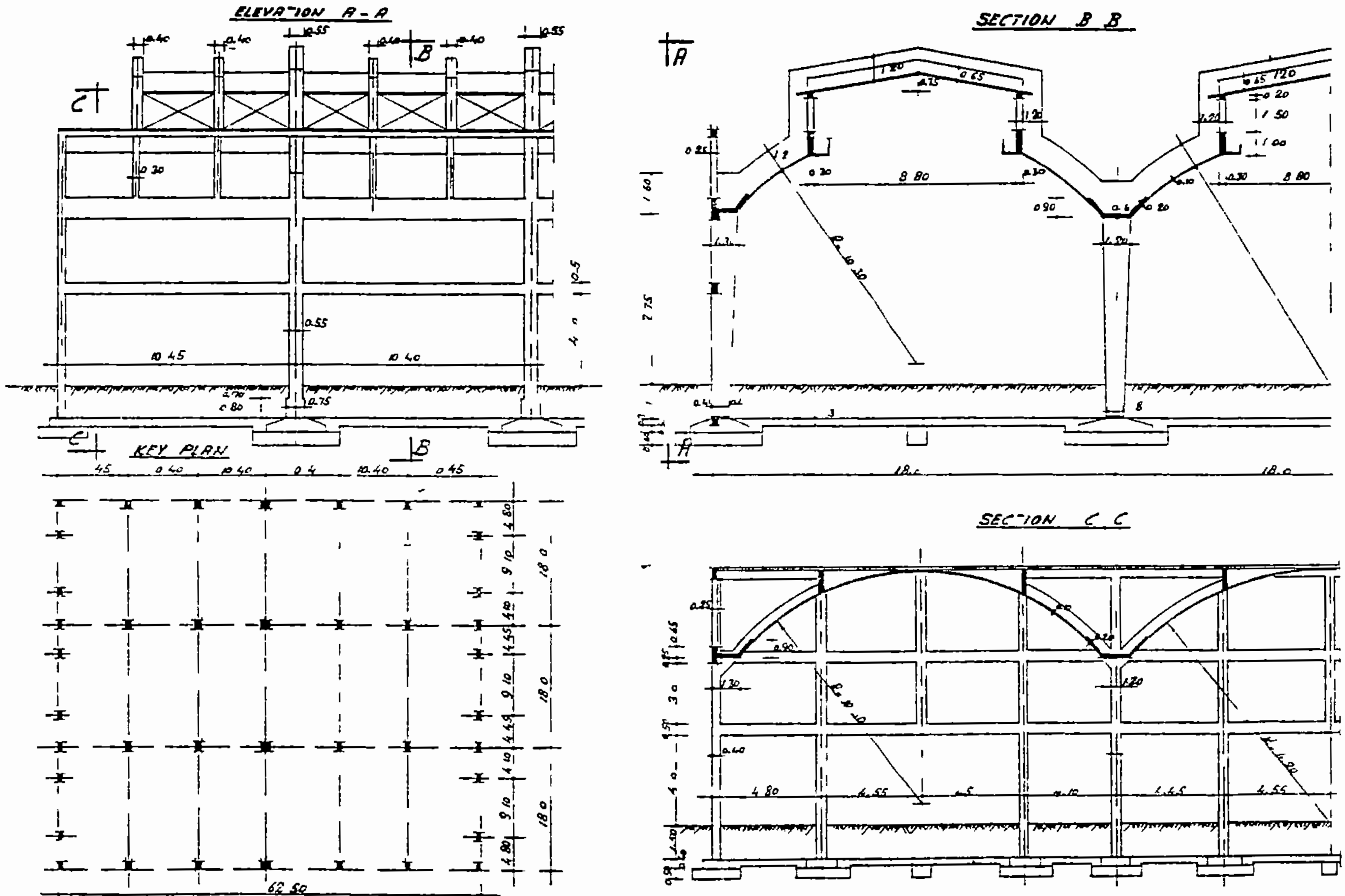
Fig VII-18

CAIRO T V MAIN STUDIO



VIERENDEEL GIRDER

F-- V-10



MISR SPW MEHALLA

DYING PLAN

Fig IV-34

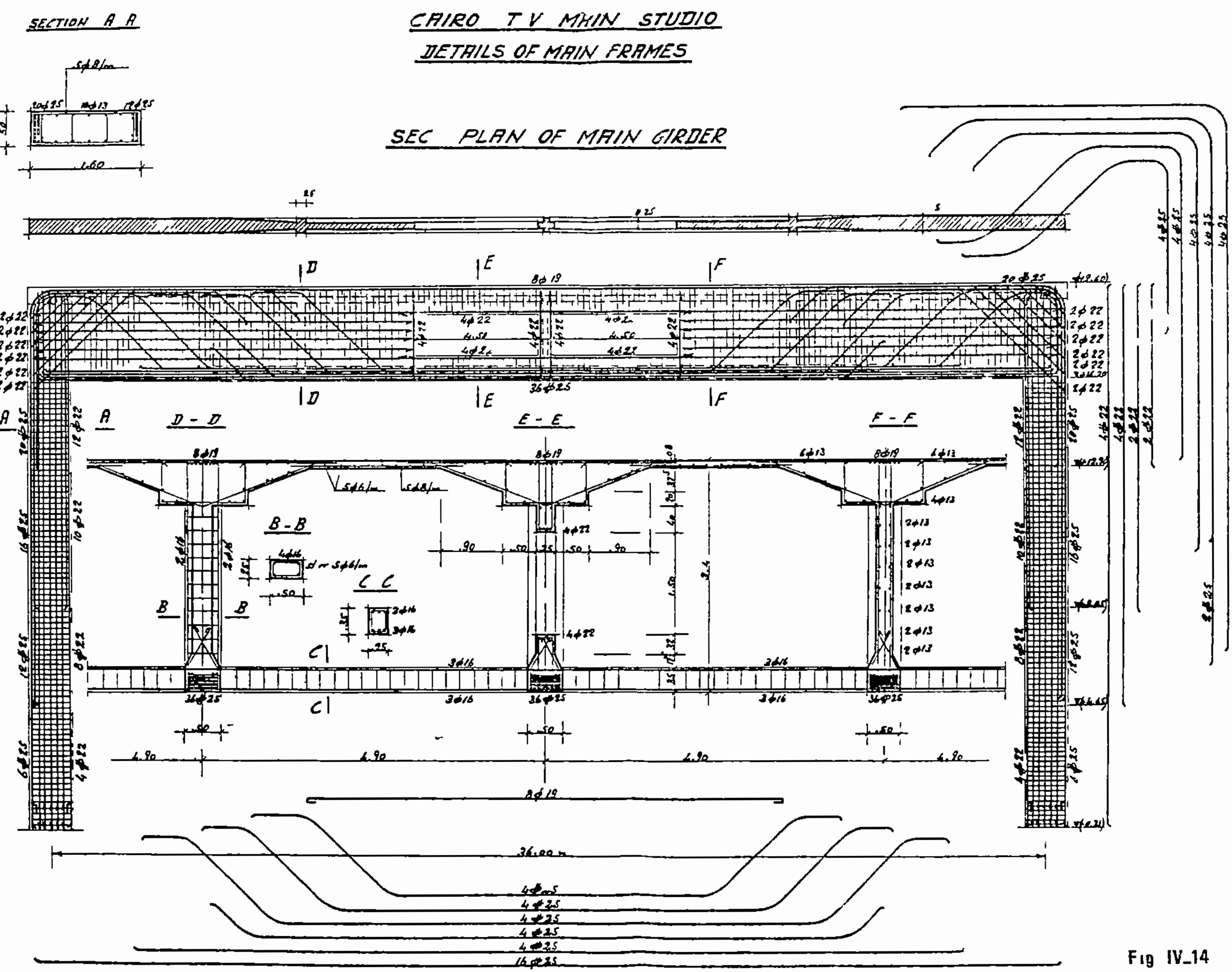
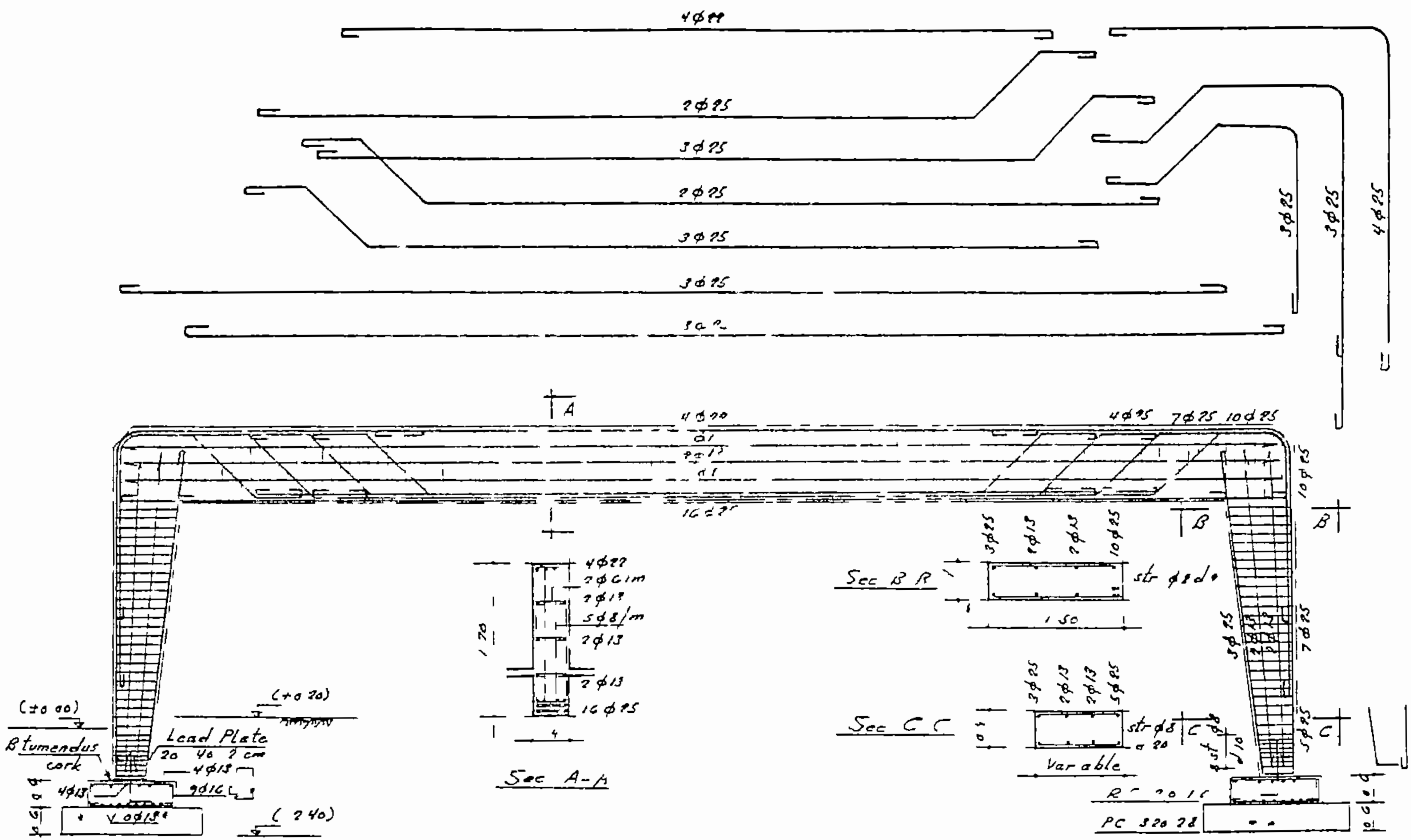

