CHAPTER 3

THE FORCE METHODS

3.1 INTRODUCTION

As presented in the previous chapters, the force methods deal with determining the unknown forces or moments in the structure. Any of the force methods can thus be used when the degree of static indeterminacy is smaller than the degree of freedom of the structure. The force method is also called the consistent deformation method because the technique used for solution depends on conditions, to solve for the unknown forces or moments, to keep the deformation in the structure consistent.

The force methods studied in this chapter are:

1. The use of unit load method.
2. The use of Castigliano's second theorem.
3. The three moment equation method.
4. The elastic center method.
5. The column analogy method.
6. The flexibility matrix method : approach-1.
7. The flexibility matrix method : approach-2.

Each of the above methods shall be demonstrated how to be used in solving statically indeterminate planar structures.

3.2 DEGREE OF STATIC INDETERMINACY AND THE PRIMARY STRUCTURES

An important step to decide whether to use the force methods or not is the determination of the degree of static indeterminacy for the structure and comparing it with its degree of freedom. This section illustrates how to determine the degree of static indeterminacy for various types of structures. From the degree of static indeterminacy one knows how many forces and moments could be released from the structure to be statically determinate. A stable statically determinate structure developed from the statically indeterminate structure is called a primary structure.

3.2.1 Plane Trusses

Truss members sustain only axial forces. The total number of forces which need to be determined in order to analyze the truss equals the number of members plus the number of reactions. The available equilibrium equations are those applicable at every frictionless joint, where the sums of horizontal and sum of vertical forces are zero. Therefore the degree of static indeterminacy (DSI) in plane trusses is determined from

$$DSI = m + r - 2J \qquad (3.1)$$

where m is number of members, r is number of reactions, and j is number of joints.

The degree of static indeterminacy represents the number of redundant forces which may contain external reactions, internal forces or both. The number of external redundants is calculated from

$$\text{Number of external redundants} = r - 3 - n \qquad (3.2)$$

in which the number 3 represents the three external equilibrium equations for the whole structure ($\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$), and n represents any additional external equilibrium equations which relate the external reactions together. The number of internal redundants is determined from

$$\text{Number of internal redundants} = \text{DSI} - \text{Number of external Redundants} \qquad (3.3)$$

Example 3.1

Determine the degree of static indeterminacy, DSI, and the number of external redundants and internal redundants in the trusses shown in Figures 3.1a and 3.1b. Select a primary structure for each truss.

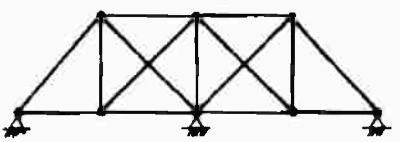


Figure 3.1a

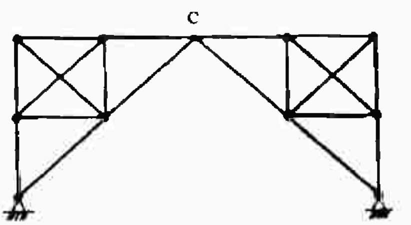


Figure 3.1b

Solution

For truss of Figure 3.1a, the number of members m = 15, the number of reactions for three hinged supports r = 6, the number of joints j = 8. Therefore, the degree of static indeterminacy for this truss is

$$\text{DSI} = 15 + 6 - 2 \times 8 = 5$$

The number of external redundants is determined from (r − 3 − 0), this gives 3 external redundant reactions. The number of internal redundants is thus (5 − 3) = 2.

The primary structure is selected such that three unknown external reactions, x_1, x_2, and x_3 are zero. The two internal redundants are assumed zero for any two unknown members forces in the truss such that the truss remains stable. A primary structure is shown in Figure 3.2.

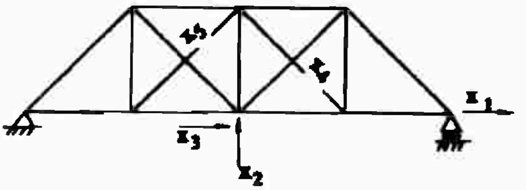


Figure 3.2

The truss of Figure 3.1a has m = 20, r = 4, j = 11, and n = 1 which represents the moment at C to relate the reactions together.

DSI = 20 + 4 − 2 × 11 = 2

The number of external redundants = 4 − 3 − 1 = 0.

Therefore, the number of the internal redundants is two.

A primary structure is shown in Figure 3.3.

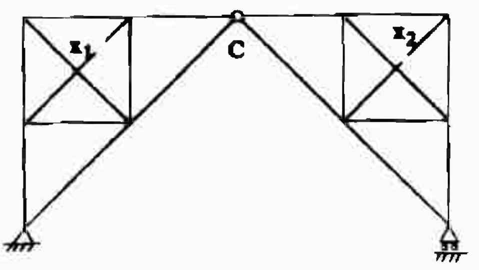


Figure 3.3

3.2.2 Space Trusses

In space trusses, each joint has three equilibrium equations. Therefore, the degree of static indeterminacy is given by

$$DSI = m + r - 3j \qquad (3.4)$$

The number of external redundants is obtained from

$$\text{Number of external redundants} = r - 6 - n \qquad (3.5)$$

where n is any additional equilibrium equations which relate the reactions together.

Example 3.2

Determine DSI in the space truss shown in Figure 3.4.

Solution

The number of members m = 20, number of reactions = 12 which represents three reactions at each hinged support, and number of joints j = 8.

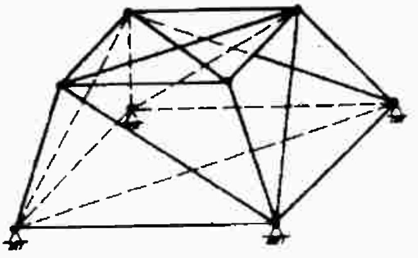


Figure 3.4

DSI = 20 + 12 – 3 × 8 =8
Number of external redundants = 12 – 6 = 6
Number of internal redundants = 8 – 6 = 2

A primary structure is shown in Figure 3.5.

3.2.3 Plane Frames

In plane frames, the member internal forces are characterized by the axial force, shear force, and bending moment. To determine the internal forces in all members of the frame, one has to apply the static equilibrium equations at every joint

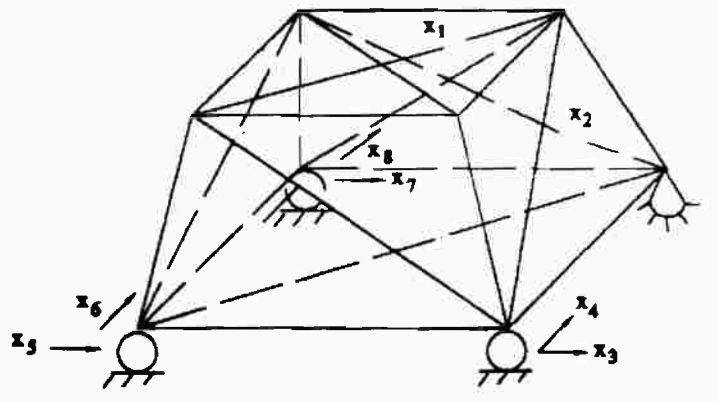


Figure 3.5

in the frame. Therefore, the DSI can be determined from

$$DSI = 3m + r - 3J - n \tag{3.6}$$

in which J is number of joints between members, m is number of members, r is number of external reactions, and n is any additional equilibrium equations.

The number of external redundants is obtained from

$$\text{Number of external redundants} = r - 3 - n_{ext} \tag{3.7}$$

in which n_{ext} is the number of static equilibrium equations which relate the external reactions together.

The number of internal redundants is the difference between DSI and the number of external redundants.

Example 3.3

Determine the degree of static indeterminacy for the plane frames shown in Figures 3.6a, 3.6b, 3.6c, 3.6d and 3.6e.

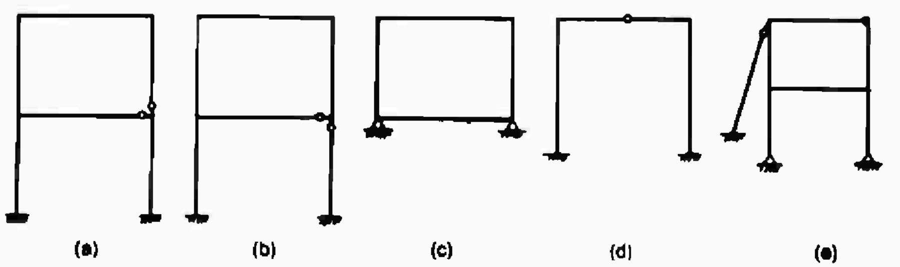


Figure 3.6

The frame of Figure 3.6a has m = 6, J = 6, r = 6, and n = 2 where $n_{ext} = 0$.

The number of external redundants = 6 – 3 = 3
The number of internal redundants = 4 – 3 = 1

A primary structure for the frame is shown in Figure 3.7.

The frame of Figure 3.6b has m = 6, J = 6, r = 6, and n = 2 where $n_{ext} = 1$.

DSI = 3 × 6 + 6 – 3 × 6 – 2 = 4
Number of external redundants = 6 – 3 – 1 = 2
Number of internal redundants = 4 – 2 = 2

A primary structure for the frame is shown in Figure 3.8.

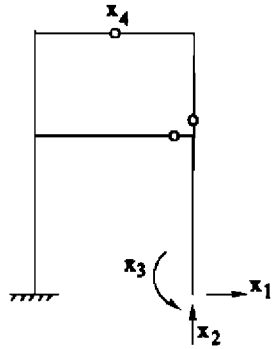


Figure 3.7

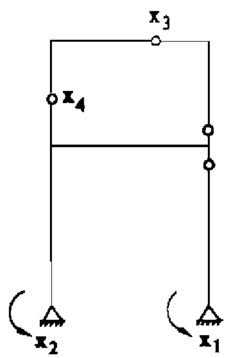


Figure 3.8

The frame of Figure 3.6c has m = 4, J = 4, r = 4, and n = 0.

DSI = 3 × 4 + 4 − 3 × 4 = 4
Number of external redundants = 4 − 3 = 1
Number of internal redundants = 4 − 1 = 3

A primary structure for the frame is shown in Figure 3.9.

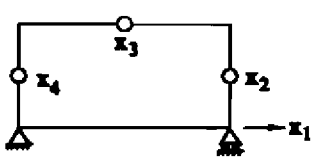


Figure 3.9

The frame of Figure 3.6d has m = 4, J = 5, r = 6, and n = 1 where $n_{ext} = 1$.

DSI = 3 × 4 + 6 – 3 × 5 – 1 = 2
Number of external redundants = 6 – 3 – 1 = 2
Number of internal redundants = 2 – 2 = 0

A primary structure for the frame is shown in Figure 3.10.

The frame of Figure 3.6e has m = 7, J = 7, r = 7 and n = 2 where $n_{ext} = 1$.

DSI = 3 × 7 + 7 – 3 × 7 – 2 = 5
Number of external redundants = 7 – 3 – 1 = 3
Number of internal redundants = 5 – 3 = 2

A primary structure is shown in Figure 3.11.

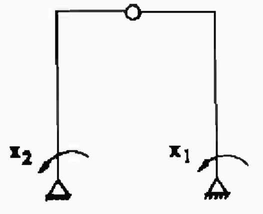


Figure 3.10

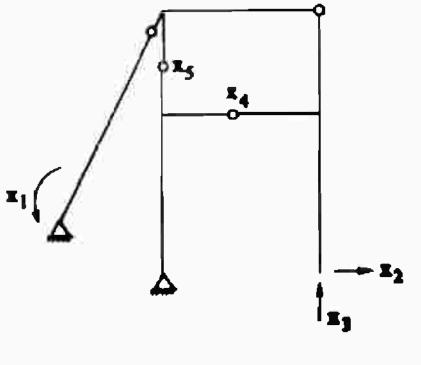


Figure 3.11

Example 3.4

Determine the DSI for the arches shown in Figures 3.12a and 3.12b.

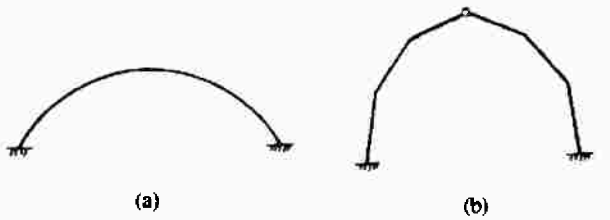


Figure 3.12

Solution

The arch of Figure 3.12a can be discretized into m members. Therefore, the number of joints is $j = m + 1$. Since $r = 6$, and $n = 0$, one has

DSI = 3m + 6 – 3 (m + 1) = 3
Number of external redundants = 6 – 3 = 3
Number of internal redundants = 3 – 3 = 0

A primary structure for the arch is shown in Figure 3.13.

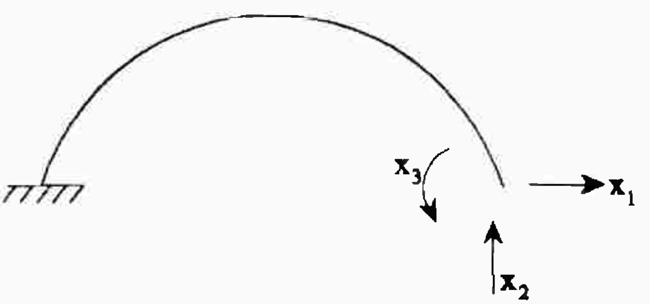


Figure 3.13

The arch of Figure 3.12b has $m = 6$, $j = 7$, $r = 6$ and $n = 1$ where $n_{ext} = 1$.

DSI = 3 × 6 + 6 – 3 × 7 – 1 = 2
Number of external redundants = 6 – 3 – 1 = 2
Number of internal redundants = 2 – 2 = 0

A primary structure for the arch is shown in Figure 3.14.

Example 3.5

Determine the DSI for the beams shown in Figures 3.15a and 3.15b.

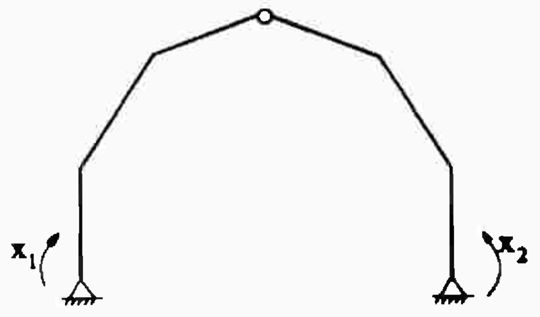


Figure 3.14

Figure 3.15

Solution

The beams can be considered as special cases of plane frames. For the beam shown in Figure 3.15a, m = 1, J = 2, r = 6 and n = 0.

DSI = 3 × 1 + 6 − 3 × 2 − 0 = 3
Number of external redundants = 6 − 3 = 3
Number of internal redundants = 3 − 3 = 0

A primary structure for the beam is shown in Figure 3.16.

Figure 3.16

The beam shown in Figure 3.15b has m = 4, J = 5, r = 9 and n = 1 where n_{ext} = 1.

DSI = 3 × 4 + 9 − 3 × 5 − 1 = 5
Number of external redundants = 9 − 3 − 1 = 5
Number of internal redundants = 5 − 5 = 0

A primary structure is shown in Figure 3.17.

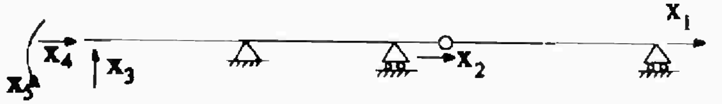


Figure 3.17

3.2.4 Space Frames

Each member in a space frame contains six internal actions represented by one axial force, two shear forces, one torsion, and two bending moments, as was shown in section 2.13. To determine the internal forces in the frame one has to apply the six static equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, $\Sigma M_x = 0$, $\Sigma M_y = 0$, $\Sigma M_z = 0$) at every joint.

The DSI in space frames can thus be determined from

$$DSI = 6\,m + r - 6\,J - n \tag{3.8}$$

where n is the number of additional static equilibrium equations.

The number of external redundants is obtained from

$$\text{Number of external redundants} = r - 6 - n_{ext} \tag{3.9}$$

where n_{ext} is the number of additional equilibrium equations relating the reactions together.

Example 3.6

Determine the DSI in the space frames shown in Figures 3.18a and 3.18b.

Solution

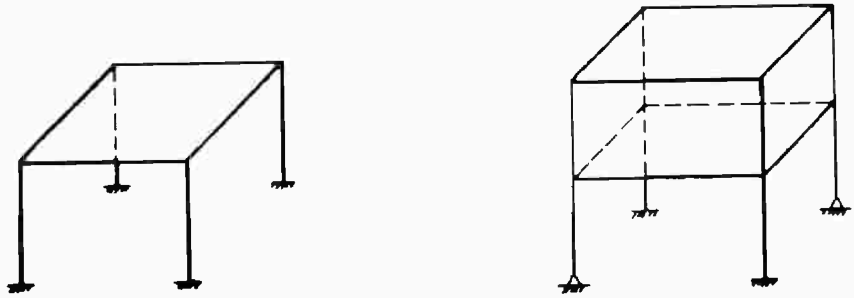
The space frame shown in Figure 3.18a has m = 9, J = 8, r = 24, and n = 0

DSI = 6 × 8 + 24 − 6 × 8 − 0 = 24
Number of external redundants = 24 − 6 = 18
Number of internal redundants = 24 − 18 = 6

A primary structure is shown in Figure 3.19.

For the frame of Figure 3.18b, m = 16, J = 12, r = 18 and n = 0.

DSI = 6 × 6 + 18 − 6 × 12 = 42



(a) (b)

Figure 3.18

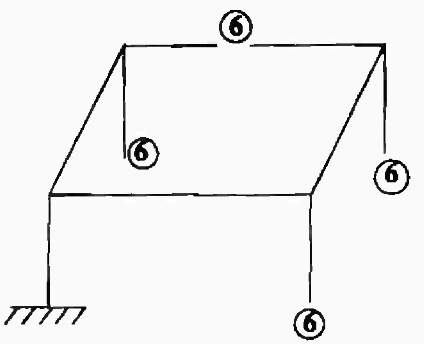


Figure 3.19

Number of external redundants = 18 – 6 = 12
Number of internal redundants = 42 – 12 = 30

A primary structure is shown in Figure 3.20.

3.2.5 Grids

Grids are plane frames where the loading is applied perpendicular to their plane. Each member in the grid contains three internal actions represented by one torsion, one shear force, and one bending moment. Therefore the DSI is obtained as

$$DSI = 3\,m + r - 3\,J - n \qquad (3.10)$$

The number of external redundants is obtained from ($r - 3 - n_{ext}$)

where n is the total number of additional equilibrium equations and n_{ext} is the number of equilibrium equations relating the reactions together.

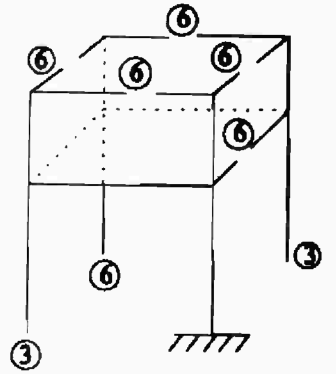


Figure 3.20

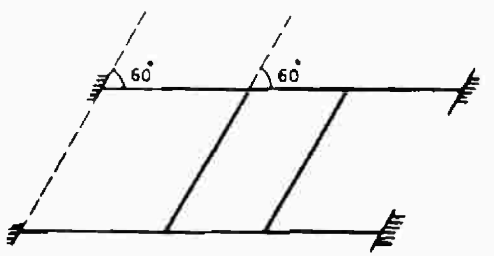


Figure 3.21

Example 3.7

Determine DSI for the grid shown in Figure 3.21.

Solution

The grid of Figure 3.21 has m = 8, J = 8, r = 12 and n = 0.

DSI = 3 × 8 + 12 − 3 × 8 = 12
Number of external redundants = 12 − 3 = 9
Number of internal redundants = 12 − 9 = 3

A primary structure for the grid is shown in Figure 3.22.

3.2.6 Frame-Truss Structures

Many structures are combinations of frames and trusses. One can separate the truss and the frame by considering the internal forces between both systems. One can also deal directly with the whole structure as a frame and consider the additional static

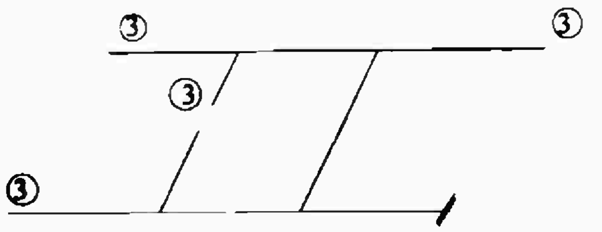


Figure 3.22

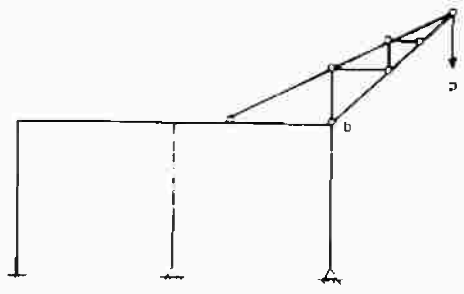


Figure 3.23

equilibrium equation due to the hinges.

Example 3.8

Determine the DSI for the structure shown in Figure 3.23.

Solution

One can separate the truss from the frame by adding the reactions at a and b as shown in Figure 3.24.

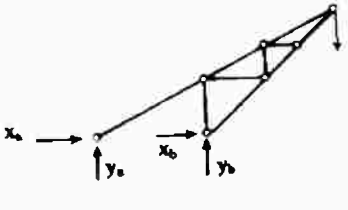


Figure 3.24

In the truss of Figure 3.24, m = 10, J = 7, and r = 4. Then, DSI = m + r – 2j = 0. This means that this part is statically determinate.

The determined reactions of the truss are considered as external forces on the frame of Figure 3.25. The frame has m = 5, J = 6, and r = 7.

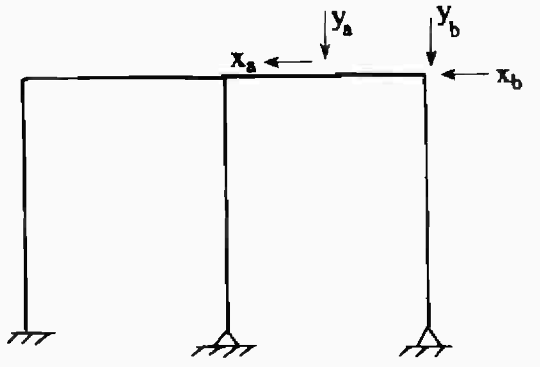


Figure 3.25

DSI = 3 × 5 + 7 – 3 × 6 = 4
Number of external redundants = 7 – 3 = 4
Number of internal redundants = 4 – 4 = 0

Another solution is to consider the whole structure as a plane frame and the hinges are additional equilibrium conditions. In this case, m = 16, J = 12, r = 7, and n = 1 + 2 + 3 + 3 + 3 + 2 + 1 = 15, where n_{ext} = 0.

DSI = 16 × 3 + 7 – 3 × 12 – 15 = 4
Number of external redundants = 7 – 3 = 4
Number of internal redundants = 4 – 4 = 0

A primary structure is shown in Figure 3.26.

3.3 THE USE OF UNIT LOAD METHOD

3.3.1 The Basic Formulation

In this method, a primary structure is selected for the statically indeterminate structure. Under the effect of the applied actions, the primary structure is subjected to deformation at the locations of the redundants. These deformation are determined by the unit load method. To keep the deformation in the actual structure consistent, the deformation due to each redundant is found by the unit load method, and the compatibility equations are written at each released redundant. This results in a system of linear simultaneous equations whose number equals the degree of static indeterminacy and the unknowns represent the magnitudes of the redundants.

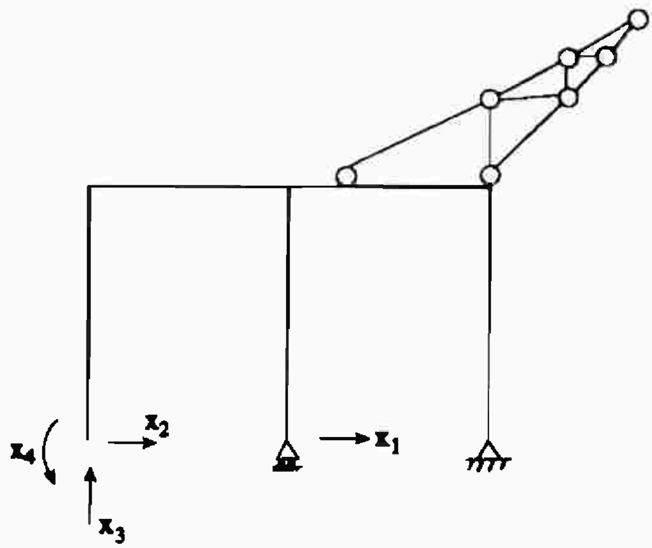


Figure 3.26

Consider, for simplicity, the plane frame shown in Figure 3.27a. The frame is externally two times statically indeterminate (m = 3, J = 4, r = 5). Therefore one has to release two reactions to form a primary structure. Possible choices are shown in Figures 3.27 b, 3.27c, and 3.27d. The released reactions shall induce deformation in x_1 and x_2 assumed directions. However, since the actual structure is restrained at the locations of these redundants, the compatibility equations are obtained by using the principle of superposition for linear elastic structures. Considering the frame of Figure 3.27b as the primary structure, one has, by the superposition principle and according to Figure 3.28, the following compatibility equations:

$$\Delta_1 = \Delta_{10} + \Delta_{11} + \Delta_{12} \tag{3.11}$$

$$\Delta_2 = \Delta_{20} + \Delta_{21} + \Delta_{22} \tag{3.12}$$

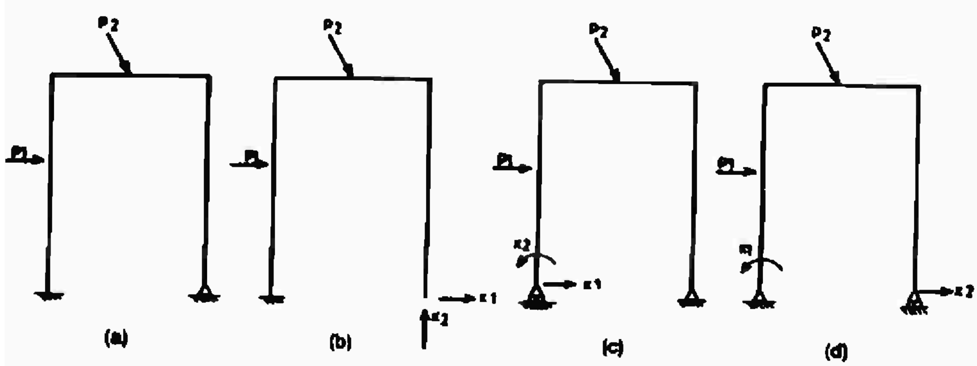


Figure 3.27

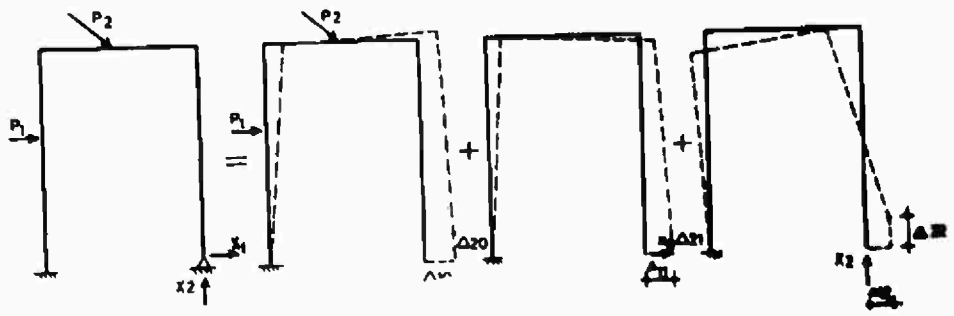


Figure 3.28

in which, Δ_1 = deflection in direction of x_1 for the statically indeterminate structure; Δ_2 = deflection in the direction of x_2 for the statically indeterminate structure; Δ_{10} = deflection in the direction of x_1 for the primary structure due to the external loads; Δ_{20} = deflection in the direction of x_2 for the primary structure due to the external loads; Δ_{11} = deflection in the direction of x_1 due to the redundant x_1 applied to the primary structure; Δ_{21} = deflection in the direction of x_2 due to the redundant x_1 applied to the primary structure; Δ_{12} = deflection in the direction of x_1 due to the redundant x_2 applied to the primary structure; Δ_{22} = deflection in the direction of x_2 due to the redundant x_2 applied to the primary structure.

To calculate the deflection Δ_{10}, a unit load is applied in the direction of x_1 and from the integration of the product of the internal actions in the primary structure (A_{xo}, A_{yo}, M_{zo}) and the internal actions due to the unit load in x_1 direction (a_{x1}, a_{y1}, m_{z1}) one obtains the value of Δ_{10}. This can be expressed, using Equation 2.43, as

$$\Delta_{10} = \int \frac{A_{xo}\, a_{x1}\, dx}{EA} + \int \frac{A_{yo}\, a_{y1}\, dx}{GA_r} + \int \frac{M_{zo}\, m_{z1}}{EI_z} dx$$

The value of Δ_{11} can be obtained from integrating the internal actions due to the unknown x_1 with the internal actions due to $x_1 = 1$. This can easily be computed by integrating the internal actions due to $x_1 = 1$ with itself times x_1. Similarly Δ_{22} can be found by integrating the internal actions due to $x_2 = 1$ with itself times x_2 as shown in Figures 3.29 a and 3.29b.

