sacrifice — whatever this may be — then we have to assume that in utilizing this sacrifice it is best adapted to the size of population, and productivity per head in this case **is bound** to be smaller for a greater population, while if it means a constant amount of physical aids to production — the same number and sizes of dock-yards, railways lines etc., the relation between the size of population and natural resources — and to throw light on this relation is the purpose of an optimum theory of population — is obscured, since the productivity of the population will depend as much on the existing amount of capital as on the natural resources. If the effect of capital is to be impounded in the **ceteris paribus** so as to isolate this relation, the statement must refer, not to a population whose size is "instantaneously varied"(1) but, as argued above, to different sizes which are maintained stationary in number for such long periods that the final optimum capital equipment, has been made and the comparison made between the two sizes at **this optimum. Directly the theory is so stated, it will be seen that it has little relevance, for no country has reached this state of bliss, and for the purpose of this study, the underdeveloped countries are so far removed from it as to make the effect of change in the size of population in the rate of capital formation the dominant aspect of the problem.**

The effect of population on employment was not discussed in this chapter because we abstracted completely from different forms of organization and "in the last analysis, unemployment of every type is a problem of organization."(2)

---

(1) Ibid, p. 107. In the third edition of Wealth, 1928, Cannan, in deference to various criticisms of his previous formulations explains that "... what we have to look for is not the best population at any particular moment of time without reference to what has gone before and what is to follow after, but the best at all moments taken together." This explanation is based on the fact technique changes and population does not actually vary instantaneously. It is not related to the supply of capital. (See op. cit., pp. 59-61).

(2) Report of the Economic Committee, Papers of the Royal Commission on Population, p. 47,

population only indirectly, through the reactions of numbers to the rate and form of capital formation, whereas it can affect the latter directly. Accordingly, an optimum theory of capital formation is needed only to set a standard by which the effects of changes in population on economic welfare can be studied. For this, we study capital formation on the assumption of a stationary population. **But when we take into consideration the effect of population on economic welfare, the optimum rate of capital formation becomes the rate which induces optimum changes in population size.**

The usefulness of this procedure can perhanps best be seen against the background of the current theory of optimum population. This was first precisely formulated by Cannan in his "Elementary Political Economy". **"At any time,"** he wrote in 1888 "The amount of labour which can be exerted on a given extent of land, consistently with the attainment of the greatest productiveness of industry at that time is definite. Assuming (what within short period is almost exactly true) that the total amount of labour exerted on a given territory increases or diminishes, we may word the law thus **-at any given time** the population which can exist on a given extent of land, consistently with the attainment of the greatest productiveness of industry possible at that time, is definite"(1).From the context, prefacing the statement of the law with the words "at any given time" is meant to exclude changes in technique, but it is not clear what are the assumptions made about capital. Professor Robbins, however, elaborating on Cannan's theory and contrasting it with earlier views on the subject, stresses the static nature of the theory, particularly with regard to capital. "Among the assumptions enclosed in that **ceteris paribus**" he writes, "is the assumption that the amount of capital remains constant."(2) But, if by a constant amount of capital is meant a constant amount of previous

(1) P. 25 The theory was first formally stated by Sidgwick, (Principles, pp. 150-1) but not so precisely. Cannan's statement of "the law of population" was later developed into a theory of optimum population in the "Wealth",.On the history and implications of the theory, cf. L. Robbins "The optimum Theory of population" published in the "London Essays in Economics", ed. Dv. H. Dalton, 1927.

(2) L. Robbins; op. cit., p. 121.

the future course of international trade and of international investment, both of which were left out of account in the previous discussion. The same difficulty arises with regard to the probable rate of population increase, though rough guess work—while more difficult for under-developed than for other countries—has a much more solid basis than in the case of income.

Because of these difficulties, great care must be taken before applying conclusions drawn from the above analysis to policy in any particular country. They do not, however, deprive it of all use, for the recognition of the existence of the problem must be the starting point of all rational policy. and it is hoped that future research would throw much more light on it, both theoretically and from the point of view of the applicability of the second and third conditions to any particulary country.

IV. Optimum rate of Capital Formation and Optimum Population.

In considering the optimum population we have already related it to optimum capital formation. This was done in two ways :

1. By considering the effect of the rate of capital formation on changes in the size of population we have studied the reaction of changes in one factor on the optimum supply of the other.

2. Focusing attention on changes in the size of population, rather than on absolute size, the effect of changes in population on the rate of capital formation was made on integral part of the optimum theory of population. This was done by the rather crude method of concentrating on the effect of changes in the size of population on its age-composition, for a very young population means a greater percentage of dependents and hence a slower rate of capital formation, as compared with the optimum rate, had population been stationary.