CAIRO T V MAIN STUDIO
DETAILS OF MAIN FRAMES
SEC PLAN OF MAIN GIRDER
SECTION A A
B - B
C C
D - D
E - E
F - F
36.00 m

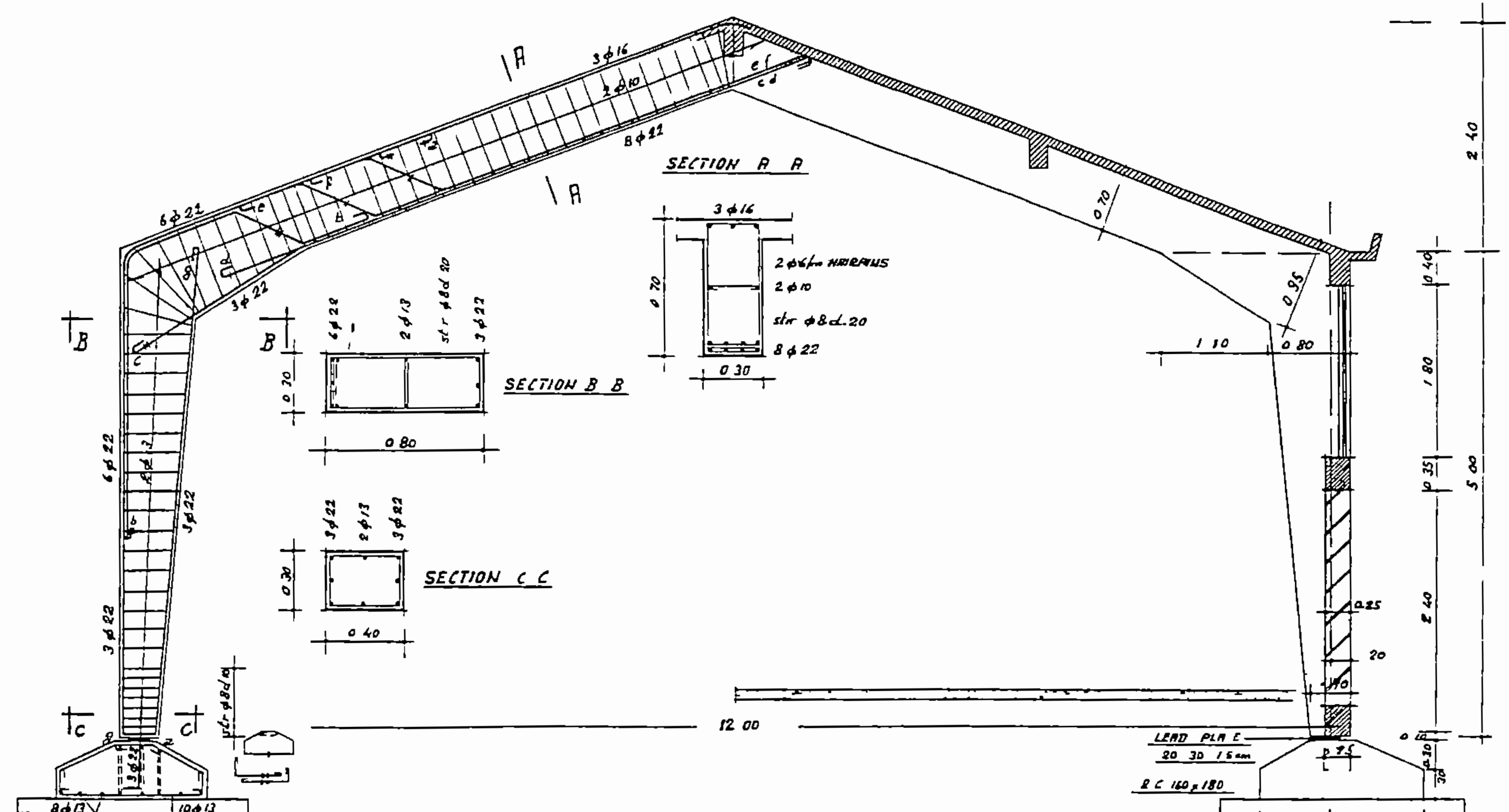
Fig IV_14



PAINTS AND CHEMICAL INDUSTRIES Co

Details of reinforcement of main frames

Fig IV-13



2 HINGED POLYGONAL FRAME

Fig IV-16

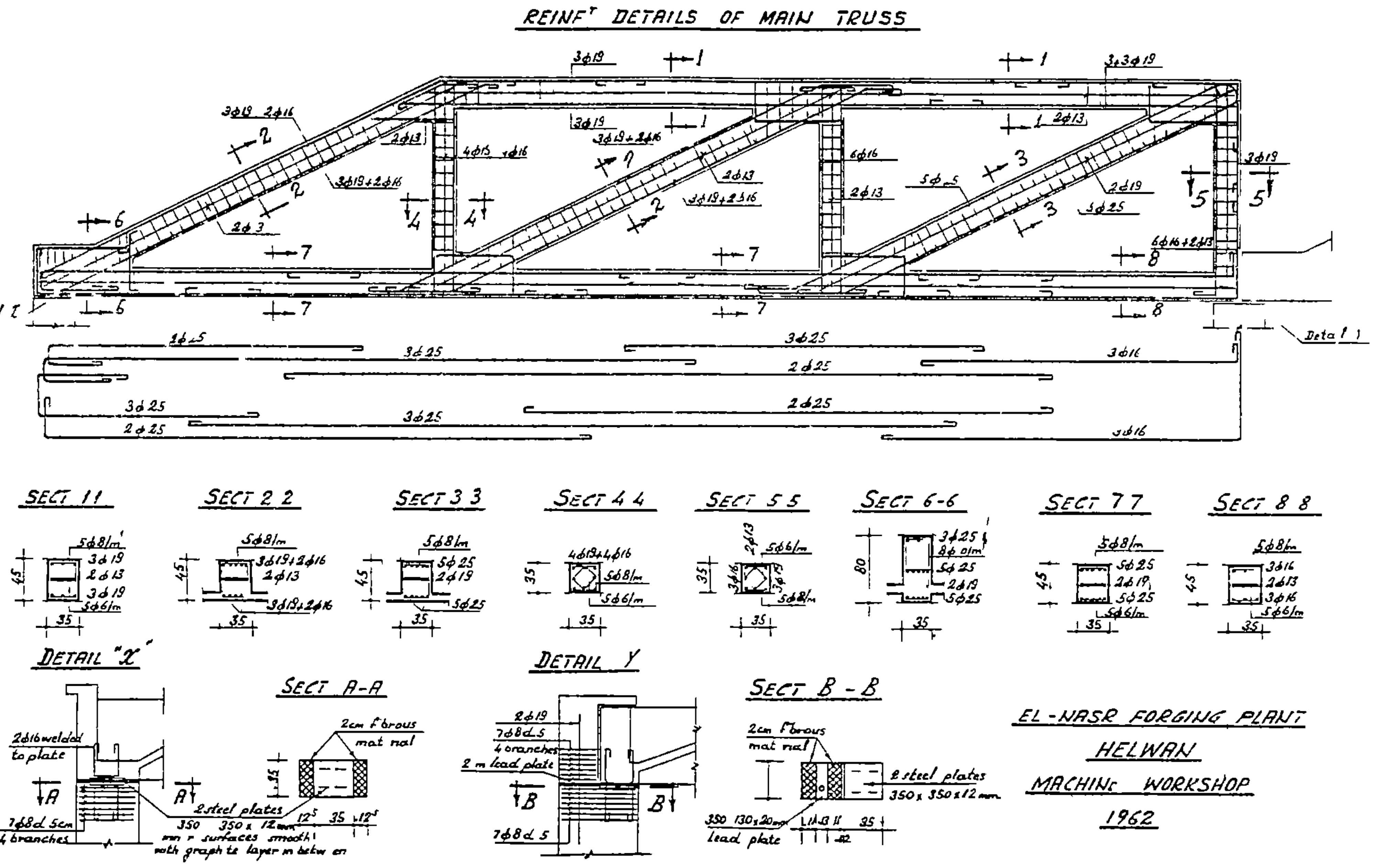

REINF^T DETAILS OF MAIN TRUSS
SECT 1 1
SECT 2 2
SECT 3 3
SECT 4 4
SECT 5 5
SECT 6-6
SECT 7 7
SECT 8 8
DETAIL "X"
SECT A-A
DETAIL Y
SECT B - B
2 steel plates
350 x 350 x 12 mm
2 cm fibrous mat rial
lead plate
EL-NASR FORGING PLANT
HELWAN
MACHINE WORKSHOP
1962

Fig VI-9

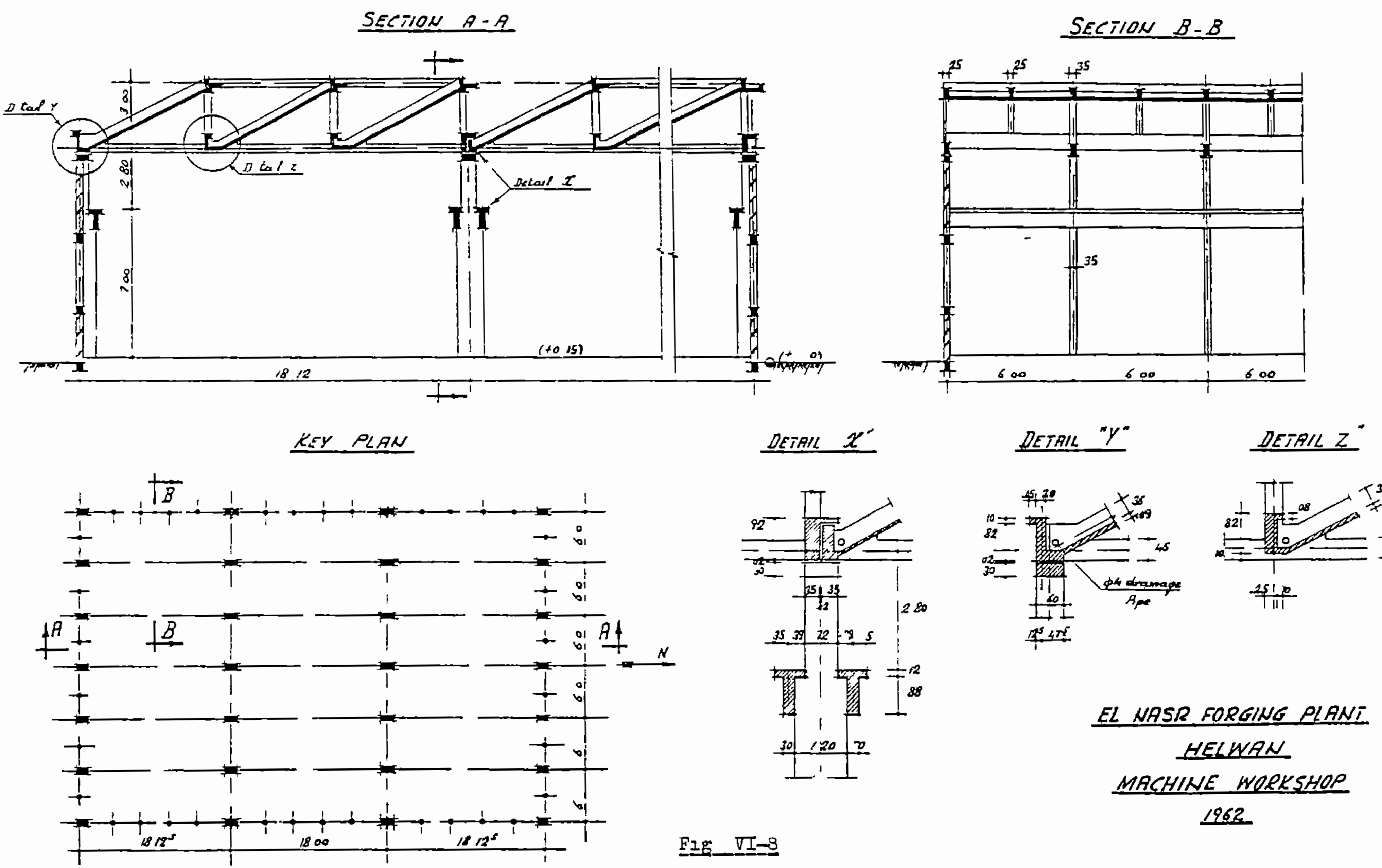

SECTION A-A
Detail Y
Detail Z
Detail X
3 00
2 80
7 00
18 12
(+0 15)
SECTION B-B
25
25
35
35
6 00
6 00
6 00
KEY PLAN
B
B
A
A
N
18 12^5
18 00
18 12^5
DETAIL "X"
DETAIL "Y"
DETAIL "Z"
ϕ4 drainage pipe
2 80
120
12
88
EL NASR FORGING PLANT
HELWAN
MACHINE WORKSHOP
1962