Therefore, Equations 3.11 and 3.12 can be written as

$$\Delta_1 = \Delta_{10} + x_1\, f_{11} + x_2\, f_{12} \tag{3.13}$$

$$\Delta_2 = \Delta_{20} + x_1\, f_{21} + x_2\, f_{22} \tag{3.14}$$

in which Δ_{i0} and f_{ij} are defined, in general, as

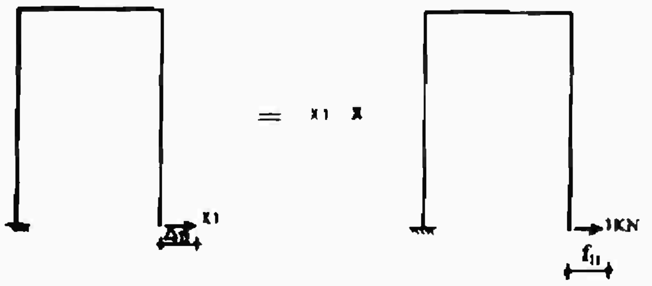


Figure 3.29a

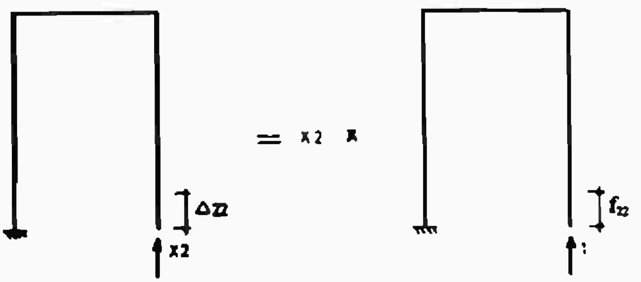


Figure 3.29b

$$\Delta_{i0} = \int \frac{A_{x0}\, a_{xi}\, dx}{EA} + \int \frac{A_{y0}\, a_{yi}\, dx}{G A_r} + \int \frac{M_{z0}\, m_{zi}\, dx}{EI_z} \tag{3.15}$$

$$f_{ij} = \int \frac{a_{xi}\, a_{xj}\, dx}{EA} + \int \frac{a_{yi}\, a_{yj}\, dx}{G A_r} + \int \frac{m_{zi}\, m_{zj}\, dx}{EI_z} \tag{3.16}$$

where a_{xi}, a_{yi} and m_{zi} are the axial force, shear force and bending moment in any member due to $x_i = 1$.

The values f_{ij} are called the flexibility coefficients since they represent the deflection at i due to unit actions at j. That is why the force method is often called the flexibility method. It is obvious that the solution of Equations 3.13 and 3.14 depends on specifying the left hand side in order to satisfy the compatibility at the boundaries of the actual structure. That is why the force method is also called the compatibility method.

For a statically indeterminate structure of n degree static indeterminate, one ends up with n linear simultaneous equations in n unknowns. These equations can be written, in general, as

$$\Delta_1 = \Delta_{10} + f_{11}\,x_1 + f_{12}\,x_2 + \ldots\ldots + f_{1n}\,x_n$$
$$\Delta_2 = \Delta_{20} + f_{21}\,x_1 + f_{22}\,x_2 + \ldots\ldots + f_{2n}\,x_n$$
$$\vdots$$
$$\Delta_n = \Delta_{no} + f_{n1}\,x_1 + f_{n2}\,x_2 + \ldots\ldots + f_{2n}\,x_n \qquad (3.17)$$

Equations 3.17 can be expressed in a matrix form as

$$\underline{\Delta} = \underline{\Delta}_0 + [f]\underline{x} \qquad (3.18)$$

where $\underline{\Delta} = [\Delta_1\ \Delta_2\ \ldots\Delta_n]^T$, $\underline{\Delta}_0 = [\Delta_{10}\ \Delta_{20}\ \ldots\Delta_{n0}]^T$, $\underline{x} = [x_1\ x_2\ \ldots x_n]^T$, and [f] is called the structural flexibility matrix which takes the form of

$$[f] = \begin{bmatrix} f_{11} & f_{12} & \ldots\ldots & f_{1n} \\ f_{21} & f_{22} & \ldots\ldots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \ldots\ldots & f_{nn} \end{bmatrix} \qquad (3.19)$$

The solution of Equation 3.18 is obtained as

$$\underline{x} = [f]^{-1}[\underline{\Delta} - \underline{\Delta}_0] \qquad (3.20)$$

The internal actions at any section in the statically indeterminate structure can now be obtained from the superposition principle as follows:

$$A_x = A_{xo} + x_1\,a_{x1} + x_2\,a_{x2} + \ldots\ldots + x_n\,a_{xn}$$
$$A_y = A_{yo} + x_1\,a_{y1} + x_2\,a_{y2} + \ldots\ldots + x_n\,a_{yn} \qquad (3.21)$$
$$M_z = M_{zo} + x_1\,m_{z1} + x_2\,m_{z2} + \ldots\ldots + x_n\,m_{zn}$$

Similarly, the deformation at any point in the structure can be obtained using the principle of superposition. For example, if it is desired to determine the horizontal sway of the frame of figure 3.28 at joint c, one uses

$$\Delta_c = \Delta_{co} + x_1\,f_{c1} + x_2\,f_{c2} \qquad (3.22)$$

where Δ_{co} is the horizontal deflection at c in the primary structure due to the applied external loading; f_{c1} is the horizontal deflection at c in the primary structure due to x_1 only and $x_1 = 1$; f_{c2} is the horizontal deflection at c in the primary structure due to x_2 only and $x_2 = 1$.

The horizontal deflection Δ_{co} is obtained by applying a horizontal unit load on the primary structure at c, and integrating the internal actions due to the applied loading (A_{xo}, A_{yo}, M_{zo}) with the internal actions due to the unit load at c (a_{xc}, a_{yc}, m_{zc}) as shown in Figure 3.30. The displacement Δ_{co} at c is obtained from

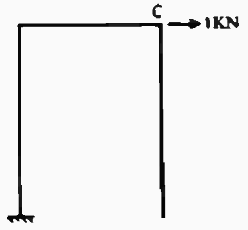


Figure 3.30

$$\Delta_{co} = \int \frac{A_{xo}\, a_{xc}\, dx}{EA} + \int \frac{A_{yo}\, a_{yc}\, dx}{G\, A_r} + \int \frac{M_{zo}\, m_{zc}\, dx}{EI_z} \qquad (3.23)$$

Similarly, any of the flexibility coefficients f_{ci} is generally obtained from

$$f_{ci} = \int \frac{a_{xi}\, a_{xc}\, dx}{EA} + \int \frac{a_{yi}\, a_{yc}\, dx}{G\, A_r} + \int \frac{m_{zi}\, m_{zc}\, dx}{EI_z} \qquad (3.24)$$

Substituting Equations 3.23 and 3.24 into Equation 3.22 and using Equation 3.21, one obtains the deflection at any point c from

$$\Delta_c = \int \frac{A_x\, a_{xc}\, dx}{EA} + \int \frac{A_y\, a_{yc}\, dx}{G\, A_r} + \int \frac{M_z\, m_{zc}\, dx}{EI_z} \qquad (3.25)$$

where A_x, A_y, and M_z are the internal actions of the statically indeterminate structure, after determining the values of the redundants x_1, x_2, etc.....; and a_{xc}, a_{yc}, and m_{zc} are the internal actions due to a unit load applied at c in the primary structure.

Since the internal actions in the indeterminate structure under certain external actions are unique, and the final deformed shape of the structure is also unique according to the uniqueness theorem presented in chapter 2, there is no limitation that one has to apply the unit load at c on a specific primary structure like the one previously chosen in determining the internal actions of the actual structure. In fact, one can select any primary structure in order to apply the unit load at c and carry out the computation according to Equation 3.25 and it turns out to have the same results. The following example illustrates this point:

Example 3.9

Determine the vertical deflection at point c of the beam shown in Figure 3.31, neglecting shear deformations. Consider EI_z is constant.

Solution

The bending moment diagram of this beam is as shown in Figure 3.32a. This moment

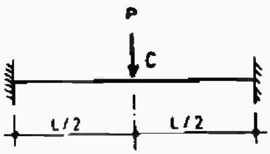


Figure 3.31

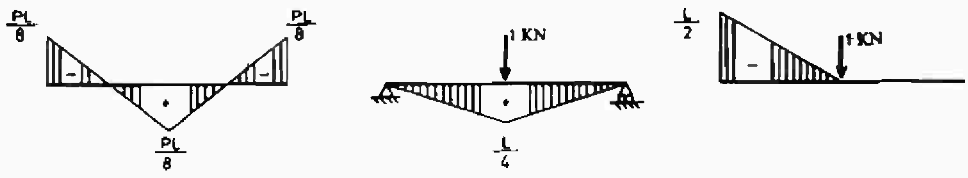


Figure 3.32

has been obtained by the consistent deformation method using any primary structure. According to the primary structure of Figure 3.32b, the deflection at c is calculated as follows:

$$\Delta_c = \int \frac{M_z\, m_{zc}\, d\ell}{EI}$$

$$= \frac{L}{2 \times 6EI} \times 2\left[\frac{2PL^2}{8 \times 4} + 0 + 0 - \frac{PL^2}{8 \times 4}\right] = \frac{PL^3}{192\,EI}$$

Using the primary structure of Figure 3.32c, the deflection Δ_c is calculated from

$$\Delta_C = \frac{L}{2 \times 6EI}\left[2 \times \frac{L}{2} \times \frac{PL}{8} + 0 - \frac{PL}{8} \times \frac{L}{2}\right] = \frac{PL^3}{192\,EI}$$

which gives the same result whatever is the chosen primary structure.

3.3.2 Applications to Beams

Example 3.10

Draw the bending moment and shear force diagrams for the beam shown in Figure 3.33 due to the applied loads, a rotation at support A, and a rise in temperature in member BC as shown. ($EI_z = 10^5$ kN.m^2, $\alpha = 10^{-5}$/°c).

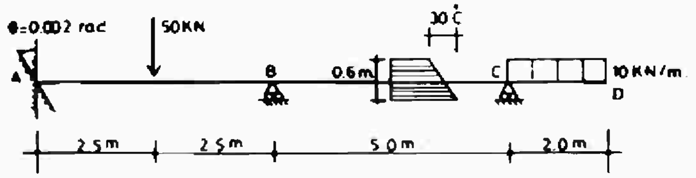


Figure 3.33

Solution

One first determines the degree of static indeterminacy for the beam in order to know the number of unknowns.

$DSI = 3m + r - 3j = 3 \times 3 + 5 - 3 \times 4 = 2$

Selecting two releases out of the five reactions such as the moment at A and the vertical reaction at C, the primary structure is the one shown in Figure 3.34 which is statically determinate and stable structure. The bending moment diagram M_{zo} is constructed after finding the reactions. The bending moment diagram can be decomposed as shown in Figure 3.34 in order to ease the integration process.

To calculate the flexibility coefficients, each of the redundant x_1, and x_2 are assumed

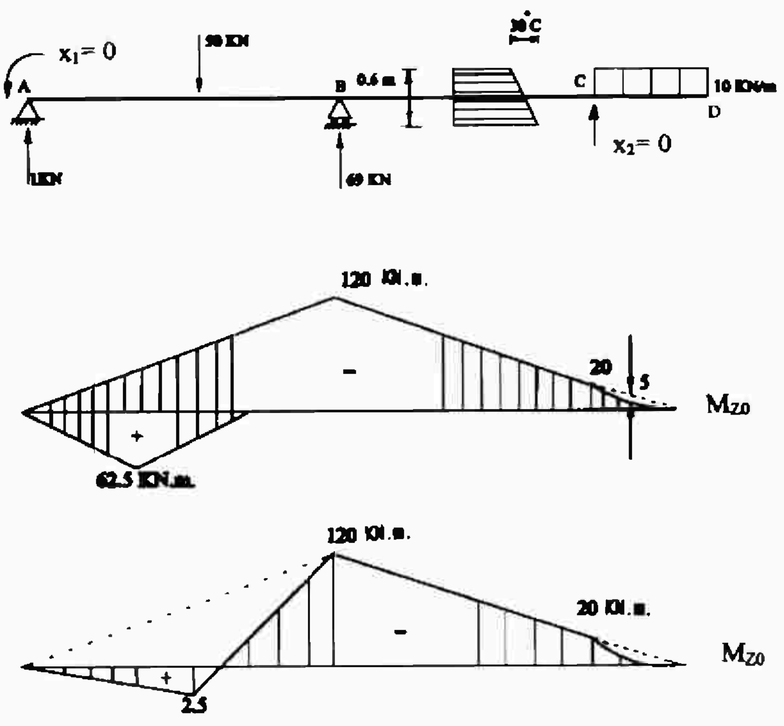


Figure 3.34

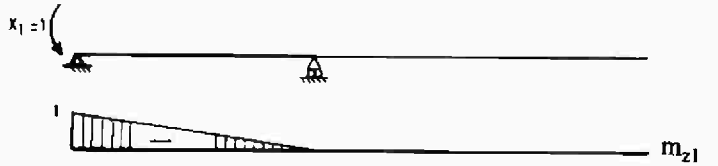


Figure 3.35

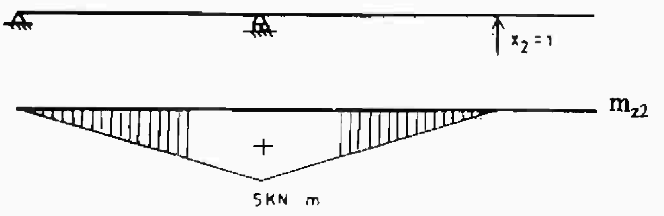


Figure 3.36

unity and the m_{z1} and m_{z2} diagrams are determined as shown, respectively, in Figures 3.35 and 3.36.

The equations of consistent deformations for this beam are

$$\Delta_{10} + f_{11}\, x_1 + f_{12}\, x_2 = \Delta_1$$
$$\Delta_{20} + f_{21}\, x_1 + f_{22}\, x_2 = \Delta_2$$

The coefficients Δ_{io} and f_{ij} are determined, neglecting the shear and axial deformations, as follows:

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, dx}{EI_z} - \alpha\left(\frac{T_1 - T_2}{h}\right) m_{z1}\, dx$$
$$= \frac{1}{EI}\left[\frac{-120 \times 5}{2} \times \frac{-1}{3} + \frac{62.5 \times 5}{2} \times \frac{-1}{2}\right] - 0 = 21.875 \times 10^{-5} \text{ rad}.$$

$$f_{11} = \int \frac{m_{z1}^2\, dx}{EI} = \frac{1}{EI} \times \frac{1 \times 5}{2} \times \frac{2}{3} = 1.666 \times 10^{-5} \text{ rad.}$$

$$f_{12} = \int \frac{m_{z1}\, m_{z2}\, dx}{EI} = \frac{5 \times 5}{2EI} \times \frac{-1}{3} = -4.1667 \times 10^{-5} \text{ m}$$

$$f_{21} = f_{12} = -4.1667 \times 10^{-5} \text{ rad.}$$

$$f_{22} = \int \frac{m_{z2}^2\, dx}{EI} = 2 \times \frac{5 \times 5}{2} \times \frac{2 \times 5}{3EI} = 83.33 \times 10^{-5} \text{ m}$$

$$\Delta_{20} = \int \frac{M_{z0}\, m_{z2}\, dx}{EI} - \alpha \left(\frac{T_1 - T_2}{h} \right) \int m_{z2}\, dx$$

$$= \frac{1}{EI} \left[\frac{-120 \times 5}{2} \times \frac{2}{3} \times 5 + \frac{62.5 \times 5}{2} \times \frac{5}{2} + \frac{5}{6} \left[2 \times (-120) \times 5 + 0 + 0 - 5 \times 20 \right] \right]$$

$$- \alpha \left(\frac{-30}{0.6} \right) \left[\frac{5 \times 5}{2} \right] = -1067.71 \times 10^{-5} \text{ m}$$

Substituting into the consistent deformation equations, where $\Delta_1 = +0.002$ rad. Since it is in the direction of x_1, one has

$$21.875 + 1.666\, x_1 - 4.1667\, x_2 = 0.002 \times 10^5 = 200$$

$$-1067.71 - 4.1667\, x_1 + 83.33\, x_2 = 0$$

Solving the above equations one obtains x_1 = 158.72 kN.m. in the assumed direction of x_1 and x_2 = 20.75 kN in the same direction of x_2. The bending moment and shear force diagrams can then be constructed either from using the superposition principle or using the static principles. Using superposition principle, Equations 3.21, the moment at points A, B, and C are calculated as follows:

$$M_A = 0 + 158.72 \times (-1) + 20.75 \times (0) = -158.72 \text{ kN.m.}$$

$$M_B = -120 + 158.72 \times (0) + 20.75 \times (5) = -16.25 \text{ kN.m.}$$

$$M_C = -20 + 0 + 0 = -20 \text{ kN.m.}$$

The bending moment is drawn on the tension side as shown in Figure 3.37.

Figure 3.37

The shear forces are obtained from the equilibrium of each member in the beam as summarized in Figure 3.38.

This problem can also be solved by choosing a primary structure which is fixed at A as shown in Figure 3.39. In this case, the deformation of the primary structure due to support rotation at A has to be determined in accordance with section 2.23. The

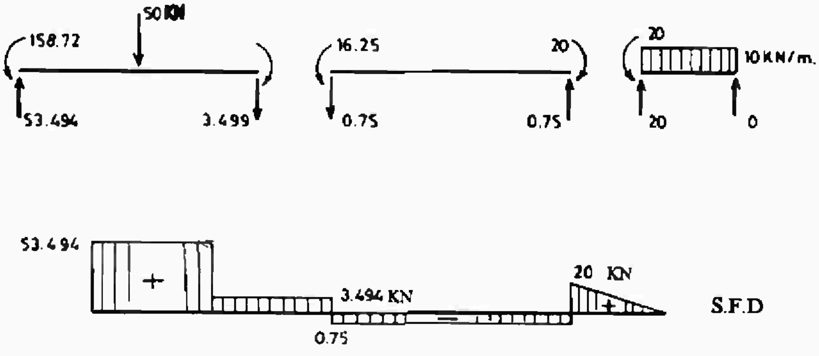


Figure 3.38

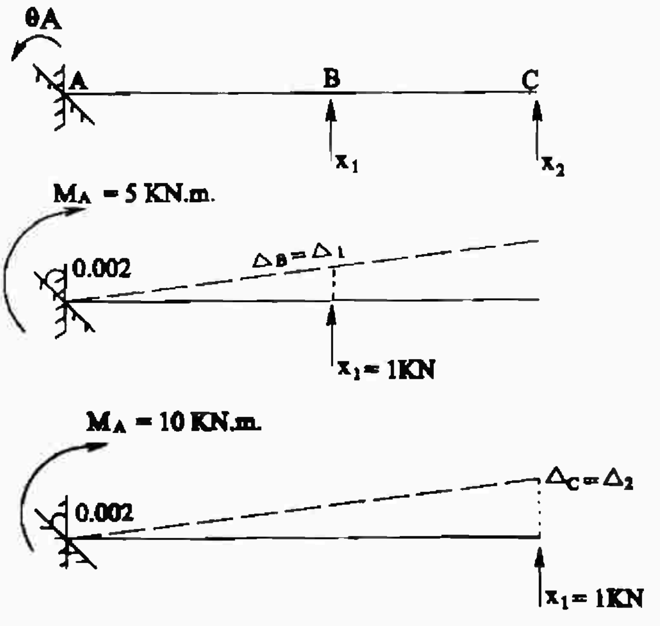


Figure 3.39

deformation at the releases x_1 and x_2 are determined by applying a virtual unit load in the directions of x_1 and x_2, respectively, as shown in Figure 3.39 and considering that the total work done is zero.

For the case x_1 = 1 kN, the work done is

$-5 \times 0.002 + 1 \times \Delta_1 = 0$; which leads to $\Delta_1 = 0.01$ m.

Similarly, when x_2 = 1 kN, the work done is

$-10 \times 0.002 + 1 \times \Delta_2$; which leads to $\Delta_2 = 0.02$ m.

The equations of consistent deformation become

$$\Delta_{10} + x_1 f_{11} + x_2 f_{12} = \Delta_1 = 0.01\,\text{m}$$

$$\Delta_{20} + x_1 f_{21} + x_2 f_{22} = \Delta_2 = 0.02\,\text{m}$$

To determine the flexibility coefficients, the diagrams M_{z0}, m_{z1}, and m_{z2} for the primary structure of Figure 3.39 are drawn as shown in Figure 3.40. The coefficients are computed using the integration of diagrams of Table 2.2 as follows:

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, d\ell}{EI}$$
$$= \frac{2.5}{6EI}\left[-170 \times 2.5 \times 2 - 120 \times 2.5\right] + \frac{2.5}{6EI}\left[-170 \times 2.5 \times 2 - 345 \times 5 \times 2 - 345 \times 2.5 - 170 \times 5\right]$$
$$= -2984.375 \times 10^{-5}\ \text{m}$$

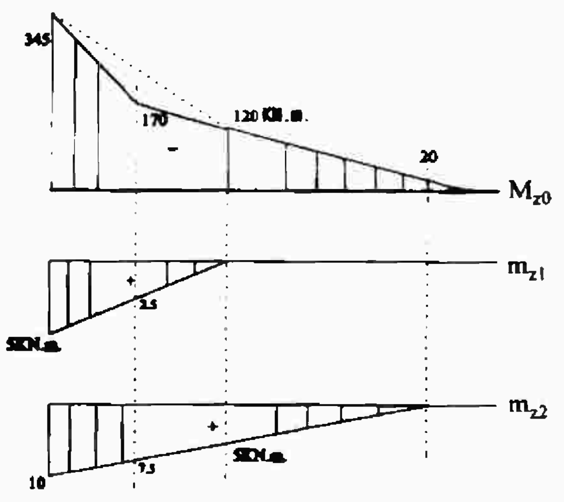


Figure 3.40

$$\Delta_{20} = \int \frac{M_{z0}\, m_{z2}\, d\ell}{EI}$$
$$= \frac{5}{6EI}\left[-120 \times 5 \times 2 - 20 \times 5\right] + \frac{2.5}{6EI}\left[-120 \times 5 \times 2 - 170 \times 7.5 \times 2 - 170 \times 5 - 120 \times 7.5\right]$$
$$+ \frac{2.5}{6EI}\left[-170 \times 7.5 \times 2 - 345 \times 10 \times 2 - 345 \times 7.5 - 170 \times 10\right] - \left(\frac{-30}{0.6}\right) \times \left(\frac{+5 \times 5}{2}\right)$$
$$= -8473.96 \times 10^{-5}\ \text{m}$$

$$f_{11} = \int \frac{m_{z1}^2 \, d\ell}{EI} = \frac{5 \times 5}{2EI} \times \frac{2}{3} \times 5 = 41.67 \times 10^{-5} \text{ m}$$

$$f_{22} = \int \frac{m_{z2}^2 \, d\ell}{EI} = \frac{10 \times 10}{2EI} \times \frac{2}{3} \times 10 = 333.33 \times 10^{-5} \text{ m}$$

$$f_{12} = \int \frac{m_{z1} \, m_{z2} \, d\ell}{EI} = \frac{5}{6EI} [2 \times 5 \times 10 + 5 \times 5] = 104.67 \times 10^{-5} \text{ m}$$

Substituting into the consistent deformation equations one has

$$-2984.375 + 41.67 \, x_1 + 104.167 \, x_2 = 1000$$
$$-9473.96 + 104.167 \, x_1 + 333.33 \, x_2 = 2000$$

Solving these equations one obtains x_1 = -4.25 kN and x_2 = 20.75 kN. The final bending moment at any point is obtained by the superposition principle as follows:

$$M = M_{zo} + x_1 \, m_{z1} + x_2 \, m_{z2}$$
$$M_A = -345 + (-4.25) \times 5 + 20.75 \times 10 = -158.75 \text{ kN.m}$$
$$M_B = -120 + (-4.25) \times 0 + 20.75 \times 5 = -16.25 \text{ kN.m}$$
$$M_C = -20 + (-4.25) \times 0 + 20.75 \times 0 = -20 \text{ kN.m}$$

which are the same results obtained before.

Example 3.11

Determine the bending moment, shear force diagrams, and the deflection at B for the beam shown in Figure 3.41. The support at B is linear elastic spring of stiffness K. ($EI_z = 10^5$ kN.m^2, $\alpha = 10^{-5}$/°c, K = 10 kN/cm).

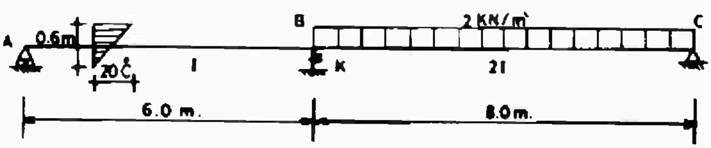


Figure 3.41

Solution

The degree of static indeterminacy for the beam is

DSI = 3m + r – 3J = 3 × 2 + 4 – 3 × 3 = 1

Select the redundant to be the reaction at B. The primary structure, reactions, and M_{z0} are shown in Figure 3.42.

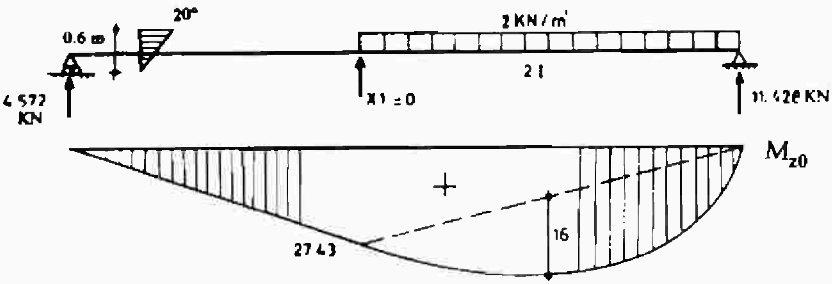


Figure 3.42

The equation of consistent deformation for the beam is

$$\Delta_{10} + f_{11}\,x_1 = \Delta_1$$

To calculate the flexibility coefficient f_{11} put x_1 = 1 kN to get m_{z1} as shown in Figure 3.43.

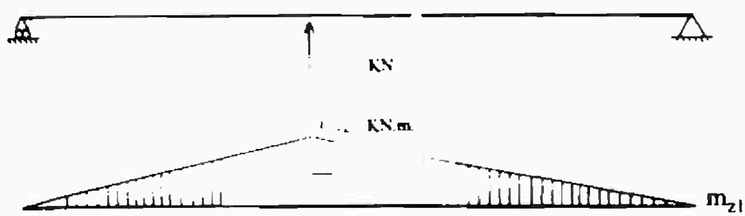


Figure 3.43

The coefficients Δ_{10} and f_{11} are determined, neglecting axial and shear deformations, from

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, dx}{EI_z} - \alpha \int \frac{(T_1 - T_2)}{h} m_{z1}\, dx$$

$$= \frac{27.430 \times 6}{2EI} \times \left(\frac{-2}{3} \times 3.428\right) + \frac{27.430 \times 8}{4EI} \times \left(\frac{-2}{3} \times 3.428\right) + \left(\frac{16 \times 8 \times \frac{2}{3}}{2EI}\right) \times \left(\frac{-3.428}{2}\right)$$

$$-\alpha\left(\frac{20}{0.6}\right) \times \left(\frac{-3.428 \times 6}{2}\right) = -43.72 \times 10^{-5} \text{ m (downward)}$$

$$f_{11} = \int \frac{m_{z1}^2\,dx}{EI} = \frac{1}{EI}\left[\frac{3.428\times 6}{2}\times\frac{2}{3}\times 3.428 + \frac{3.428\times 8}{2}\times\frac{2}{3}\times\frac{3.428}{2}\right] = 39.17\times 10^{-5}\ \text{m}$$

Substituting into the consistent deformation equation, where $\Delta_1 = (-x_1/K)$ one obtains

$$-43.799 + 39.17\,x_1 = 10^5\left(\frac{-x_1}{1000}\right) = -100x_1$$

The value of x_1 is determined to be 0.314 kN in the assumed direction.

The deflection at B could be determined from the reaction and spring's stiffness or using the unit load principle as before. Using the former method one has

$$\text{The deflection at B} = \frac{x_1}{K} = \frac{0.314}{10} = 0.0314\text{cm (downward)}$$

The bending moment is obtained using the superposition principle as follows:

$$M_Z = M_{z0} + x_1\, m_{z1}$$

$$M_B = +27.432 - 3.428\times 0.315 = 26.352\ \text{kN.m}$$

The bending moment and shear force diagrams are, respectively, given in Figures 3.44 and 3.45.