This way of relating the two concepts is somewhat unsystematic, but it is the only way which would make them relevant to welfare theorising. Policy — and welfare theory must always be subservient to the possibilities of policy — can affect the size of

ber, however, that it is not a very great extent-general education, perhaps with special stree on education in the social sciences, should be carried beyond the point determined by economic considerations alone,(1) particularly in the higher stages of education, since, in addition to the fact that probably only at these stages can education instil in the individual some rationality of behaviour, such education will also raise the age of marriage. The force of this argument will be increased if we remember the gain from the dissimination of the habit of late marriage, by the power of example, through the less educated classes of society. Another, though less certain, example is that of infant mortality. The preservation of human life is an end in itself, and to the extent that, even after a redistribution of wealth, in the manner and subject to the qualification set forth in the previous chapter, parents may spend less of their income than is necessary for preserving the life of their infants, there is a case, on ethical grounds, for state interference with private expenditure so as to secure that more is spent for this purpose than private individuals are willing to do. But apart from this, if it is accepted that a large number of children is desired partly as an insurance against total loss, public expenditure to reduce infant mortality may be a less wasteful method of achieving the same object and at the same time a means of reducing the birth-rate. A third example is that of old-age pensions. To the extent that a large number of children is desired as a provision for old-age, direct pensions may lead to a reduction in the birth-rate.

The third condition refers to the rate of rise in national income in relation to the rate of population increase. Here again there is no available method—theoretical or practical — of knowing the rate of rise in income which would correspond to the optimum rate of capital formation. Much will also depend on

(1) State encouragement of education has for long been advocated by economists on cultural and economic grounds. Professor Mead, «Economic Analysis and Policy» pp. 214 et suite, treats it as a question of investment in which the wealthier groups of the community have a monopoly, and argues in detail that the best use of resources will be achieved if the marginal return on capital invested in education is equal to its return elsewhere. The point here is that there is a case for investing in education beyond this point.

and clarity about the meaning of the alternatives must not be obscured by altogether evading the issue.

The theoretical clarity of the first condition is not matched by a similar clarity in the other two, which refer to the course of future events. It is not at all easy to determine what would be the effect on the form of population increase of an immediate rise in the standard of living due to capital accumulation. Much will depend on the form of capital accumulation. It is probable that, if this form is left to a very great extent to individual preferences, the increase in the average age of population due to a reduction in the death rate will be offset by a decrease due to an increase in the total birth-rate, even though the specific-age birth-rate may not rise, with the result of an abnormal age distribution similar to that which was obtained in Victorian England. Those under-developed countries which experience now a "swarming period" corresponding to that of early and middle nineteenth-century England, already have a population in which the ratio of the under-fifteens to the total population is similar to that of the English swarming period and in some of them it is much higher.(1)

Because the choice referred to in the previous paragraph is a choice between two great evils, there is a presumption for the State interfering with the form of the rise in the standard of living, in as much as this interference may alter the form of population increase. A conspicuous example is education. We do not refer here merely to propaganda for and the spreading of knowledge of birth control. To the extent that people are influenced towards birth control by irrational motives, we must remem-

(1) Compare the table in page 62 in the Report of the Economics Committee (paper of the Royal Commission on population), London, 1950, where figures are given for the age distribution of the population of England and Wales, 1841-1944, with table 104, page 318 of the United Nations : "Survey of Asia and the far East, 1949", where figures for the present age distribution in that region are given. While in England the figure for the dependents under 15 never rose beyond 36.5% of total population and was considerably smaller over the whole period, it shows an average of about 40% for the latter region. Figures for old age dependents are not easy to compare. In England the over 65 rose continuously from 4.4% in 1841 to 10% in 1944, while in Asia the over 60 are around 5%.

possible by a large size of population, are offset by diminishing returns from natural resources, and it is probably true, the size of possible gain from an increase in population will be reduced in case (1), the certainty of gain will be reduced to a mere probability and the size of the gain will also be reduced in case (2), and the size of the loss will be increased in case (3).

It may seem paradoxical to say that an increase in population due to a reduction in death-rates leads to a reduction in the standard of living or puts an obstacle to its raising, but we refer here, not to the generation in which the increase occurred, but to the following generation. Moreover, it is possible, with the achievements of medical science, to reduce the death-rate at the expense of the standard of living. The increase in numbers which would otherwise have been abortive is thus given a lease of life, at the cost of an increasingly lower standard.