Fig VI-8

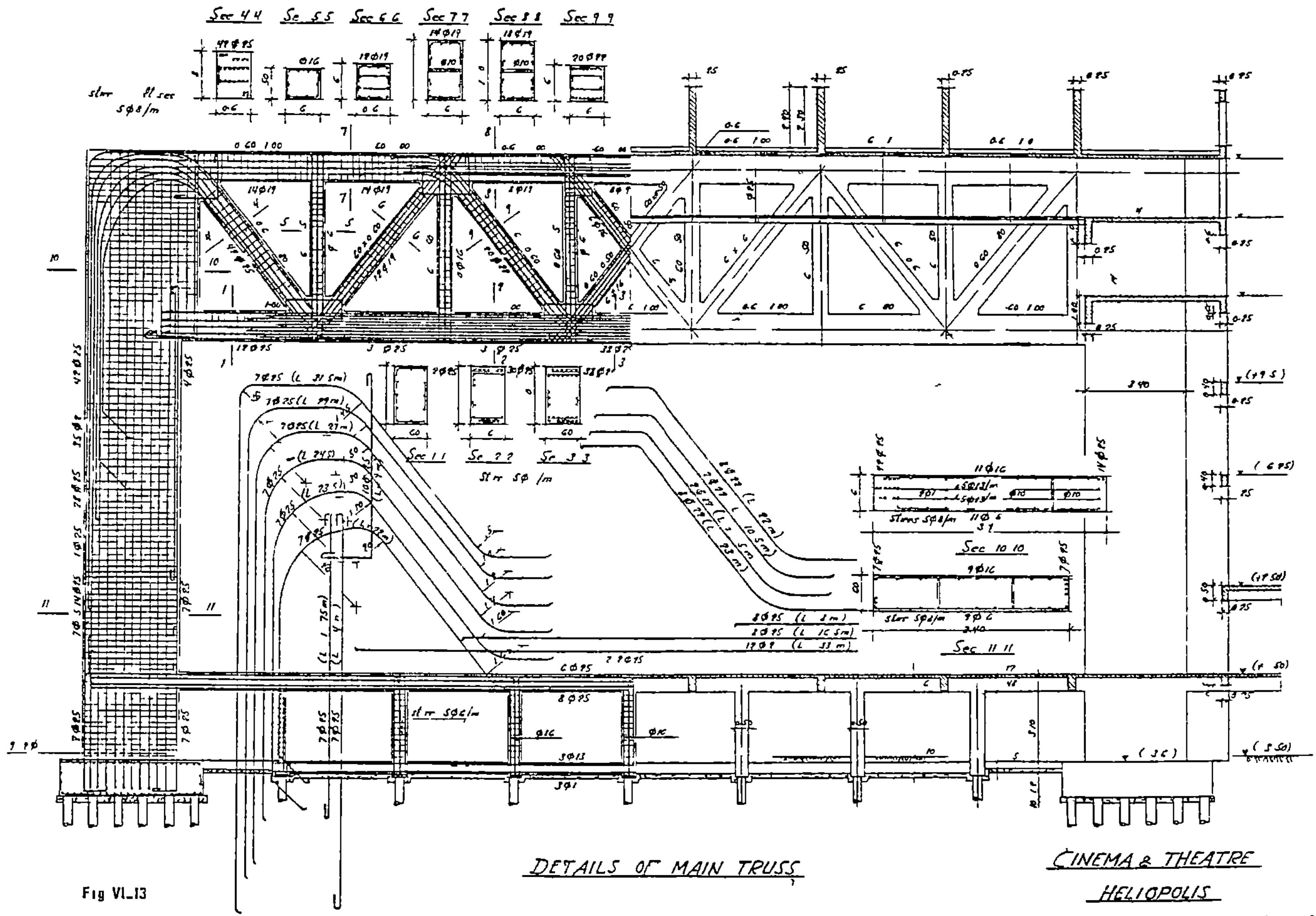


Fig VI_13

DETAILS OF MAIN TRUSS

CINEMA & THEATRE

HELIOPOLIS

DETAILS OF ROOF OF STAGE-3

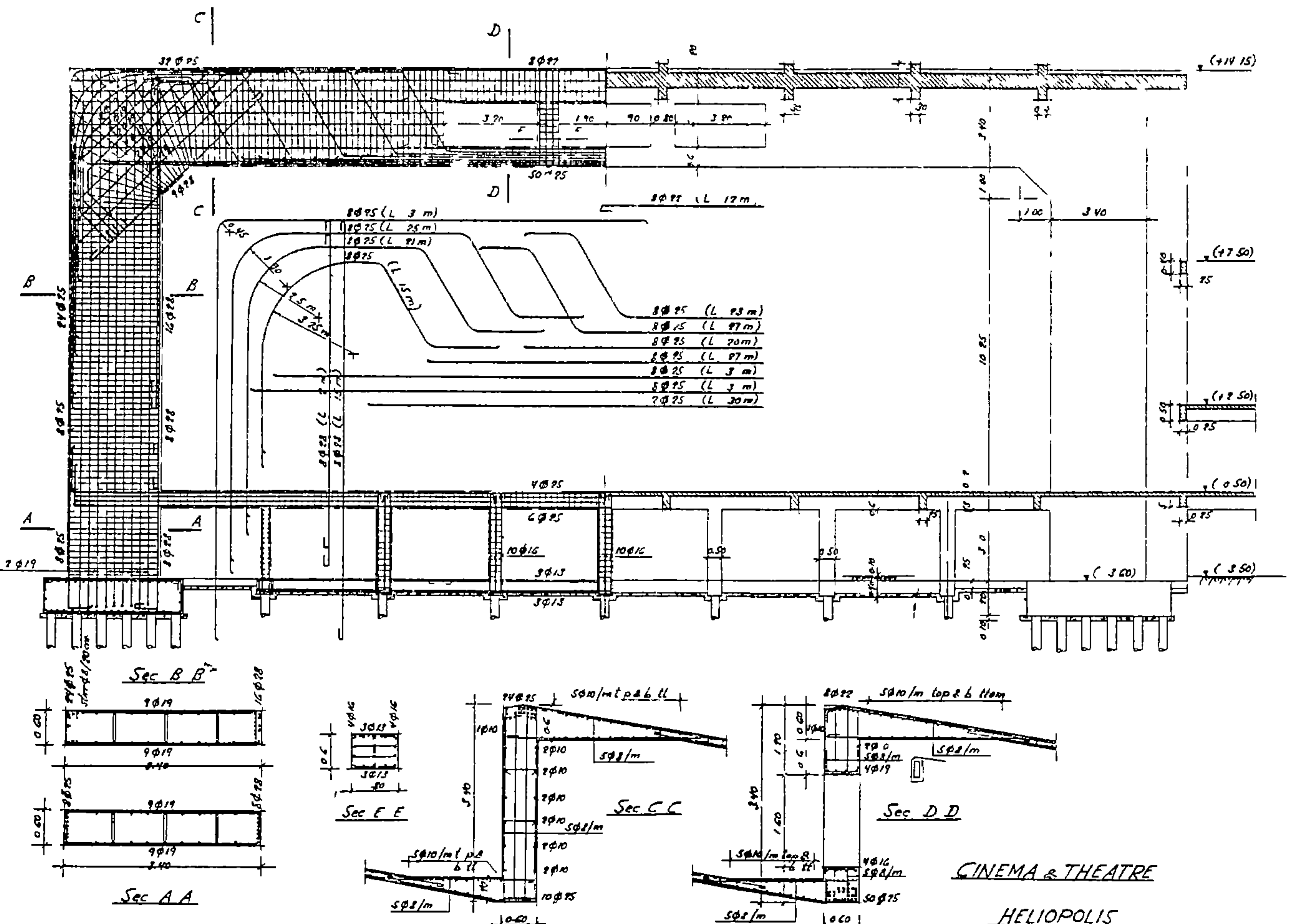


Fig VI 12

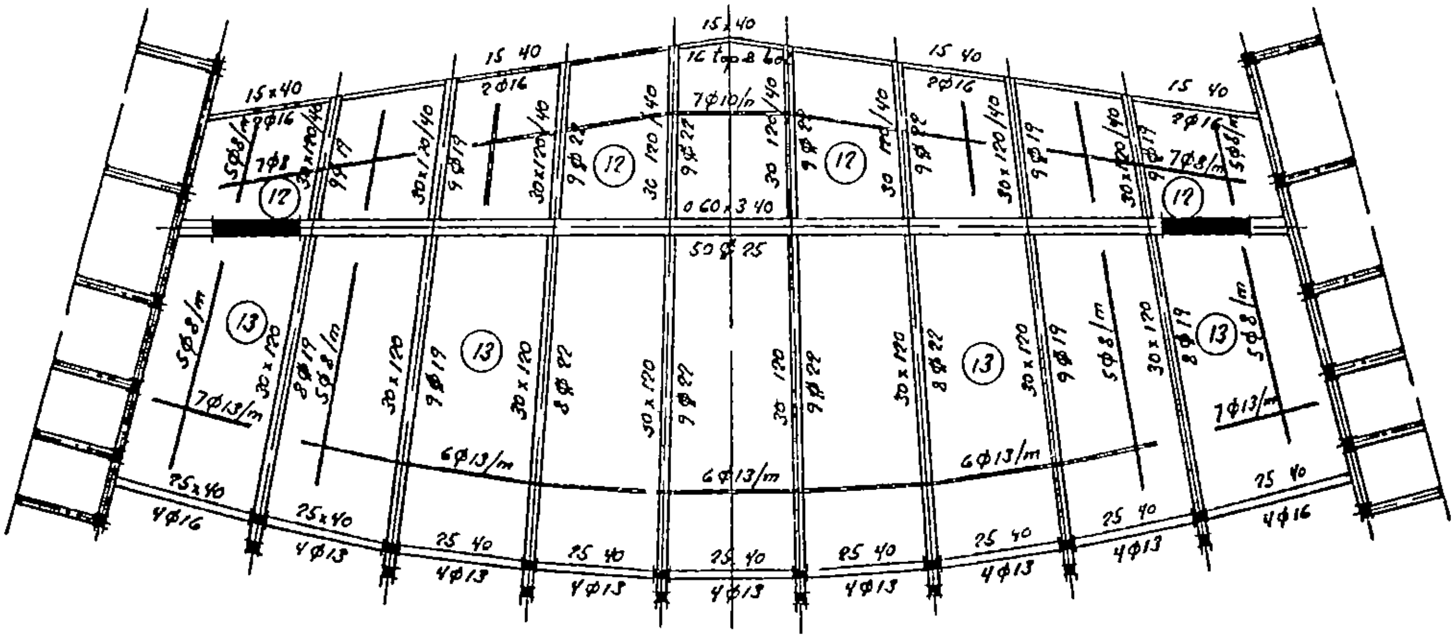
DETAILS OF MAIN FRAME

CINEMA & THEATRE

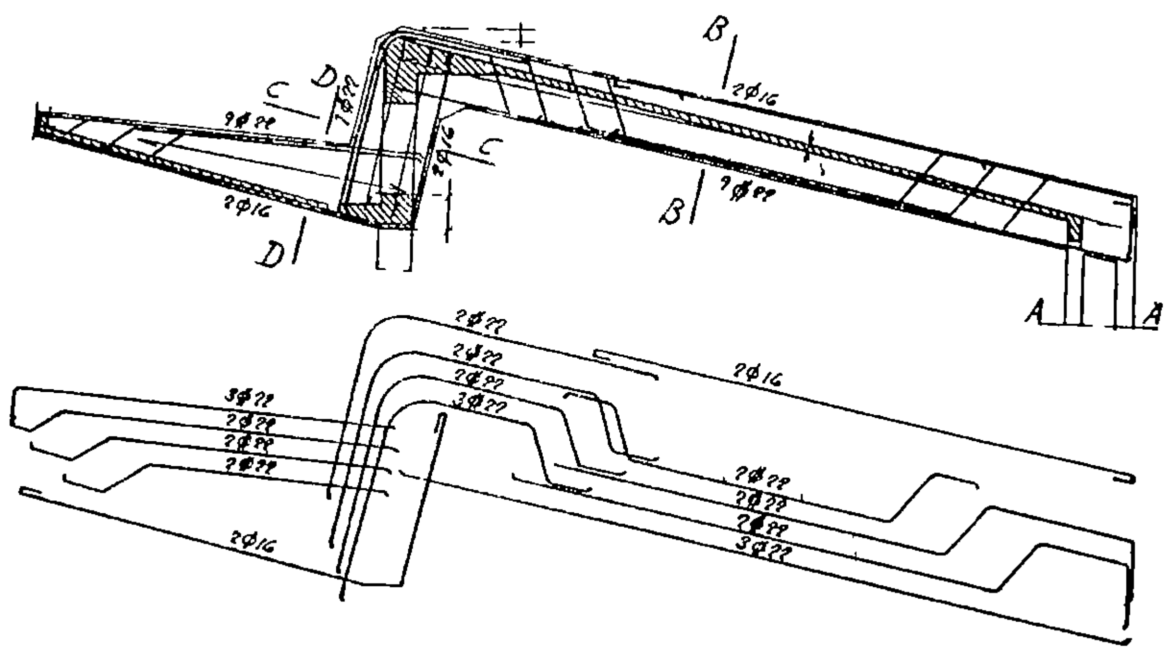
HELIOPOLIS

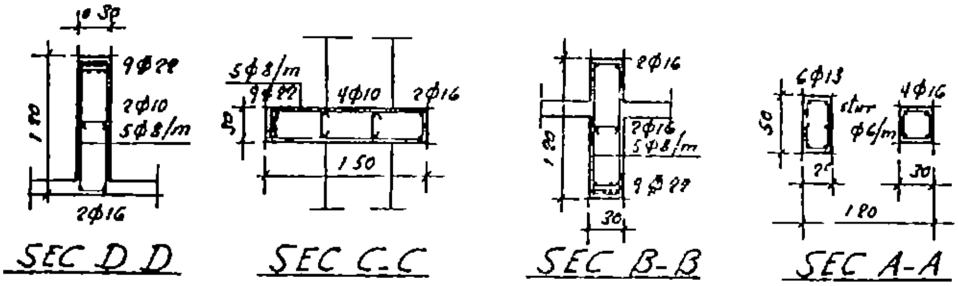
DETAILS OF ROOF OF STAGE-2

a Plan of roof of stage



b- Dela ls of cross girders





CINEMA & THEATRE
HELIOPOLIS
DETAILS OF ROOF
OF STAGE -1-

Fig VI 11

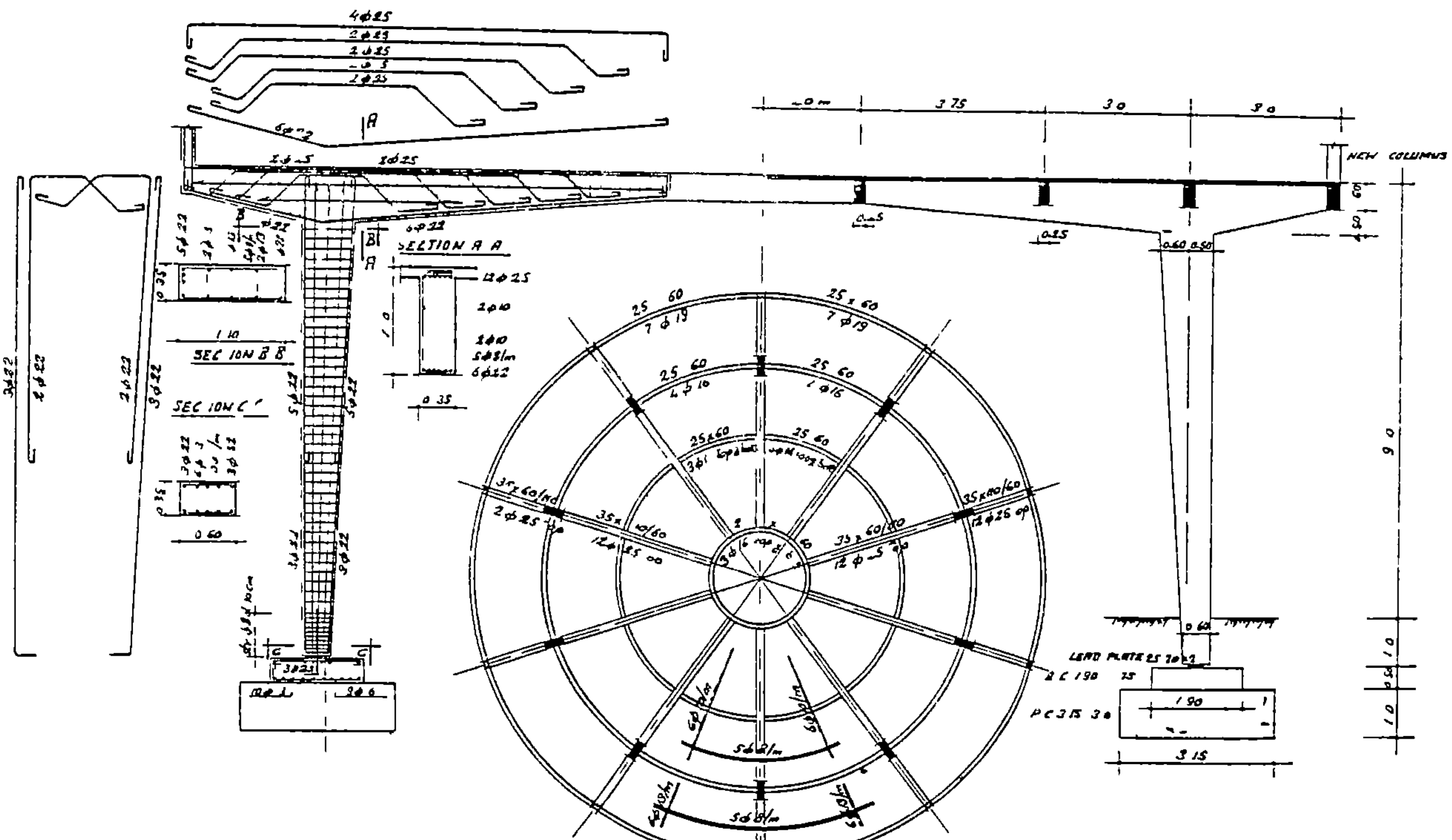


Fig. II-73

FRAMES SUPPORTING PLANETARIUM
CAIRO

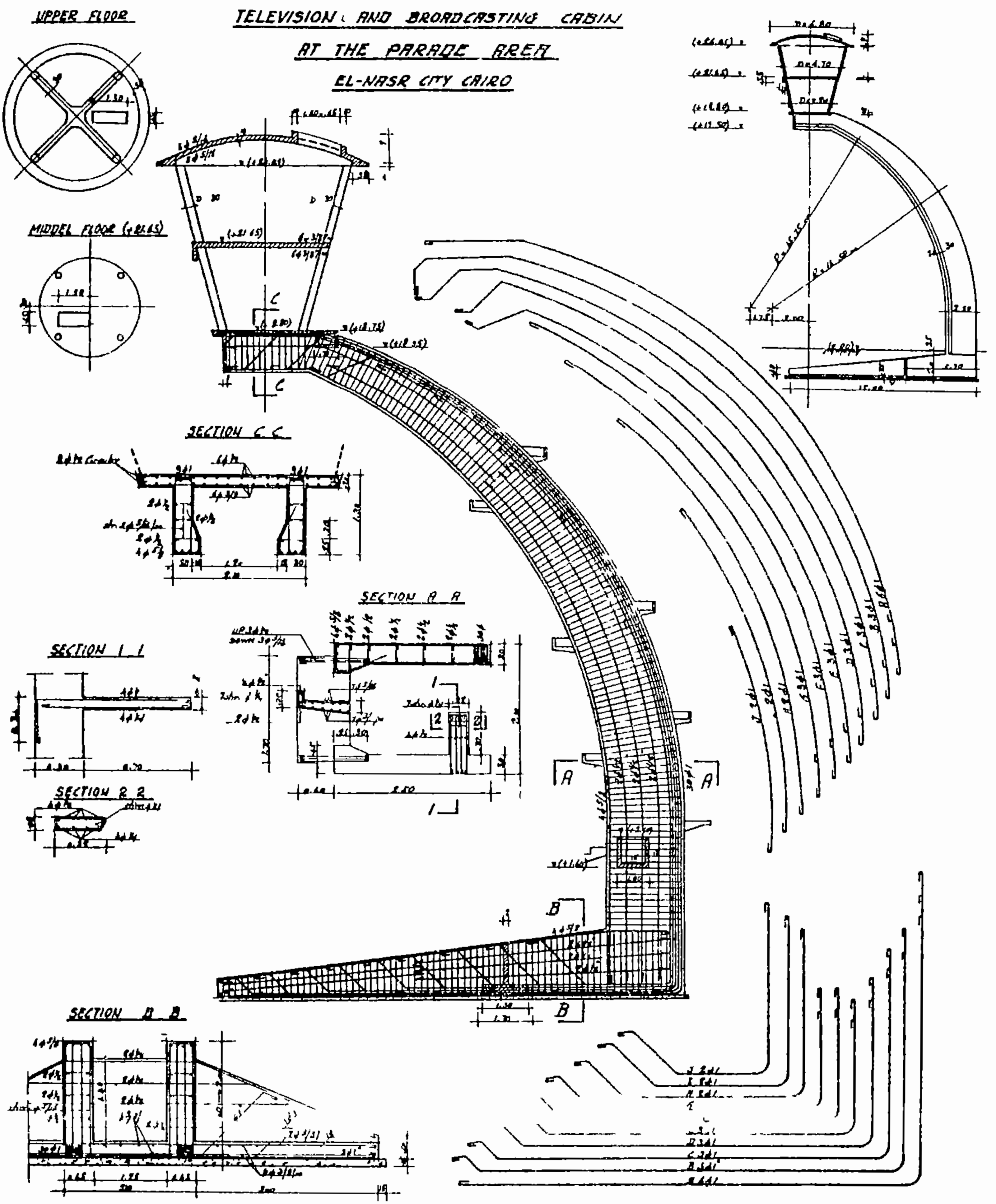
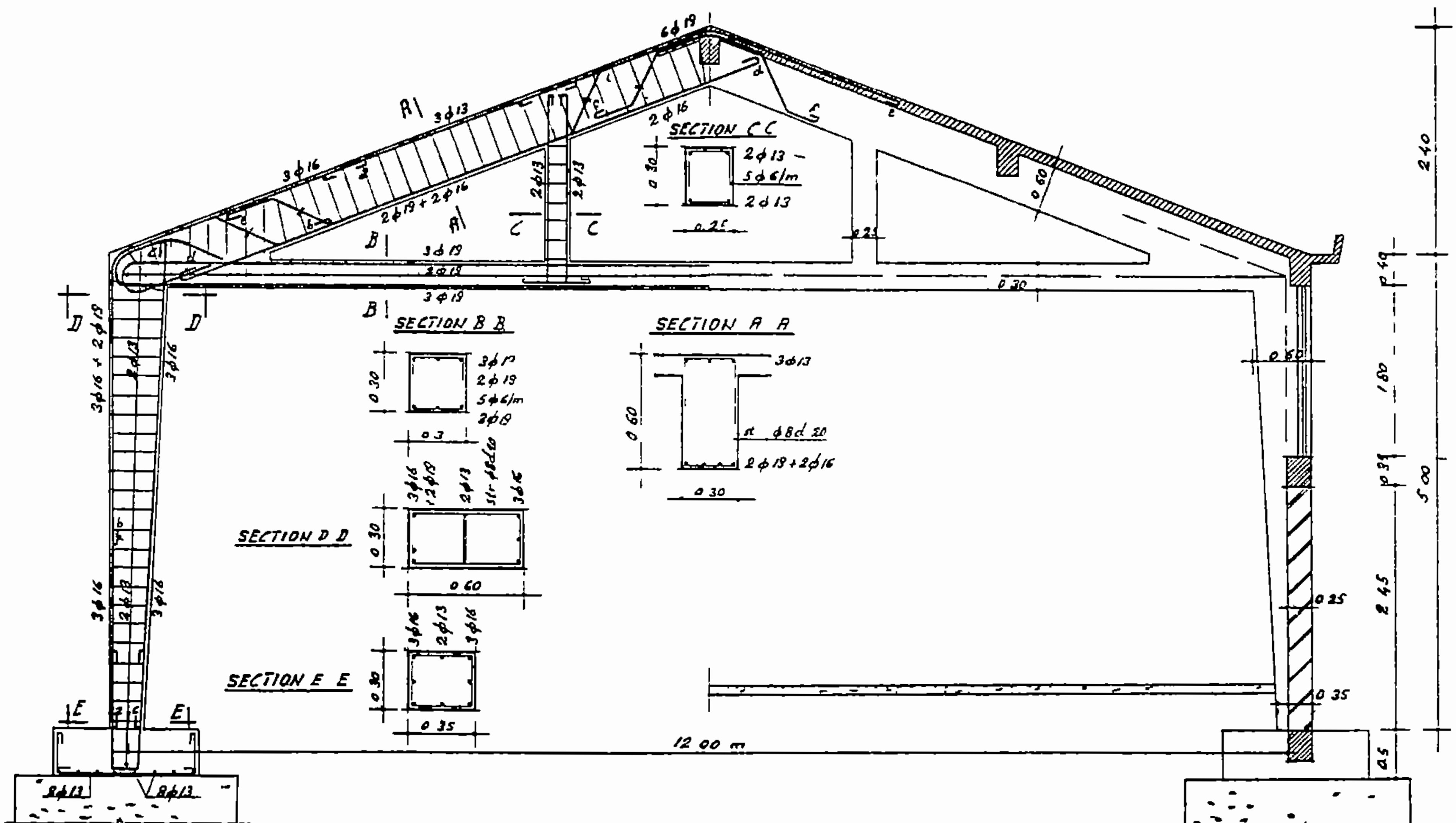
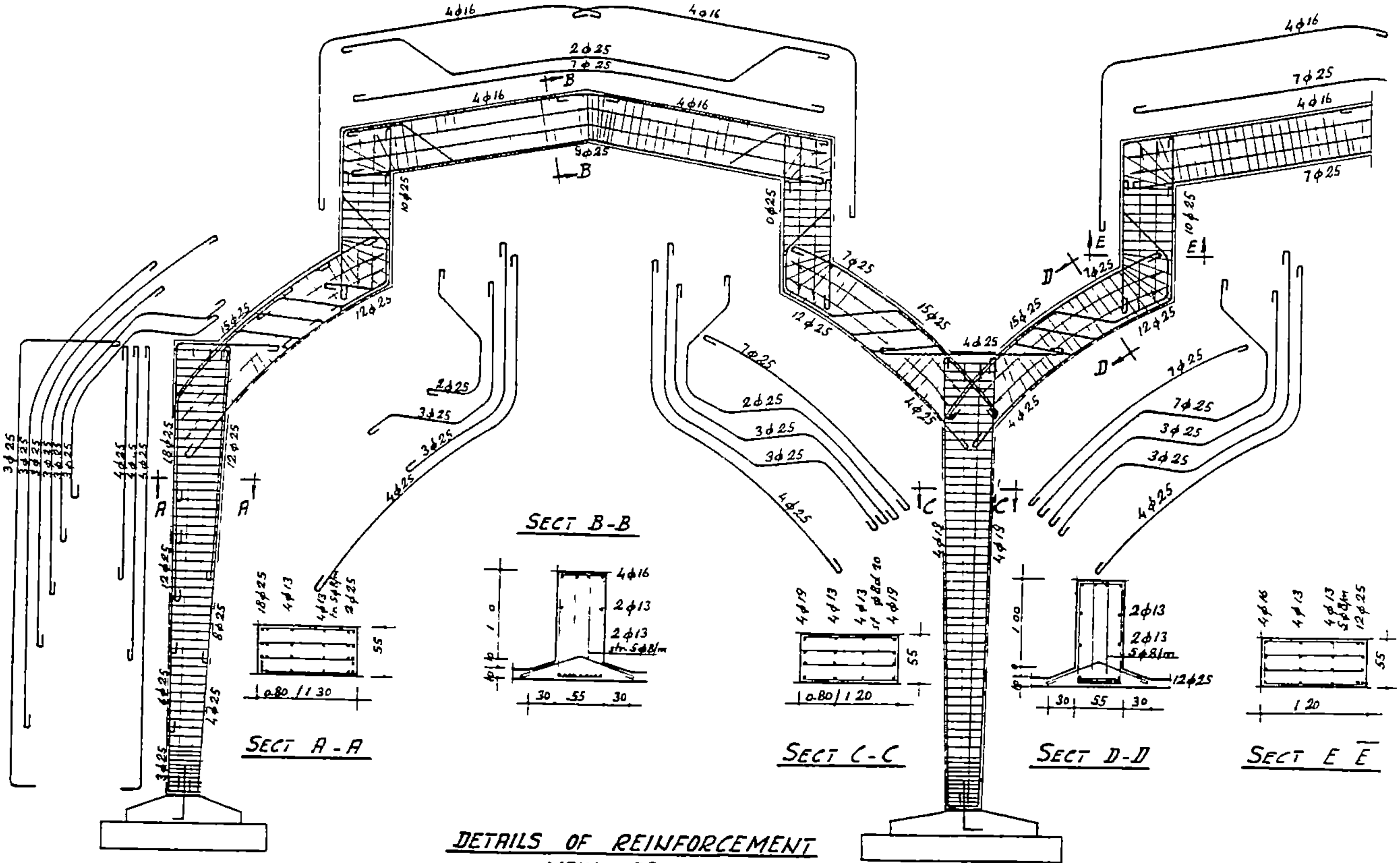