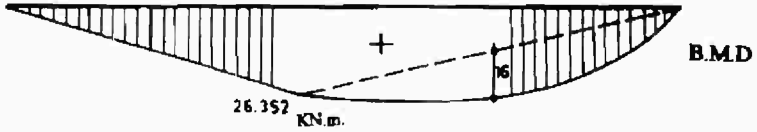


Figure 3.44

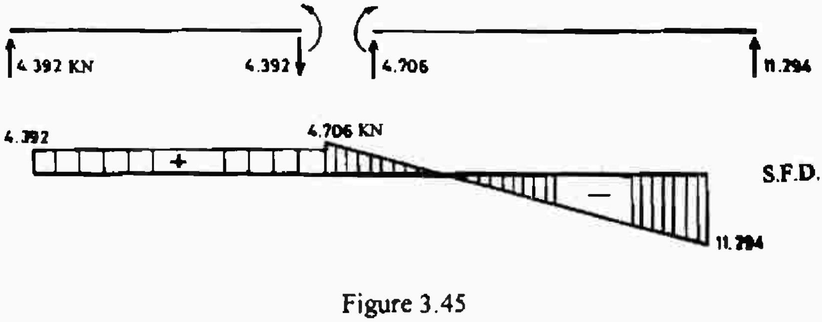


Figure 3.45

The deflection at B can also be found by the unit load method as shown below:

$$\Delta_B = \int \frac{M_t\, m_{xB}\, dx}{EI}$$

$$= \frac{26.352 \times 6}{2EI} \times \frac{-2}{3} \times 3.428 + \frac{26.352 \times 8}{4EI} \times \frac{-2}{3} \times 3.428$$

$$+ \frac{16 \times 8 \times \frac{2}{3}}{2EI} \times \frac{-3.428}{2} - \alpha \times \frac{20}{0.6} \times \frac{-3.428 \times 6}{2} = -0.031445 \text{ cm}$$

which indicates that it is in an opposite direction to the assumed direction of x_1.

3.3.3 Applications to Frames

Example 3.12

Determine bending moment, axial force, and shear force diagrams for the frame shown in Figure 3.46 due to the loads and a rise in temperature in member BC. Determine also the horizontal sway of the frame at joint B. ($EI = 10^5$ kN.m^2, $EA = 50 \times 10^5$ kN, $\alpha = 10^{-5}/°c$).

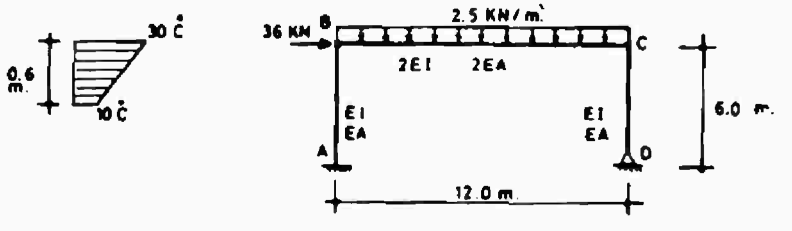


Figure 3.46

Solution

The degree of static indeterminacy is

DSI = 3m + r − 3j = 3 × 3 + 5 − 3 × 4 = 2

Select the two redundants to be the reactions of the hinged support at D. The axial force and bending moment diagrams are obtained for the primary structure as shown in Figure 3.47.

The equations of consistent deformation for the frame are

$$\Delta_{10} + f_{11}\, x_1 + f_{12}\, x_2 = \Delta_1$$

$$\Delta_{20} + f_{21}\, x_1 + f_{22}\, x_2 = \Delta_2$$

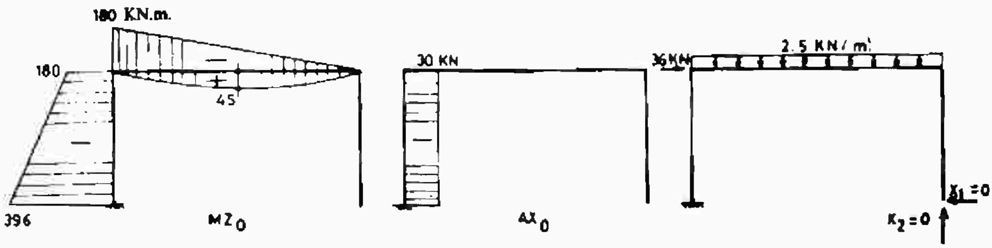


Figure 3.47

To determine Δ_{10}, Δ_{20}, and the flexibility coefficients, the redundants x_1 and x_2 are assumed unity. The bending moment and axial force diagrams for each unit redundant are shown in Figures 3.48 and 3.49.

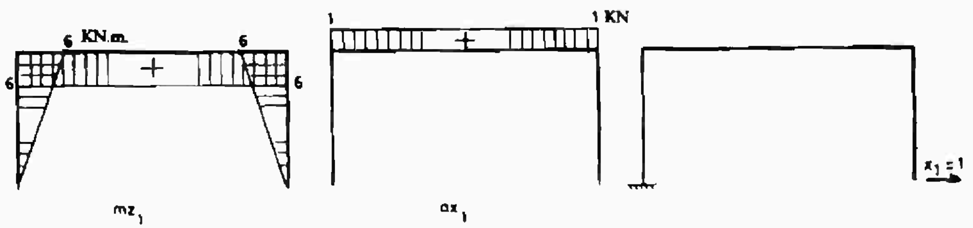


Figure 3.48

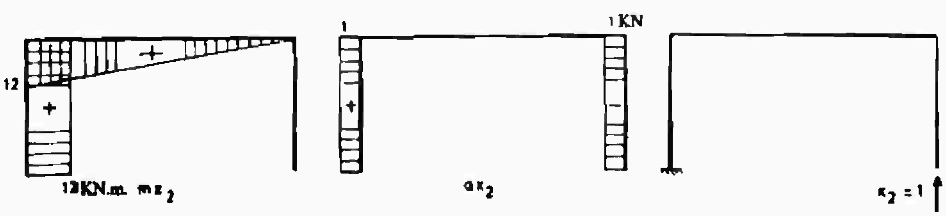


Figure 3.49

The coefficients Δ_{i0} and f_{ij} are calculated as follows:

$$\Delta_{i0} = \int \frac{M_{z0}\, m_{zi}\, dx}{EI_z} + \int \frac{A_{x0}\, a_{xi}\, dx}{EA} - \alpha \int \left(\frac{T_1 - T_2}{h} \right) m_{zi}\, dx + \alpha \int \left(\frac{T_1 + T_2}{2} \right) a_{xi}\, dx$$

$$\Delta_{10} = \frac{6}{2EI_z}\left(\frac{-180\times 12}{2} + \frac{2}{3}\times 45\times 12\right) + \frac{6}{6EI_z}\left(2\times 6\times(-180) + 0 - 396\times 6\right) + 0$$
$$-\alpha\left(\frac{20}{0.6}\right)\times(6\times 12) + \alpha\left(\frac{40}{2}\right)\times(1\times 12) = -8856\times 10^{-5}\ \text{m}$$

$$\Delta_{20} = \left[\frac{-180\times 12}{4EI}\times(+8) + \frac{2}{3}\times\frac{45\times 12}{2EI}\times(+6)\right] - \left(\frac{180+396}{2EI}\right)\times 12\times 6 - \frac{30\times 6}{EA}$$
$$-\alpha\left(\frac{20}{0.6}\right)\times\left(12\times\frac{12}{2}\right) + 0 = -26379.6\times 10^{-5}\ \text{m}$$

$$f_{ij} = \int\frac{m_{zi}\, m_{zj}\, dx}{EI} + \int\frac{a_{xi}\, a_{xj}\, dx}{EA}$$

$$f_{11} = \int\frac{m_{z1}^2\, dx}{EI} + \int\frac{a_{x1}^2\, dx}{EA}$$
$$= \frac{6\times 12\times 6}{2EI} + 2\times\frac{6\times 6}{2EI}\times\frac{2}{3}\times 6 + \frac{1\times 12}{2EA} = 360.12\times 10^{-5}\ \text{m}$$

$$f_{22} = \int\frac{m_{z2}^2\, dx}{EI} + \int\frac{a_{x2}^2\, dx}{EA}$$
$$= \frac{12\times 12}{2\times 2EI}\times 8 + \frac{12\times 6\times 12}{EI} + \frac{1\times 6}{EA}\times 2 = 1152.24\times 10^{-5}\ \text{m}$$

$$f_{12} = f_{21} = \int\frac{m_{z1}\, m_{z2}\, dx}{EI} + \int\frac{a_{x1}\, a_{x2}\, dx}{EA}$$
$$= \frac{12\times 12}{4EI}\times 6 + \frac{6\times 6}{2EI}\times 12 + 0 = 432\times 10^{-5}\ \text{m}$$

Substituting into the consistent deformation equations, where $\Delta_1 = \Delta_2 = 0$, one obtains the following final forms:

$360.12\, x_1 + 432\, x_2 = 8856$
$432\, x_1 + 1152.24\, x_2 = +26379.6$

The solution of the above equations gives $x_1 = -5.219$ kN and $x_2 = 24.851$ kN.

The bending moments in the actual structure are obtained using the superposition principles as follows.

$$M_Z = M_{z0} + x_1\, m_{z1} + x_2\, m_{z2}$$
$$M_A = -396 + 12\times 24.851 = -97.788\ \text{kN.m}$$
$$M_B = -180 + 6\times(-5.219) + 24.851\times 12 = +86.886\ \text{kN.m}$$
$$M_C = 0 + 6\times(-5.219) = -31.314\ \text{kN.m}$$
$$M_D = 0 + 0 = 0$$

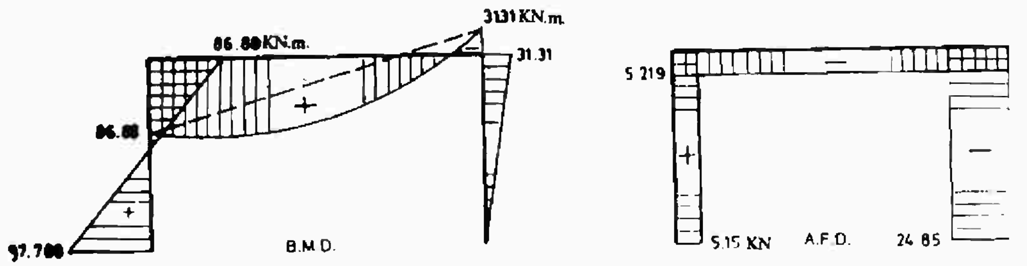


Figure 3.50

The axial force in each member can similarly be obtained. The bending moment and axial force diagrams are shown in Figure 3.50.

To determine the horizontal sway at joint B, one applies a unit horizontal load at B for an arbitrary primary structure, and determine m_{zB} and a_{xB} as shown in Figure 3.51.

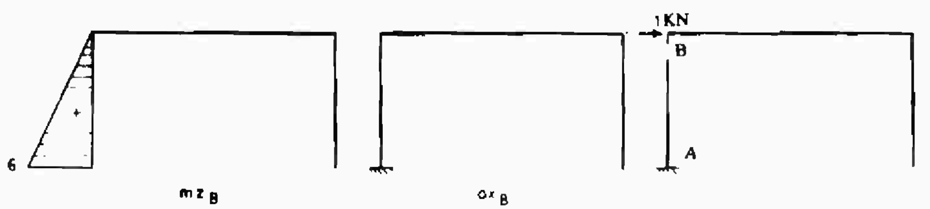


Figure 3.51

$$\Delta_B = \int \frac{M_z\, m_{zB}\, dx}{EI} + \int \frac{A_x\, a_{xB}\, dx}{EA}$$

$$= \frac{6}{6EI}\left[-6 \times (-97.788) \times 2 + 0 + 86.88 \times (-6)\right] + 0 = 652.176 \times 10^{-5}\ \text{m}$$

which means that joint B is translated in the horizontal right direction a value of 0.652 cm.

Example 3.13

Neglecting the axial deformations, determine the bending moment and shear force diagrams for the frame shown in Figure 3.52 due to the applied loads and a vertical settlement at B of 1 cm downward. ($EI = 10^5$ kN.m^2).

Solution

The degree of static indeterminacy is computed from

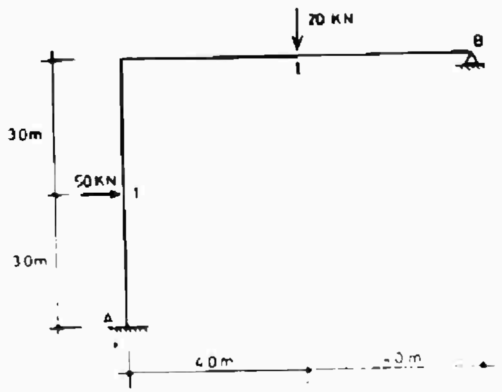


Figure 3.52

DSI = 3m + r − 3j = 3 × 2 + 5 − 3 × 3 = 2

Select the redundants to be the two reactions at B. The bending moment diagram for the primary structure, M_{z0}, is shown in Figure 3.53.

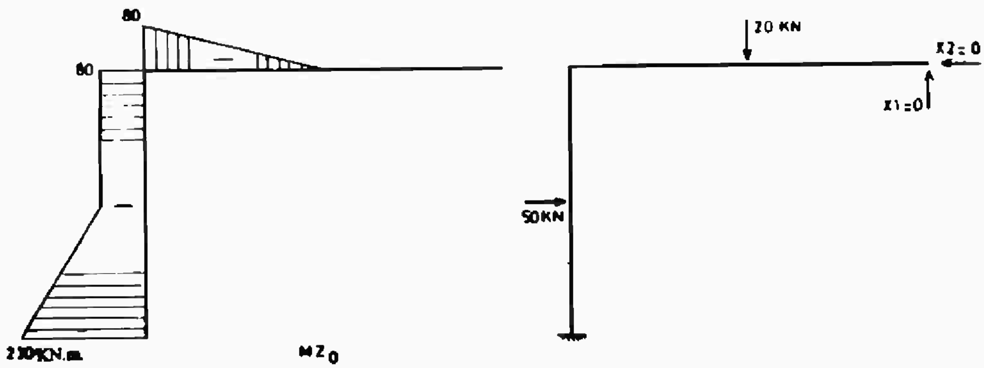


Figure 3.53

The equations of consistent deformations are

$$\Delta_{10} + f_{11}\, x_1 + f_{12}\, x_2 = \Delta_1$$
$$\Delta_{20} + f_{21}\, x_1 + f_{22}\, x_2 = \Delta_2$$

where Δ_1 = -0.01 m and Δ_2 = 0.

To determine the coefficients of these equations let x_1 and x_2 be unity in turn The bending moments m_{z1} and m_{z2} are shown in Figures 3.54 and 3.55 respectively.

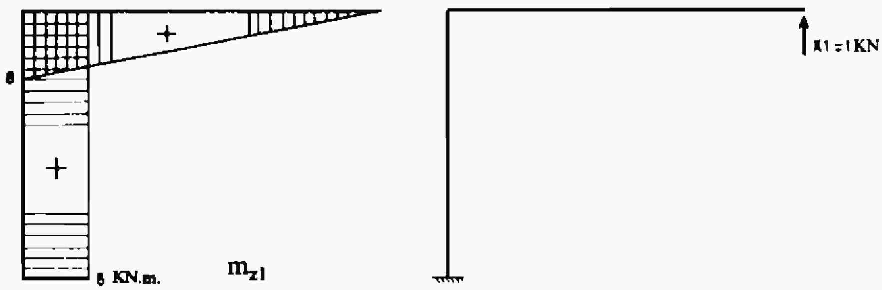


Figure 3.54

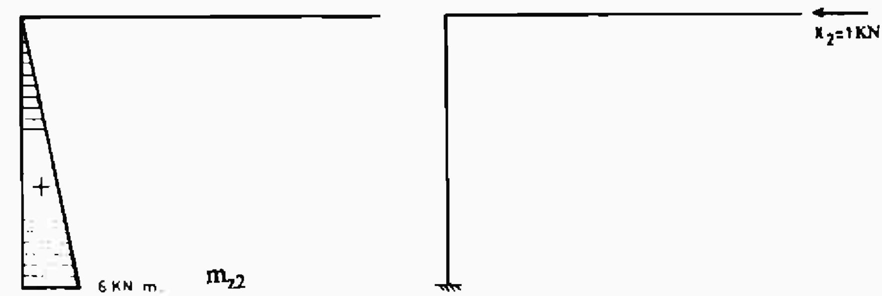


Figure 3.55

The coefficients Δ_{i0} and f_{ij} are calculated as follows:

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, dx}{EI}$$

$$= \frac{-4}{6EI}(8\times 80\times 2+80\times 4)+\frac{1}{EI}(-80\times 3\times 8)+\frac{-3}{EI}\times\left(\frac{80+230}{2}\right)\times 8=-6706.667\times 10^{-5}\ \text{m}$$

$$\Delta_{20} = \int \frac{M_{z0}\, m_{z2}\, dx}{EI}$$

$$= \frac{1}{EI}\left(-80\times\frac{3\times 3}{2}\right)+\frac{3}{6EI}(-80\times 3\times 2-230\times 6\times 2-80\times 6-3\times 230)=-2565\ \times 10^{-5}\ \text{m}$$

$$f_{11}=\frac{1}{EI}\left(\frac{8\times 8}{2}\times\frac{2}{3}\times 8+8\times 6\times 8\right)=554.667\times 10^{-5}\ \text{m}$$

$$f_{22}=\frac{6\times 6}{2EI}\times\frac{2}{3}\times\ 6=70\ \times 10^{-5}\ \text{m}$$

$$f_{12} = \frac{6 \times 6}{2EI} \times 8 = 144 \times 10^{-5} \text{ m}$$

Substituting into the consistent deformation equations one obtains

$$-6706.667 + 554.667\,x_1 + 144\,x_2 = -10^5 \times 0.01 = -1000$$
$$-2565 + 144\,x_1 + 72\,x_2 = 0$$

The solution of the above equations gives $x_1 = 2.1625$ kN and $x_2 = 31.3$ kN.

The final bending moment can be obtained either by the superposition principle as presented before, or by using statics equilibrium equations. The bending moment diagram is shown in Figure 3.56.

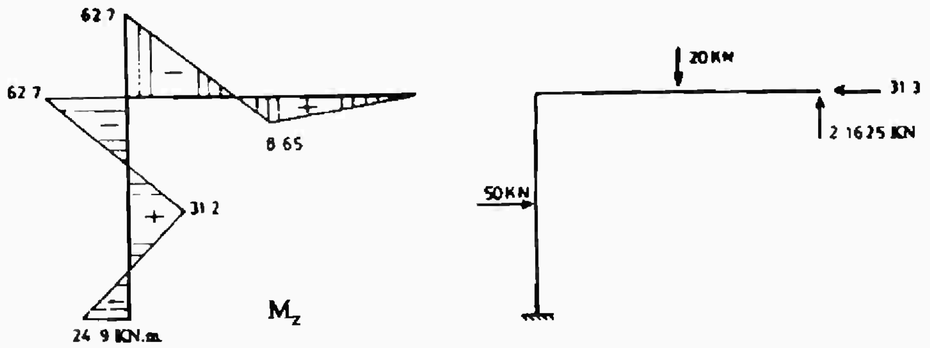


Figure 3.56

Example 3.14

Determine the bending moment diagram for the frame shown in Figure 3.57 due to a vertical settlement at D of 5 cm downward. Consider EI = 10^5 kN.m^2 and ignore axial force effect.

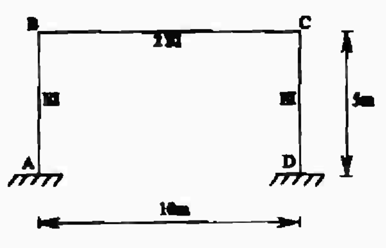


Figure 3.57

Solution

The frame is externally three times statically indeterminate. The general consistent deformation equations are

$$\Delta_{10} + x_1 f_{11} + x_2 f_{12} + x_3 f_{13} = \Delta_1$$
$$\Delta_{20} + x_1 f_{21} + x_2 f_{22} + x_3 f_{23} = \Delta_2$$
$$\Delta_{30} + x_1 f_{31} + x_2 f_{32} + x_3 f_{33} = \Delta_3$$

The primary structure, shown in Figure 3.58, is chosen such that the reactions at D are released. The moment diagrams M_{z0}, m_{z1}, m_{z2} and m_{z3} are constructed in order to determine the flexibility coefficients and deformation as follows:

$\Delta_{10} = 0$; $\Delta_{20} = 0$; $\Delta_{30} = 0$.

$$f_{11} = \int \frac{m_{z1}^2 \, d\ell}{EI} = \frac{5 \times 5}{2EI} \times \frac{2}{3} \times 5 \times 2 + \frac{5 \times 10 \times 5}{2EI} = 208.33 \times 10^{-5} \text{ m/kN}$$

$$f_{12} = \int \frac{m_{z1} \, m_{z2} \, d\ell}{EI} = \frac{10 \times 10}{2EI} \times \left(\frac{-5}{2} \right) - \frac{5 \times 5}{2EI} \times 10 = -250 \times 10^{-5} \text{ m/kN}$$

$$f_{22} = \int \frac{m_{z2}^2 \, d\ell}{EI} = \frac{10 \times 10}{2 \times 2EI} \times \frac{2}{3} \times 10 + \frac{10 \times 5}{EI} \times 10 = 666.67 \times 10^{-5} \text{ m/kN}$$

$$f_{33} = \int \frac{m_{z3}^2 \, d\ell}{EI} = \frac{5}{EI} + \frac{5}{EI} + \frac{10}{2EI} = 15 \times 10^{-5} \text{ rad/kN.m}$$

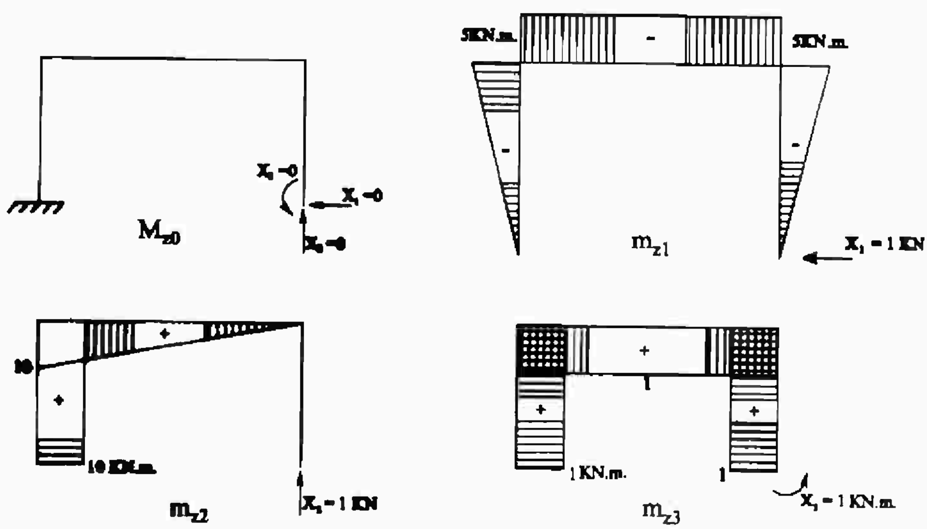


Figure 3.58

$$f_{23} = \frac{10 \times 10}{2 \times 2\,EI} + \frac{5 \times 10}{EI} = 75 \times 10^{-5} \text{ m/kN}$$

Substituting into the consistent deformation equations where $\Delta_1 = 0$, $\Delta_2 = -0.05$m, and $\Delta_3 = 0$, one obtains

$$208.33\,x_1 - 250\,x_2 - 50\,x_3 = 0$$
$$-250\,x_1 + 666.67\,x_2 + 75\,x_3 = -5000$$
$$-50\,x_1 + 75\,x_2 + 15\,x_3 = 0$$

Solving the above equations one obtains $x_1 = 0$, $x_2 = -17.14$ kN, and $x_3 = 85.71$ kN.m. The bending moment in the frame is obtained as follows:

$$M_Z = M_{z0} + x_1\,m_{z1} + x_2\,m_{z2} + x_3\,m_{z3}$$
$$M_A = 0 + 0 - 17.14 \times 10 + 1 \times 85.71 = -85.71 \text{ kN.m}$$
$$M_B = 0 + 0 - 17.14 \times 10 + 1 \times 85.71 = -85.71 \text{ kN.m}$$
$$M_C = 0 + 0 - 17.14 \times 0 + 1 \times 85.71 = 85.71 \text{ kN.m}$$
$$M_D = 0 + 0 - 17.14 \times 0 + 1 \times 85.71 = 85.71 \text{ kN.m}$$

The bending moment diagram is given in Figure 3.59.

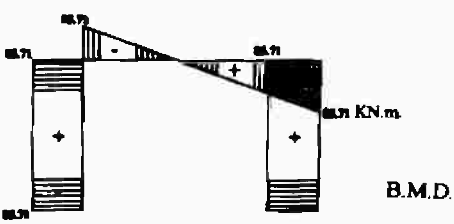


Figure 3.59

This example can be solved by considering the primary structure shown in Figure 3.60 in which the internal actions at the middle of member BC are used as redundants. In this case, the right hand side of the equations of consistent deformations represent the relative deformations at the location of the redundants. Therefore, $\Delta_1 = 0$, $\Delta_2 = -0.05$ m, and $\Delta_3 = 0$. The flexibility coefficients are obtained using m_{z1}, m_{z2}, and m_{z3} given in Figure 3.61 as follows:

$$f_{11} = \int \frac{m_{z1}^2\,d\ell}{EI} = \frac{5 \times 5}{2EI} \times \frac{2}{3} \times 5 \times 2 = 83.33 \times 10^{-5} \text{ m/kN}$$

$$f_{22} = \int \frac{m_{z2}^2\,d\ell}{EI} = \frac{5 \times 5}{2EI} \times 5 \times 2 + \frac{5 \times 5}{2} \times \frac{2}{3} \times \frac{5}{2EI} \times 2 = 291.67 \times 10^{-5} \text{ m/kN}$$

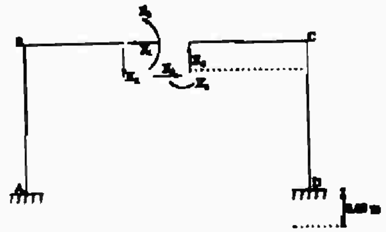


Figure 3.60

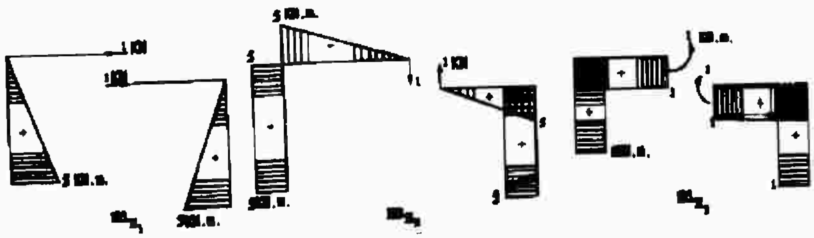


Figure 3.61

$$f_{12} = \int \frac{m_{x1}\, m_{x2}\, d\ell}{EI} = 0$$

$$f_{13} = \int \frac{m_{x1}\, m_{x3}\, d\ell}{EI} = \frac{5 \times 5}{2EI} \times 1 + \frac{5 \times 5}{2EI} \times 1 = 25 \times 10^{-5} \text{ rad/kN.m}$$

$$f_{23} = \int \frac{m_{x2}\, m_{x3}\, d\ell}{EI} = 0$$

$$f_{33} = \int \frac{m_{x3}^2\, d\ell}{EI} = \frac{1 \times 5}{EI} \times 2 + \frac{5 \times 1}{EI} \times 2 = 15 \times 10^{-5} \text{ rad/kN.m}$$

The equations of consistent deformation become

$83.33\, x_1 + 25\, x_3 = 0$
$291.67\, x_2 = -5000$
$25\, x_1 + 15\, x_3 = 0$

The solution of these equations is $x_1 = 0$, $x_3 = 0$, and $x_2 = 17.14$ kN. The final bending moment is obtained as follows:

$$M_x = M_{x0} + x_1\, m_{x1} + x_2\, m_{x2} + x_3\, m_{x3}$$
$$M_A = 0 + 0 + 17.14(-5) + 0 = -85.71 \text{ kN.m}$$

$M_B = 0 + 0 + 17.14(-5) + 0 = -85.71\,kN.m$
$M_C = 0 + 0 + 17.14(5) + 0 = 85.71\,kN.m$
$M_D = 0 + 0 + 17.14(5) + 0 = 85.71\,kN.m$

which are the same results as obtained before in Figure 3.59.

3.3.4 Applications to Trusses

Example 3.15

Determine the member forces in the truss shown in Figure 3.62 due to the applied loads and a vertical settlement at B of 1 cm downward (EA = 10^6 kN for all members).

Solution

The degree of static indeterminacy is computed from

DSI = m + r − 2j = 7 + 4 − 2 x 5 = 1
Number of external redundants = 4 − 3 = 1

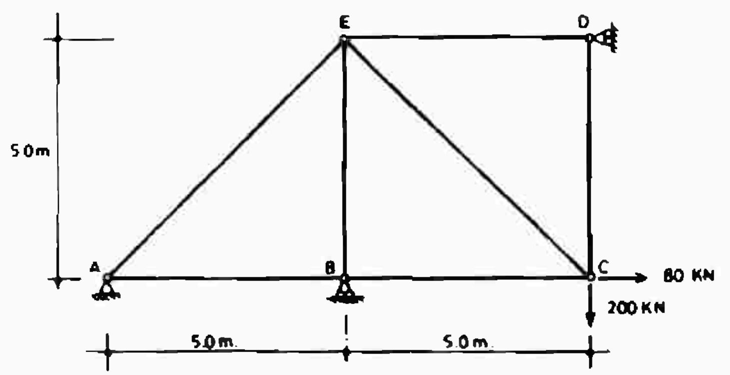


Figure 3.62

Select the released redundant to be at support B to form the primary structure shown in Figure 3.63. The released redundant is arbitrary but by this choice one saves much time in calculating the corresponding displacements due to the settlement at B. The reactions for the primary structure and the internal forces are also shown in Figure 3.63.