It follows from the above considerations that :—

(1) **if it is accepted that the State should have an interest in future generations equal to its interest in the present generation, (2) if population increase is likely to be of the kind which defeats itself or raises greater obstacles for future generations, (3) and if the optimum rate of capital formation is not likely to increase the national income at a rate greater than the rate of population increase, if these three conditions are fulfilled, and they are all very big ifs, there is a presumption for capital formation at a rate greater than the optimum rate, the latter understood in the sense defined above.**

The first condition undoubtedly entails some interference with the individual's preferences and interests. It implies even the possibility of deliberately sacrificing present lives - by preferring forms of capital accumulation which lead to a rise in income at a certain point in the future, so steep that it would overtake population increase and bring about that level where it would react on the birth rate, to other forms which would lead to an immediate reduction in the death-rates, for the sake of saving future lives. How far this is totalitarianism, will be discussed in a concluding chapter, but the necessity of choice

stationary population will be able better to utilize its opportunities. Since the populations in which we are interested are likely to be continuously increasing during that period in the future for which it is profitable to generalize, we may neglect the possibility — and it is very remote — of gain which may accrue to them, when they become stationary, from having larger numbers.

The generalization about an increase in sacrifice resulting from population increase is not absolute: much depends on the way the population increase affects the average age of the population. If the increase accrues from a reduction in the specific-age death-rates, there may be an increase in the average age of population, the extent of this increase depending on the rate of reduction and on the age affected, particularly since, if it affects mainly the reproductive age, this reduction will lead to an increase in the total birth-rate, even though the specific - age birth-rates may not change, and this will partly offset the increase in the average age of the population. If the increase in population occurs through an increase in the specific-age birth-rates — as may well happen if temporary prosperity leads to a reduction in the marriage-age—it will lead to a reduction in the average age of population.

Accordingly

(1) **An increase in population due to the first cause, if it is not accompanied by an increase in the specific-age birth-rates, may or may not be an obstacle to maintaining the standard of living or to raising it, according to its secondary effect on the total birth rate.**

(2) **It is almost certain to facilitate both if it is accompanied by a decrease in the specific-age birth-rates.**

(3) **An increase due to the second cause is certain to be an obstacle to both.**

In all this we have abstracted from the effect of the law of decreasing returns from natural resources. If it is true that, for the countries with which we are concerned, the point has been reached where any further gains from specialization etc, made.

ii. The relation between changes in the size of population and economic welfare.

(I) As to the first aspect, two opposing forces are here at work; the law of diminishing returns from natural resources and the law of increasing returns from organization — both these laws used in their wider, classical, sense. The way in which each of them affects economic welfare is well-known, but two points may be emphasized here.

1. The fact that capital may increase man's ability to utilize existing natural resources does not mean that it raises the optimum size of population, for in determining the optimum we assume that the best known methods are fully utilized. Thus the existence of unutilized natural resources and the possibilities, for example, of land reclamation, irrigation schemes, etc..., do not mean that there is no over-population.

2. The same thing applies to changes in technical knowledge, except in so far as an increased number of people may be necessary to take full advantage of the change.

It is impossible to determine theoretically whether, according to the two laws referred to above, any particular country is over populated or not. Comparison of natural resources in relation to population in different countries and the corresponding levels of income — vague as such comparison necessarily must be — may give an indication, and by this standard there is no doubt that the old-established low-income countries with which we are concerned are over-populated.

(II) Any doubt left about this is removed when we remember that, in practice, the question is not one of comparing two absolute sizes, each maintained indefinitely until the best known methods are utilized, but of increase from one size to another. To secure a given rise of income at a certain point of time in the future by the application of better methods of production will always mean a greater sacrifice — since this application will normally take the form of capital formation — incurred by each generation, if population is continuously increasing. It may be that if population increases and then becomes stationary, the

1 — **Capital formation is essentially a process of diverting resources from production for current consumption to production for increased future consumption.**(1)

2 — **The sacrifice from any given act of capital formation is temporary whereas the benefit is permanent.**(2)

3 — **In calculating the optimum rate of capital formation and in deciding the method, account must be taken of the whole process, i.e. until the final optimum is reached.**

III. Optimum Population.

The concept of optimum population, as well as its counterparts : over- and under- population, is one of those highly significant (from a welfare point of view) concepts which, as we shall see in the following section, do not gain in clarity by applying to them the equilibrium technique, and is best approached by analysing in general terms the relation between the population factor and economic welfare.

Two aspects may be distinguished :

i. The relation between the **size** of population and economic welfare.

---

(1) Because we assumed that income was composed of one commodity, the only resource which needed to be taken into account was labour. When many commodities are introduced, the services of land acquire a scarcity value and enter into our calculations. This however, does not effect the analysis of the nature of the process of capital formation.