TELEVISION AND BROADCASTING CABIN
AT THE PARADE AREA
EL-NASR CITY CAIRO
UPPER FLOOR
MIDDEL FLOOR (+21.65)
SECTION C C
SECTION A A
SECTION 1 1
SECTION 2 2
SECTION B B

Fig IV 59

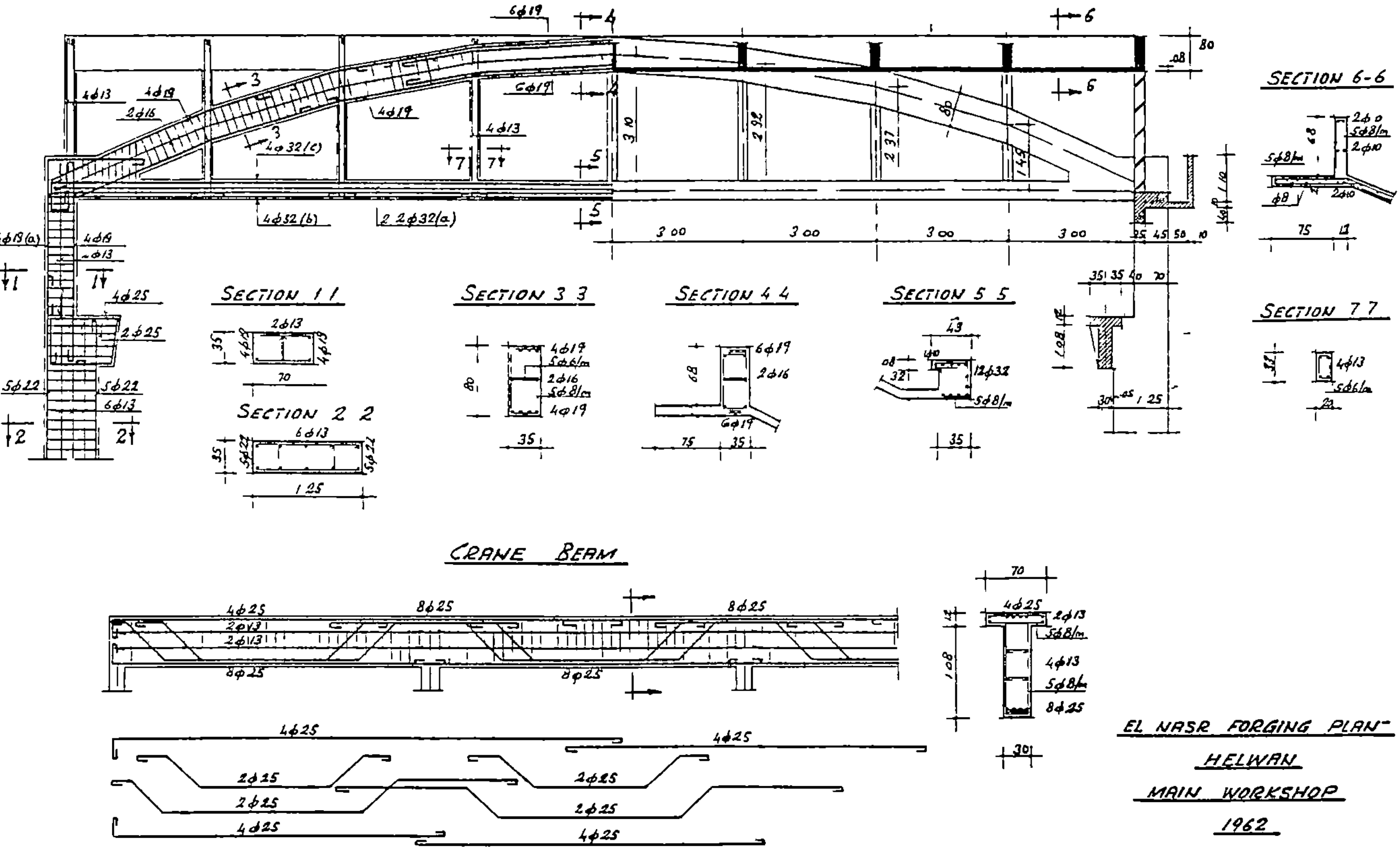


2-HINGED FRAME WITH A TIE

Fig. IV-23

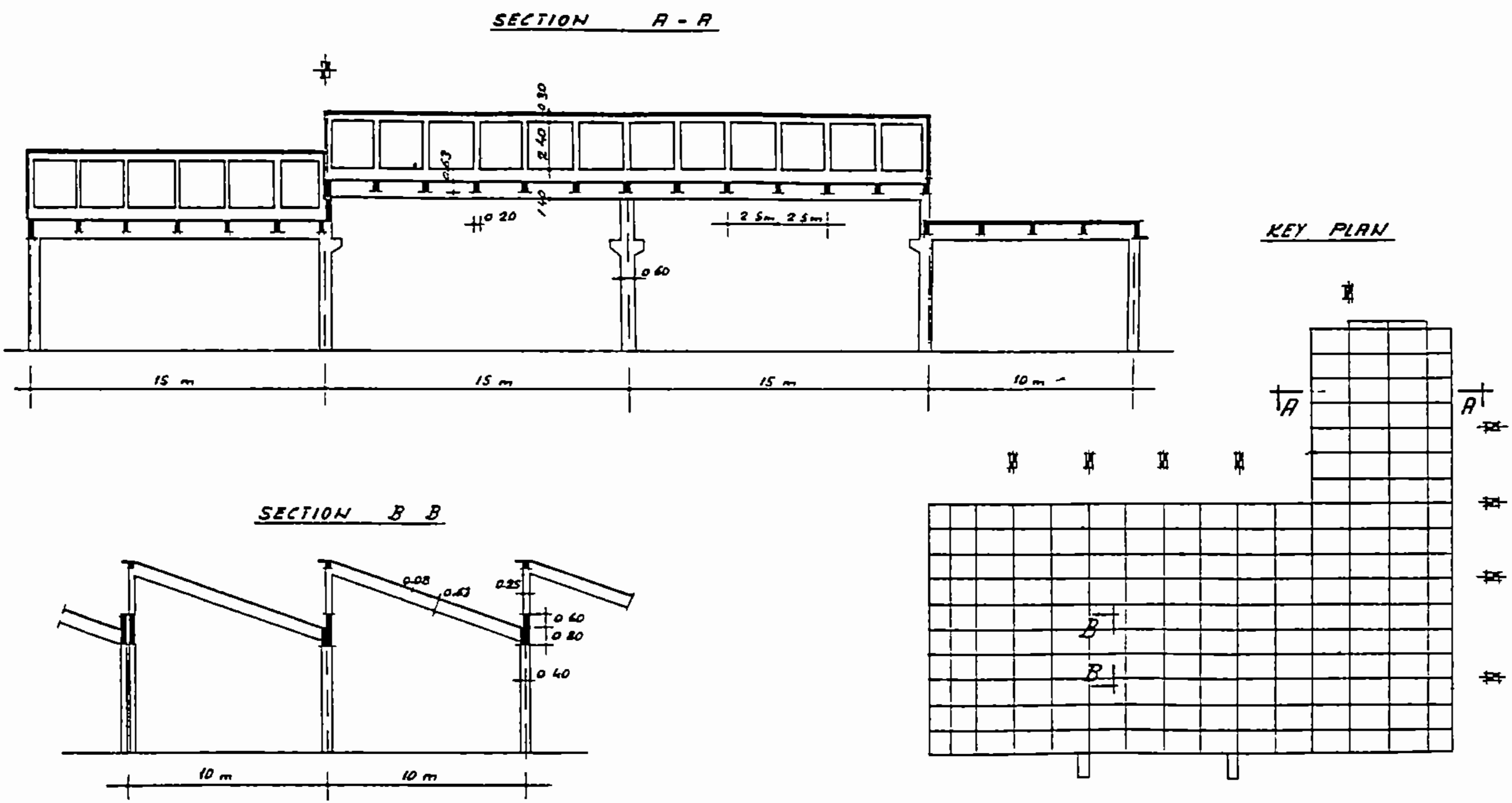


DETAILS OF REINFORCEMENT
MAIN FRAMES
Fig IV-35



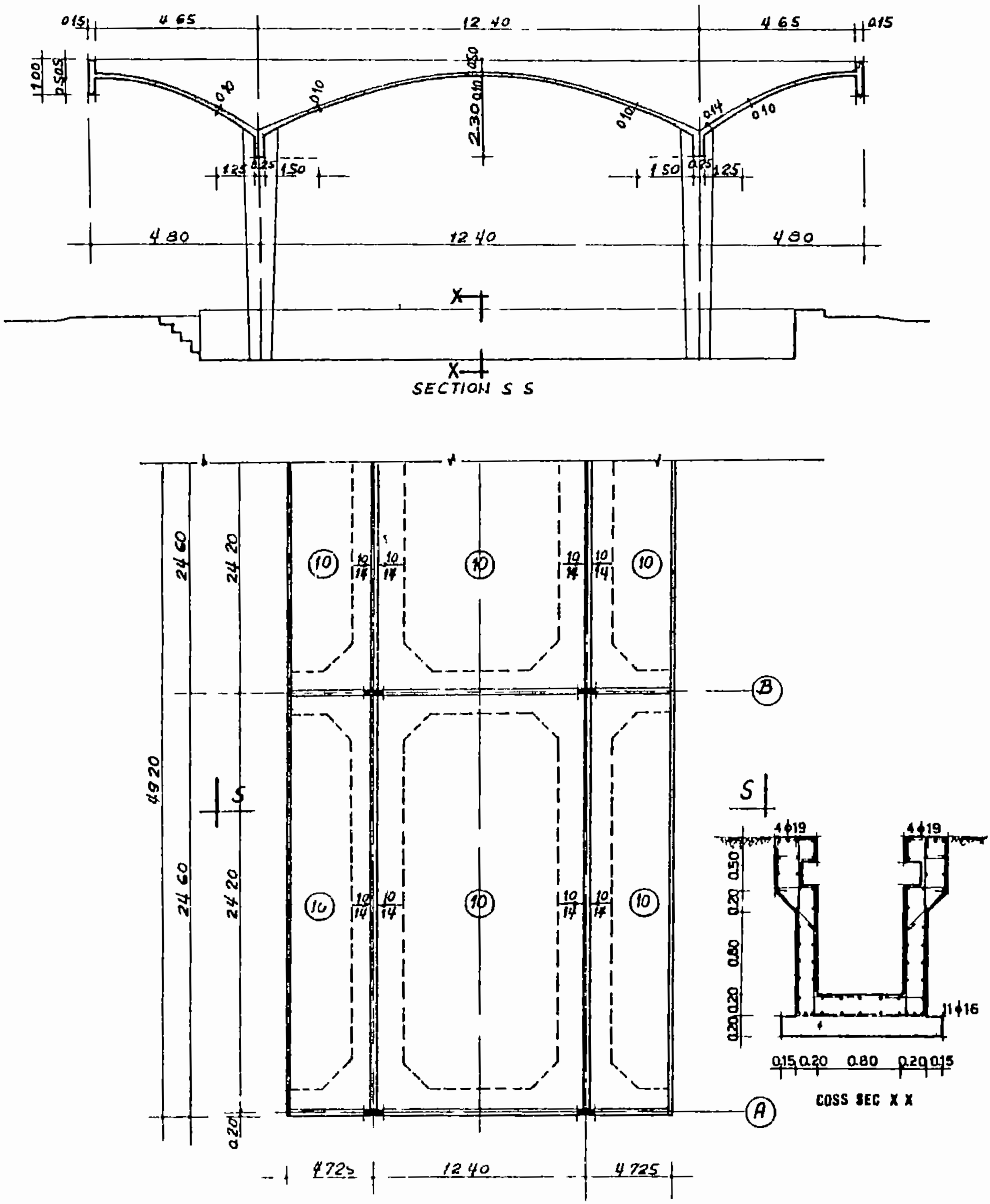
EL NASR FORGING PLANT
HELWAN
MAIN WORKSHOP
1962