The equation of consistent deformation is

$$\Delta_{10} + f_{11}\,x_1 = \Delta_1 = -0.01$$

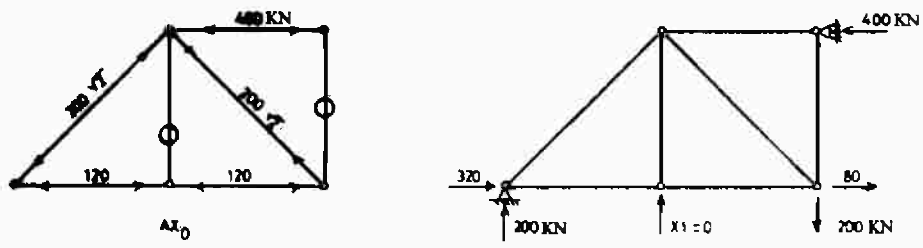


Figure 3.63

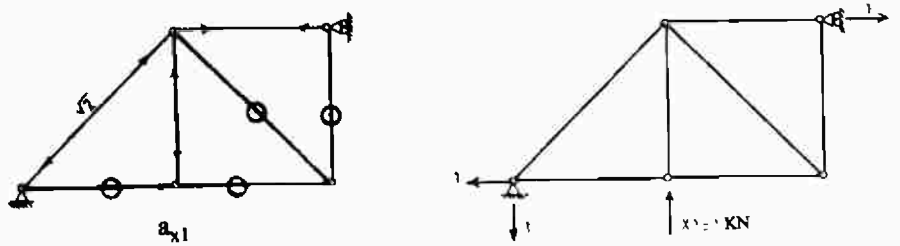


Figure 3.64

To determine Δ_{10} and f_{11} apply $x_1 = 1$ kN on the primary structure and determine a_{x1} as shown in Figure 3.64. The values of Δ_{10} and f_{11} are thus calculated as follows:

$$\Delta_{10} = \sum \frac{A_{x0}\, a_{x1}\, L}{EA}$$
$$= \frac{1}{EA}\left[(-400)(+1)\times 5 + 0 + 0 + 0 + 0 + \left(-200\sqrt{2}\right)\left(\sqrt{2}\right)\left(5\sqrt{2}\right)\right]$$
$$= -4828.43\times 10^{-6}\ \text{m}$$

$$f_{11} = \sum \frac{a_{x1}^2\, L}{EA}$$
$$= \frac{1}{EA}\left(1^2\times 5 + 1^2\times 5 + \left(\sqrt{2}\right)^2 \times \left(5\sqrt{2}\right)\right) = 24.1242\times 10^{-6}\ \text{m}$$

Substituting into the consistent deformation equation and solving for x_1 one obtains $x_1 = -214.215$ kN.

The member forces can thus be found using the superposition as follows:

$$A_x = A_{x0} + a_{x1}\, x_1$$

The values of the member forces are indicated in Figure 3.65.

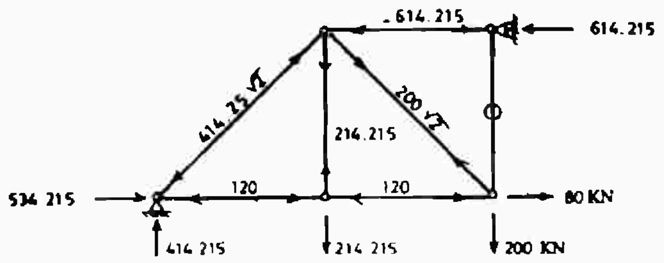


Figure 3.65

Example 3.16

Determine the member forces and the horizontal displacement of support C for the truss shown in Figure 3.66 due to the applied loads and a uniform rise in temperature for members ED and DC of 20°c. (EA = 1×10^6 kN for all members, α = 10-5/°c).

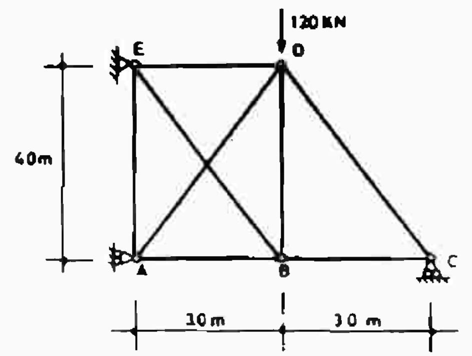


Figure 3.66

Solution

The degree of static indeterminacy is computed as follows:

DSI = $m + r - 2j = 8 + 4 - 2 \times 5 = 2$
Number of external redundants = $4 - 3 = 1$
Number of internal redundants = $2 - 1 = 1$

Select one redundant among the reactions and one redundant among the indeterminate members. The reactions and internal forces for the chosen primary structure are shown in Figure 3.67.

The equations of consistent deformations are

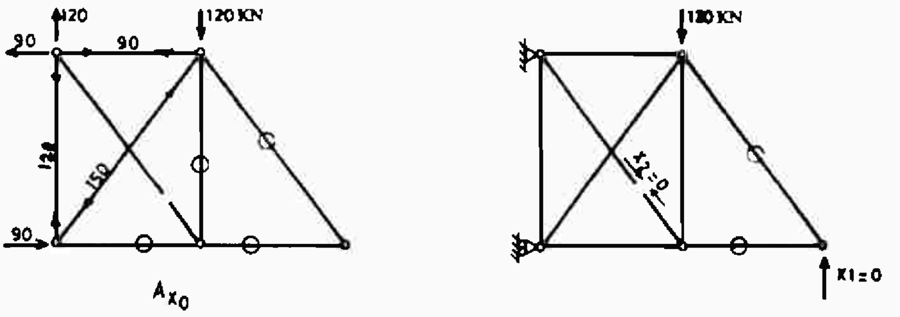


Figure 3.67

$\Delta_{10} + f_{11}\, x_1 + f_{12}\, x_2 = \Delta_1 = 0$

$\Delta_{20} + f_{21}\, x_1 + f_{22}\, x_2 = \Delta_2 = 0$

To determine the coefficients $\Delta_{10}, \Delta_{20}, f_{11}, f_{12}$, and f_{22} apply, in turn, a unit oad for x_1 and x_2 to determine the member forces a_{x1} and a_{x2} as shown in Figure 3.68.

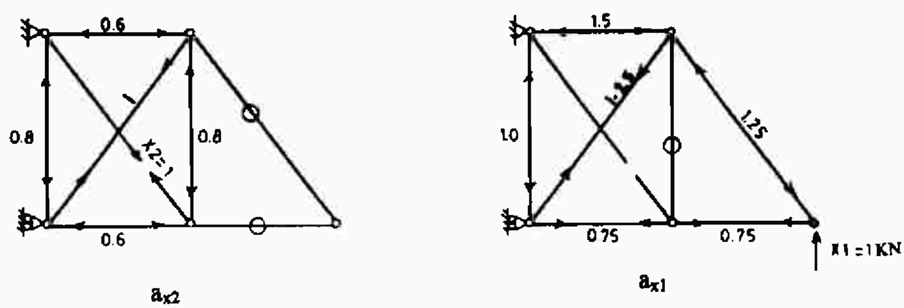


Figure 3.68

$$\Delta_{10} = \Sigma\, \frac{A_{x0}\, a_{x1}\, L}{EA} + \Sigma\, \alpha\,(T)\, a_{x1}\, L$$

$$= \frac{1}{EA}\left[+90\times(-1.5\times3) - 150\times1.25\times5 + 120\times4\times(-1)\right] + \alpha\ \times 20\times\ \left[-1.5\times3 - 1.25\times5\right]$$

$$= -306.125\times10^{-5}\ \text{m}$$

$$\Delta_{20} = \Sigma\, \frac{A_{x0}\, a_{x2}\, L}{EA} + \Sigma\, \alpha\,(T)\, a_{x2}\, L$$

$$= \frac{-150\times1\times5}{EA} + \frac{90\times(-0.6)\times3}{EA} + \frac{120\times(-0.8)\times4}{EA} + \alpha\times20\times\ \left[-0.6\times3\right]$$

$$= -100.8\times10^{-5}\ \text{m}$$

$$f_{11} = \Sigma \frac{a_{x1}^2 L}{EA}$$

$$= \frac{1}{EA}\left[1.25^2 \times 5 \times 2 + 0.75^2 \times 3 \times 2 + 1.5^2 \times 3 + 1^2 \times 4\right] = 1.4875 \times 10^{-5}\ \text{m}$$

Notice here that in calculating f_{22}, member, EB which is subjected to x_2 = 1 kN is considered in the summation as follows:

$$f_{22} = \Sigma \frac{a_{x2}^2 L}{EA}$$

$$= \frac{1}{EA}\left[0.6^2 \times 3 \times 2 + 0.8^2 \times 4 \times 2 + 1^2 \times 5 \times 2\right] = 0.864 \times 10^{-5}\ \text{m}$$

$$f_{12} = \Sigma \frac{a_{x1}\, a_{x2}\, L}{EA}$$

$$= \frac{1}{EA}\left[-0.6 \times (-1.5) \times 3 + 0.8 \times 4 \times 1 - 0.6 \times 0.75 \times 3 + 1.25 \times 1 \times 5\right] = 0.54 \times 10^{-5}\ \text{m}$$

Substituting into the compatibility equations one has

$-306.125 + 1.4875\, x_1 + 0.54\, x_2 = 0$

$-100.8 + 0.54\, x_1 + 0.864\, x_2 = 0$

The solution is obtained as x_1 = 211.42 kN and x_2 = -15.49 kN.

The member forces are obtained by the superposition principle using the relation $A_X = A_{X0} + a_{x1}\, x_1 + a_{x2}\, x_2$, and the results are shown in Figure 3.69. To determine the horizontal displacement at C apply a unit horizontal load at C for the primary structure, as shown in Figure 3.70, and obtain a_{xc}.

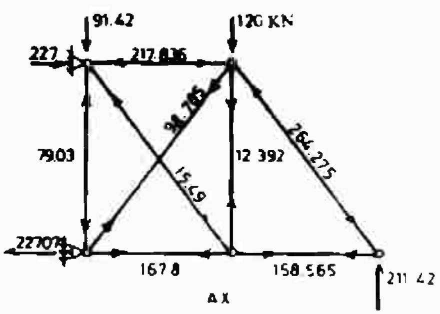


Figure 3.69

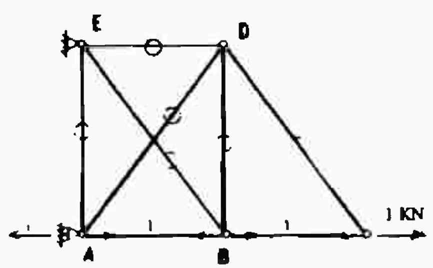


Figure 3.70

$$\Delta_C = \Sigma \frac{a_{xc}\, A_x\, L}{EA}$$
$$= \frac{(167.8 + 158.565) \times 3}{2 \times 10^6} = 0.0489 \text{ cm}$$

which means that the displacement of C is in the same direction as the unit load.

3.3.5 Applications to Frame-Truss Structures

Example 3.17

Draw the bending moment and axial force diagrams for the structure shown in Figure 3.71 due to:

(a) a vertical load of 40 kN at joint H.

(b) a rise in temperature of 20°C for members CD and DE. (Consider EI = 20000 kN.m^2, EA = 20000 kN, a = 10^{-5}/°c).

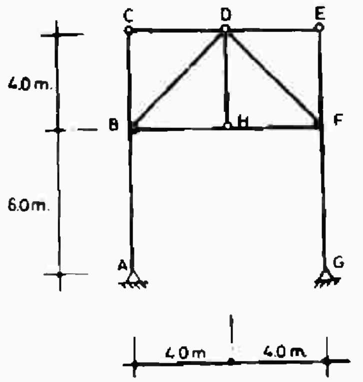


Figure 3.71

Solution

Considering the structure as a plane frame, the degree of static indeterminacy is calculated as follows:

m = 11, J = 8, r = 4, and n = 6 + 6 = 12
DSI = 3m + r – 3J – n = 3 × 11 + 4 – 3 × 8 – 12 = 1
Number of external redundants = r – 3 – $n_{ext.}$ = 4 – 3 – 0 = 1

Select a primary structure as shown in Figure 3.72. The M_{xo} and A_{xo} diagrams are then determined.

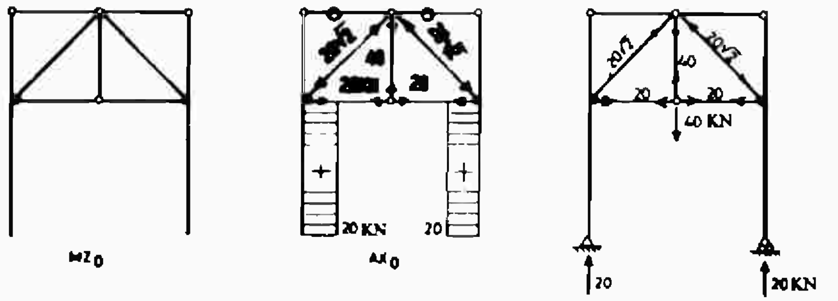


Figure 3.72

Part a

The diagrams for M_{zo} and A_{xo} are shown in the Figure 3.72.

The equation of consistent deformation is

$$\Delta_{10} + f_{11}\, x_1 = \Delta_1 = 0$$

To determine Δ_{10} and f_{11} apply a horizontal unit load at G and get a_{x1} and m_{z1} as shown in Figure 3.73.

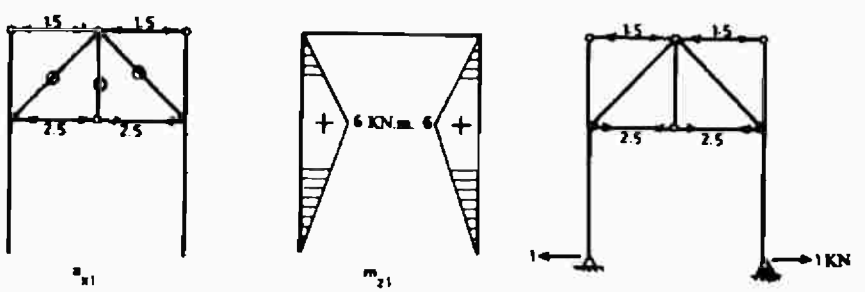


Figure 3.73

The coefficients Δ_{10} and f_{11} are calculated as follows:

$$\Delta_{10} = \int \frac{M_{zo}\, m_{z1}\, dx}{EI} + \int \frac{A_{xo}\, a_{x1}\, dx}{EA}$$

$$= 0.0 + \frac{1}{EA}\left[20 \times 2.5 \times 4 \times 2\right] = \frac{400}{EA} = 200 \times 10^{-4} \text{ m}$$

$$f_{11} = \int \frac{m_{z1}^2\,dx}{EI} + \int \frac{a_{x1}^2\,dx}{EA}$$

$$= \frac{1}{EI}\left(\frac{6\times6}{2}\times\frac{2}{3}\times6\times2+\frac{6\times4}{2}\times\frac{2}{3}\times6\times2\right)+\frac{1}{EI}\left[2.5^2\times4\times2+1.5^2\times4\times2\right]$$

$$= \frac{240}{EI}+\frac{68}{EA} = 154\times10^{-4}\ \text{m}$$

The value of x_1 is found to be x_1 = -1.2987 kN. The final internal forces are calculated using the superposition principle. The axial force and bending moment are shown in Figure 3.74.

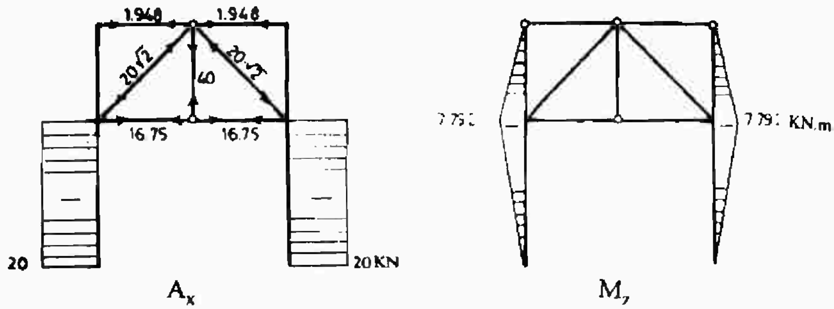


Figure 3.74

Part (b)

The deflection Δ_{10} due to temperature change is calculated as follows:

$$\Delta_{10} = \int \alpha\, T\, a_{x1}\, dx$$

$$= 20\alpha\left[-1.5\times4\times2\right] = -240\times10^{-5}\ \text{m}$$

Substituting into the consistent deformation equation one gets x_1 = 0.1558 kN. The final bending moment and axial force diagrams are shown in Figure 3.75.

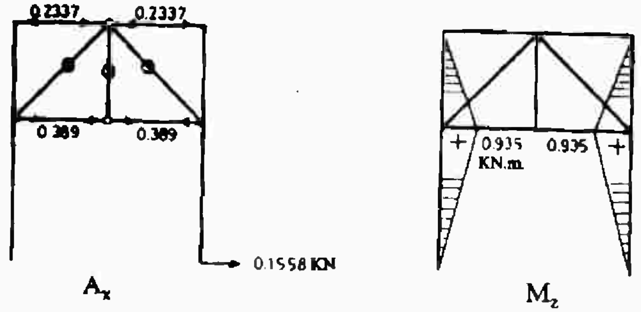


Figure 3.75

Example 3.18

Determine the bending moment and axial force diagrams for the structure shown in Figure 3.76 and determine the horizontal displacement of support C due to the applied loads. Then solve the problem without the cable AC and compare the results. (EI = 10^5 kN.m^2, EA = 0.5 × 10^5 kN).

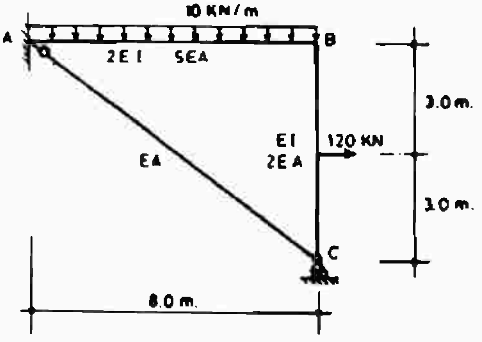


Figure 3.76

Solution

The degree of static indeterminacy is

DSI = 3 m + 4 − 3J − n
= 3 × 3 + 4 − 3 × 3 − 2 = 2
Number of external redundants = 4- 3 = 1
Number of internal redundants = 2 − 1 = 1

Select a primary structure as the one shown in Figure 3.77 and determine A_{xo} and M_{zo}.

The equations of consistent deformations are

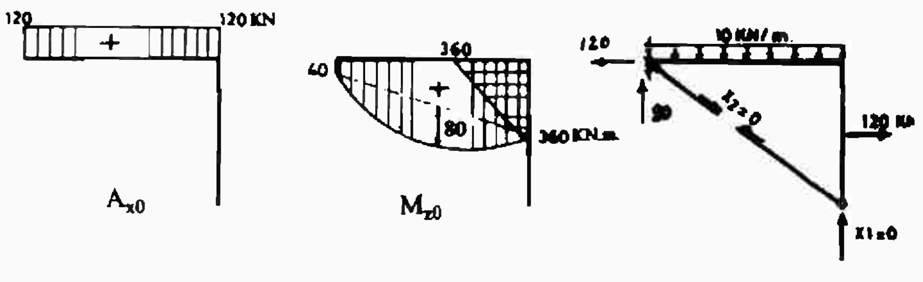


Figure 3.77

$$\Delta_{10} + f_{11} X_1 + f_{12} X_2 = \Delta_1 = 0$$
$$\Delta_{20} + f_{21} X_1 + f_{22} X_2 = \Delta_2 = 0$$

The coefficients $\Delta_{10}, \Delta_{20}, f_{11}, f_{12}$, and f_{22} are obtained by constructing the diagrams m_{z1}, a_{x1}, m_{z2}, and a_{x2} which are shown in Figures 3.78 and 3.79.

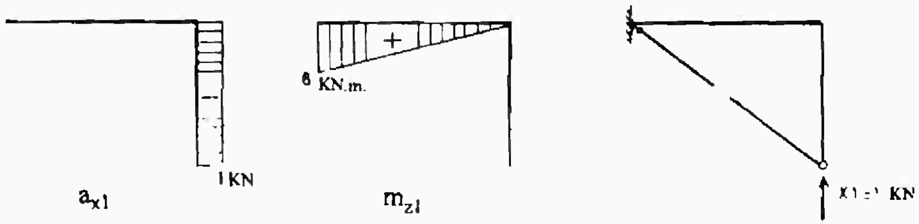


Figure 3.78

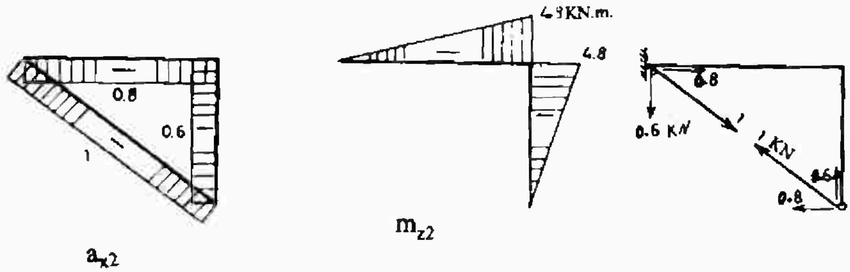


Figure 3.79

The coefficients Δ_{io} and f_{ij} are found as follows

$$\Delta_{10} = \int \frac{M_{zo}\, m_{z1}\, dx}{EI} + \int \frac{A_{xo}\, a_{x1}\, dx}{EA}$$
$$= \frac{1}{2EI}\left[\frac{2}{3} \times 80 \times 8 \times 4 + \frac{40 \times 8}{2} \times \frac{2}{3} \times 8 + \frac{360 \times 8}{2} \times \frac{8}{3}\right] + 0 = 3200 \times 10^{-5} \text{ m}$$

$$\Delta_{20} = \int \frac{M_{zo}\, m_{z2}\, dx}{EI} + \int \frac{A_{xo}\, a_{x2}\, dx}{EA}$$
$$= \frac{3}{6EI}[-4.8 \times 360 \times 2 - 2.4 \times 360] + \frac{1}{5EA}[120 \times (-0.8) \times 8]$$
$$+ \frac{1}{2EI}\left[\frac{2}{3} \times 80 \times 8 \times (-2.4) + \frac{40 \times 8}{2} \times \frac{(-4.8)}{3} + \frac{360 \times 8}{2} \times \frac{2}{3} \times (-4.8)\right] + 0 = -5411.2 \times 10^{-5} \text{ m}$$

$$f_{11} = \int \frac{m_{x1}^2\, dx}{EI} + \int \frac{a_{x1}^2\, dx}{EA}$$

$$= \frac{1}{2EI}\left(8 \times \frac{8}{2} \times \frac{2}{3} \times 8\right) + \frac{1}{2EA}(1 \times 1 \times 6) = 91.33 \times 10^{-5} \text{ m}$$

$$f_{12} = \int \frac{m_{z1}\, m_{z2}\, dx}{EI} + \int \frac{A_{x1}\, a_{x2}\, dx}{EA}$$

$$= \frac{1}{2EI}\left[\frac{8 \times 8}{2} \times \frac{(-4.8)}{3}\right] + \frac{1}{2EA}(0.6 \times 1 \times 6) = -22 \times 10^{-5} \text{ m}$$

$$f_{22} = \int \frac{m_{z2}^2\, dx}{EI} + \int \frac{a_{x2}^2\, dx}{EA}$$

$$= \frac{1}{2EI}\left[\frac{4.8 \times 8}{2} \times \frac{2}{3} \times 4.8\right] + \frac{1}{EI}\left[\frac{4.8 \times 6}{2} \times \frac{2}{3} \times 4.8\right] + \frac{1}{5EA}\left[(-0.8)^2 \times 8\right]$$

$$+ \frac{1}{2EA}\left[(-0.6)^2 \times 6\right] + \frac{1}{EA}\left[1^2 \times 10\right] = 101.008\ 10^{-5} \text{ m}$$

Substituting into the consistent deformation equations one obtains

$$3200 + 91.33x_1 - 22x_2 = 0$$
$$-5411.2 - 22x_1 + 101.008x_2 = 0$$

The corresponding solution is $x_1 = -23.358$ kN and $x_2 = 48.48$ kN.

The final bending moment and axial forces are determined using the superposition principle as follows:

$M_A = 40 - 8 \times 23.358 + 0 = -146.86$ kN.m
$M_B = 360 - 4.8 \times 48.48 = 127.296$ kN.m
$M_c = 0$
$A_{xa} = +120 - 0.8 \times 48.48 = 81.216$ kN
$A_{xb} = 0 - 1 \times (-23.358) - 0.6 \times 48.48 = -5.73$ kN

The bending moment and axial force diagrams are shown in Figure 3.80.

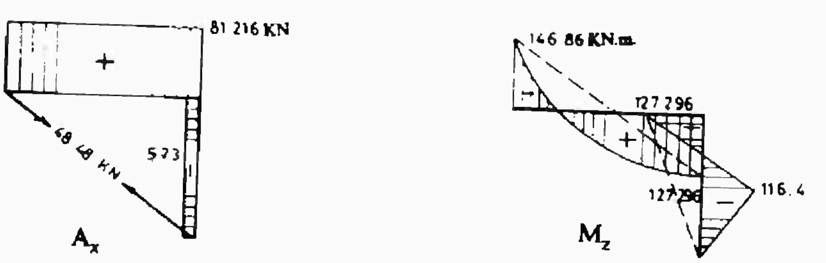


Figure 3.80

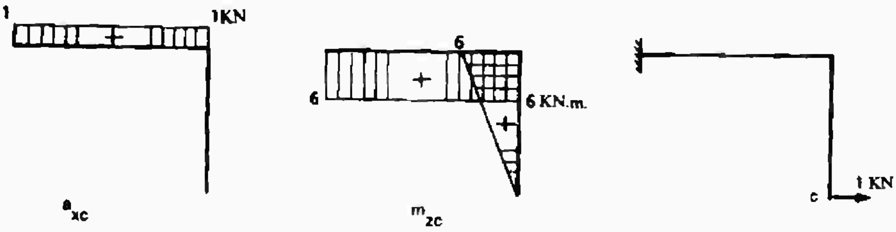


Figure 3.81

To determine Δ_c, apply a unit horizontal load at c to determine m_{zc} and a_{xc} as shown in Figure 3.81. The deflection at C is obtained using the unit method. In this case, the moment diagram of Figure 3.80 can be decomposed, as shown in Figure 3.80', for easy integration, as follows:

$$\Delta_c = \int \frac{M_z\, m_{zc}\, dx}{EI} + \int \frac{A_x\, a_{xc}\, dx}{EA}$$

$$= \frac{1}{2EI}\left[\frac{2}{3}\times\frac{10\times 8^2\times 8\times 6}{8} + \left(\frac{-146.86+127.296}{2}\right)\times 8\times 6\right]$$

$$+\frac{1}{EI}\left[\frac{127.296\times 6}{2}\times 4 - \frac{120\times 6}{4}\times\frac{6}{2}\times 3\right] + \frac{1}{5EA}\left[81.216\times 8\times 1\right] = 0.01212 \text{ m}$$

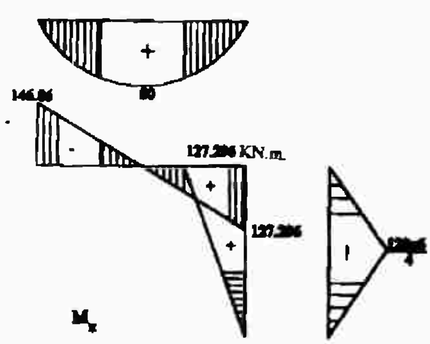


Figure 3.80'

Solving this problem again without the cable AC, we find that the structure is one degree statically indeterminate. The coefficients of the consistent deformation equation are as computed before, where

$\Delta_{10} = 3200\times 10^{-5}$ m ; $f_{11} = 91.33\times 10^{-5}$ m

The solution of the consistent deformation equation gives x_1 = -35.038 kN. The bending moment and axial force diagrams are shown in Figure 3.82. It is obvious that the moment and axial forces are larger in the frame without the cable. The displacement Δ_c is calculated as follows:

$$\Delta_c = \frac{1}{5EA} \times 120 \times 8 \times 1 + \frac{1}{2EI}\left[\left(\frac{-170.228 + 360}{2}\right) \times 8 \times 6 + \frac{2}{3} \times 10 \times \frac{8^2}{8} \times 6\right]$$

$$+ \frac{3}{2EI}[2 \times 6 \times 360 + 360 \times 3] = 0.10921 \text{ m}$$

which is very large as compared with the displacement when using the bracing cable Ac.

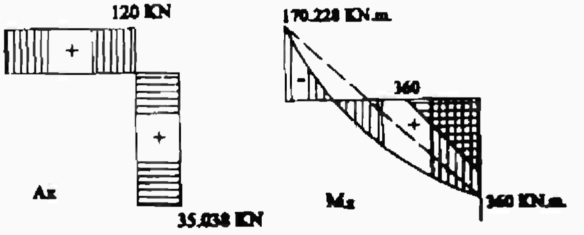


Figure 3.82

3.4 THE USE OF CASTIGLIANO'S SECOND THEOREM

3.4.1 The Basic Formulations

It was shown in section 2.21 that Castigliano's second theorem is the basic tool for the force (flexibility) method. This theorem is stated again as "the partial derivative of strain energy with respect to a certain action gives the displacement in the direction of this action". The action could be a real load or a virtual load. The value of the load is substituted after taking the derivative of the strain energy with respect to specific action.