(2) Had this been immediately apparent from the traditional analysis of the theory of capital, Prof. Schumpeter's thesis (Theory of Economic Development) that in a stationary community the rate of interest is bound to reach zero would have gained more acceptance. The objection to the explanation of interest, which is permanent, as a reward for saving which is a single act, would have also gained more force. The realization of this underlines Keynes' theory of diminishing marginal efficiency of capital and the theories of secular stagnation which stemmed from it, all of which explaining unemployment by the refusal of the rate of interest to fall continuously.

through the utility and the capital productivity functions, but not through the productivity of equipment-making labour in different periods. When we come to the case of a community, this assumption could no longer be maintained, for whether an equipment of a given productivity will take, say, 30 days or more in the second period, will normally depend on the arrangements made for producing the equipment in the first period. These in their turn have to be related to plans for the next period, and so on. All the elements of the process of capital formation, up to the final optimum, have to be determined together, as one set. There is no available technique, in current economic theory, which would take account of this. The equilibrium analysis partly subsumes it under the heading of external economies, but, it will be argued in chapter V, it does not adequately solve the problem and we may well keep its importance in mind.

The complication referred to in the previous paragraph is not likely to alter our conclusion about the rate of capital formation. If anything, it is likely to substantiate it, for the increase in productivity resulting from capital formation in any given period will accrue not only to the succeeding period, as was the case with the isolated individual, but also, through its effect on the facilities for producing future capital equipment, to all the succeeding periods.

The rate of capital accumulation, however, is only a side line. It was not the main purpose from constructing the above model, for the inference as to the application of the conclusion drawn about it to the real world is not conclusive; too much was left out of account. The main object was to stress some aspects of capital formation which are not immediately apparent from the usual methods of analysis. In this respect there is no difference between capital formation for an isolated individual and for a community. These aspects can be summarised as follows :

it is immaterial from the point of view of any one generation whether the increase in satisfaction from a given sacrifice accrues to it or to a future generation.

We turn now to other differences which interest us here. For the isolated individual, the diversion of resources from production for immediate consumption to capital formation means dividing his time between the two occupations, with no change in productivity per hour as the number of hours worked in each occupation changes—an hour spent on current consumption, with the help of any given equipment, will always yield the same number of units of product, whether he devotes six, seven or eight hours to current consumption. For a community, it means diverting a man from one activity to another, and the productivity of the man in either activity will depend on the number of men employed in it. To put it in another way, the productivity per man of any given capital equipment will not—in normal cases—solely depend on the size of the capital equipment, as the productivity per hour did in the case of the individual, but, also on the number of men employed on it. Since this can be allowed for in drawing up the capital productivity function—the function which represents the relation between size of any equipment and the productivity per man—it does not alter our calculations about the rate of capital formation. But it would affect the **method** of capital formation, **for, whereas the individual would maximize his total utility through time by reaching the optimum size gradually through sub-optimum sizes, the community may reach it, not by gradually increasing the size of sub-optimum equipment units with which all those who produce for current consumption work, but by gradually increasing the number of optimum capital equipment units.**

There is also another major difference. In the case of the isolated individual, we may assume as a first approach, that the productivity of the input used for capital formation is not affected at each stage by what happened at the previous stage, e.g. 30 hours labour on equipment making would always produce an equipment of a given productivity irrespective of the shape or size of the equipment of the previous period. All the periods of the process of capital formation were related to each other

II. Capital Formation for a Community

Before stepping from the case of an isolated immortal to a mortal community, we must refer to two essential differences :

(i) In the case of an individual all sacrifices and all satisfactions are referred to one single unit. This applies also to the process of calculation itself. For a community, the sacrifices may refer to one group of people and the benefits to another. Moreover, the calculations of the savers may be in terms different from those used by those who utilize the sacrifices to increase future production. In the Ramsey-Allen model, the question is solved along traditional lines, with the rate of interest representing at one and the same time the marginal disutility of saving and the marginal productivity of capital. It will be argued in Chapter 6 that the rate of interest has no exact sense in either capacity, and the discussion will be continued in terms of satisfactions derived from different amounts of income at different points of time. This will bring out more immediately the nature of capital formation as a process of diverting resources from production for current satisfaction to production for future increased satisfaction. For this we assume a community of like minded individuals similarly situated, and interested only in organizing production so as to maximize satisfaction through time.(1)

(ii) We cannot balance the satisfactions of one generation against those of another. Here again the question is not solved by any known method of analysis and in order to isolate, in this section, the more important aspects of the process of capital formation, the simplest way out is to assume with Ramsey that

(1) We may assume here that the community is either run by an omniscient dictator or composed of omniscient individuals. The first assumption was made by Prof. Hayek ("The Pure Theory of Capital", chapters XII to XVII.) "This assumption allows us to investigate the influence of the technological data in their simplest form without having to take account of the differences in aims of a multiplicity of persons and the effects of a different distribution of resources between them." Hayek op. cit. p. 155. It also enables us to abstract completely from the complications of organization.

models here considered has much to do with this. For, with mortals, we cannot balance the gains of a present generation against the losses of a future one. It would serve some purpose, however, particularly since we will argue that there may be a case for over-riding the preferences of one generation in the interest of another, to see if the above conclusions rest on more solid grounds than mathematical deduction.