Fig VII-19



FACTORY 135 HELWAN

Fig VI-22

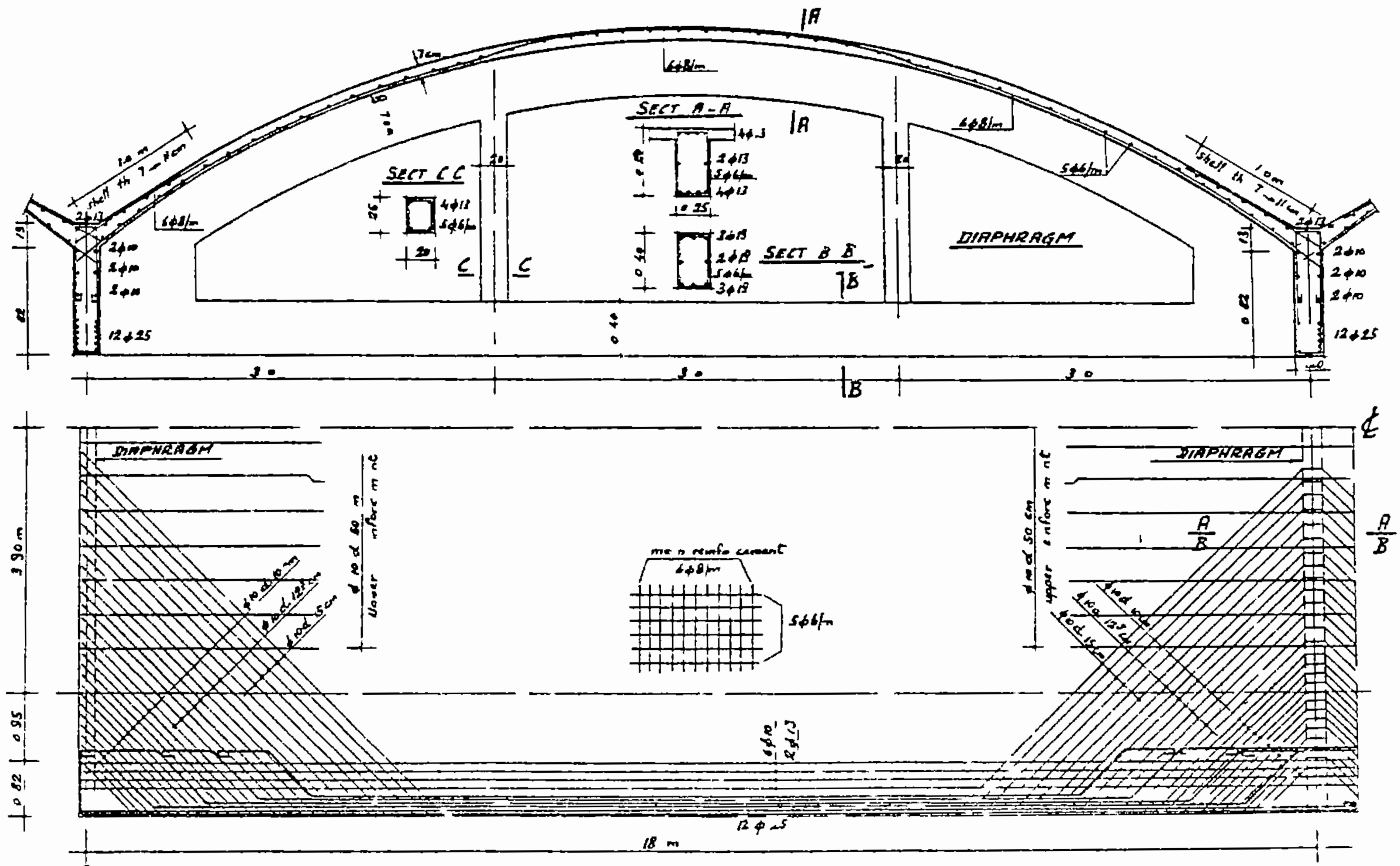


PUBLIC TRANSPORTATION ORGANIZATION

GESR EL SUEZ GARAGE

INSPECTION SHED

Fig XII 74 a



INTERMEDIATE CYLINDRICAL SHELL 9 x 18 m

Fig. XII-73

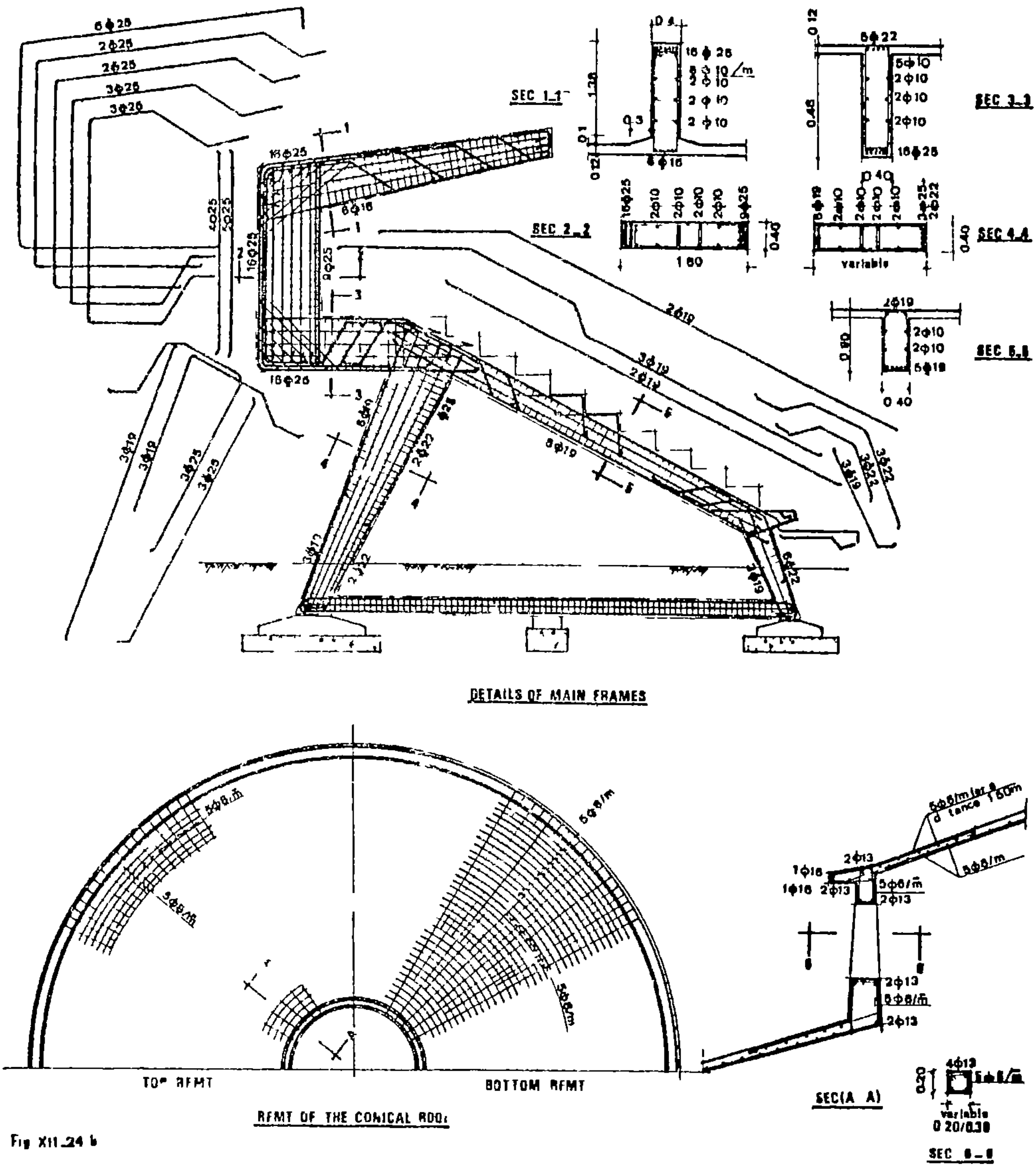
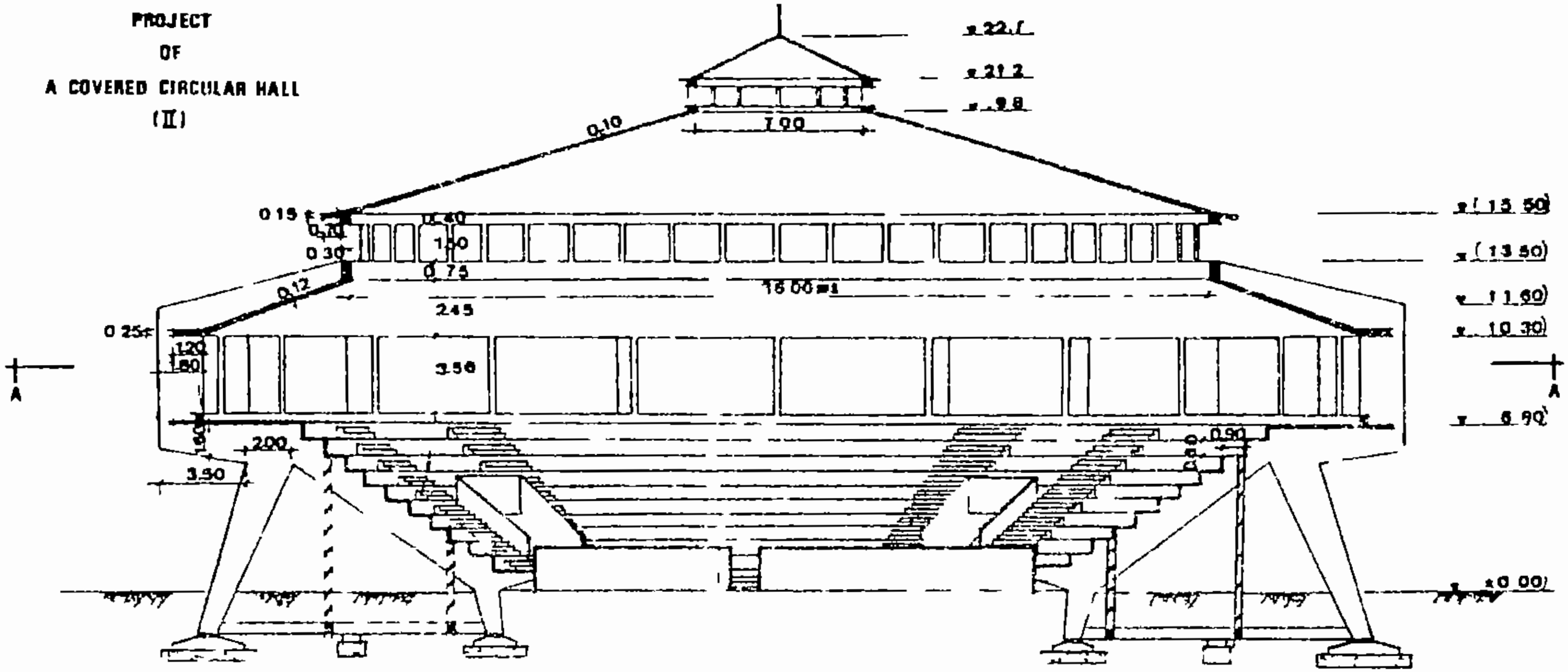

SEC 1_1
SEC 2_2
SEC 3_3
SEC 4_4
SEC 5_5
DETAILS OF MAIN FRAMES
TOP RFMT
BOTTOM RFMT
RFMT OF THE CONICAL ROOF
SEC(A A)
SEC B_B

Fig XII_24 b

PROJECT
OF
A COVERED CIRCULAR HALL
(II)



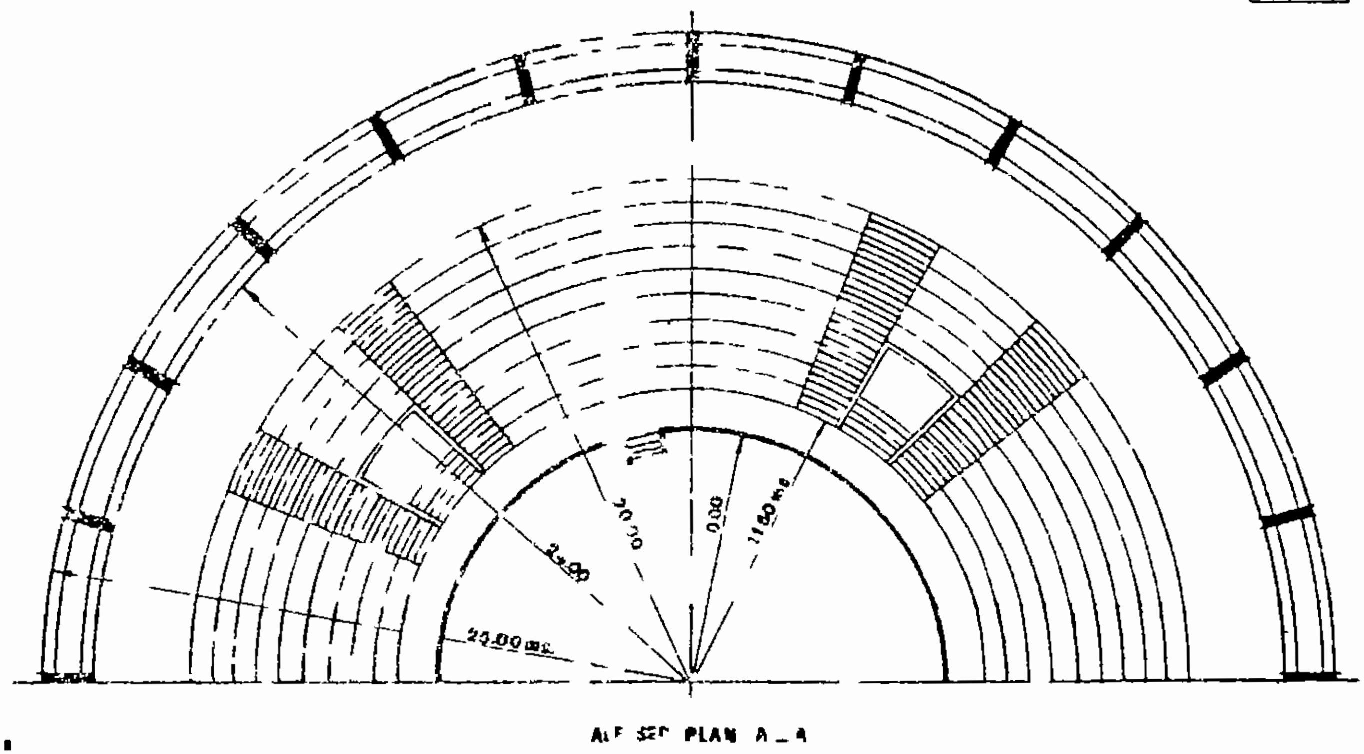
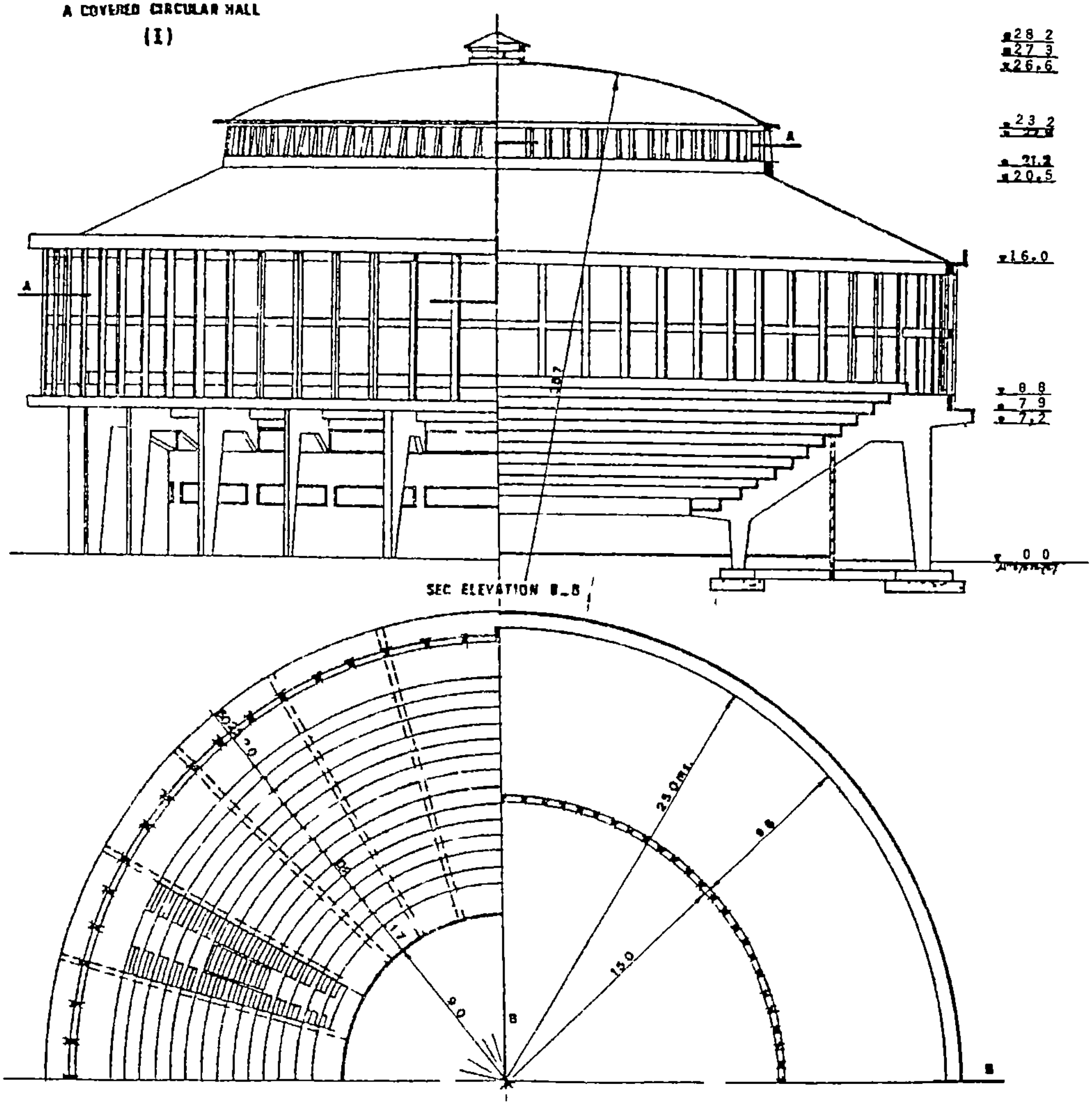


Fig XII 24 a

PROJECT
OF
A COVERED CIRCULAR HALL
(I)