The theorem could be used in solving statically indeterminate structure if one can formulate the strain energy in terms of the redundants. The partial derivative of the strain energy, (in general, called the complementary strain energy) with respect to each redundant gives the deformation in the direction of the redundant. By applying the boundary conditions for the deformation at the released redundants, one obtains a set of linear simultaneous equations in terms of the unknown redundants, which can be solved.

Consider, for example, the two times statically indeterminate frame shown in Figure 3.83. Selecting x_1 and x_2 as the redundants, one can find the strain energy for the frame in terms of P_1, P_2, x_1, and x_2 because the internal actions in the members

(A_x, A_y, and M_z) are functions of these actions. The strain energy is obtained from the formula

$$U = \sum_{k=1}^{3} \left(\int_0^{L_k} \frac{A_x^2 \, dx}{2EA} + \int_0^{L_k} \frac{A_y^2 \, dx}{2GA_r} + \int_0^{L_k} \frac{M_z^2 \, dx}{2EI} \right)$$
$$= U(p_1, p_2, x_1, x_2) \qquad (3.26)$$

in which k represents k^{th} member in the frame.

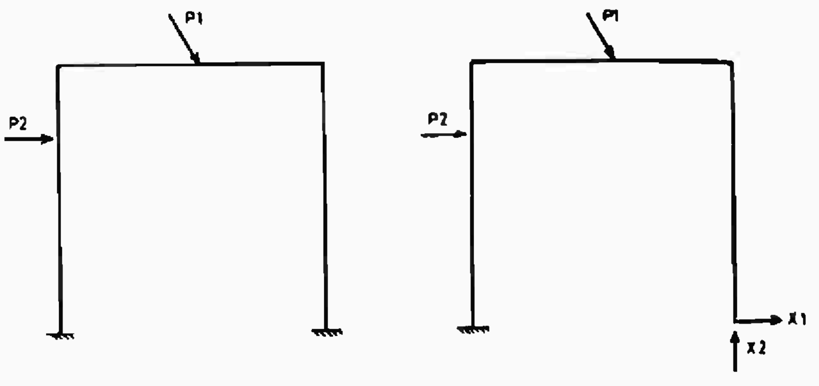


Figure 3.83

By applying Castigliano's second theorem one obtains

$$\Delta_1 = \frac{\partial U}{\partial x_1} = f_1(P_1, P_2, x_1, x_2) \qquad (3.27)$$

$$\Delta_2 = \frac{\partial U}{\partial x_2} = f_2(P_1, P_2, x_1, x_2) \qquad (3.28)$$

in which Δ_i is the deformation in direction of x_i, and f_1, f_2 are symbols for functions.

By specifying the boundary conditions for Δ_1 and Δ_2 at the redundants locations, Equation 3.27 and 3.28 are solved to obtain the values of the redundants x_1 and x_2. For example, if the hinged support in Figure 3.83 is not subjected to any movements, then Δ_1 and Δ_2 are equal to zero. If there are settlements in the hinged support, one has to substitute the values of these settlements for Δ_1 and Δ_2 in Equations 3.27 and 3.28, respectively. By finding the redundants x_1 and x_2, the structure becomes statically determinate and the internal actions can easily be determined in all members.

In order to determine the deformation at any point c, one may use the approach of the unit load method after determining the final internal actions in the structure. One can also use Castigliano's second theorem by applying a virtual load at point c and write the strain energy in terms of this virtual load.

3.4.2 Application to Beams

Example 3.19

Solve Example 3.11 using Castigliano's second theorem.

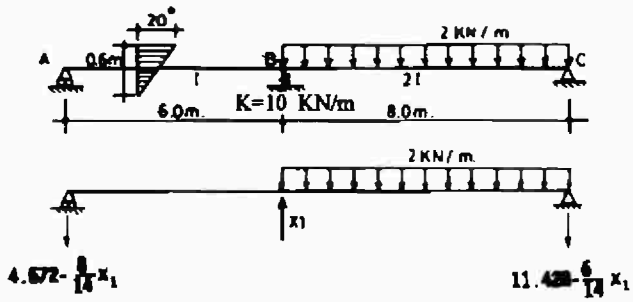


Figure 3.84

Solution

Using the reaction at B as a redundant denoted as x_1, the reactions at A and C are determined in terms of the redundant as shown in Figure 3.84. The bending moments at any section are defined as follows:

For member BC,

$$M_1(\zeta) = \left(\frac{-6}{14}x_1 + 11.428\right)\zeta - 2\times\frac{\zeta^2}{2} \quad ; \quad \text{for } 0 \le \xi \le 8$$

For member AB,

$$M_2(\eta) = \left(4.572 - \frac{8}{14}x_1\right)\eta \quad ; \quad \text{for } 0 \le \eta \le 6$$

The strain energy in the beam is given by

$$U = \int_0^8 \frac{M_1^2(\zeta)\,d\zeta}{4EI} + \int_0^6 \frac{M_2^2(\eta)\,d\eta}{2EI} + \frac{1}{2}\times\frac{1}{K}\times x_1^2 - \alpha\left(\frac{20}{0.6}\right)\int_0^6 M_2(\eta)\,d\eta$$

Using Castigliano's second theorem, one has

$$\frac{\partial U}{\partial x_1}=\int_0^8 \frac{M_1(\zeta)}{2EI}\left(\frac{-6}{14}\zeta\right)d\zeta+\int_0^6 \frac{M_2(\eta)}{EI}\left(\frac{-8}{14}\eta\right)d\eta+\frac{1}{K}x_1-\alpha\left(\frac{20}{0.6}\right)\int_0^6\left(\frac{-8\eta}{14}\right)d\eta$$

$$=\int_0^8 \frac{-3\zeta}{14EI}\left[11.428\zeta-\frac{6}{14}x_1\zeta-\zeta^2\right]d\zeta+\int_0^6 \frac{-4\eta}{7EI}\left[4.572\eta-\frac{4x_1\eta}{7}\right]d\eta$$

$$+\frac{1}{k}x_1-\alpha\left(\frac{20}{0.6}\right)\int_0^6\left(-\frac{8}{14}\eta\right)d\eta$$

$$=\frac{-3}{14EI}\left[\frac{11.428\zeta^3}{3}-\frac{6x_1\zeta^3}{14\times3}-\frac{\zeta^4}{4}\right]\Bigg|_0^8-\frac{4}{7EI}\left[\frac{4.572\eta^3}{3}-\frac{4x_1\zeta^3}{7\times3}\right]\Bigg|_0^6$$

$$+\frac{1}{K}x_1-\alpha\left(\frac{20}{0.6}\right)\left[\frac{-8}{14}\times\frac{\eta^2}{2}\right]\Bigg|_0^6$$

$$=10^{-5}\left[-417.938+15.675x_1+219.428-188.105+23.51x_1+342.833\right]+\frac{1}{1000}x_1=0$$

which leads to $139.185\,x_1 - 43.78 - 0$ from which $x_1 = 0.315$ kN as was determined in example 3.11.

3.4.3 Application to Frames

Example 3.20

Determine the bending moment diagram for the frame shown in Figure 3.85 (as Example 3.12) due to the loads, and a rise in temperature in member BC. Determine also the horizontal sway of joint B. ($EI = 10^5$ kN.m^2, $EA = 50 \times 10^5$ kN, $\alpha = 10^{-5}/°c$).

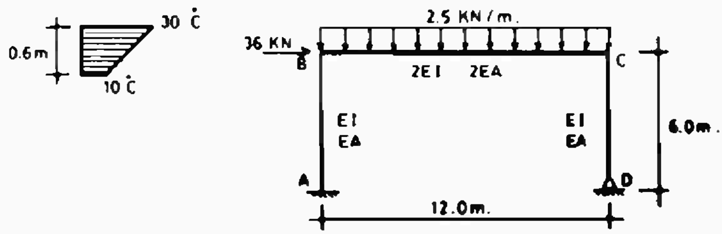


Figure 3.85

Solution

The frame is two degree statically indeterminate as was determined before in example 3.12. The redundants x_1 and x_2 are taken at D as shown in Figure 3.86. The internal forces in the frame as functions of the redundants x_1 and x_2 are determined as follows:

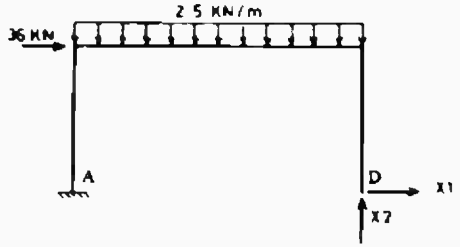


Figure 3.86

For member CD where $0 \le x \le 6$,

$$M_z = x \times x_1 \quad , \frac{\partial M_z}{\partial x_1} = x \quad , \frac{\partial M_z}{\partial x_2} = 0$$

$$A_x = -x_2 \quad , \frac{\partial A_x}{\partial x_1} = 0 \quad , \frac{\partial A_x}{\partial x_2} = -1$$

For member BC where $0 \le x \le 12$,

$$M_z = 6x_1 + x \times x_2 - \frac{2.5x^2}{2} \quad , \frac{\partial M_z}{\partial x_1} = 6 \quad , \frac{\partial M_z}{\partial x_2} = x$$

$$A_x = +x_1 \quad , \frac{\partial A_x}{\partial x_1} = 1 \quad , \frac{\partial A_x}{\partial x_2} = 0$$

For member AB where $0 \le x \le 6$,

$$M_z = 12x_2 + (6-x) \times x_1 - \frac{2.5 \times 12^2}{2} - 36x \quad , \frac{\partial M_z}{\partial x_1} = 6 - x \quad , \frac{\partial M_z}{\partial x_2} = 12$$

$$A_x = -30 + x_2 \quad , \frac{\partial A_x}{\partial x_1} = 0 \quad , \frac{\partial A_x}{\partial x_2} = 1$$

The strain energy is obtained by substituting as follows:

$$U = \left[\int_0^6 \frac{M_z^2\,dx}{2EI} + \int_0^6 \frac{A_x^2\,dx}{2EA} \right]_{CD} + \left[\int_0^{12} \frac{M_z^2\,dx}{4EI} + \int_0^{12} \frac{A_x^2\,dx}{4EA} \right]_{BC} + \left[\int_0^6 \frac{M_z^2\,dx}{2EI} + \int_0^6 \frac{A_x^2\,dx}{2EA} \right]_{AB}$$

$$+ \left[20\alpha \int_0^{12} x_1\,dx - \alpha \times \frac{20}{0.6} \times \int_0^{12} \left(6x_1 + x \times x_2 - \frac{2.5x^2}{2} \right) dx \right]_{BC}$$

Using Castigliano's second theorem one obtains

$$\frac{\partial U}{\partial x_1}=\int_0^6\frac{x^2 x_1}{EI}dx+\int_0^{12}\frac{6}{2EI}\left(6x_1+x\,x_2-\frac{2.5\,x^2}{2}\right)dx+\int_0^{12}\frac{x_1\,dx}{2EA}$$

$$+\int_0^6\frac{(6-x)}{EI}\left(12x_2+6x_1-x\,x_1-\frac{2.5\times12^2}{2}-36x\right)dx+0+240\alpha-2400\,\alpha$$

$$=\left[\frac{x^3 x_1}{3EI}\right]_0^6+\frac{1}{EI}\left[18x\,x_1+\frac{3}{2}x^2\,x_2-\frac{7.5\,x^3}{2\times3}\right]_0^{12}+\frac{6\,x_1}{EA}$$

$$+\frac{6}{EI}\left[12x_2\,x+6x_1\,x-\frac{x^2\,x_1}{2}-\frac{2.5\times12^2\,x}{2}-36\times\frac{x^2}{2}\right]_0^6$$

$$-\frac{1}{EI}\left[12x_2\frac{x^2}{2}+6x_1\times\frac{x^2}{2}-\frac{x^3\,x_1}{3}-\frac{2.5\times12^2\,x^2}{4}-36\times\frac{x^3}{3}\right]_0^6-2160\alpha$$

$$=\frac{72\,x_1}{EI}+\frac{1}{EI}\left[216\,x_1+216x_2-2160\right]+\frac{6x_1}{EA}$$

$$+\frac{1}{EI}[432x_2+216x_1-108x_1-6480-3888]$$

$$-\frac{1}{EI}[216x_2+108x_1-72x_1-3240-2592]-2160\alpha$$

$$=360.12x_1+432x_2-8856=0 \qquad \text{(a)}$$

$$\frac{\partial U}{\partial x_2}=+\left[\int_0^6\frac{x_2}{EA}dx\right]_{CD}+\left[\int_0^{12}\frac{x}{2EI}\left(6x_1+x\,x_2-\frac{2.5x^2}{2}\right)dx\right]_{BC}$$

$$+\left[\int_0^6\frac{12}{EI}\left(12x_2+6x_1-x\,x_1-2.5\times\frac{12^2}{2}-36x\right)dx+\int_0^6\frac{(-30+x_2)}{EA}dx\right]_{AB}$$

$$+\left[-\frac{20}{0.6}\alpha\int_0^{12}x\,dx\right]_{BC}$$

$$=\frac{6x_2}{EA}+\frac{1}{EA}[215x_1+288x_2-3240]+\frac{12}{EI}[72x_2+36x_1-18x_1-1080-648]$$

$$+\frac{6x_2}{EA}-\frac{180}{EA}-2400\alpha$$

$$=432x_1+1152.24x_2-26379.6=0 \qquad \text{(b)}$$

Notice that the two equations (a) and (b) are the same as the equations obtained by the consistent deformation method in Example 3.12. Solving equations (a) and (b) one obtains the same results obtained previously which are x_1 = -5.219 kN and x_2 = 24.851 kN.

To determine the horizontal displacement at joint B, one applies a virtual horizontal load P at B as shown in Figure 3.87. By using Castigliano's theorem, we need, out of the strain energy terms, only the terms which are function of the load P. It is obvious from Figure 3.87 that only the moment in member AB is function of P. Therefore, the strain energy expression can be written as follows:

$$U = \left[\int_0^6 \frac{M_z^2\,dx}{2EI}\right]_{AB} + \text{terms not function of P}$$

where

$$M_z(x) = 24.85 \times 12 - 2.5 \times 12 \times 6 - 5.219 \times (6 - x) - (P + 36)\,x\,.$$

The substitution into Castigliano's second theorem gives $\Delta_B = \partial U/\partial P$ where P is zero.

$$\Delta = \int_0^6 -x\left(86.886 + 5.219x - Px - 36x\right)dx$$

$$= \left[-86.886 \times \frac{x^2}{2} - 5.219 \times \frac{x^3}{3} + 36 \times \frac{x^3}{3}\right]_0^6 = 652.20 \times 10^{-5}\ \text{m}$$

which was obtained by substituting P = 0 for the virtual load. This result is the same as in Example 3.12.

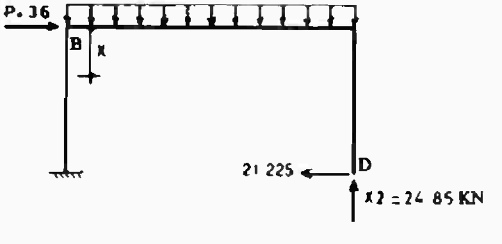


Figure 3.87

3.4.4 Application to Trusses

Example 3.21

Determine the member forces and the horizontal deflection at support C for the truss shown in Figure 3.88 (Example 3.15) due to the applied loads and a uniform rise in temperature for members ED and DC of 20°c. (EA = 2 × 10^6 kN, α = 10^{-5}/°c).

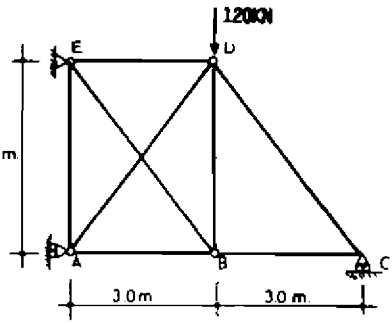


Figure 3.88

Solution

The truss is two degree statically indeterminate, one external and one internal redundants, as was shown in example 3.15.

Select the redundants to be x_1 and x_2 as shown in Figure 3.89. The members forces are determined as function of x_1 and x_2. The strain energy function is then obtained as follows:

$$U=\sum \frac{A_x^2 L}{2EA}$$

$$=\left[(1.25x_1)^2\times\frac{5}{2EA}\right]_{CD}+\left[(0.75x_1)^2\times\frac{3}{2EA}\right]_{BC}+\left[(0.8x_2)^2\times\frac{4}{EA}\right]_{BD}+\left[\frac{x_2^2\times 5}{2EA}\right]_{EB}$$

$$+\left[\frac{(150-x_2-1.25x_1)^2\times 5}{2EA}\right]_{AD}+\left[\frac{(90-0.6x_2-1.5x_1)^2\times 3}{2EA}\right]_{ED}+\left[\frac{(0.75x_1-0.6x_2)^2\times 3}{2EA}\right]$$

$$+\left[\frac{(120-0.8x_2-x_1)^2}{2EA}\right]_{AE}+\left[20\alpha\times 3\times(90-0.6x_2-1.5x_1)\right]_{ED}+\left[20\alpha\times 5\times(-1.25x_1)\right]_{DC}$$

in which the letters shown at the end of each term indicates the contribution of that member.

Using Castigliano's second theorem one has

$$\frac{\partial U}{\partial x_1}=\frac{1}{EA}\left[5\times(1.25x_1)\times 1.25+3\times(0.75x_1)\times 0.75-5\times 1.25\times(150-x_1-1.25x_1)\right.$$
$$-1.5\times 3\times(90-0.6x_1-1.50x_1)+(0.75x_1-0.6x_2)\times 3\times 0.75$$
$$\left.-4\times(120-0.8x_2-x_1)\right]-20\alpha\times 3\times 1.5-20\alpha\times 5\times 1.25=0$$

This leads to

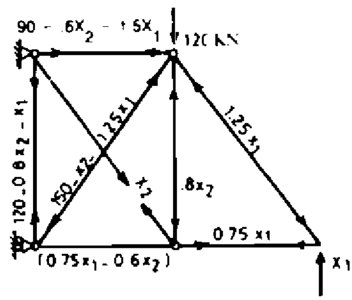


Figure 3.89

$$29.75x_1 + 10.8x_2 = 6122.5 \qquad \text{(a)}$$

Similarly,

$$\frac{\partial U}{\partial x_2} = \frac{1}{EA}\big[4 \times 0.8 \times (0.8x_2) + 5x_2 - 5 \times (150 - x_2 - 1.25x_1) - 3 \times 0.6 \times (90 - 0.6x_2 - 1.5x_1)$$

$$-3 \times 0.6 \times (0.75x_1 - 0.6x_2) - 0.8 \times 4 \times (120 - 0.8x_2 - x_1)\big] - 0.6 \times 60\alpha = 0$$

This leads to

$$10.8x_1 + 17.28x_2 = 2016 \qquad \text{(b)}$$

Notice that the two equations (a) and (b) are exactly the same as the consistent deformation equations obtained in example 3.15. Solving these equations one obtains the values of x_1 = 211.42 kN and x_2 = -15.49 kN.

To determine the horizontal displacement of C apply a horizontal virtual load P at C as shown in Figure 3.90.

In obtaining the strain energy, we may only be concerned with the terms which are function of P. The strain energy is given by

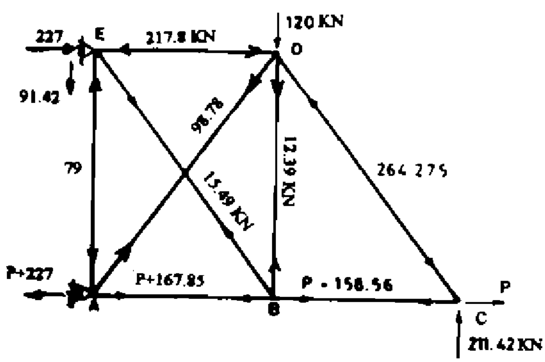


Figure 3.90

$$U=\frac{(P+158.565)^2\times 3}{2EA}+\frac{(P+158.565+0.6\times 15.49)^2\times 3}{2EA}+\text{terms not function of P}$$

According to Castigliano's second theorem one has

$$\Delta_C=\frac{\partial U}{\partial P}=\frac{3}{EA}\times(158.565)+\frac{3}{EA}\times(158.565+0.6\times 15.49)=0.0489\,\text{cm}$$

which indicates that the displacement of C is in the same direction as the assumed virtual load P. This is the same result obtained in example 3.16.

3.4.5 Application to Frame-Truss Structures

Example 3.22

Determine the bending moment diagram for the structure shown in Figure 3.91 (Example 3.17) using Castigliano's second theorem.

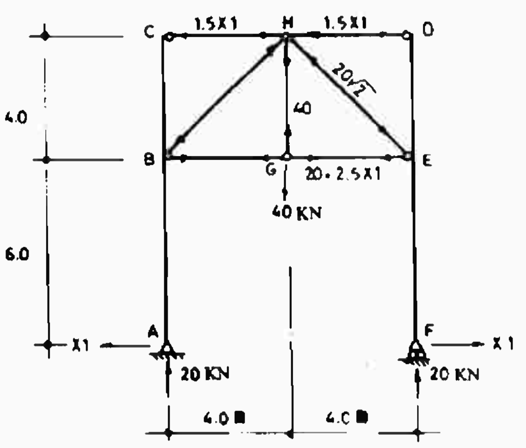


Figure 3.91

Solution

The structure is one degree statically indeterminate. The redundant x_1 is chosen at F. The strain energy is obtained as follows:

$$U=\sum\int\frac{M_z^2\,dx}{2EI}+\sum\int\frac{A_x^2\,dx}{2EA}$$

For members AB and EF

$$M_z = x(x_1) \qquad \text{for} \quad 0 \le x \le 6$$

For members BC and DE

$$M_z = x(1.5x_1) \qquad \text{for} \quad 0 \le x \le 4$$

For members AB and EF

$$A_x = -20 \qquad \text{for} \quad 0 \le x \le 6$$

The magnitudes of axial forces in the truss are shown in Figure 3.91. Substituting, one has

$$U = \left[2\int_0^6 \frac{x^2 x_1^2 \, dx}{2EI} \right]_{AB,\,EF} + \left[2\int_0^4 \frac{1.5^2 x^2 x_1^2}{2EI} dx \right]_{BC,\,DE}$$
$$+ \left[\frac{1}{2EA} \left[(1.5x_1)^2 \times 4 \times 2 + \left(20\sqrt{2}\right)^2 \times 4\sqrt{2} \times 2 + (40)^2 \times 4 + (20 + 2.5x_1)^2 \times 4 \times 2 \right] \right]_{\text{truss membe}}$$

$$\frac{\partial U}{\partial x_1} = 2\int_0^6 \frac{x^2 x_1 \, dx}{EI} + 2\int_0^4 \frac{1.5^2 x^2 x_1 \, dx}{EI}$$
$$+ \frac{1}{2EA}\left[16 \times (1.5x_1) \times 1.5 + 8 \times 2 \times (20 + 2.5x_1) \times 2.5\right]$$
$$= \frac{2x_1}{EI}\left[\frac{x^3}{3}\right]_0^6 + \frac{2 \times 1.5^2 x_1}{EI}\left[\frac{x^3}{3}\right]_0^4 + \frac{1}{2EA}\left[1.5^2 \times 16x_1 + 16 \times 2.5 \times (20 + 2.5x_1)\right] = 0$$

This leads to x_1 = -1.298 kN which is the same result obtained in example 3.16.

3.5 THE THREE MOMENT EQUATION

3.5.1 Introduction

This method is a special consistent deformation method in which the analyst is obliged to select specific redundants and also neglects the axial force and shear force effects. The three moment equation is, thus, most suitable for continuous beams. However, it could be used for some frames which are not subjected to sidesway. The method considers the redundants to be the bending moments at the intermediate and fixed supports in order to make the primary structure consists of a set of simply supported beams.

3.5.2 Derivation of the Equation

Consider the continuous beam shown in Figure 3.92. By selecting the redundants to be the unknown moments at the supports, the primary structure consists of simply supported beams as shown in Figure 3.93. Each redundant is assumed to be

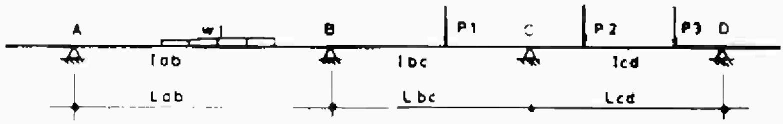


Figure 3.92

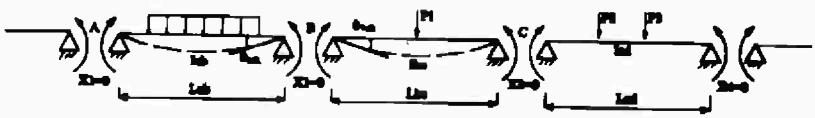


Figure 3.93

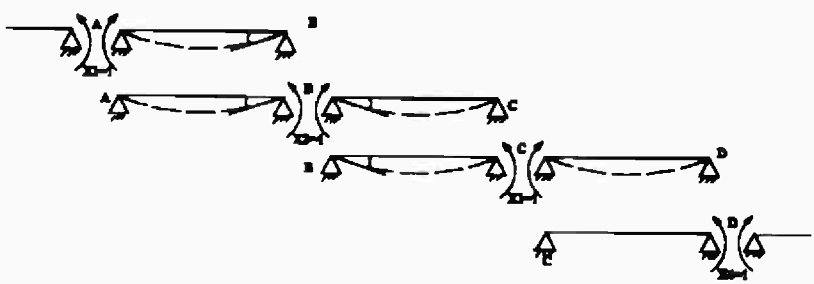


Figure 3.94

unity, as shown in Figure 3.94, in order to determine the flexibility coefficients and the supports relative rotation in the primary structure.

For the consistent deformation at each released continuous or fixed support, the sum of the relative rotations of the primary structure must equal the relative rotation of the same support in the actual structure. For example, the equations of consistent deformations at supports B and C are, respectively,

$$\theta_{B0} + f_{21} x_1 + f_{22} x_2 + f_{23} x_3 = \theta_B \qquad (3.29a)$$

$$\theta_{C0} + f_{32} x_2 + f_{33} x_3 + f_{34} x_4 = \theta_C \qquad (3.29b)$$

where θ_{BO} is the deformation at B in the primary structure obtained from integrating M_{z0} and m_{z1}, and f_{ij} are the flexibility coefficients, each is obtained from integrating m_{zi} with m_{zj}. Therefore, the moment diagrams of Figures 3.93 and 3.94 are used to determine these coefficients.

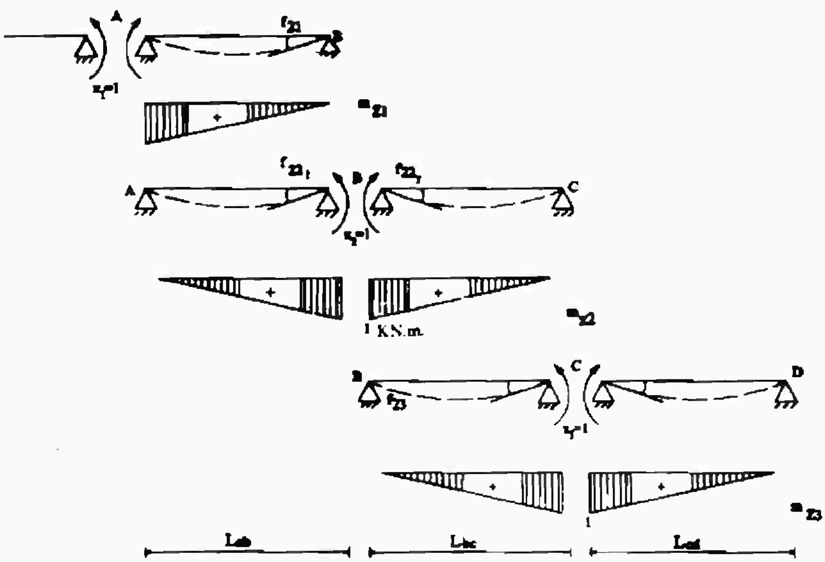


Figure 3.95

Considering, for example, the consistent deformation at support B. The flexibility coefficients are obtained using the moment diagrams shown in Figure 3.95 as follows:

$$f_{21} = \int \frac{m_{z2}\, m_{z1}\, dx}{EI} = \frac{L_{ab}}{6EI_{ab}} \quad (3.30a)$$

$$f_{22} = \int \frac{m_{z2}^2\, dx}{EI} = \frac{L_{ab}}{3EI_{ab}} + \frac{L_{bc}}{3EI_{cb}} \quad (3.30b)$$

$$f_{23} = \int \frac{m_{z2}\, m_{z3}\, dx}{EI} = \frac{L_{bc}}{6EI_{cb}} \quad (3.30c)$$

Substituting into equation 3.29a, and realizing that $x_1 = M_A$, $x_2 = M_B$, and $x_3 = M_C$ one obtains

$$M_A\left(\frac{L_{ab}}{6EI_{ab}}\right) + 2M_B\left(\frac{L_{ab}}{6EI_{ab}} + \frac{L_{bc}}{6EI_{bc}}\right) + M_C\left(\frac{L_{bcb}}{6EI_{bc}}\right) = \theta_B - \theta_{B0} \quad (3.31)$$

where θ_{B0} represents the rotation at support B in the primary structure which is obtained from integrating the moment diagram of Figure 3.93 with the moment diagram m_{z2} of Figure 3.95. It can be expressed as $\theta_{B0} = \theta_{B0\,L} + \theta_{B0\,r}$ to show the contribution of the left side and the right side of the support B.