During the earlier steps of capital formation, because capital productivity rises at the earlier steps at a greater rate than at later steps, a given amount of input, though, due to the psychological law of diminishing marginal utility of income, it means a greater current sacrifice, it also means a greater future gain in terms of physical and psychological units than at a later stage. For example, if our individual reaches the optimum by three steps, in period 1 making a small size equipment which would increase his productivity from one unit per hour to 4, in period 2 using this equipment for current consumption and making at the same time medium-size equipment which would increase it to 6, and in period 3 using it for current consumption and making the optimum which would increase it still to 7, then a seventh hour's work if applied to production for immediate consumption will produce for him one (physical) unit in the first period, 4 in the second, 6 in the third and 7 after reaching the optimum, while had the same hour been applied to capital formation (in which case it would be the fourth hour, considering that he has only 10 hours to divide between current consumption and equipment making), it would yield him an increase in productivity equal to 4 units per hour during the second period, to 6 units (i.e. the increase is at a decreased rate) during the third, and to 7 units for the rest of his life, so that we may expect him, for example, if he devotes only six hours to current consumption, thus giving up the utility of the 7th to 10th units and gaining 3 more units for every hour he works for current consumption during the second period, to devote seven hours to current consumption in the second period,thus giving up the utility of the 29th to the 40th units and gaining 2 extra units for every hour he works for immediate consumption during the third period, rather than devote only six hours of immediate consumption giving up the utility of the 25th through to the 40th units.

standards, so that we can speak simply of quantities of capital, consumption and labour without discussing their particular forms (p. 543). Lastly, he assumes that the community will be governed by the same motives as regards accumulation, so that there is no chance of one generation's savings being selfishly consumed by another generation. In short, he assumes a community of immortals similar in all respects to our isolated individual.

He then shows that :

$$\frac{dc}{dt} = \frac{B - (U(x) - V(a))}{U(x)},$$

where $\frac{dc}{dt}$ is the rate of saving (which he identifies with the rate of capital formation), B is the maximum obtainable rate of enjoyment (i.e. the rate of enjoyment when capital reaches a final optimum size, which he calls Bliss) U (x) the rate of total utility of consumption actually enjoyed and u (x) its marginal rate, and V (a) the total disutility of labour, i.e. he shows that the optimum amount will be saved when the rate of saving multiplied by the marginal utility of a unit of present consumption is equal to the excess of one year's satisfaction when capital equipment is of the (final) optimum size minus the one year's satisfaction at present standards.(1) He also shows that, like what has been called above the optimum rate of capital formation, this rate is altogether independent of the production function, except in so far as this determines Bliss.

Professor R.G.D. Allen, in a modified version of Mr. Ramsay's model, shows that, if an optimum rate of saving is followed, capital would increase at a **decreasing** rate until it becomes stationary at the optimum.(2) His conclusion, however, applies to the rate at which the amount of capital goods increases, while the conclusion derived from the isolated individual model refers to the rate at which **input** is diverted for capital formation rather than used for immediate consumption. Both conclusions contradict the often assumed, though never rigorously proved, view that as countries grow richer, they will be able to afford to divert more of their resources to capital formation, **and should do so,** and no doubt the unrealistic assumptions of immortality in the

(1) p. 546.

(2) Mathematical analysis for Economists P. 540.

0 and 1[1] and even proposes an ingenious scheme for measuring it empirically.[2] His method, however, is open to serious objections[3] and we are left in the dark about it. In our use of it, however, we always compare it with $\frac{X_r}{a}$ This is the ratio of input used for capital formation to total input and on any realistic account, is likely to be a **small** fraction. On the other hand, an elasticity of the marginal utility of income with the value of a small fraction means that the marginal utility of income is declining very quickly, which seems to be contrary to immediate experience, and it may thus not be unplausible to suppose that the latter will always be greater than the former, with the result that as our individual moves from one step to another and thus rises from one income level to a higher one, the rate of capital formation—if he is to maximize utility through time—will slow down.

That the optimum rate of capital formation will be diminishing is confirmed—as we shall see in a moment — by another line of approach. The main point from the above model, however, is to illustrate the nature of capital formation. But before turning to this we may refer to F. Ramsay's treatment of the same problem, in his well-known article «A Mathematical Theory of Saving»[4]

Mr. Ramsay, also thinking in terms of a moving optimum, that is, a rate which maximizes utility through time, sets out to find how much of its income a nation should save. He assumes a community which goes on living forever, without changing in its capacity for enjoyments, or in its aversions to labour; that enjoyments and sacrifices at different times can be calculated independently and that there is no autonomous change in technical knowledge. Particularly, he assumes no time-preference «a practice which is ethically indefensible and rises merely from the weakness of the imagination». He also neglects the differences between different kinds of goods and different kinds of labour, and supposes them to be expressed in terms of fixed

(1) «Towards a Dynamic Economics» p. 51.
(2) Op. cit. 43-44.
(3) c.f. J. De V. Graff; «Mr. Harrod on Lump Saving», Economica February 1950 P. 86.
(4) Economic Journal, December 1928.