SEC ELEVATION B_B

HALF SEC PLAN A_A

Fig XII_23

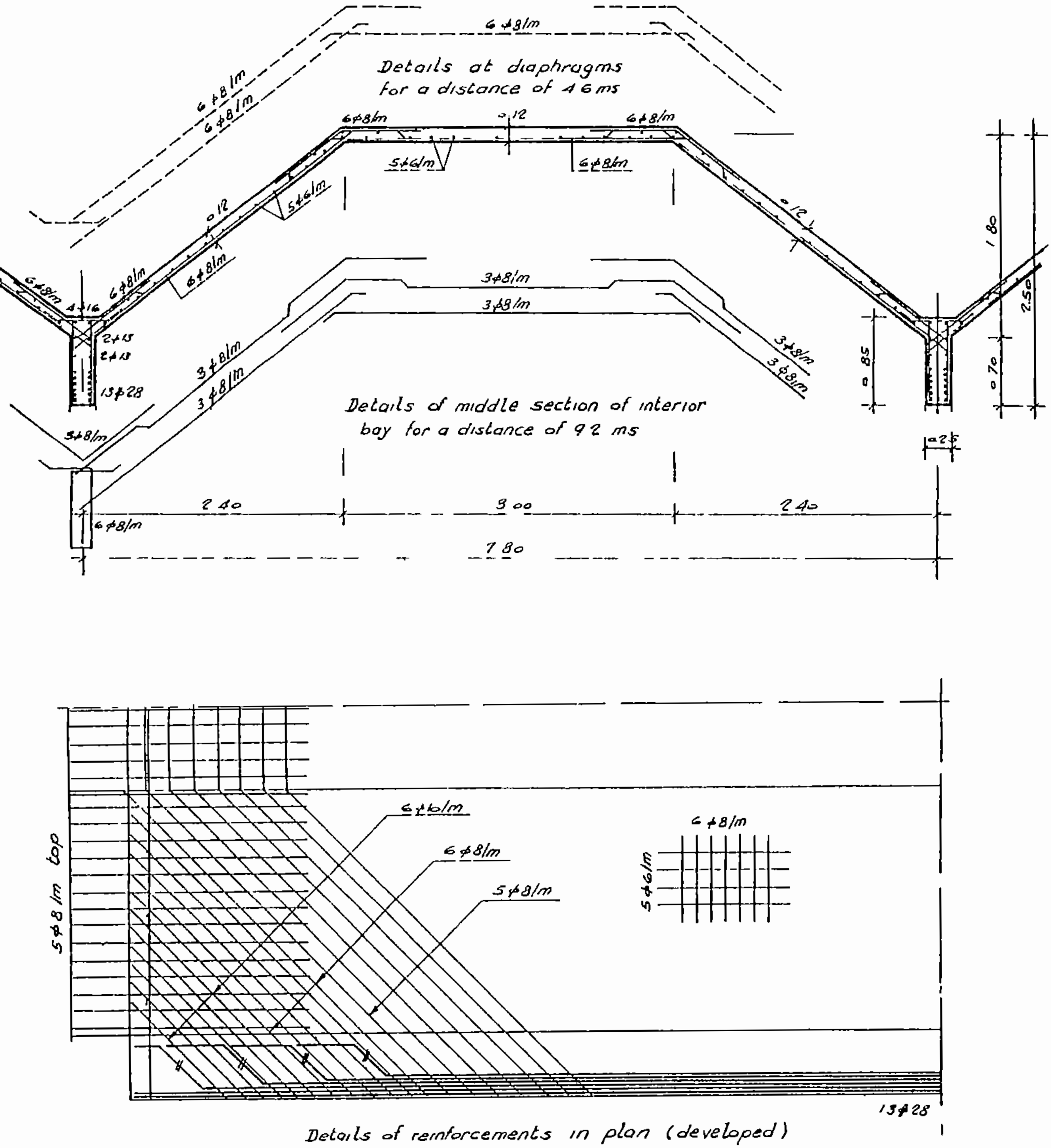
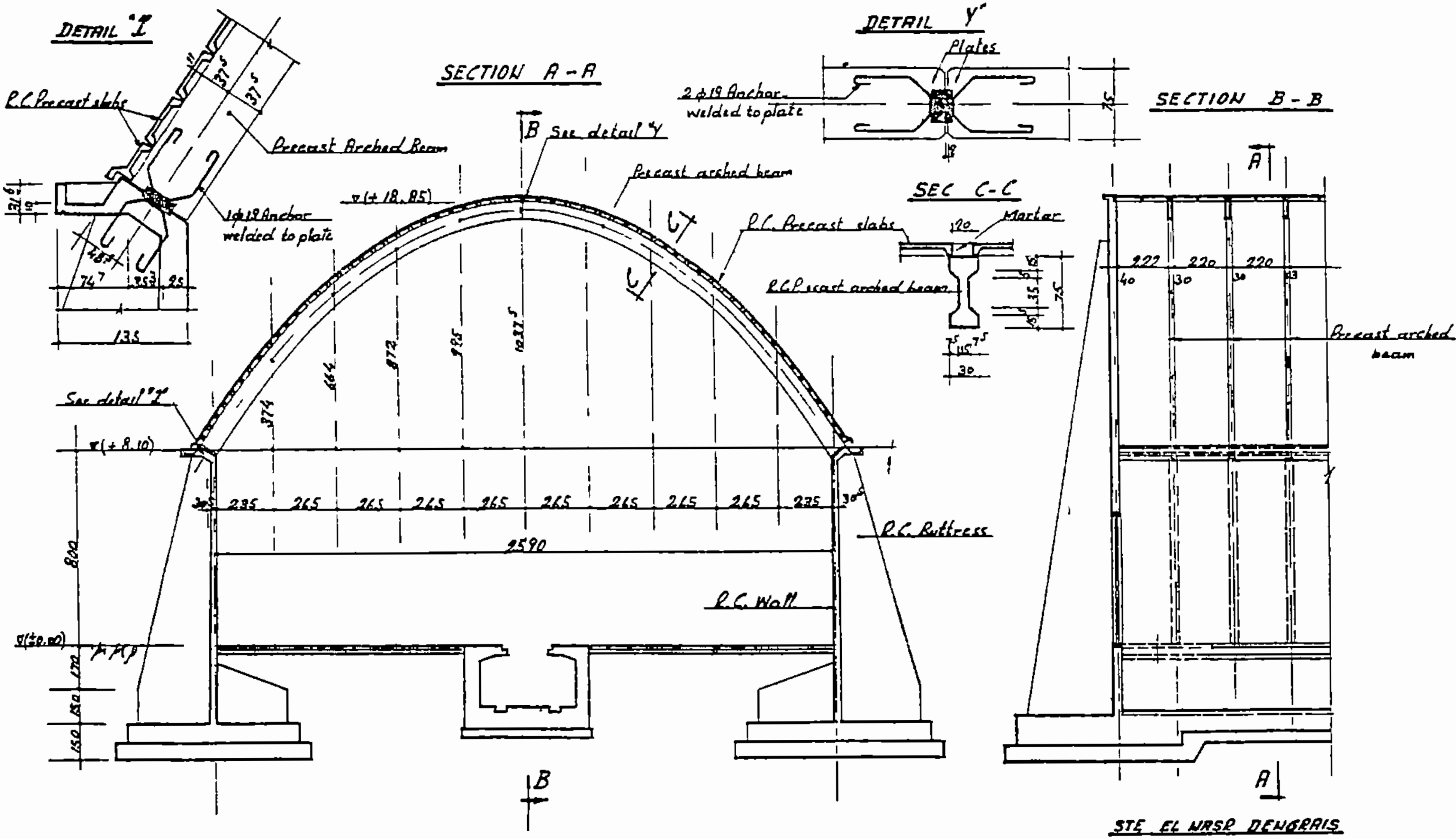
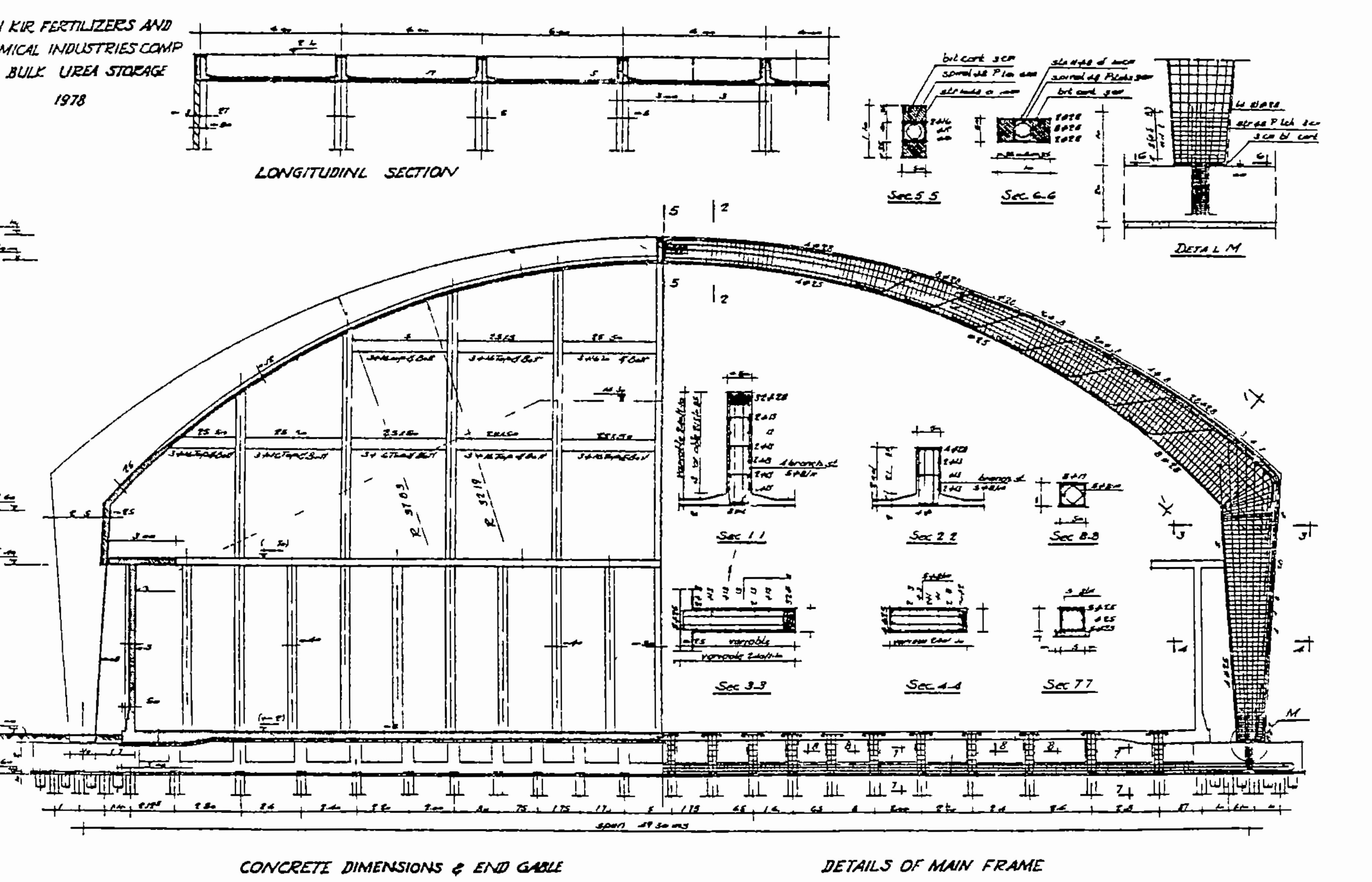

6 φ8/m
Details at diaphragms
for a distance of 4 6 ms
6 φ8/m
6 φ8/m
6φ8/m
∘12
6φ8/m
5φ6/m
6φ8/m
∘12
5φ6/m
∘12
6φ8/m
6φ8/m
4φ16
2φ13
2φ13
13φ28
3φ8/m
3φ8/m
3φ8/m
3φ8/m
3φ8/m
3φ8/m
1 80
2.50
∘85
∘70
Details of middle section of interior
bay for a distance of 9 2 ms
5φ8/m
∘25
2 40
3 00
2 40
6 φ8/m
7 80
6φ6/m
6 φ8/m
5φ8/m
6 φ8/m
5φ6/m
5φ8/m top
13φ28
Details of reinforcements in plan (developed)

Fig XI-19



STE EL NASR DENGRAIS
ET D INDUSTRIES CHIMIQUES
SUEZ
AMMONIUM NITRATE LIMESTONE SILO

Fig. VIII-7



KIR FERTILIZERS AND
MICAL INDUSTRIES COMP
BULK UREA STORAGE
1978
LONGITUDINL SECTION
Sec 5 5
Sec 6 6
DETAIL M
Sec 1 1
Sec 2 2
Sec 8 8
Sec 3 3
Sec 4 4
Sec 7 7
R 3703
R 3219
CONCRETE DIMENSIONS & END GABLE
DETAILS OF MAIN FRAME

Fig VIII-9

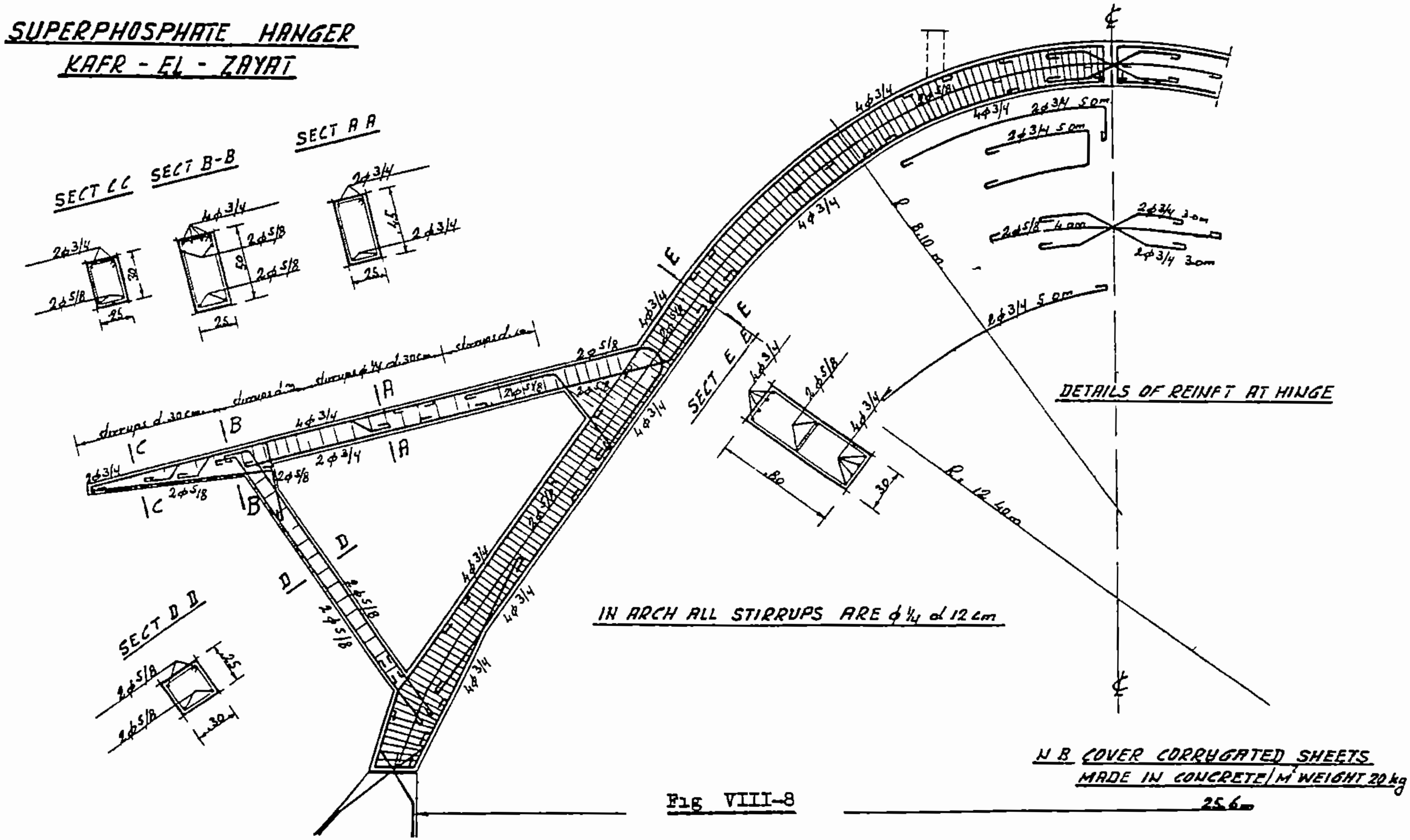

SUPERPHOSPHATE HANGER
KAFR - EL - ZAYAT
SECT C C
SECT B-B
SECT A A
SECT D D
SECT E E
2φ3/4
2φ5/8
4φ3/4
25
30
45
80
DETAILS OF REINFT AT HINGE
IN ARCH ALL STIRRUPS ARE φ1/4 d 12 cm
N B COVER CORRUGATED SHEETS
MADE IN CONCRETE/M² WEIGHT 20 kg
25.6 m