The relative rotation θ_B, in the actual structure is zero if the support is not subjected to any settlement. However, if the interior support B, for example, is subjected to a downward settlement, as shown in Figure 3.96, then θ_B is obtained by summing up the angles made by members AB and BC with the original position of these members. This gives

$$\theta_B = \frac{y_B}{L_{ab}} + \frac{y_B}{L_{bc}} \qquad (3.32)$$

In general, if supports A, B and C are displaced downward with specific quantities y_A, y_B, and y_C as shown in Figure 3.97, then the relative rotation at B is

$$\theta_B = \frac{y_B - y_A}{L_{ab}} + \frac{y_B - y_C}{L_{bc}} \qquad (3.33)$$

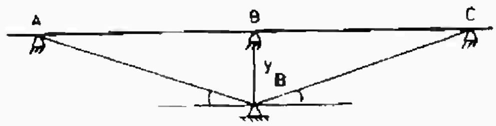


Figure 3.96

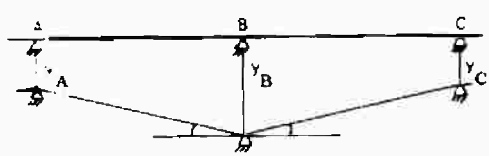
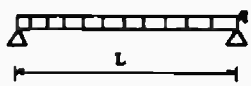
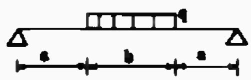
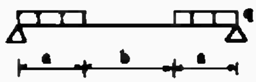
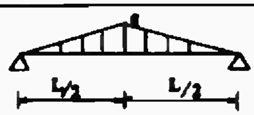
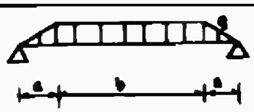
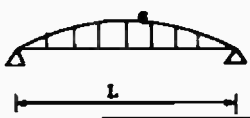
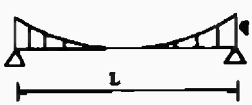
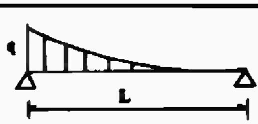
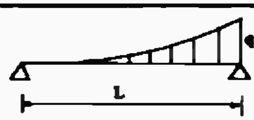
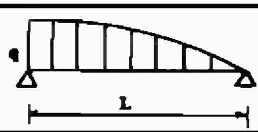


Figure 3.97

Tables 3.1 provide the rotations at the supports of simply supported beams due to various cases of loading to be used in solving problems by the three moment equation method.

Table 3.1 Elastic Reactions

No.	Case of Loading	Elastic reaction = given values × L/6 EI	
		at left support	at right support
1		$\frac{q \times L^2}{4}$	$\frac{q \times L^2}{4}$
2		$\frac{q \times b \times L}{8}\left(3 - \frac{b^2}{L^2}\right)$	$\frac{q \times b \times L}{8}\left(3 - \frac{b^2}{L^2}\right)$
3		$\frac{q \times a^2}{2}\left(3 - \frac{2a}{L}\right)$	$\frac{q \times a^2}{2}\left(3 - \frac{2a}{L}\right)$
4		$\frac{5}{32} \times q \times L^2$	$\frac{5}{32} \times q \times L^2$
5		$\frac{q}{4}\left[L^2 - a^2\left(2 - \frac{a}{L}\right)\right]$	$\frac{q}{4}\left[L^2 - a^2\left(2 - \frac{a}{L}\right)\right]$
6		$\frac{q \times L^2}{5}$	$\frac{q \times L^2}{5}$
7		$\frac{q \times L^2}{20}$	$\frac{q \times L^2}{20}$
8		$\frac{q \times L^2}{12}$	$\frac{q \times L^2}{15}$
9		$\frac{q \times L^2}{15}$	$\frac{q \times L^2}{12}$
10		$\frac{11 \times q \times L^2}{60}$	$\frac{q \times L^2}{6}$

*Reference 9.

Table 3.1 Elastic Reactions (Cont'd…)

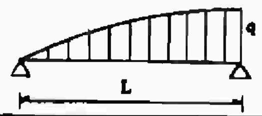
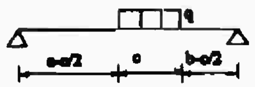
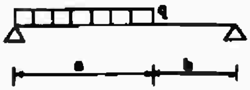
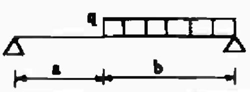
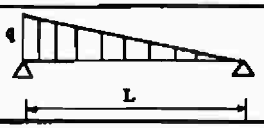
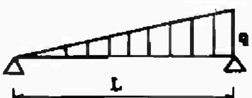
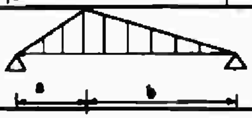
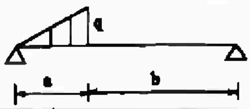
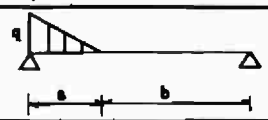
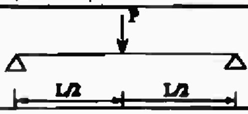
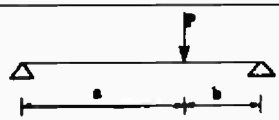
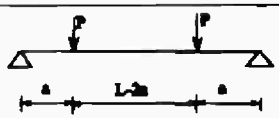
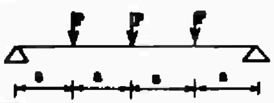
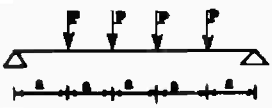
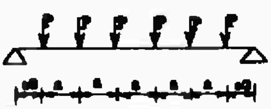
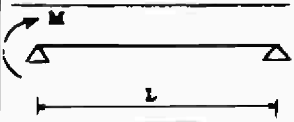
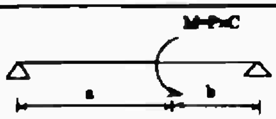
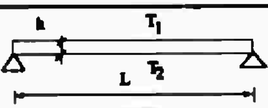
No.	Case of Loading	Elastic reaction = given values × L/6 EI	
		at left support	at right support
11		$\frac{q \times L^2}{6}$	$\frac{11 \times q \times L^2}{60}$
12		$\frac{q \times a \times b \times c}{L^2}\left(L + a - \frac{c^2}{4a}\right)$	$\frac{q \times a \times b \times c}{L^2}\left(L + a - \frac{c^2}{4b}\right)$
13		$\frac{q \times a^2}{4}\left(2 - \frac{a}{L}\right)^2$	$\frac{q \times a^2}{4}\left(2 - \frac{b^2}{L^2}\right)$
14		$\frac{q \times b^2}{4}\left(2 - \frac{b^2}{L^2}\right)$	$\frac{q \times b^2}{4}\left(2 - \frac{b}{L}\right)^2$
15		$\frac{8 \times q \times L^2}{60}$	$\frac{7 \times q \times L^2}{60}$
16		$\frac{7 \times q \times L^2}{60}$	$\frac{8 \times q \times L^2}{60}$
17		$\frac{q(L + b)}{60L}\left(7L^2 - 3b^2\right)$	$\frac{q(L + a)}{60L}\left(7L^2 - 3a^2\right)$
18		$q \times a^2\left(\frac{1}{3} - \frac{a}{4L} + \frac{a^2}{20L^2}\right)$	$q \times a^2\left(\frac{1}{6} - \frac{a^2}{20L^2}\right)$
19		$q \times a^2\left(\frac{2}{3} - \frac{3a}{4L} + \frac{a^2}{5L^2}\right)$	$\frac{q \times a^2}{15}\left(5 - 3 - \frac{a^2}{L^2}\right)$
20		$\frac{3 \times P \times L}{8}$	$\frac{3 \times P \times L}{8}$

Table 3.1 Elastic Reactions (Cont'd...)

No.	Case of Loading	Elastic reaction = given values × L/6 EI	
		at left support	at right support
21		$\frac{P \times a \times b\,(b+L)}{L^2}$	$\frac{P \times a \times b\,(a+L)}{L^2}$
22		$3 \times P \times a\left(1-\frac{a}{L}\right)$	$3 \times P \times a\left(1-\frac{a}{L}\right)$
23		$\frac{15 \times P \times L}{16}$	$\frac{15 \times P \times L}{16}$
24		$\frac{P \times L\,(n^2-1)}{4n}$	$\frac{P \times L\,(n^2-1)}{4n}$
25		$\frac{P \times L}{8}\left(2n+\frac{1}{n}\right)$	$\frac{P \times L}{8}\left(2n+\frac{1}{n}\right)$
26		2M	M
27		$P \times c\left(1-3\frac{b^2}{L^2}\right)$	$P \times c\left(3\frac{a^2}{L^2}-1\right)$
28		$-3 \times E \times I \times \alpha_t \times \frac{(T_1-T_2)}{h}$	$-3 \times E \times I \times \alpha_t \times \frac{(T_1-T_2)}{h}$

3.5.3 Beams with Fixed Supports

If the beam has a fixed support, the moment at the support is unknown. One can derive the three moment equation in the same way derived for continuous supported beams. Consider the continuous beam shown in Figure 3.98. The primary structure and the bending moments M_{z0}, m_{z1}, and m_{z2} are shown in Figure 3.99. The consistent deformation at the fixed support A is

$$\theta_{A0} + f_{11}\,x_1 + f_{12}\,x_2 = \theta_A \tag{3.34}$$

The flexibility coefficients f_{11} and f_{12} are obtained as follows:

$$f_{11} = \int \frac{m_{z1}\, m_{z1}\, dx}{EI} = \frac{L_{ab}}{3EI_{ab}}$$

$$f_{12} = \int \frac{m_{z1}\; m_{z2}\; dx}{EI} = \frac{L_{ab}}{6EI_{ab}}$$

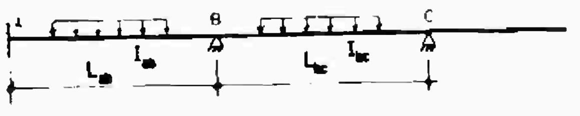


Figure 3.98

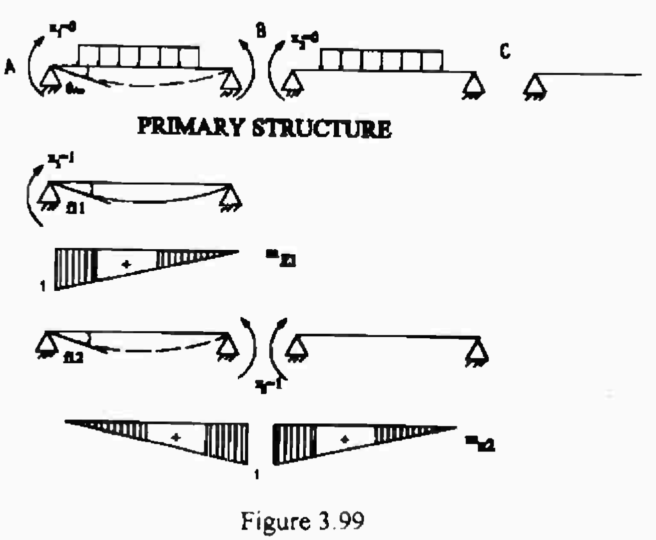


Figure 3.99

Therefore, Equation 3.34 can be written as

$$0+2M_A\left(\frac{L_{ab}}{6EI_{ab}}\right)+M_B\left(\frac{L_{ab}}{6EI_{ab}}\right)=\theta_A-\theta_{A0} \tag{3.35}$$

which shows that it is a special form of Equation 3.31.

The angle of rotation θ_A here represents the relative angle of rotation at A, in the actual structure, which could be due to settlement or rotations at the support A. If a settlement occurs at A as shown in Figure 3.100, θ_A is obtained using Equation 3.33 as follows:

$$\theta_A=\frac{y_A-y_B}{L_{ab}} \tag{3.36}$$

If the fixed support A was subjected to an angular rotation, one must substitute its value and appropriate sign in the three moment equation. In the derivation of the three moment equation, the redundants x_i were assumed to provide tension in the bottom side and compression in the top side of the beam. This assumption is considered throughout solving any problem by this method. Thus, the sign is positive if the angle of rotation provides tension on the bottom side as shown in Figure 3.101.

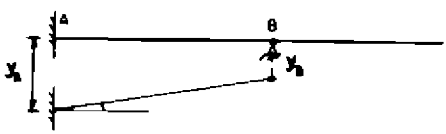


Figure 3.100

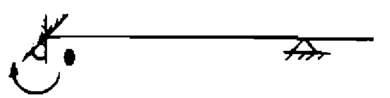


Figure 3.101

Example 3.23

Determine the bending moment diagram due to a clockwise unit rotation at A in the beam shown in Figure 3.102 (EI is constant).

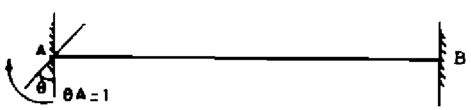


Figure 3.102

Solution

Applying the three moment equation at A one has

$$2M_A\left(\frac{L_{ab}}{6EI_{ab}}\right)+M_B\left(\frac{L_{ab}}{6EI_{ab}}\right)=\theta_A-\theta_{A0}$$

where $\theta_{A0}=0$ due to the absence of any loading on the beam, and $\theta_A=+1$ due to the angular rotation at A. Substituting, one obtains

$$2M_A+M_B=\frac{6EI}{L} \qquad \text{(a)}$$

Similarly, applying the three moment equation at B one obtains

$$M_A+2M_B=0 \qquad \text{(b)}$$

Solving equations (a) and (b) one gets

$$M_B=-\frac{2EI}{L} \quad ; \quad M_A=\frac{4EI}{L}$$

The bending moment diagram is shown in Figure 3.103. This result is important and should be remembered because it shall be used in the stiffness matrix method. The values of M_A which results in a unit rotation at A is called the stiffness coefficient.

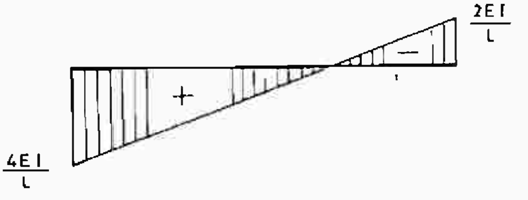


Figure 3.103

Example 3.24

Determine the bending moment diagram for the beam shown in Figure 3.104 due to the settlement of support A a unit value downward (EI is constant).

Applying the three moment equation at A one obtains

$$2M_A\left(\frac{L_{ab}}{6EI_{ab}}\right)+M_B\left(\frac{L_{ab}}{6EI_{ab}}\right)=\theta_A-\theta_{A0}$$

Figure 3.104

where $\theta_{A0} = 0$ due to the absence of any loading on the beam, and $\theta_A = (1 - 0)/L$. This gives

$$2M_A + M_B = \frac{6EI}{L^2} \qquad \text{(a)}$$

Similarly, applying the three moment equation at B one obtains

$$M_A\left(\frac{L}{6EI}\right) + 2M_B\left(\frac{L}{6EI}\right) = \theta_B - \theta_{B0}$$

where $\theta_{B0} = 0$ and $\theta_B = (0 - 1)/L$. This gives

$$M_A + 2M_B = \frac{-6EI}{L^2} \qquad \text{(b)}$$

Solving equations (a) and (b) one obtains

$$M_A = \frac{6EI}{L^2} \quad ; \quad M_B = \frac{-6EI}{L^2}$$

The bending moment diagram is shown in Figure 3.105. This result should also be remembered because it shall mostly be used in the stiffness matrix method.

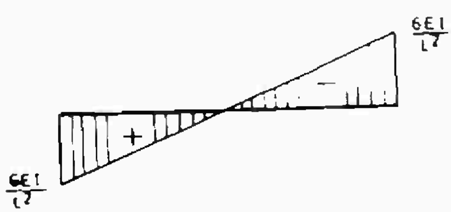


Figure 3.105

3.5.4 Numerical Applications

Example 3.25

Determine the bending moment and shear force diagrams for the beam shown in Figure 3.106 due to the applied loads, a rise in temperature in member BC and an anticlockwise rotation of support A of 0.002 rad. Consider EI = 10^5 kN.m^2 and α = 10^{-5}/°c.

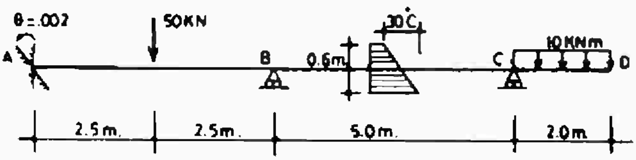


Figure 3.106

Solution

In this beam, the moment on member CD is determinate. The unknown bending moments are M_A and M_B because the degree of static indeterminacy is two determined from

DSI = 3 × 3 + 5 − 3 × 4 = 2

Applying the three moment equation at the locations of the two redundant moments M_A and M_B, respectively one has

$$2M_A\left(\frac{5}{6EI}+0\right)+M_B\left(\frac{5}{6EI}\right)=\theta_A-\theta_{A0}$$

where $\theta_A = -0.002$ rad ; $\theta_{A0} = \dfrac{50\times 5^2}{16EI}$

This leads to

$$2M_A+M_B=\left(\frac{6EI}{5}\right)\left(0.002-\frac{50\times 25}{16EI}\right)=-333.75 \text{ kN.m.} \qquad \text{(a)}$$

The three moment equation at B gives

$$M_A\left(\frac{5}{6EI}\right)+2M_B\left(\frac{5}{6EI}+\frac{5}{6EI}\right)+M_C\left(\frac{5}{6EI}\right)=\theta_B-\theta_{B0}$$

where $\theta_B = 0$; $\theta_{B0} = \dfrac{50 \times 5^2}{16EI} + \left(-\alpha\left(\dfrac{-30}{0.6}\right)\left(\dfrac{1 \times 5}{2}\right)\right)$

which results from θ_{B0L} (Table 3.1 case No. 20) plus θ_{B0r} (Table 3.1 case No. 28).

Substituting, one obtains

$$M_A + 4M_B = -223.75 \text{ kN.m.} \qquad \text{(b)}$$

Solving (a) and (b) one obtains

$M_A = -16.25$ kN.m ; $M_B = -158.75$ kN.m.

The negative sign indicates that the moment makes tension on the top side and compression on the bottom side. The bending moment and shear force diagrams are shown in Figure 3.107 which are the same as in Example 3.10.

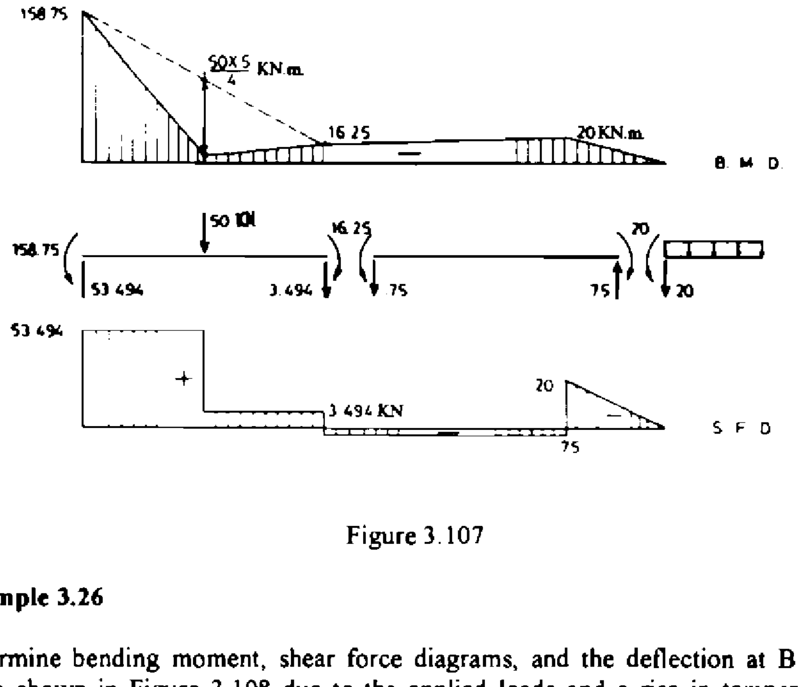


Figure 3.107

Example 3.26

Determine bending moment, shear force diagrams, and the deflection at B for the beam shown in Figure 3.108 due to the applied loads and a rise in temperature in member AB (EI = 10^5 kN.m^2, $\alpha = 10^{-5}$/°c, spring constant K = 10 kN/cm).

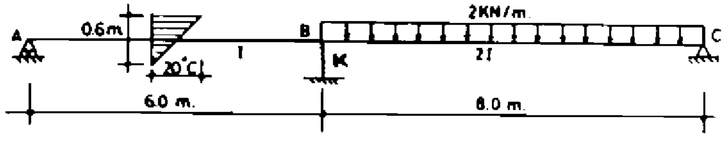


Figure 3.108

Solution

This beam is one degree statically indeterminate and has unknown moment at B since $M_A = M_C = 0$. The three moment equation at B gives

$$0 + M_B\left(\frac{6}{6EI} + \frac{8}{12EI}\right) + 0 = \theta_B - \theta_{B0}$$

where θ_B, θ_{B0} are obtained according to Figure 3.109 as

$$\theta_B = \frac{y_B}{6} + \frac{y_B}{8}$$

$$\theta_{B0} = -\alpha\left(\frac{20}{0.6}\right) \times \frac{1 \times 6}{2} + \frac{1}{2EI} \times \frac{2 \times 8^3}{24} = -100\alpha + \frac{64}{3EI}$$

Figure 3.109

Substituting into the three moment equation, one has

$$2M_B\left(\frac{20}{12EI}\right) = \left(\frac{y_B}{6} + \frac{y_B}{8}\right) - \left[-100\alpha + \frac{21.333}{EI}\right]$$

$$20M_B = 1.75y_B\, EI + 472 \qquad \text{(a)}$$

Since y_B is related to M_B by the spring reaction relation at B, using Figure 3.110 one has

$$y_B = \frac{R_B}{K} = \frac{1}{1000}\left(8 - \frac{M_B}{6} - \frac{M_B}{8}\right) \qquad \text{(b)}$$

The assumed deformation at B indicates that the spring is under compression and the reaction at B is upward. Therefore R_B is taken as upward reaction.

Solving equations (a) and (b) one obtains

$M_B = 26.351$ kN.m

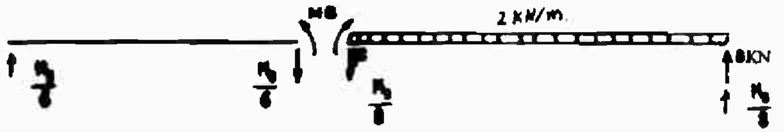


Figure 3.110

$$y_B = \frac{1}{1000}\left(8 - \frac{M_B}{6} - \frac{M_B}{8}\right) = 0.315 \times 10^{-3}\ \text{m} = 0.0315\,\text{cm}$$

The positive value of y_B indicates that the deformed shape of the beam is downward as assumed. The moment and shear force diagrams are shown in Figure 3.111.

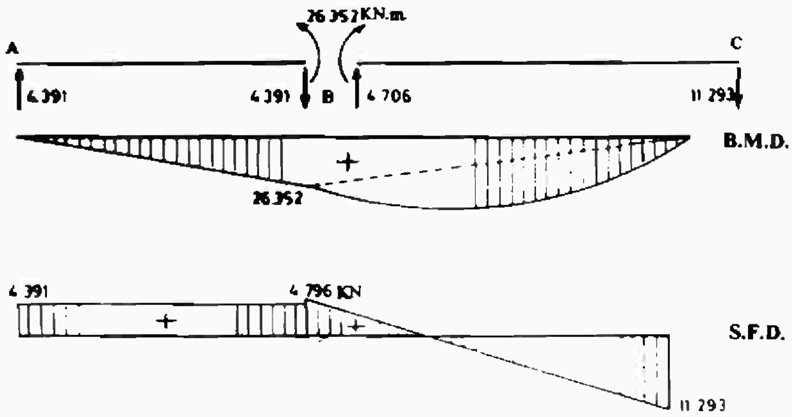


Figure 3.111

3.6 THE ELASTIC CENTRE METHOD

3.6.1 Introduction

This method is applicable, in general, to structures having three degrees statically indeterminacy such as closed rings, closed frames, arches with fixed supports, and portal frames with fixed supports. It is considered as a special application of the consistent deformation method using the virtual work principle. It has been developed in order to ease solving the three simultaneous consistent deformation equations before the wide spread of computer utilization. The redundants are chosen at a certain location called the elastic centre in order to obtain three independent consistent deformation equations, which can easily be solved. This method depends on the assumption that the axial force effect is small and can be neglected.

3.6.2 Determination of the Elastic Centre

Consider for example, the three degrees statically indeterminate portal frame shown in Figure 3.112. In general, the equations of consistent deformation for three degree static indeterminate structures are as follows:

$$\begin{aligned} \Delta_{10} + f_{11}\, x_1 + f_{12}\, x_2 + f_{13}\, x_3 &= \Delta_1 \\ \Delta_{20} + f_{21}\, x_1 + f_{22}\, x_2 + f_{23}\, x_3 &= \Delta_2 \\ \Delta_{30} + f_{31}\, x_1 + f_{32}\, x_2 + f_{33}\, x_3 &= \Delta_3 \end{aligned} \qquad (3.37)$$

To convert Equations 3.37 into three independent equations, the flexibility coefficients f_{ij} for $i \neq j$ must be zero. In this case Equations 3.37 become

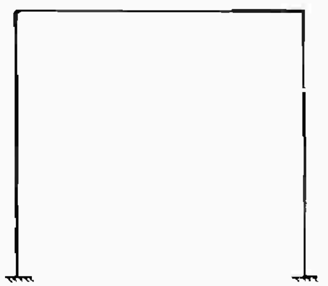


Figure 3.112

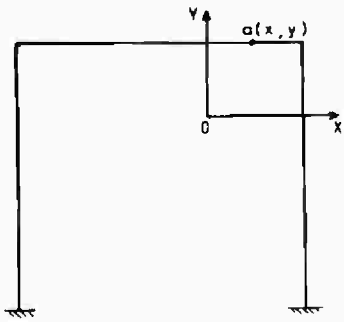


Figure 3.113

$$\begin{aligned} \Delta_{10} + f_{11}\, x_1 &= \Delta_1 \\ \Delta_{20} + f_{22}\, x_2 &= \Delta_2 \\ \Delta_{30} + f_{33}\, x_3 &= \Delta_3 \end{aligned} \qquad (3.38)$$

which can easily be solved for x_1, x_2, and x_3.

Let the elastic centre location to be at point O as shown in Figure 3.113. The three redundants x_1, x_2, and x_3 must be applied at point O in order to obtain Equations 3.38. Because point O is not connected with the frame, and it is not known yet and in order to make this discussion realistic, point O is connected with any section in the frame by rigid members having infinite moment of inertia. The location of the connection is arbitrary. Figure 3.114 show different choices for the connection between the elastic centre and the frame members.

Considering Figure 3.114a, the bending moments at any point with coordinates (x,y) with respect to point O and due to the redundants $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$, can be obtained. Directions of x_1, x_2, and x_3 are assumed to produce positive bending moments such that

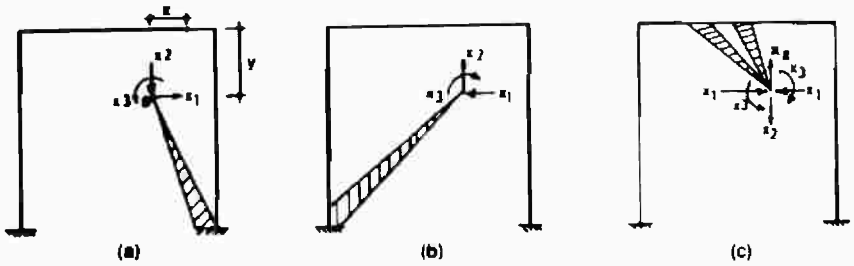


Figure 3.114

$$m_{z1} = y \qquad (3.39a)$$

$$m_{z2} = x \qquad (3.39b)$$

$$m_{z3} = 1 \qquad (3.39c)$$

Substituting Equations 3.39 into the expressions of the flexibility coefficients $f_{ij}, i \neq j$ and with ignoring the axial forces effect, one obtains

$$f_{12} = f_{21} = \int \frac{m_{z1}\, m_{z2}\, d\ell}{EI} = \int \frac{y\, x\, d\ell}{EI} \qquad (3.40)$$

$$f_{23} = f_{32} = \int \frac{m_{z2}\, m_{z3}\, d\ell}{EI} = \int \frac{x\, d\ell}{EI} \qquad (3.41)$$

$$f_{13} = f_{31} = \int \frac{m_{z1}\, m_{z3}\, d\ell}{EI} = \int \frac{y\, d\ell}{EI} \qquad (3.42)$$

One realizes that to make Equations 3.41 and 3.42 equal zero, the elastic centre O must lie at the centroid of the areas $(d\ell/EI)$ of the frame members including the imposed infinite moment of inertia members. These members, however, do not affect on the location of the elastic centre since their $(d\ell/EI)$ is zero, due to infinite EI. To make Equation 3.40 equals zero, the axes x and y must pass through the centroid of the areas $(d\ell/EI)$ and their directions are in the direction of the principal axes. Therefore, the location of the elastic centre must be the centroid of the areas $(d\ell/EI)$ of the frame members and the redundants x_1 and x_2 are located at the elastic centre in the direction of principal axes of the areas $(d\ell/EI)$.