Following Professor R.G.D. Allen's notation,(1) the elasticity η of a function

$y = f(x)$ at the point x is $\frac{E y}{E x} = \frac{x}{y} \frac{d y}{d x}$

in our case elasticity $= \frac{OL}{PL} \times - \frac{PL}{LM} = - \frac{OL}{LM}$

Putting thus $\eta = \frac{OL}{LM}$

$$\frac{d x_r}{d n_{r-1}} = - \eta \frac{(y r) + \frac{x r}{a}}{n_{r-1} / a}$$

since n_{r-1} is always positive,

$\frac{dx_r}{dn_{r-1}}$ will de positive when $\eta (y r) < \frac{x r}{a}$

« « = o « « = $\frac{x_r}{a}$

« « negative « « > $\frac{x_r}{a}$

and since nr - 1 in increasing from one step to another, then in the first case xr (that part of his total input he devotes to capital formation) will be increasing from one step to another; in the second it will be stationery and in the third it will be decreasing. It follows that, computing the elasticity of the MU curve we can say that at the zones for which the elasticity of the marginal utility of income is less than $\frac{x_r}{a}$ the rate of capital formation will be increasing from one step to another, where $\eta = \frac{x_r}{a}$ it will be constant, and for the zone where it is bigger than $\frac{x_r}{a}$ it will be decreasing.

It is very difficult to evaluate the elasticity of the marginal utility of income, even under the most heroic of simplifying assumptions. Mr. Harrod assumes that it is likely to be between

(1) cf, "Mathematical Analysis for Economists", P. 251.

whence $F\ (n_{r-1}\overline{a-x_r}) - F\ (n_m) + n_{r-1}\,x_r\,F\ (n_{r-1}\overline{a-x_r}) = 0$

differentiating both sides with respect to n_{r-1} we get

$$F\ (n_{r-1}\overline{a-x_r})\left\{-n_{r-1}\frac{dx_r}{dn_{r-1}} + \overline{a-x_r}\right\} + n_{r-1}\frac{dx_r}{dn_{r-1}}\ F\ (n_{r-1}\overline{a-x_r}) + x_r\,F\ (n_{r-1}\overline{a-x_r}) + n_{r-1}\,x\,F\ (n_{r-1}\overline{a-x_r})\left\{-n_{r-1}\frac{dx_r}{dn_{r-1}} + \overline{a-x_r}\right\} = 0$$

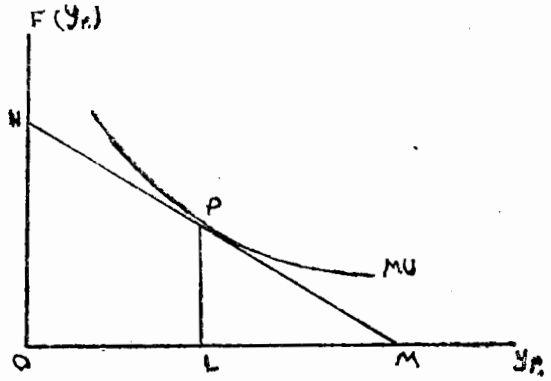
hence $$\frac{dx_r}{dn_{r-1}} = \frac{a\,F\ (n_{r-1}\overline{a-x_r}) + n_{r-1}\,x_r\,\overline{a-x_r}\ F\ (n_{r-1}\overline{a-x_r})}{n^2_{r-1}\overline{a-x_r}\ F\ (n_{r-1}\overline{a-x_r})}$$

Putting $n_{r-1}\overline{a-x_r} = y_r$, we can put this equation in the form

$$\frac{dx_r}{dn_{r-1}} = \frac{a\,F\ (y_r)}{n_{r-1}\ y_r\ F\ (y_r)} - \frac{x_r\ y_r\ F\ (y_r)}{n_{r-1}\ y_r\ F\ (y_r)} + \frac{a\,F\ (y_r)}{F\ (y_r)} + x_r\ y_r$$

$$= \frac{a\ F\ (y_r)}{y_r\ F\ (y_r)} + x_r$$

$$n_{r-1}$$



from diagram 2 :

$$\frac{F\ (y_r)}{F\ (y_r)} = \frac{PL}{PL} = -\frac{\overline{LM}}{\overline{LM}}$$

therefore, $$\frac{dx_r}{dn_{r-1}} = \frac{\frac{aLO}{LM} + x_r}{n_{r-1}}$$

when there is a possibility of making intermediate sizes, for the product of an hours's work in terms of consumption units changes with the step reached in capital formation, and it may be that he would maximize his satisfaction through time if, making intermediate sizes, he would raise his consumption at each successive step. He will be more likely to reach the optimum by steps and raise the level of his consumption at each successive step, the greater is the slope of his marginal utility curve and the smaller is the slope of the curve which traces the relation between the size of the equipment and the number of units per hour he produces with its help. To find the conditions under which he would proceed by steps, the number of steps, and the rate of capital formation at each step would require a mathematical technique which is beyond the reach of the present student.(1) An interesting conclusion, however, could be reached by simpler methods.