Fig VIII-8

COPPER FACTORY
ALEXANDRIA

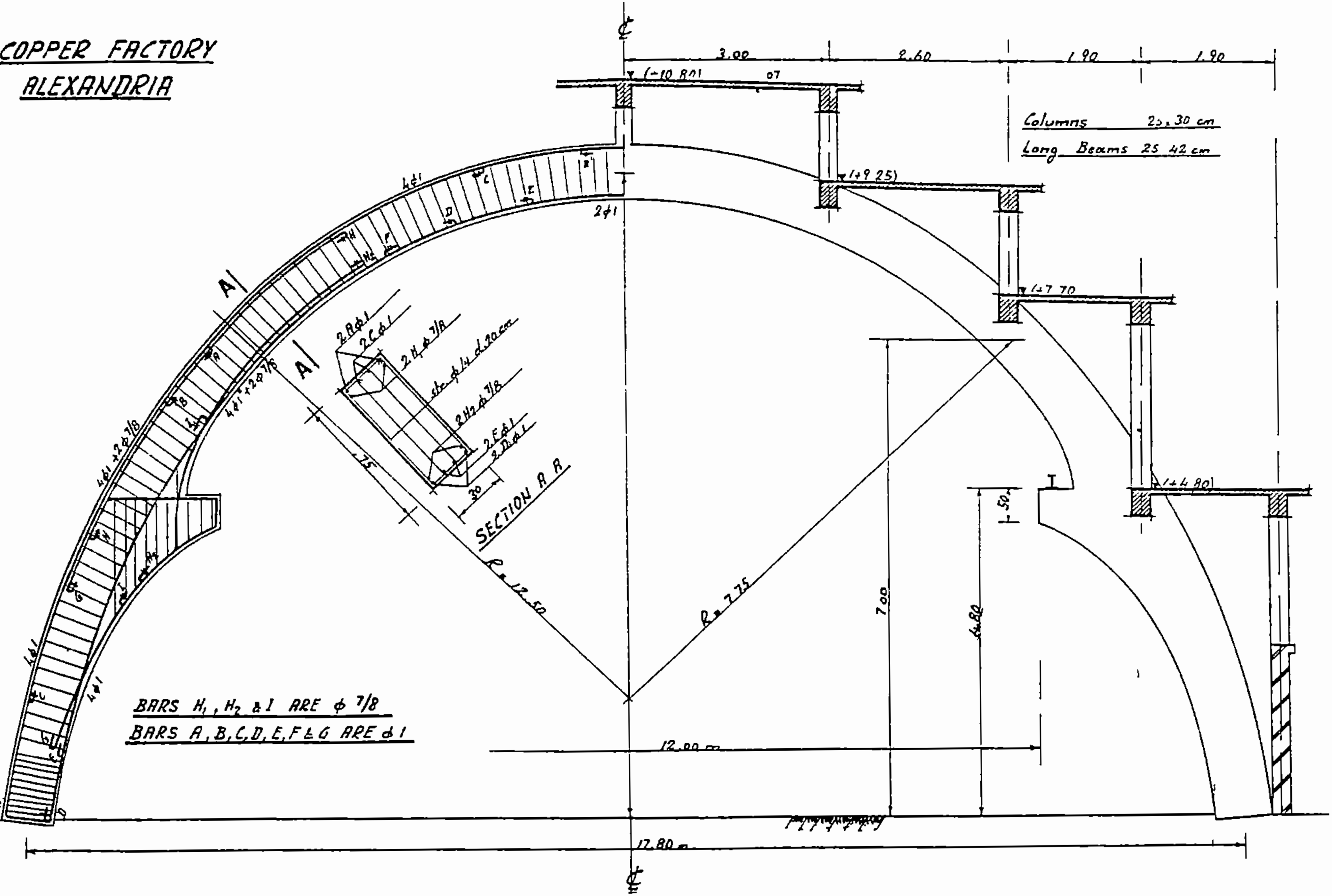
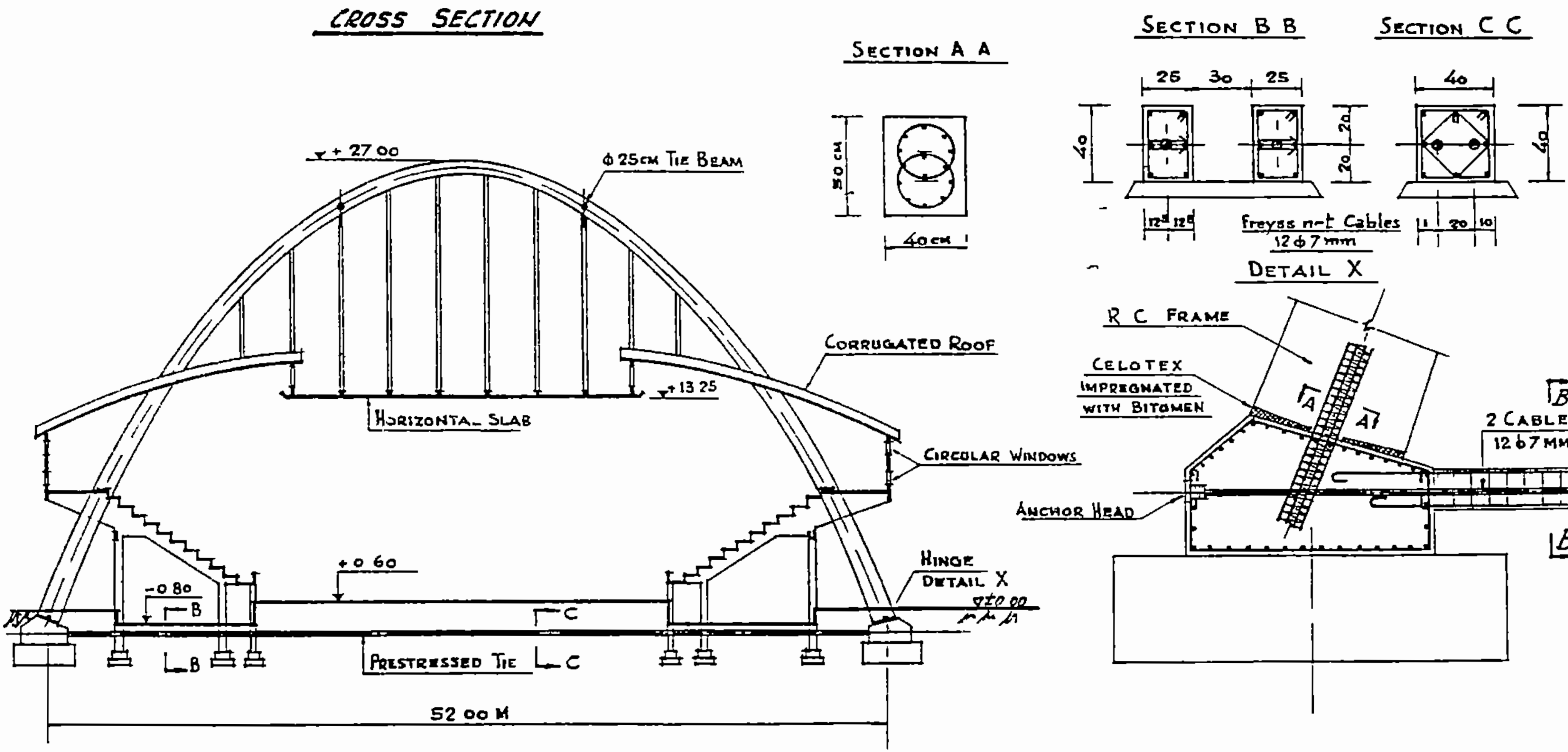
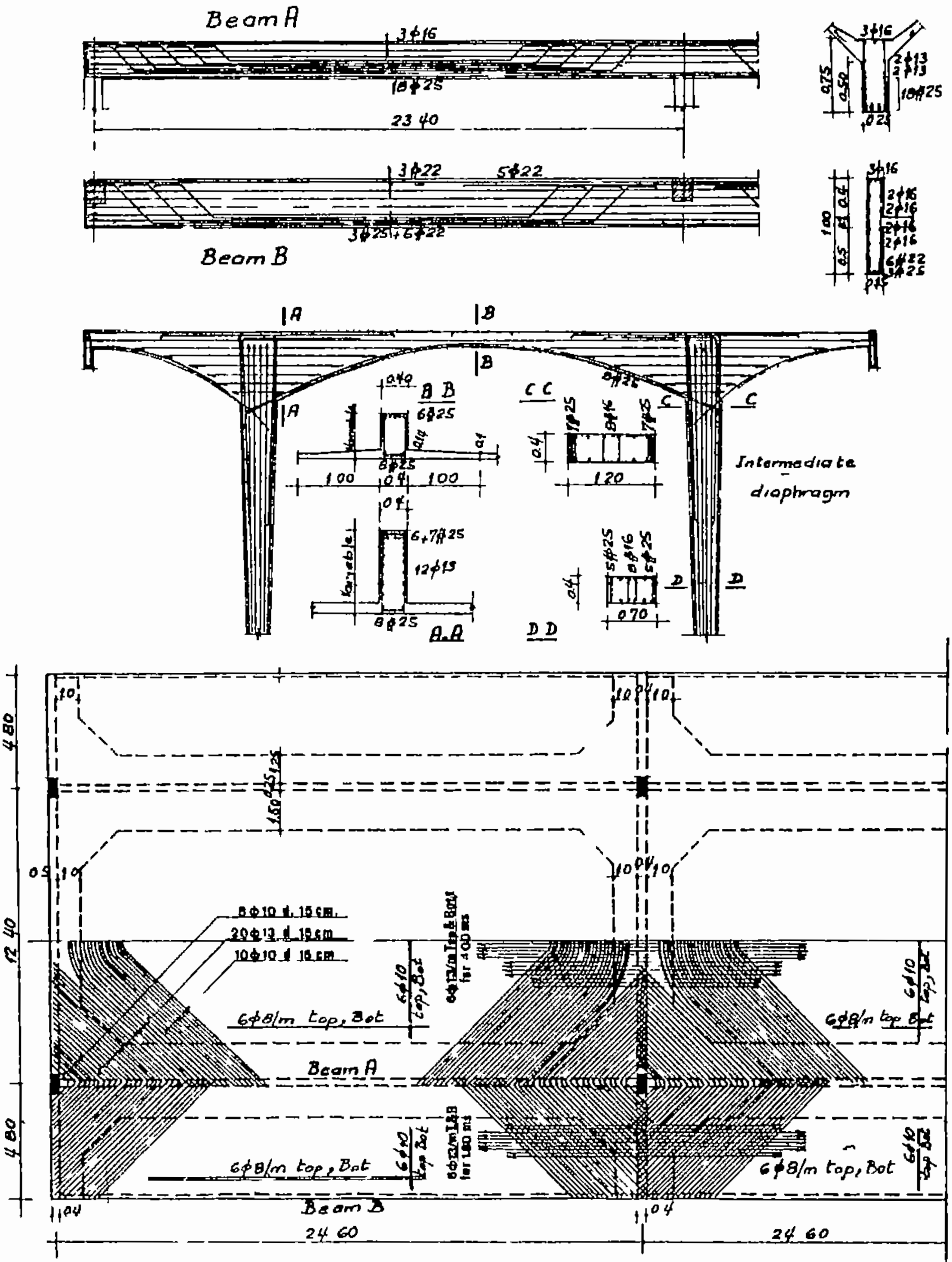


Fig VIII-13



FACULTY OF POLICE
COVERED GYMNASUIM 1964

Fig VIII-15

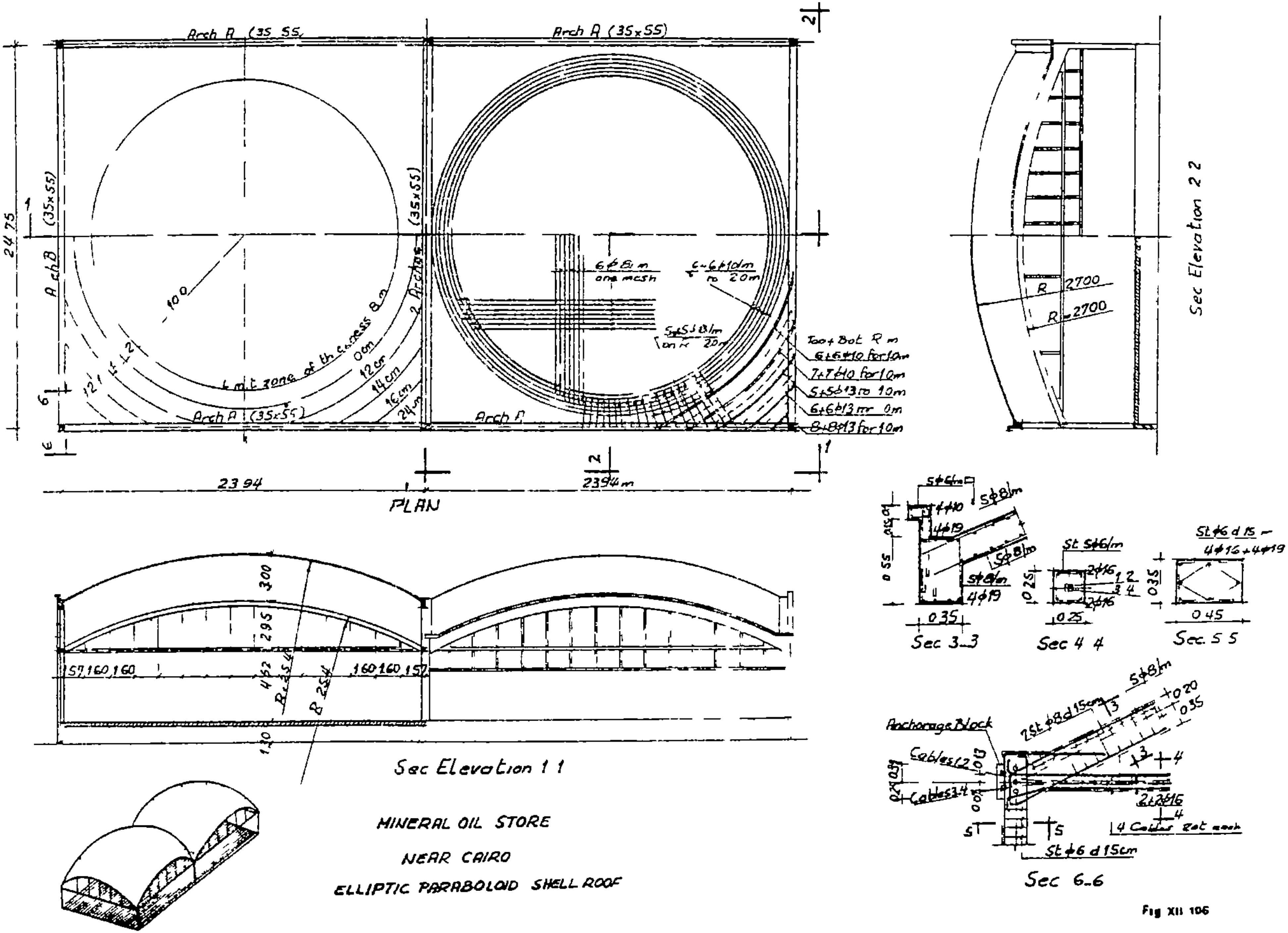


PUBLIC TRANSPORTATION ORGANIZATION

GESR EL-SUEZ GARAGE

INSPECTION SHED

Fig XII 74 b



Arch A (35 55)
Arch A (35x55)
A ch B (35x55)
24 75
(35x55)
2 Arches
100
4 m t zone of th cness 8 m
0 cm
12 cm
14 cm
16 cm
24 cm
Arch A (35x55)
Arch A
6φ8 m one mesh
6-6φ10 m to 20 m
5+5φ10 m on r 20 m
Top + Bot R m
6+6φ10 for 10 m
7+7φ10 for 10 m
5+5φ13 to 10 m
6+6φ13 mr 0 m
8+8φ13 for 10 m
23 94
23 94 m
PLAN
Sec Elevation 2 2
R 2700
R 2700
300
295
4 52
R 25 4
8 25 4
1 30
157 160 160
160 160 157
Sec Elevation 1 1
MINERAL OIL STORE
NEAR CAIRO
ELLIPTIC PARABOLOID SHELL ROOF
Sec 3 3
Sec 4 4
Sec 5 5
0 35
0 25
0 45
0 55
St φ6 d 15 -
4φ16 + 4φ19
St 5φ6/m
5φ8/m
4φ10
4φ19
Anchorage Block
Cables 2
Cables 3 4
2 St φ8 d 15 cm
St φ6 d 15 cm
4 Cables 2 at each
2+2φ16
Sec 6 6