Example 3.27

Determine the location of the elastic centre and the directions of the redundants x_1 and x_2, for the frames shown in Figures 3.115a and 3.115b.

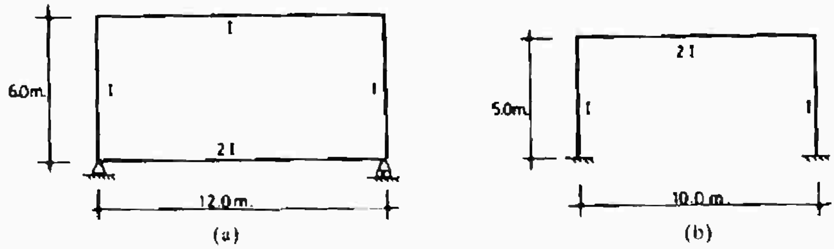


Figure 3.115

Solution

(a) The frame shown in Figure 3.115a is drawn as a tubular section, where the thickness is (1/EI) for each member in the frame. This is shown in Figure 3.116.

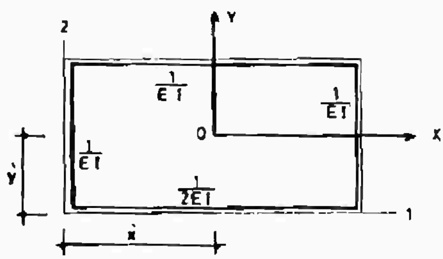


Figure 3.116

To determine the centroid of these areas, the moment of the areas is taken about any two arbitrary perpendicular axes, such as axes 1-1 and 2-2 shown in Figure 3.116. The first moment of areas about these axes are, respectively,

$$M_{1-1} = \frac{6}{EI} \times 3 \times 2 + \frac{12}{EI} \times 6 = \frac{108}{EI}\ kN^{-1}$$

$$M_{2-2} = \frac{12}{EI} \times 6 + \frac{12}{2EI} \times 6 + \frac{6}{EI} \times 12 = \frac{180}{EI}\ kN^{-1}$$

The total area of the cross section is

$$A = \frac{12}{EI} + \frac{6}{EI} + \frac{6}{EI} + \frac{12}{2EI} = \frac{30}{EI}\ kN^{-1}\ m^{-1}$$

From statics principles, the locations of the centroid $\bar{x}$ and $\bar{y}$ from the axes 2-2 and 1-1 are respectively,

$$\bar{x} = \frac{M_{2\text{-}2}}{A} = \frac{180}{30} = 6 \text{ m}$$

$$\bar{y} = \frac{M_{1\text{-}1}}{A} = \frac{108}{30} = 3.6 \text{ m}$$

To determine the directions of the principal axes, select any two arbitrary axes, x and y, passing through the elastic center O. Calculating the moments of inertia I_x, I_y, and I_{xy} for the tubular section one has

$$I_{xy} = 0$$

$$I_x = \frac{6^3}{12EI} \times 2 + \frac{12}{2EI} \times 3^2 + \frac{12}{EI} \times 3^2 = \frac{198}{EI} \text{ m/kN}$$

$$I_y = \frac{12^3}{12 \times 2EI} + \frac{12^3}{12EI} + \frac{6}{EI} \times 6^2 + \frac{6}{EI} \times 6^2 = \frac{648}{EI} \text{ m/kN}$$

The angle, θ, which determines the direction of the principle axis with respect to x-axis is obtained from

$$\tan 2\theta = \frac{-2I_{xy}}{I_x - I_y} = 0$$

The principal axes are thus along the chosen x and y axes. This result was obvious from the beginning because of the symmetry of the areas $d\ell/EI$ about the elastic centre.

(b) From Figure 3.115b, the tubular section is shown in Figure 3.117.

Taking the moment of areas about axes 1-1 and 2-2, one obtains, respectively,

$$M_{1\text{-}1} = \frac{5}{EI} \times 2.5 \times 2 + \frac{10}{2EI} \times 5 = \frac{50}{EI} \text{ kN}^{-1}$$

$$M_{2-2} = \frac{10}{2EI} \times 5 + \frac{5}{EI} \times 10 = \frac{75}{EI} \text{ kN}^{-1}$$

The total area of the cross section is

$$A = \frac{10}{2EI} + \frac{5}{EI} + \frac{5}{EI} = \frac{15}{EI} \text{ kN}^{-1} \text{ m}^{-1}$$

The location of the elastic centre is at

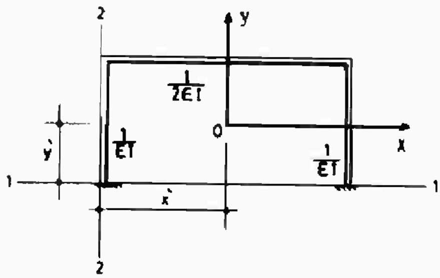


Figure 3.117

$$x' = \frac{M_{2-2}}{A} = 5 \text{ m} \quad ; \quad y' = \frac{M_{1-1}}{A} = \frac{50}{15} = 3.33 \text{ m}$$

From the symmetry of $\frac{d\ell}{EI}$ areas about the elastic centre, the directions of the redundants x_1 and x_2 should be along the chosen axes x and y.

3.6.3 Numerical Applications

By determining the location of the elastic centre, and the directions of the principal axes, one knows the locations and direction of the redundants x_1, x_2, and x_3 which make $f_{ij} = 0$ for $i \neq j$. The process of analysis by the consistent deformation method is continued as usual. A primary structure is chosen, the bending moment diagram M_{z0} is obtained, and the three bending moment diagrams m_{z1}, m_{z2}, and m_{z3} due to the unit values for the redundants x_1, x_2, x_3, respectively, are determined. The coefficient Δ_{io} and f_{ij} can be calculated as usual by the integration of diagrams. One finally obtains three independent linear equations which can easily be solved for the values of the redundants x_1, x_2, and x_3. The following examples illustrate the application of this method:

Example 3.28

Determine the bending moment and shear force diagrams for the frame shown in Figure 3.118 using the elastic centre method ($EI = 10^5$ kN.m^2). Then, determine the bending moment due to vertical settlement at D of 5 cm downward.

Solution

The elastic centre of this frame was determined in the previous example. The location and directions of the three redundants are shown in Figure 3.119. The elastic centre is connected with joint D by a member of infinite moment of inertia. The bending moment diagrams for the primary structure due to the applied loads and unit values of the redundants are shown in Figure 3.120. The bending moment diagram M_{z0} was drawn decomposed for easing the integration.

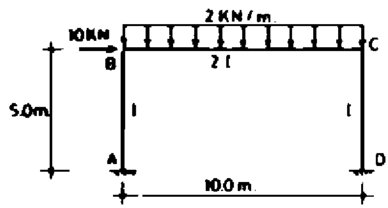


Figure 3.118

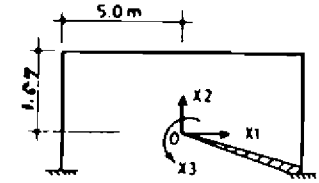


Fig. 3.119

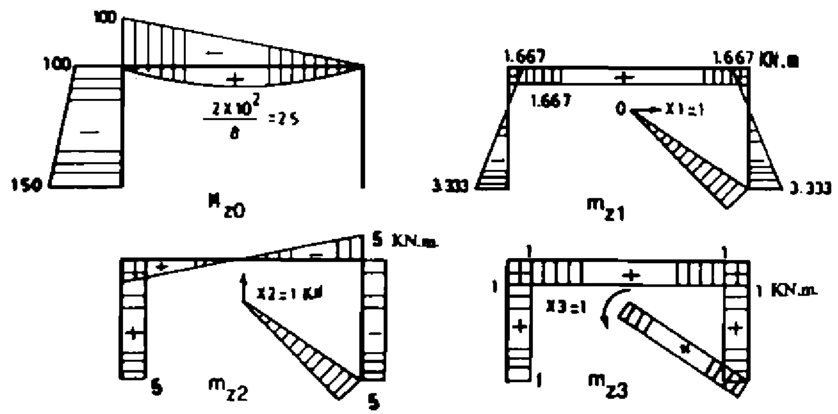


Figure 3.120

It is obvious from the figures of m_{z1}, m_{z2} and m_{z3} that $f_{ij} = 0$ for $i \neq j$. The equations of consistent deformation are

$$\Delta_{10} + f_{11}\, x_1 = \Delta_1 = 0$$
$$\Delta_{20} + f_{22}\, x_2 = \Delta_2 = 0$$
$$\Delta_{30} + f_{33}\, x_3 = \Delta_3 = 0$$

The flexibility coefficients and Δ_{io} are calculated as follows:

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, dx}{EI} = \frac{1}{2EI}\left[\frac{2}{3}\times 25\times 10\times 1.667 - \frac{100\times 10}{2}\times 1.667\right]$$
$$+\frac{5}{6EI}\left[150\times 3.333\times 2 - 100\times 1.667\times 2 + 333.3 - 150\times 1.667\right] = \frac{346.957}{EI}\ \text{m}$$

$$f_{11} = \int \frac{m_{z1}^2\, dx}{EI} = \frac{1.667^2\times 10}{2EI} + \frac{2\times 5}{6EI}\left[3.333^2\times 2 + 2\times 1.667^2 - 2\times 3.333\times 1.667\right]$$
$$= \frac{41.666}{EI}\ \text{m/kN}$$

Therefore, the value and direction of x_1 is obtained from

$$x_1 = \frac{-\Delta_{10}}{f_{11}} = \frac{-346.957}{41.666} = -8.327 \text{ kN}$$

which indicates it is in an opposite direction to the assumed x_1 direction. Similarly, for finding x_2 and x_3 one has

$$\Delta_{20} = \int \frac{M_{z0}\, m_{z2}\, dx}{EI} = 0 + \frac{10}{2EI \times 6}\left[-100 \times 5 \times 2 + 0 + 100 \times 5\right]$$
$$+ \frac{5}{6EI}\left[-100 \times 5 \times 2 - 150 \times 5 \times 2 - 100 \times 5 - 150 \times 5\right] = -\frac{3541.66}{EI} \text{ m}$$

$$f_{22} = \int \frac{m_{z2}^2\, dx}{EI} = \frac{10}{12EI}\left[5^2 \times 2 + 5^2 \times 2 - 5^2 \times 2\right] + \frac{5 \times 5 \times 5}{EI} \times 2 = \frac{291.667}{EI} \quad \text{m/kN}$$

$$x_2 = \frac{-\Delta_{20}}{f_{22}} = 12.143 \text{ kN}$$

$$\Delta_{30} = \int \frac{M_{z0}\, m_{z3}\, dx}{EI}$$
$$= \frac{1}{2EI}\left[\frac{-1000}{2} + \frac{2}{3} \times 25 \times 10\right] + \frac{1}{EI}\left[\left(\frac{-100 - 150}{2}\right) \times 5\right] = \frac{-791.67}{EI} \text{ rad}$$

$$f_{33} = \frac{1 \times 5}{EI} \times 2 + \frac{1 \times 10}{2EI} \times 1 = \frac{15}{EI} \quad \text{rad/(kN.m)}$$

$$x_3 = \frac{-\Delta_{30}}{f_{33}} = 52.778 \text{ kN.m.}$$

The final bending moment diagram is obtained by calculating the moment at A, B, C, and D using the superposition principle as follows:

$$M_z = M_{z0} + x_1\, m_{z1} + x_2\, m_{z2} + x_3\, m_{z3}$$
$$M_A = -150 - 3.333 \times (-8.327) + 12.143 \times (5) + 52.778 \times (1) = -8.753 \quad \text{kN.m}$$
$$M_B = -100 + 1.667 \times (-8.327) + 5 \times (12.143) + 52.778 = -0.388 \quad \text{kN.m}$$
$$M_C = 0 + 1.667 \times (-8.327) - 5 \times (12.143) + 52.778 = -21.818 \quad \text{kN.m}$$
$$M_D = 0 - 3.333 \times (-8.327) - 5 \times (12.143 \times 5) + 52.778 = +19.816 \quad \text{kN.m}$$

The bending moment and shear force diagrams are shown in Figure 3.121.

In the case of settlement at D, as shown in Figure 3.122, one has Δ_{10}, Δ_{20}, and Δ_{30} are zero since no loads are applied on the frame. The consistent deformation equations become

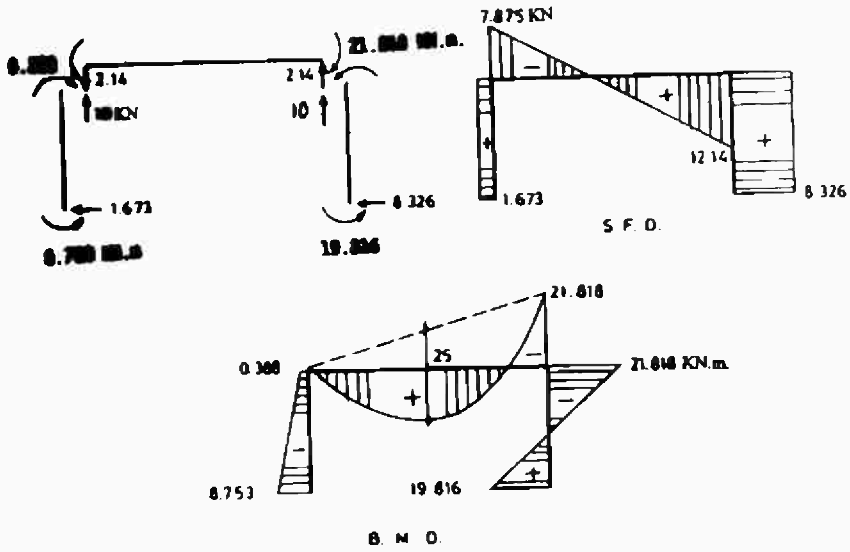


Figure 3.121

$$0 + f_{11}\,x_1 = \Delta_1 = 0$$

$$0 + f_{22}\,x_2 = \Delta_2 = -0.05\ \text{m}$$

$$0 + f_{33}\,x_3 = \Delta_3 = 0$$

which gives $x_1 = x_3 = 0$.

$$x_2 = \frac{\Delta_2}{f_{22}} = \frac{-0.05\,EI}{291.667} = -17.14\ \text{kN}$$

The final bending moment diagram is obtained by superposition principle as

$$M_z = M_{z0} + x_2\,m_{z2}$$
$$M_{zA} = 0 - 17.14 \times (5) = -85.7\ \text{kN.m}$$
$$M_{zB} = 0 - 17.14 \times (5) = -85.7\ \text{kN.m}$$
$$M_{zD} = 0 - 17.14 \times (-5) = 85.7\ \text{kN.m}$$

The bending moment diagram is shown in Figure 3.123 which gives the same results as example 3.14.

Example 3.29

Determine the bending moment diagram for the frame shown in Figure 3.124 using the method of elastic centre (EI = 10^5 kN.m^2).

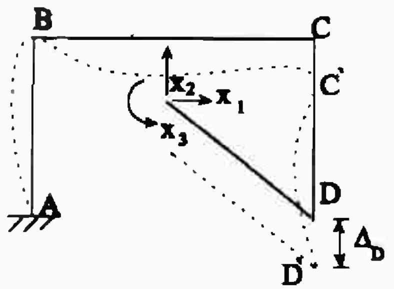


Figure 3.122

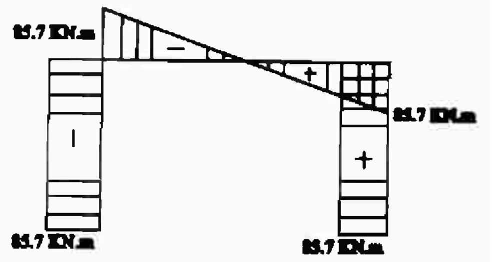


Figure 3.123

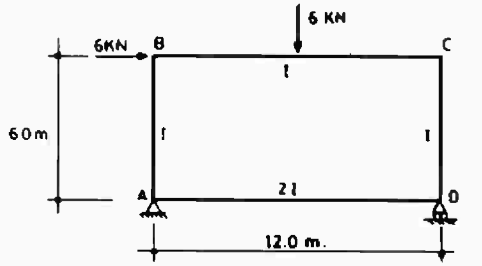


Figure 3.124

Solution

This frame is three times statically indeterminate since, m = 4, J = 4, r = 3, n = 0 and

DSI = 4 × 3 + 3 − 4 × 3 = 3
Number of external redundants = r − 3 = 0
Number of internal redundants = 3 − 0 = 3

The elastic centre of this frame has been determined in Example 3.25. By cutting any section in the frame and connecting the cut section by two infinitely rigid members to the elastic centre, one obtains the primary structure shown in Figure 3.125. Due to the symmetry of the frame about the vertical axis, the directions of the redundants coincide with the principal axes x and y.

The bending moments for the primary structure due to loading and unit values of the redundant are as shown in Figures 3.125 and 3.126.

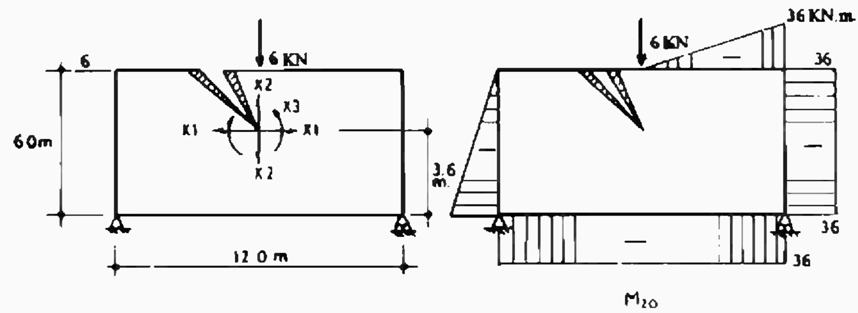


Figure 3.125

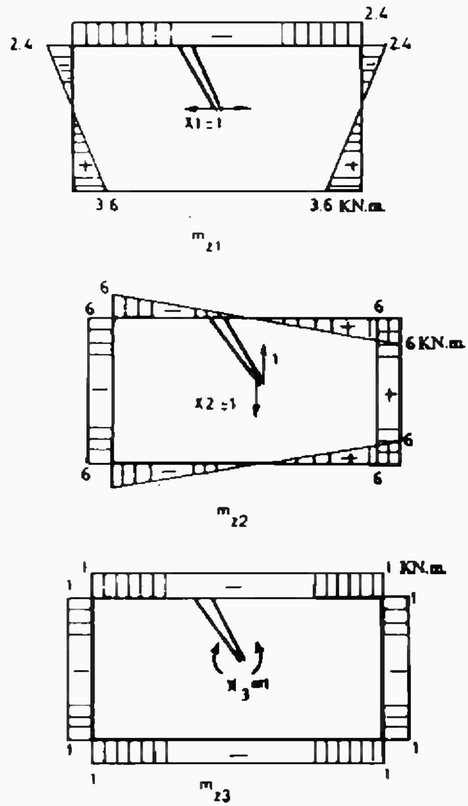


Figure 3.126

For the selected primary structure, the equations of consistent deformations are

$$\Delta_{10} + f_{11}\, x_1 = \Delta_1 = 0$$
$$\Delta_{20} + f_{22}\, x_2 = \Delta_2 = 0$$
$$\Delta_{30} + f_{33}\, x_3 = \Delta_3 = 0$$

in which the coefficients are determined as follows:

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, dx}{EI}$$

$$= \frac{1}{2EI}\left[-36 \times 12 \times 3.6\right] + \frac{36 \times 6}{2EI} \times 2.4 + \frac{6}{EI}\left[\frac{3.6 - 2.4}{2}\right] \times (-36)$$

$$+ \frac{6}{6EI}(-36 \times 2 \times 3.6 + 2.5 \times 36)$$

$$= \frac{-777.6}{EI} + \frac{259.2}{EI} + \frac{-129.6}{EI} + \frac{-172.8}{EI} = \frac{-820.8}{EI} \text{ m}$$

$$f_{11} = \int \frac{m_{z1}^2\, dx}{EI} = \frac{2.4^2 \times 12}{EI} + \frac{3.6^2 \times 12}{2EI} + \frac{6 \times 2}{6EI}\left[2.4^2 \times 2 + 3.6^2 \times 2 - 2.4 \times 3.6 \times 2\right]$$

$$= \frac{146.88}{EI} + \frac{40.32}{EI} = \frac{187.2}{EI} \text{ m/kN}$$

$$x_1 = -\frac{\Delta_{10}}{f_{11}} = 4.385 \text{ kN}$$

$$\Delta_{20} = \int \frac{M_{z0}\, m_{z2}\, dx}{EI}$$

$$= \frac{-36 \times 6}{2EI} \times (-6) + 0 - \frac{36 \times 6 \times 6}{EI} - \frac{36 \times 6}{2EI} \times \left(\frac{2}{3} \times 6\right) = \frac{64.8 - 1296 - 432}{EI} = \frac{-1080}{EI} \text{ m}$$

$$f_{22} = \int \frac{m_{z2}^2\, dx}{EI} = \frac{6^2 \times 6}{EI} \times 2 + \frac{12}{6EI}\left(6^2 \times 2 + 36 \times 0\right) + \frac{12}{12EI}\left(6^2 \times 2 + 6^2 \times 2 - 6^2 \times 2\right)$$

$$= \frac{432}{EI} + \frac{144}{EI} + \frac{72}{EI} = \frac{648}{EI} \text{ m/kN}$$

$$x_2 = -\frac{\Delta_{20}}{f_{22}} = 1.666 \text{ kN}$$

$$\Delta_{30} = \int \frac{M_{z0}\, m_{z3}\, dx}{EI}$$

$$= \frac{+36 \times 6}{2EI} + \frac{36 \times 6}{EI} + \frac{36 \times 12}{2EI} + \frac{36 \times 6}{2EI} = \frac{648}{EI} \text{ rad}$$

$$f_{33} = \int \frac{m_{z3}^2\, dx}{EI} = \frac{1 \times 12}{EI} + \frac{1 \times 6}{EI} \times 2 + \frac{1 \times 12}{2EI} = \frac{30}{EI} \text{ rad/(kN.m)}$$

$$x_3 = \frac{-\Delta_{30}}{f_{33}} = \frac{-648}{30} = -21.6 \text{ kN.m}$$

The final bending moment is obtained from the superposition principle as

$$M = M_{z0} + x_1\, m_{z1} + x_2\, m_{z2} + x_3\, m_{z3}$$
$$M_A = -36 + 3.6 \times (4.385) - 6 \times (1.666) - 1 \times (-21.6) = -8.6 \text{ kN.m}$$
$$M_B = 0 - 2.4 \times (4.385) - 6 \times (1.666) - 1 \times (-21.6) = 1.074 \text{ kN.m}$$
$$M_C = -36 - 2.4 \times (4.385) + 6 \times (1.666) + 21.6 = -14.928 \text{ kN.m}$$
$$M_D = -36 + 3.6 \times (4.385) + 6 \times (1.666) + 21.6 = 11.388 \text{ kN.m}$$

The bending moment and shear force diagrams are shown in Figure 3.127.

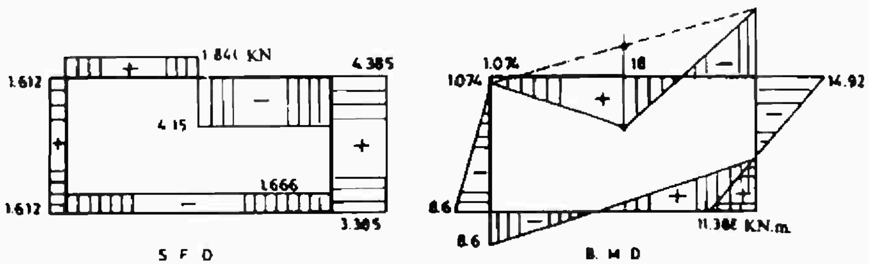


Figure 3.127

Example 3.30

Determine the bending moment diagram for the frame shown in Figure 3.128 using the elastic centre method.

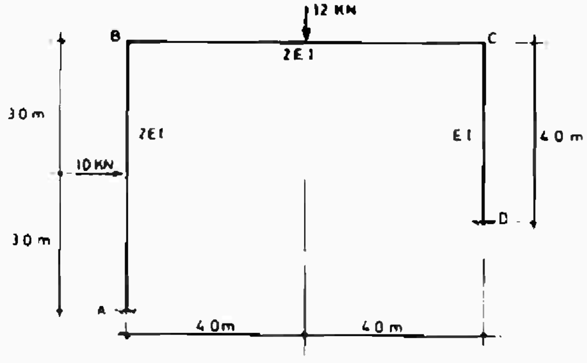


Figure 3.128

Solution

Determine the elastic centre by taking moments of the areas $\frac{d\ell}{EI}$ about lines AB and BC.

$$\bar{x} = \frac{\frac{8}{EI}\times 4 + \frac{4}{EI}\times 8}{\frac{6}{2EI}+\frac{8}{2EI}+\frac{4}{EI}} = \frac{48}{11} = 4.2636 \text{ m}$$

$$\bar{y} = \frac{\frac{4}{EI}\times 2 + \frac{6}{2EI}\times 3}{\frac{11}{EI}} = \frac{17}{11} = 1.545 \text{ m}$$

To determine the orientation of the principal axes, one calculates the values of I_x, I_y, I_{xy} and for the areas $\frac{d\ell}{EI}$ about x-y axes, using Figure 3.129. The calculated values are

$$I_x = \frac{31.06}{EI} \text{ m/kN} \quad ; \quad I_y = \frac{131.878}{EI} \text{ m/kN} \quad ; \quad I_{xy} = \frac{10.1819}{EI} \text{ m/kN}$$

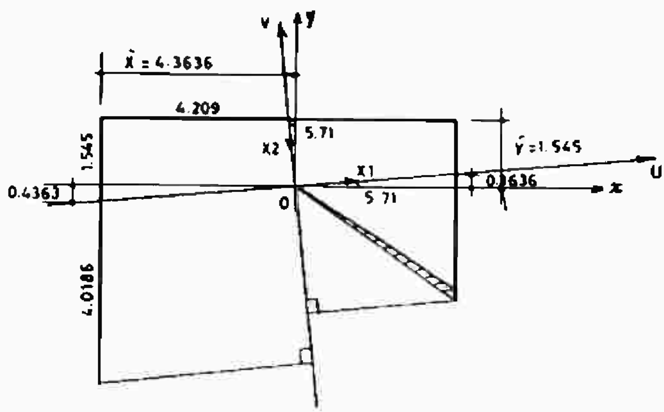


Figure 3.129

The direction of the principal axis is determined from

$$\tan 2\alpha = \frac{-2 I_{xy}}{I_x - I_y} = \frac{-2\times 10.1819}{31.06 - 131.878} = 0.2019858$$

$2\alpha = 11.419°$; $\alpha = 5.7096°$

The principal axes are called u and v in Figure 3.129. The bending moment M_{z0} for the primary structure is determined as shown in Figure 3.130. The bending moments m_{z1}, m_{z2}, and m_{z3} can also determined as shown in Figure 3.131.