Adopting the previous notation, and assuming that he takes a number of steps (m) then the number of units (n_r) he produces with the help of a given equipment (of the size Tr), will be determined by the capital productivity function, and we have,

$$U = \sum_{r=1}^{m} \frac{T_r}{X_r} \left\{ F\left(n_r - 1^{a-x_r}\right) - F\left(n_m a\right) \right\} + \text{constant}$$

$$\therefore \frac{\delta u}{\delta x_r} = - \frac{T_r}{x_r} F\left(n_r - \overline{1^{a-x_r}}\right) - F\left(n_m\right) \Bigg\}$$

$$- \frac{T_r}{X_r} n_r - 1 F\left(n_r - \overline{1^{a-x_r}}\right)$$

For a minimum equate the right hand side with O.

(1) It can be shown, however, that, given the number of steps he takes the size of each successive equipment, the amount of work he gives to capital formation during each period and the number of units he consumes during that period can all be determined.

at the maximum,

$$\frac{dU}{dX} = \frac{T}{X} 2 F (na) - \frac{T}{X} 2 F (n_o \sqrt{a - x}) -$$

$$n_o \frac{T}{X} F = (n_o \sqrt{a - x}) = O$$

and the roots of this equation in x give the rate of capital formation which will maximize his utility through time.

Assume now that instead of having only one size of capital equipment, he can choose between different sizes, (size here is defined in terms of the number of hours necessary to make it), with different degrees of increase in productivity. There will be one size which will maximize the amount he produces per hour (allowing for maintenance) and there is no doubt that, given an indefinite length of time, he will make that particular size, sometime in the future. The question is whether he will proceed directly to make this optimum size, or whether he will make first an intermediate size or sizes, each of which is larger than the preceding one, until he reaches the optimum size.

If we assume that he consumes a constant amount during the whole period of capital formation, i.e. until he builds the optimum size, his choice will depend solely on the productivity of varying sizes. If the increase in his productivity (i.e. in the number of units he produces per hour) is very great at the initial sizes, he will reach the optimum size much quicker if he makes first a small-size equipment with the help of which he produces the units he currently consumes, for he will thus be able to devote a greater amount of his time to capital formation. This however, is not the only reason why capital formation by steps may be preferable. In the case of one size capital equipment considered before, the amount he consumed was assumed to be constant during the whole period of capital formation because - in addition to the regularity of wants - an hour's work, whether used for production for current consumption or for production for future consumption, had a constant value in terms of its physical product throughout the whole period of capital formation. An hour's work meant either one unit of the commodity or 1/30th of the equipment throughout. This is not the case

The rate which maximizes the difference between his gain and his loss will be the optimum rate of Capital formation.

This can be generalised and mathematically solved.[1]

Let a be the number of hours he works per day,

n_0 the number of units per hour he produces before making any equipment,

n the number of units per hour he produces with the help of the equipment.

T the total number of hours it takes him to make the equipment.

x the number of hours he works every day on making it.

Then $\frac{T}{X}$ will be the number of days he will work on making the capital equipment.

Define F (y) = the total utility of the number of units he consumes per day.

In each day (a-x) n_0 units are produced and consumed during the capital formation period. Hence the utility from them is

$$\frac{T}{X} F (n_0 \overline{a - x})$$

$(B \frac{T}{X})$ is the number of days he lives after finishing the equipment, where B is the number of days he lives from the instant he starts making it. After finishing it he produces na units each day, hence $(B\frac{T}{X})$ F (na) = his total utility after making it.