Fig XII 106

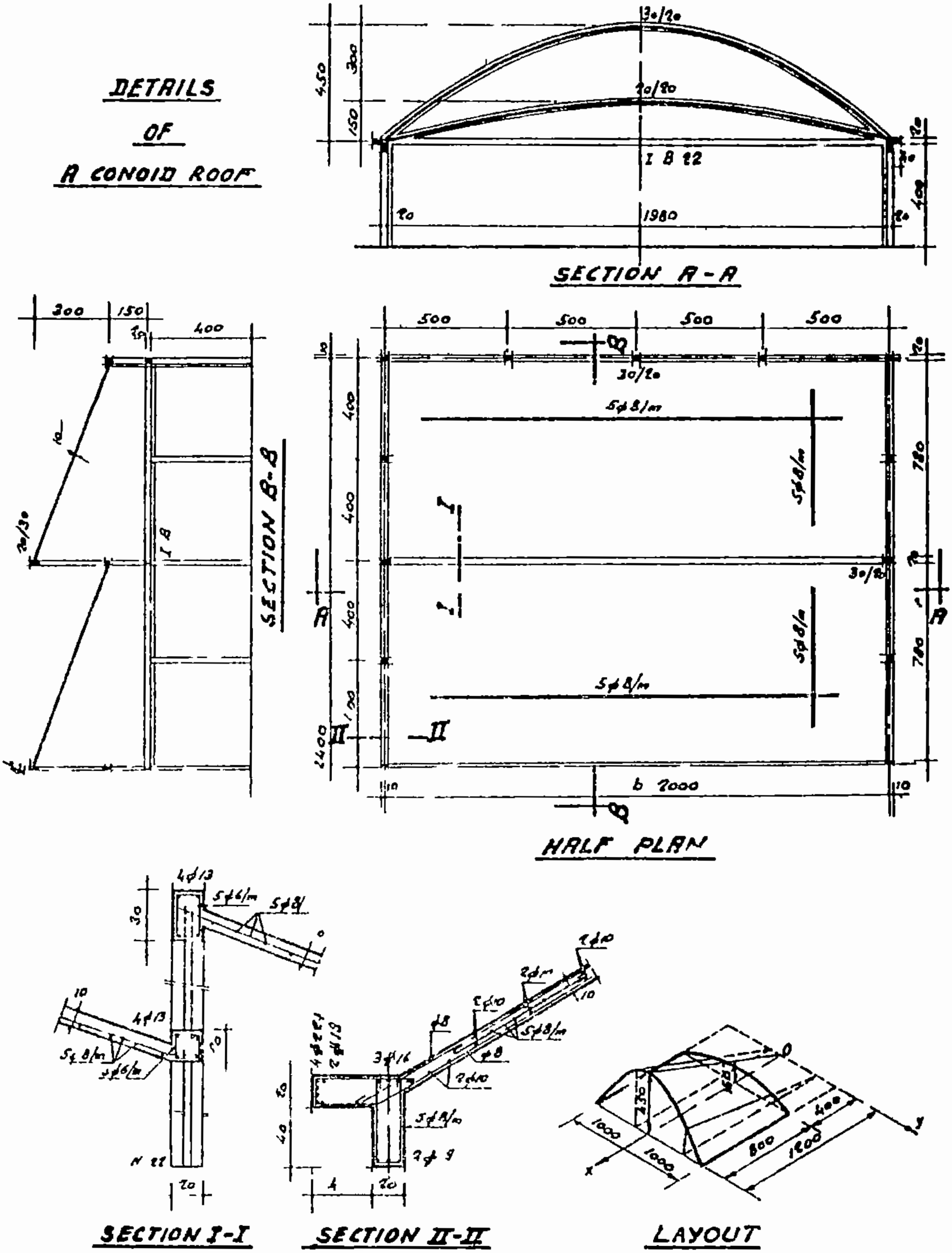


Fig XII-110

PREPARED BY
PROF DR M HILAL

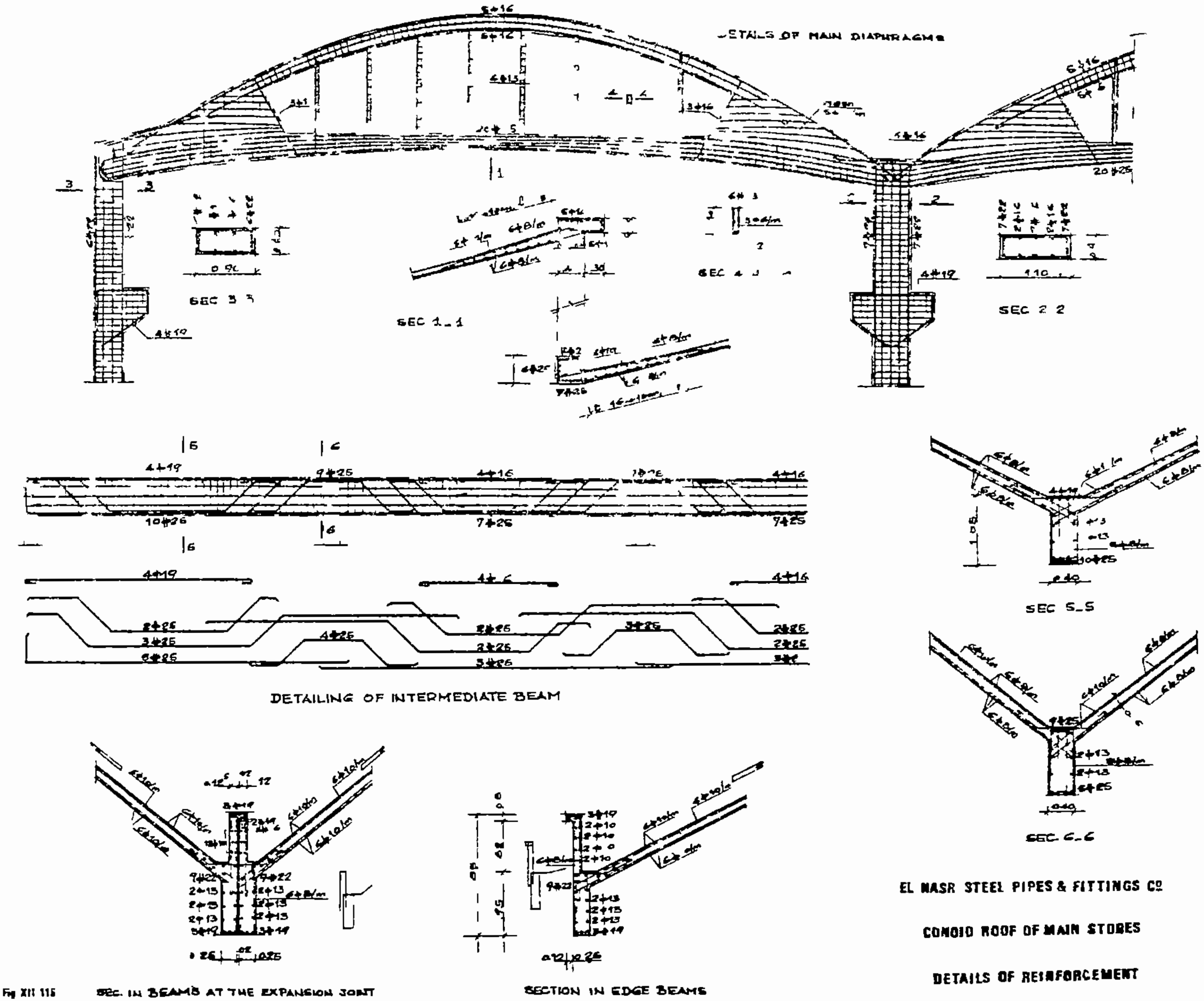
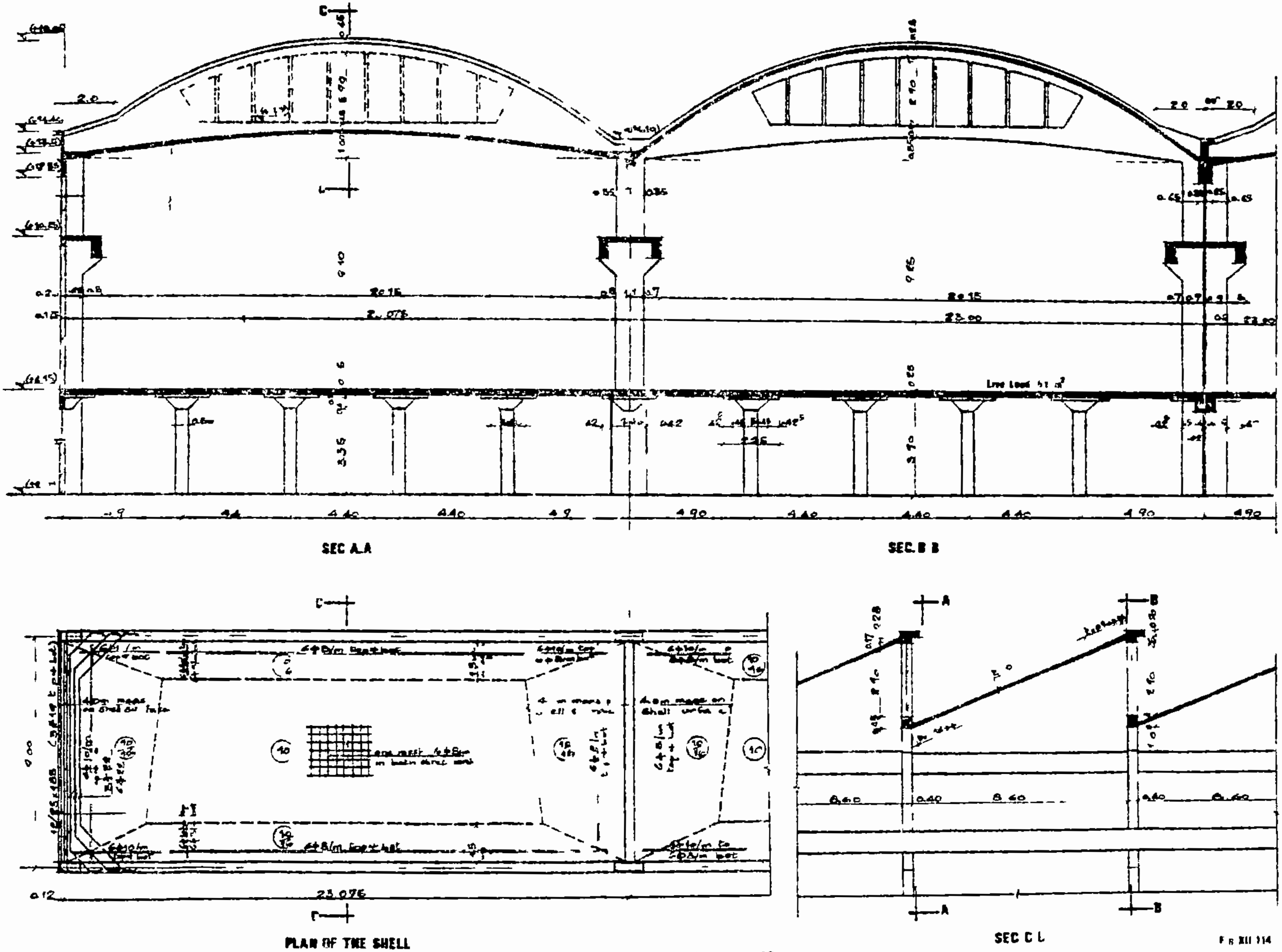

DETAILS OF MAIN DIAPHRAGMS
SEC 3 3
SEC 1_1
SEC 4 4
SEC 2 2
DETAILING OF INTERMEDIATE BEAM
SEC 5_5
SEC 6_6
SEC. IN BEAMS AT THE EXPANSION JOINT
SECTION IN EDGE BEAMS
EL NASR STEEL PIPES & FITTINGS Co
CONOID ROOF OF MAIN STORES
DETAILS OF REINFORCEMENT

Fig XII 115



EL NASR STEEL PIPES & FITTINGS Co

CONOID ROOF OF MAIN STORES

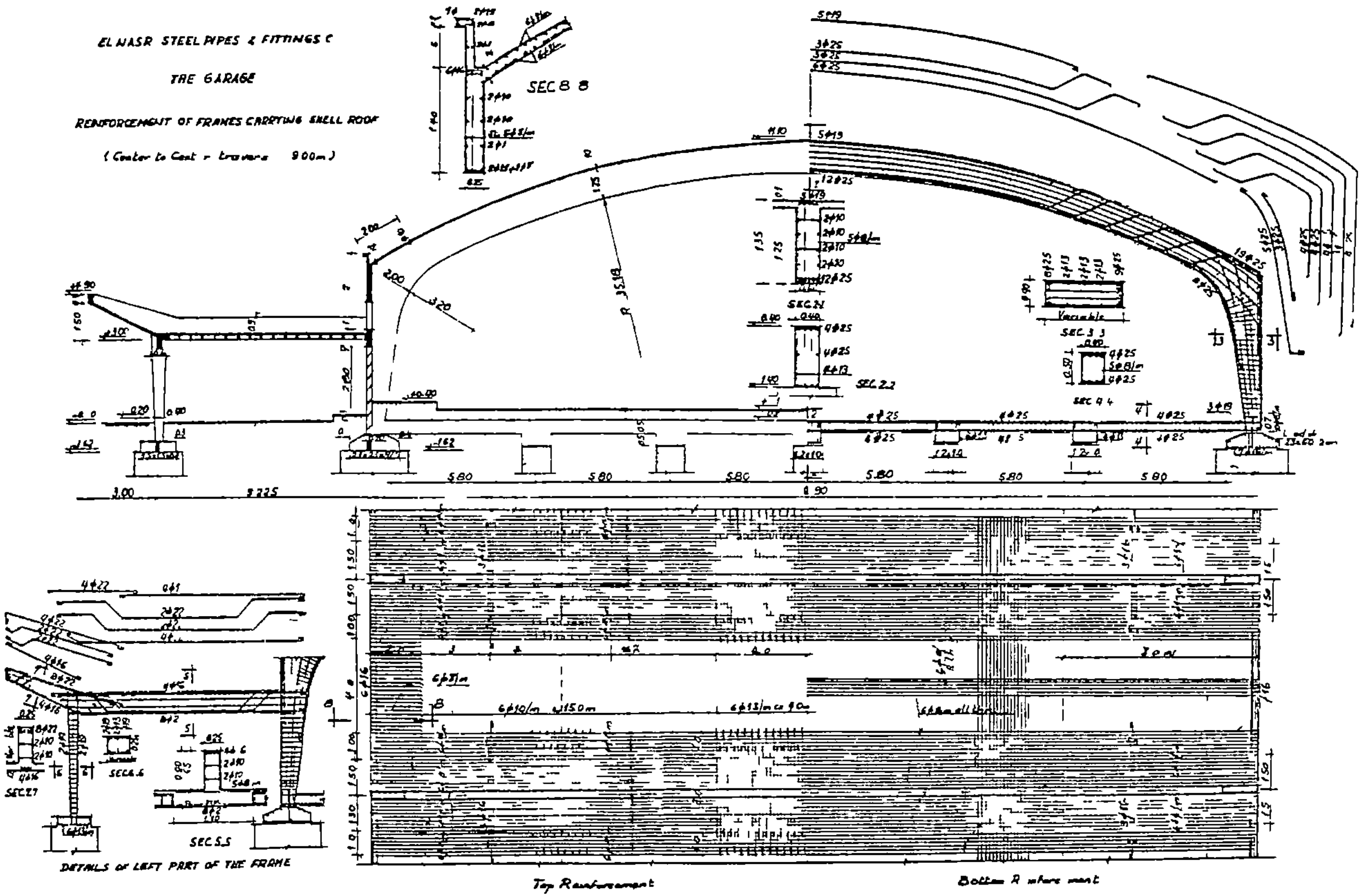
F R XII 114

EL NASR STEEL PIPES & FITTINGS C

THE GARAGE

REINFORCEMENT OF FRAMES CARRYING SHELL ROOF

(Center to Cent r travers 9.00m)



PLAN OF SHORT SHELL

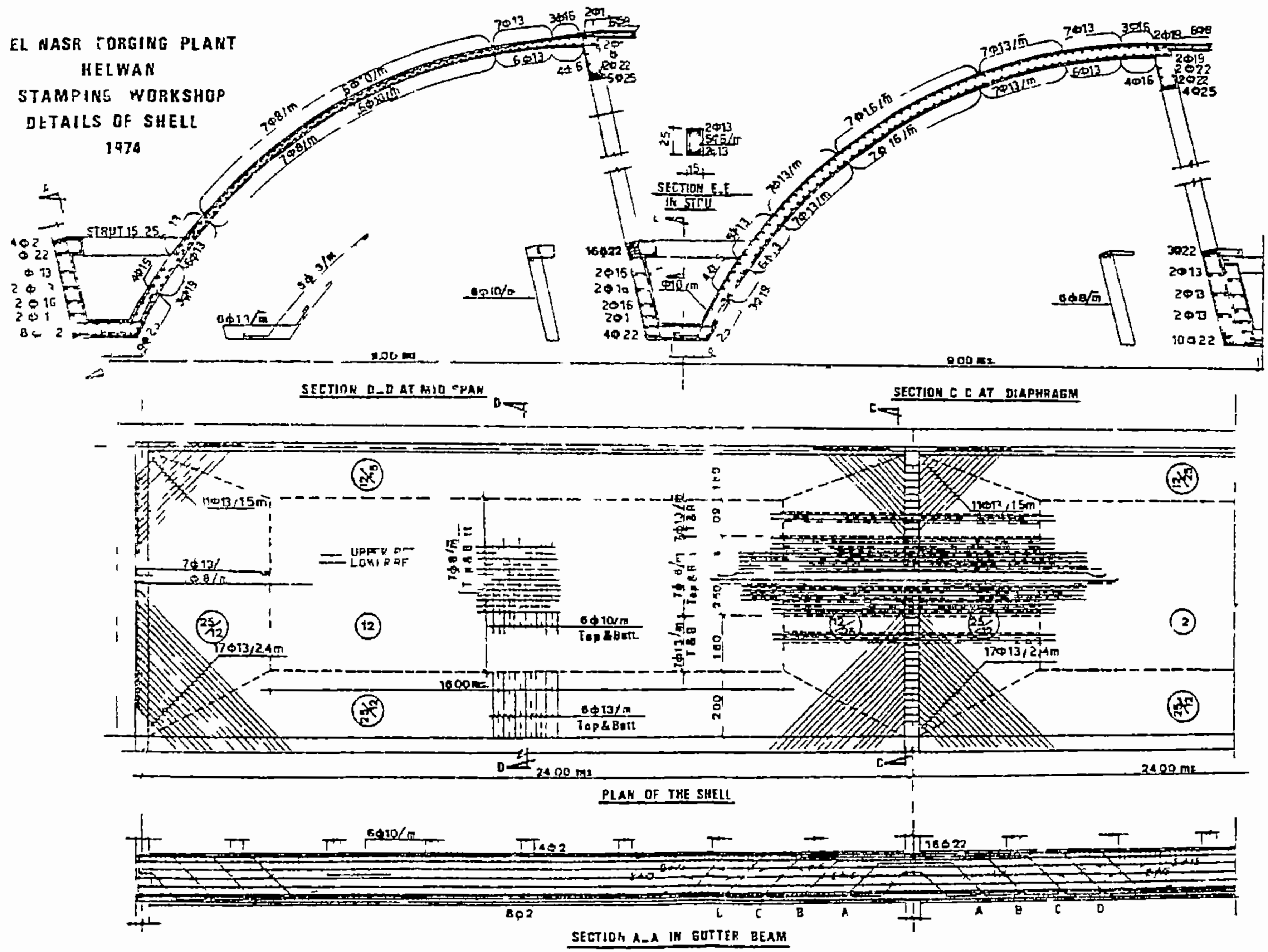

EL NASR FORGING PLANT
HELWAN
STAMPING WORKSHOP
DETAILS OF SHELL
1974
SECTION D_D AT MID SPAN
SECTION C C AT DIAPHRAGM
SECTION E.E IN SITU
PLAN OF THE SHELL
SECTION A_A IN GUTTER BEAM
STRUT 15 25
9.00 ms
24.00 ms
16.00 ms
6Φ10/m Top & Bott.
6Φ13/m Top & Bott.
17Φ13/2.4m
11Φ13/15m
16Φ22
8Φ2
6Φ10/m

Fig XII 82