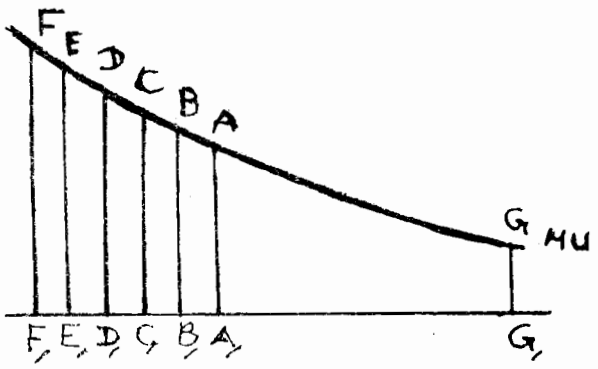
Call U the total utility during his whole life, i.e. the sum we want to maximize.

$$U = (B \frac{T}{X}) F (na) + \frac{T}{X} F (n_0 \overline{a - x})$$

$$= \frac{T}{X} (- F (na) + F (n_0 \overline{a - x})) + \text{constant.}$$

(1) The writer is grateful to Mr. Fernando Sadek, now of the WHO, for assisting in preparing the mathematical part contained in these notes.

areas AB, BC, etc. being the utilities of the 10th unit 9th unit, etc., five being the minimum number he consumes per day. AG is the additional utility per day he gains from the increase in the number of units he will be able to produce after making the equipment.



Column I below shows the number of hours he works per day on equipment making, column II the corresponding sacrifice in utility during the period of equipment making, and column III the gain in utility after it has been made. Thus, since the equipment takes him thirty hours to make, he will sacrifice AB, for 30 days and gains AG, for the rest of his life if he devotes 1 hour per day to equipment making, while if he devotes 2 hours per day, he will sacrifice AC, (which is more than double AB,) for 15 days and gains, since he will finish making the equipment 15 days earlier than before, AG, for the same period as before plus AG, for 15 days.

Column I	*Column II*	*Column III*
		AG_1 x the number of days
1.	AB_1 x 30	he lives after 30 days.
2.	AC_1 x 15	» + AG_1 x 15
3.	AD_1 x 10	» + AG_1 x 20
4.	AE_1 x 7.5	» + AG_1 x 22.5
5.	AF_1 x 6	» + AG_1 x 24.

no economic significance, and work accordingly will be the only "input" he uses in the process of production. In the ten hours, he produces ten units of the commodity, at the rate of one unit per hour.

Suppose now that one day it dawns on him that if he builds a certain equipment, which would take him 30 hours labour of standard intensity to build, he will be able to produce, with the help of this equipment, and allowing for its maintenance, 2 units of the commodity per hour. We assume also that he expects to live so long that there is no doubt that it will pay him to make the equipment. His problem is how to divide his input (the ten hours' work) between immediate consumption and equipment making in the most rational way i.e., so as to maximize through time the total utility he derives from these ten hours.

Let us call the amount he currently consumes of the commodity x his income. If the marginal utility of different amounts of x were the same he would maximize his utility through time if he works 10 hours a day on making the equipment from the moment the idea occurs to him. Since marginal utility cannot be assumed to be constant, and, moreover, since we assume that it is necessary for him to consume a certain amount of this commodity every day in order to maintain his life and vigour, we infer that he will divide his ten hours between production for immediate consumption and production for (increased) future consumption, that is, equipment making or capital formation. During the period of capital formation, he will arrange his consumption of his commodity so as to equate the marginal utilities he derives from consuming it every day of the period. Given the regularity of the daily rythm of his want for it, this he will do by consuming the same amount every day of the period. It remains to be known how many units of his total input he will use every day in production for direct consumption and how many for making equipment.

It is sufficient to know the shape of his marginal utility curve to solve his problem. Suppose that the MU curve in diagram 1 represents that section of his marginal utility curve beyond the minimum necessary to maintain his efficiency, the

SOME NOTES ON OPTIMUM RATE OF CAPITAL ACCUMULATION AND OPTIMUM POPULATION*

Professor Dr. FAWZY MANSOUR

Head of the Economics Department, Faculty of Law, Ein-Shams
Rector of the High Institute for Socialist Studies

1. The meaning of the optimum Rate of Capital Accumulation as Illustrated by the Case of an Isolated Individual.

Neither for the community, nor for an isolated individual, are there economic ends and non-economic ends. Economic theory, however, for pragmatic reasons, traditionally concerns itself with those ends which can be satisfied through exchange. In the case of an isolated individual there is no exchange, hence we cannot differentiate between economic and non-economic ends or activities. For the sake of simplicity, however, we will assume that a certain commodity corresponds to what an individual in an exchange economy would buy with his income and that the amount of effort he is willing to spend on obtaining this commodity is constant, say 10 hours work of standard intensity, whatever the productivity of an hour's "work" may be. Since there is only one commodity to produce, his natural resources will not enter into his calculations of alternatives and hence will have

---

(*) These notes are part of an unpublished study which was written twenty years ago, under the title "Economic Development for Underdeveloped Countries, within the Framework of an International Economy." By the time it was completed, the writer had lost confidence in the analytical techniques employed in it He believes however, that some parts of it, including the present one, may contain some hints which may have a certain, though limited, degree of validity which makes them worth salvaging. No modification of the original text has been attempted.