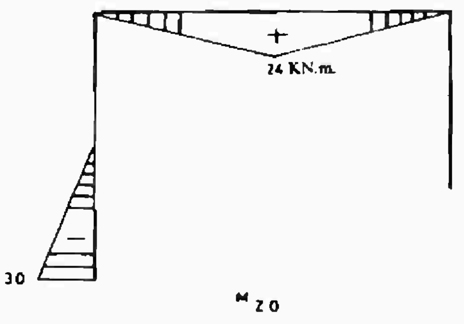


Figure 3.130

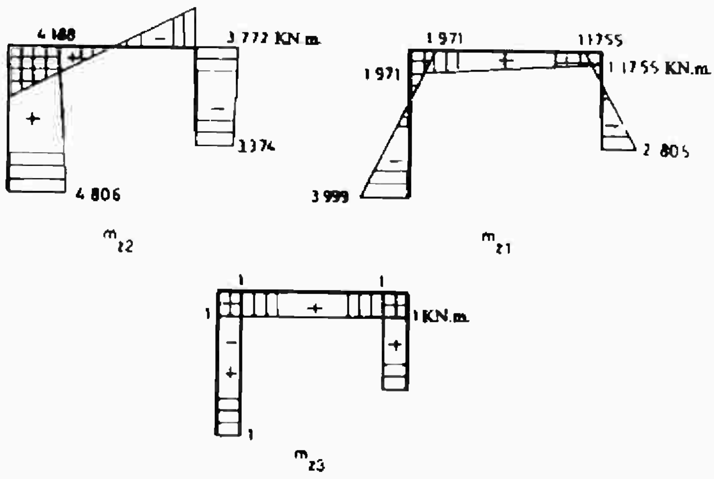


Figure 3.131

The coefficients of the consistent deformation equations are calculated as follows:

$$\Delta_{10} = \int \frac{M_{z0}\, m_{z1}\, d\ell}{EI}$$

$$= \frac{4}{2EI \times 6}\left(24 \times 1.573 \times 2 + 0 + 1.1755 \times 24\right)$$

$$+ \frac{4}{12EI}(24 \times 1.573 \times 2 + 24 \times 1.971)$$

$$+ \frac{3}{2EI \times 6}\left(2 \times 30 \times 3.999 + 30 \times 1.014\right) = \frac{143.098}{EI} \text{ m}$$

$$f_{11}=\int\frac{m_{z1}^2\,d\ell}{EI}=\frac{4}{6EI}\left[2.805^2\times 2+1.1755^2\times 2+2\times 1.1766\times(-2.805)\right]$$
$$+\frac{8}{6\times 2EI}\left(1.1755^2\times 2+1.971^2\times 2+2\times 1.971\times 1.1755\right)$$
$$+\frac{6}{6\times 2EI}\left[1.971^2\times 2+3.99^2\times 2-2\times 1.971\times 3.99\right]=\frac{30.042}{EI}\ \text{m/kN}$$

$$x_1=-\frac{\Delta_{10}}{f_{11}}=-4.763\ \text{kN}$$

$$\Delta_{20}=\int\frac{M_{z0}\,m_{z1}\,d\ell}{EI}$$
$$=\frac{4}{2EI\times 6}\left(0.208\times 24\times 2-3.772\times 24\right)+\frac{4}{2EI\times 6}\left(0.208\times 24\times 2+4.188\times 24\right)$$
$$+\frac{3}{6EI\times 2}\left(-30\times 2\times 4.806-4.497\times 30\right)=\frac{-95.8335}{EI}\ \text{m}$$

$$f_{22}=\int\frac{m_{z2}^2\,d\ell}{EI}=\frac{4}{6EI}\left(3.374^2\times 2+3.772^2\times 2+2\times 3.372\times 2\times 3.772\right)$$
$$+\frac{8}{2EI\times 6}\left(3.772^2\times 2+4.188^2\times 2-2\times 3\ 772\times 4.188\right)$$
$$+\frac{6}{2EI\times 6}\left(4.188^2\times 2+4.806^2\times 2+2\times 4.188\times 4.806\right)=\frac{133.176}{EI}\ \text{m/kN}$$

$$x_2=-\frac{\Delta_{20}}{f_{22}}=0.7195\ \text{kN}$$

$$\Delta_{30}=\int\frac{M_{z0}\,m_{z3}\,d\ell}{EI}$$
$$=\frac{24\times 8}{2\times 2EI}\times 1-\frac{30\times 3}{2\times 2EI}\times 1=\frac{25.5}{EI}\ \text{rad}$$

$$f_{33}=\int\frac{m_{z3}^2\,d\ell}{EI}=\frac{1\times 4}{EI}+\frac{1\times 8}{2EI}+\frac{1\times 6}{2EI}=\frac{11}{EI}\ \text{rad/(kN.m)}$$

$$x_3=\frac{-\Delta_{30}}{f_{33}}=-2.3181\ \text{kN.m}$$

The final bending moment is obtained using the superposition principle as

$$M=M_{z0}+x_1\,m_{z1}+x_2\,m_{z2}+x_3\,m_{z3}$$

$$M=M_{z0}-4.763\,m_{z1}+0.7195\,m_{z2}-2.3181$$

$$M_A = -30 - 4.763(-3.999) + 0.7195(4.806) - 2.3181 = -9.813 \text{ kN.m}$$

$$M_B = 0 - 4.763(1.971) + 0.7195(4.188) - 2.3181 = -8.692 \text{ kN.m}$$

$$M_C = 0 - 4.7631(1.1755) + 0.7195(-3.772) - 2.3181 = -1.631 \text{ kN.m}$$

$$M_D = 0 - 4.763(-2.805) + 0.7195(-3.374) - 2.3181 = +8.615 \text{ kN.m}$$

The bending moment diagram is shown in Figure 3.132.

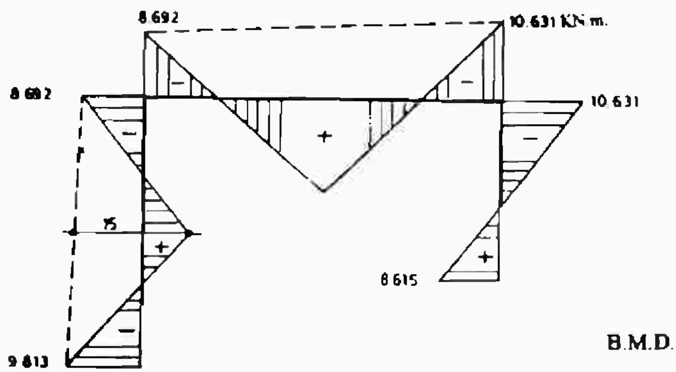


Figure 3.132

3.7 THE COLUMN ANALOGY METHOD

3.7.1 Introduction

As was presented in Example 3.29, the application of the elastic centre method is tedious in the case that any of the principal axes is not an axis of symmetry for the areas $(d\ell/EI)$. The column analogy method is developed from the elastic centre method. The method considers the areas $(d\ell/EI)$ as a cross section of a column and the bending moment of the primary structure M_{z0} as the applied pressure. The method can easily be applied to asymmetrical frames, fixed ended beams, and also to structures with one or two degrees of static indeterminacy.

3.7.2 Derivation of the Method

Consider the frame shown in Figure 3.133 where point O is the elastic centre and x, y are two arbitrary axes in the direction of the redundants x_1 and x_2. The bending moment m_{z1}, m_{z2} and m_{z3} due to unit values for the redundants x_1, x_2, and x_3 about a general point of coordinates (x, y) with respect to the elastic centre are respectively

$$m_{z1} = y \tag{3.43}$$

$$m_{z2} = x \tag{3.44}$$

$$m_{z3} = 1 \tag{3.45}$$

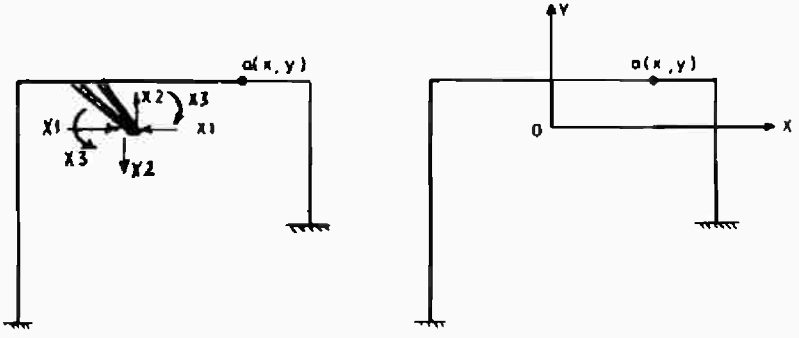


Figure 3.133

Notice that directions of x_1, x_2, and x_3 are assumed to give positive bending moment at the arbitrary point.

The flexibility coefficient f_{11} can then be calculated as follows:

$$f_{11} = \int \frac{m_{z1}^2\, d\ell}{EI} = \int y^2 \frac{d\ell}{EI} \tag{3.46}$$

which can be considered as the moment of inertia of the areas $(d\ell/EI)$ about the x-axis and is denoted by I_x.

Similarly, the flexibility coefficients f_{22}, and f_{33} are obtained as follows:

$$f_{22} = \int \frac{m_{z2}^2\, d\ell}{EI} = \int x^2 \frac{d\ell}{EI} \tag{3.47}$$

$$f_{33} = \int 1^2 \frac{d\ell}{EI} \tag{3.48}$$

where f_{22} is equal to the moment of inertia of the areas $(d\ell/EI)$ about y-axis and is denoted by I_y and f_{33} is the total areas of $(d\ell/EI)$ and is denoted by A.

The other flexibility coefficients f_{12}, f_{23}, and f_{31} are given by

$$f_{12} = \int xy \frac{d\ell}{EI} = I_{xy} \tag{3.49}$$

$$f_{23} = \int y \frac{d\ell}{EI} = 0 \tag{3.50}$$

$$f_{31} = \int x \frac{d\ell}{EI} = 0 \tag{3.51}$$

It is obvious that f_{23} and f_{31} are zero because the redundants are applied at the elastic centre which is the centroid of the section. Equation 3.49 represents the product moment of inertia of the area $(d\ell/EI)$ about the two cartesian axes x and y.

Substituting Equations 3.46 to 3.51 into the equations of consistent deformation, Equations 3.37, with considering the relative displacements Δ_1, Δ_2, and Δ_3 at the elastic centre are zero, one obtains

$$\Delta_{10} + I_x\, x_1 + I_{xy}\, x_2 = \Delta_1 = 0 \tag{3.52}$$

$$\Delta_{20} + I_{xy}\, x_1 + I_y\, x_2 = \Delta_2 = 0 \tag{3.53}$$

$$\Delta_{30} + A\, x_3 = \Delta_3 = 0 \tag{3.54}$$

The coefficients Δ_{10}, Δ_{20}, and Δ_{30} are determined as follows:

$$\Delta_{10} = \int M_{z0}\, m_{z1}\, \frac{d\ell}{EI} = \int y\, M_{z0}\, \frac{d\ell}{EI} \tag{3.55}$$

$$\Delta_{20} = \int M_{z0}\, m_{z2}\, \frac{d\ell}{EI} = \int x\, M_{z0}\, \frac{d\ell}{EI} \tag{3.56}$$

$$\Delta_{30} = \int M_{z0}\, m_{z3}\, \frac{d\ell}{EI} = \int M_{z0}\, \frac{d\ell}{EI} \tag{3.57}$$

The bending moment M_{z0} is regarded as a pressure applied on the areas $(d\ell/EI)$. Therefore, Δ_{30} is analogous to an axial force called N applied on the section. The coefficients Δ_{10} and Δ_{20} are thus the bending moments of N about x and y axes respectively. Equations 3.55 to 3.57 can then be written as

$$\begin{aligned} \Delta_{10} &= M_x \\ \Delta_{20} &= M_y \\ \Delta_{30} &= N \end{aligned} \tag{3.58}$$

Solving Equations 4.52 to 4.54 and using Equations 3.58, one obtains the magnitudes of the redundants x_1, x_2, and x_3 as follows:

$$x_1 = -\left(\frac{M_x\, I_y - M_y\, I_{xy}}{I_x\, I_y - I_{xy}^2}\right) \tag{3.59}$$

$$x_2 = -\left(\frac{M_y\, I_x - M_x\, I_{xy}}{I_x\, I_y - I_{xy}^2}\right) \tag{3.60}$$

$$x_3 = -\frac{N}{A} \tag{3.61}$$

If Δ_1, Δ_2, and Δ_3 are not zero, they must be transformed into equivalent values in terms of bending moments M_x, M_y, and axial force N. This shall be shown in section 3.7.4. The final bending moment at any point (x, y) is determined from the superposition principle as follows:

$$\begin{aligned} M_z(x,y) &= M_{z0} + x_1\, m_{z1} + x_2\, m_{z2} + x_3\, m_{z3} \\ &= M_{z0} + x_1\, x + x_2\, y + x_3 \end{aligned} \tag{3.62}$$

Substituting Equations 3.59 to 3.61 into Equation 3.62 one obtains

$$M_z(x,y) = M_{z0} - \left(\frac{M_x\, I_y - M_y\, I_{xy}}{I_x\, I_y - I_{xy}^2}\right) y - \left(\frac{M_y\, I_x - M_x\, I_{xy}}{I_x\, I_y - I_{xy}^2}\right) x - \frac{N}{A} \tag{3.63}$$

It is obvious that the terms in Equation 3.63 except the term M_{z0} represent the expression for normal stress at any point (x, y) in a column whose cross section area is the sum of the areas $(d\ell/EI)$. The sign conventions considered here are N is positive for positive M_{z0}. Equation 3.63 is only suitable for the assumed directions of x_1, x_2, and x_3 in Figure 3.125.

3.7.3 Hinges in the Column Analogy

The frictionless hinge can be represented by a section whose moment of inertia is zero. Therefore its $(d\ell/EI)$ is infinity. It means then the elastic centre lies at the hinge location which is the centroid of the infinite area. If the structure contains more than one hinge, then the elastic centre shall lie somewhere on the line connecting the two hinges. It is obvious that a structure with three hinges is statically determinate and there is no need to solve it by the column analogy method.

The roller support can be considered as a link connected with two hinges. Therefore, the elastic centre lies along the line of the reaction. The exact location of the elastic centre in this case is not important since it is not needed to solve the problem. Examples for the location of the elastic centre for structures with hinges and rollers are shown in Figure 3.134.

The column analogy formula in the case of structure with a hinge becomes

$$M_z(x,y) = M_{z0} - \left(\frac{M_x\, I_y - M_y\, I_{xy}}{I_x\, I_y - I_{xy}^2}\right) y - \left(\frac{M_y\, I_x - M_x\, I_{xy}}{I_x\, I_y - I_{xy}^2}\right) x \tag{3.64}$$

where A in this case is infinity and the term N/A is dropped out. The column analogy formula used in the case of a roller whose reaction, for example, along y axis becomes

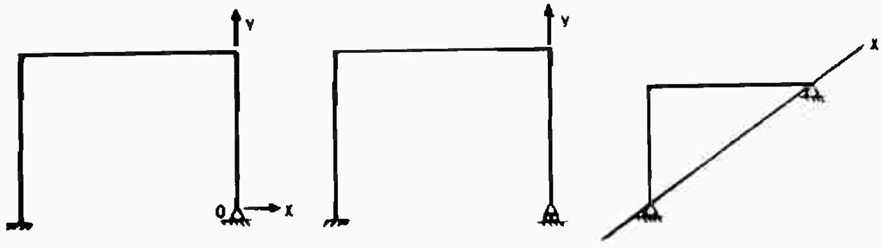


Figure 3.134

$$M_z(x,y) = M_{z0} - \left(\frac{M_y}{I_y}\right)x \tag{3.65}$$

since both A and I_x are infinity.

3.7.4 Settlements and Temperature Effects

From Equations 3.55 to 3.57, it is obvious that the displacements Δ_{10} and Δ_{20} were expressed as bending moment M_x and M_y and the rotation Δ_{30} as an axial force N. The bending moment is considered about the axis along which the deflection occurs. The axial force is considered positive if it causes tension on the bottom fibers of the member. One can thus consider the settlement as bending moment and the support rotation as axial force, in the column analogy method.

Consider for example the frame shown in Figure 3.133. If the support D has displaced values of Δ_1, Δ_2, and Δ_3 in the same directions assumed for the redundants, then one can consider the following bending moments and axial forces for the settlements in the support:

$$\begin{aligned} \Delta_1 &= M_{xs} \\ \Delta_2 &= M_{ys} \\ \Delta_3 &= N_s \end{aligned} \tag{3.66}$$

where the subscript s indicates settlement effect.

These values are substituted in the right hand side of equations 3.52 to 3.54. In the case of temperature, the deformation Δ_{io} is given by

$$\Delta_{io} = \alpha\left(\frac{T_1 + T_2}{2}\right)\int a_{xi}\, d\ell - \alpha\left(\frac{T_1 - T_2}{h}\right)\int m_{zi}\, d\ell \tag{3.67}$$

where a_{xi} is the axial force and m_{zi} is the bending moment due to redundant $x_i = 1$.

Neglecting the axial deformation term, one obtains

$$\Delta_{10} = -\alpha\left(\frac{T_1 - T_2}{h}\right)\int m_{z1}\, d\ell$$
$$= -\alpha\left(\frac{T_1 - T_2}{h}\right)\int y\, d\ell$$
$$\Delta_{20} = -\alpha\left(\frac{T_1 - T_2}{h}\right)\int m_{z2}\, d\ell \qquad (3.68)$$
$$= -\alpha\left(\frac{T_1 - T_2}{h}\right)\int x\, d\ell$$
$$\Delta_{30} = -\alpha\left(\frac{T_1 - T_2}{h}\right)\int m_{z3}\, d\ell$$
$$= -\alpha\left(\frac{T_1 - T_2}{h}\right)\int d\ell$$

where, the integrals are taken only for members subjected to temperature change. The integrals which contain x or y are evaluated as the length of the member times the ordinate of its centroid with respect to elastic centre. One should note that axial deformation due to temperature changes could be included in the column analogy using Equation 3.67, but this requires the determination of the axial force a_{xi} in the members subjected to temperature. This point shall be illustrated in example 3.33.

3.7.5 Numerical Examples

Example 3.31

Determine the bending moment diagram in the beam shown in Figure 3.135 due to:

(a) a vertical downward settlement D at A
(b) a unit clockwise rotation at A.

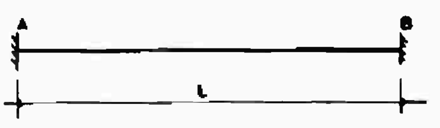


Figure 3.135

Solution

One first determines the properties of the column analogy which is shown in Figure 3.136. It is obvious that the elastic centre O is at the middle of the member AB. The properties of the column are:

$$A = \frac{L}{EI}\ \ kN^{-1}\, m^{-1}\ \ ;\ \ I_y = \frac{L^3}{12EI}\ \ kN^{-1}\, m^{-1}\ \ ;\ \ I_x = 0\ \ ;\ \ I_{xy} = 0$$

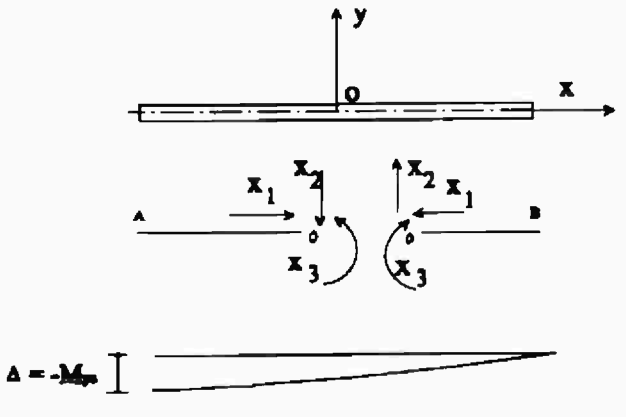


Figure 3.136

The settlement is expressed as a moment in the direction of x_2 for part A0 as shown in Figure 3.136. The forces and moments applied on the column cross section are:

$M_{ys} = +\Delta$, $N = 0$, $M_x = 0$, $M_{x0} = 0$

$M_y = M_{ys} = +\Delta$

Substituting into the column analogy equation one obtains

$$M_A = 0 - \frac{M_y}{I_y} x = -\frac{+\Delta}{\left(\frac{L^3}{12EI}\right)} \times \left(\frac{-L}{2}\right) = \frac{+6EI\Delta}{L^2} \text{ kN m}$$

$$M_B = 0 - \frac{+\Delta}{\left(\frac{L^3}{12EI}\right)} \times \left(\frac{L}{2}\right) = \frac{-6EI\Delta}{L^2} \text{ kN m}$$

The signs of the moment here follow the sign conventions presented in Chapter 2.

The bending moment diagram is shown in Figure 3.137, which is the same as the results of Example 3.24.

(c) The clockwise rotation is considered as an axial force N_s at A as shown in Figure 3.138. The sign is opposite to x_3 in part A0, therefore one has

$N_s = -1$ rad ; $N = N_s = -1$ rad

The moments are obtained using Figure 3.136 as follows.

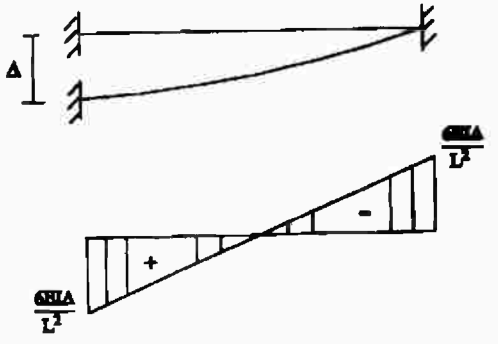


Figure 3.137

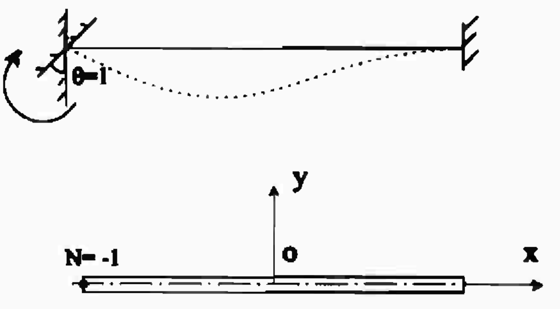


Figure 3.138

$$M_y = N\,x = -1 \times \left(\frac{-L}{2}\right) = \frac{L}{2}\ \text{kN.m}$$

Substituting into the column analogy equation one obtains

$$M_A = -\frac{N}{A} - \frac{M_y}{I_y}\,x$$

$$= -\frac{(-1)}{\left(\frac{L}{EI}\right)} - \frac{\left(\frac{L}{2}\right)}{\left(\frac{L^3}{12EI}\right)}\left(\frac{-L}{2}\right) = +\frac{EI}{L} + \frac{3EI}{L} = \frac{4EI}{L}\ \text{kN.m}$$

$$M_B = \frac{1}{\left(\frac{L}{EI}\right)} - \frac{\left(\frac{L}{2}\right)}{\left(\frac{L^3}{12EI}\right)}\left(\frac{L}{2}\right) = -\frac{2EI}{L}\ \text{kN.m}$$

The bending moment diagram is given in Figure 3.139 which is the same as the results of Example 3.23.

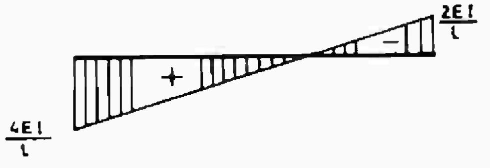


Figure 3.139

Example 3.32

Determine the bending moment diagram for the frame shown in Figure 3.140 due to the applied loading shown.

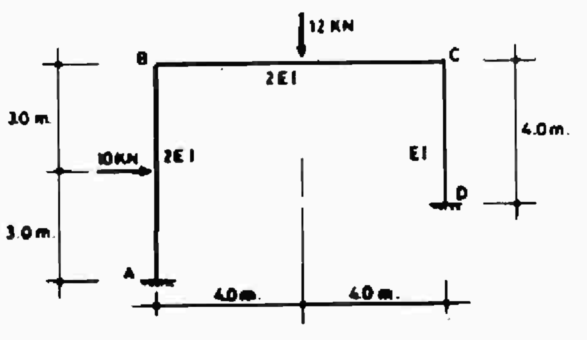


Figure 3.140

Solution

This example was solved before, example 3.30, using the elastic centre method. It shall be solved here again using the column analogy. One first determines the elastic centre location at O and the properties of the column cross section shown in Figure 3.141.

$$A = \frac{6}{2EI} + \frac{8}{2EI} + \frac{4}{EI} = \frac{11}{EI}\ \text{kN}^{-1}\ \text{m}^{-1}$$

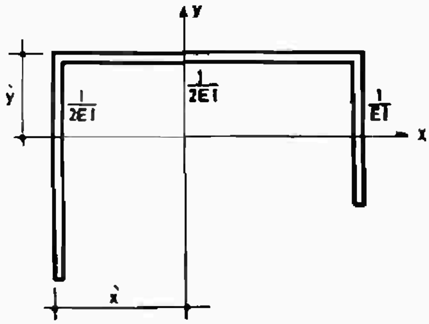


Figure 3.141

$$\bar{x} = \frac{\frac{8}{2EI}\times 4 + \frac{4}{EI}\times 8}{\frac{11}{EI}} = \frac{48}{11} = 4.3636 \text{ m}$$

$$\bar{y} = \frac{\frac{4}{EI}\times 2 + \frac{6}{2EI}\times 3}{\frac{11}{EI}} = \frac{17}{11} = 1.545 \text{ m}$$

$$I_x = \frac{8}{2EI}(1.545)^2 + \left(\frac{4}{EI}\times\frac{4^2}{12}\right) + \frac{4}{EI}(2-1.545)^2 + \frac{6}{2EI}\times\frac{6^2}{12} + \frac{6}{2EI}(3-1.545)^2$$
$$= \frac{31.06}{EI} \text{ kN}^{-1}\text{ m}$$

$$I_y = \frac{8}{2EI}(4-4.3636)^2 + \frac{4}{EI}(8-4.3636)^2 + \frac{8}{2EI}\left(\frac{8^2}{12}\right) + \frac{6}{2EI}(4.3636)^2$$
$$= \frac{131.878}{EI} \text{ kN}^{-1}\text{ m}$$

$$I_{xy} = \frac{4}{EI}(8-4.3636)(-2+1.545) + \frac{8}{2EI}(-0.3636)(1.545) + \frac{6}{2EI}(-4.3636)(-3+1.545)$$
$$= \frac{10.1819}{EI} \text{ kN}^{-1}\text{ m}$$

The normal forces and moments applied on the column due to the loading M_{z0} of Figure 3.142 are obtained as follows:

$$N_1 = \frac{24\times 8}{2}\times\frac{1}{2EI} = \frac{48}{EI}$$

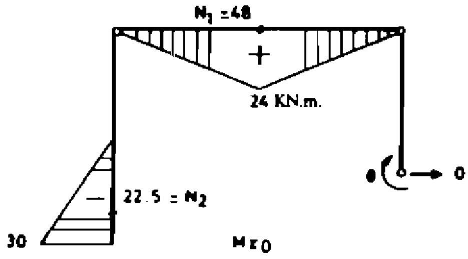


Figure 3.142

$$N_2 = -\frac{30 \times 3}{2} \times \frac{1}{2EI} = -\frac{22.5}{EI}$$

$$N = N_1 + N_2 = \frac{48}{EI} - \frac{22.5}{EI} = \frac{25.5}{EI}$$

$$M_x = N_1 \times 1.545 + N_2 \times (-6 + 1.545 + 1) = \frac{151.897}{EI} \text{ m}$$

$$M_y = N_1 (4 - 4.3636) + N_2 (-4.3646) = \frac{80.728}{EI} \text{ m}$$

Substituting into the column analogy equation, Equation 3.64, one has

$$M_z = M_{z0} - \frac{N}{A} - \left(\frac{M_x I_y - M_y I_{xy}}{I_x I_y - I_{xy}^2} \right) y - \left(\frac{M_y I_x - M_x I_{xy}}{I_x I_y - I_{xy}^2} \right) x$$

where

$$I_x I_y - I_{xy}^2 = 4096.13 - 103.6711 = 3992.249 \text{ kN}^{-1} \text{ m}$$

$$M_x I_y + M_y I_{xy} = 19209.908 \text{ kN}^{-1} \text{ m}$$

$$M_y I_x + M_x I_{xy} = 960.8116 \text{ kN}^{-1} \text{ m}$$

Therefore, the bending moment is calculated as follows:

$$M_z = M_{z0} - \frac{25.5}{11} - 4.8118\, y - 0.24066\, x$$

$$M_A = -30 - \frac{25.5}{11} - 0.24066 (-4.3636) - 4.8118\ (-6 + 1.545) = -9.831 \text{ kN.m}$$

$$M_B = 0 - \frac{25.5}{11} - 0.24066 (-4.3636) - 4.8118\ (+1.545) = -8.702 \text{ kN.m}$$

$$M_C = 0 - \frac{25.5}{11} - 0.24066 (8 - 4.3636) - 4.8118\ (1.545) = -10.627 \text{ kN.m}$$

$$M_D = 0 - \frac{25.5}{11} - 0.24066(8 - 4.3636) - 4.8118\ (-4 + 1.545) = +8.6195\ \text{kN.m}$$

The bending moment diagram is shown in Figure 3.143, which is the same as obtained in Example 3.30.

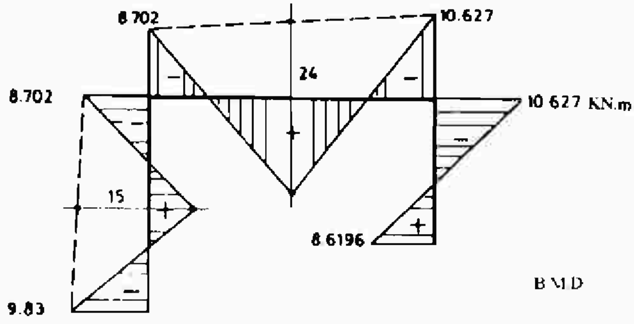


Figure 3.143

Example 3.33

Determine the bending moment diagram for the frame shown in Figure 3.144 due to the applied loads and a rise in temperature for member BC shown (EI = 10^5 kN.m^2, α = 10^{-5}/°c).

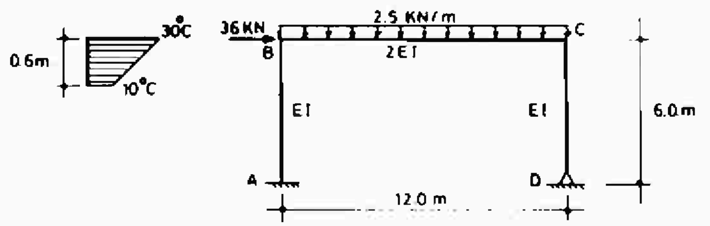


Figure 3.144

Solution

The elastic centre is at the hinge D. The properties of the column cross section are obtained using Figures 3.145 and 3.146 as follows:

$A = \infty$

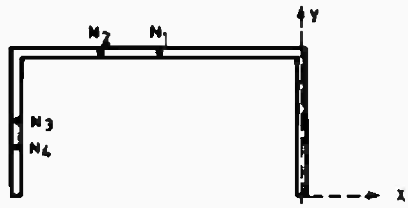


Figure 4.145

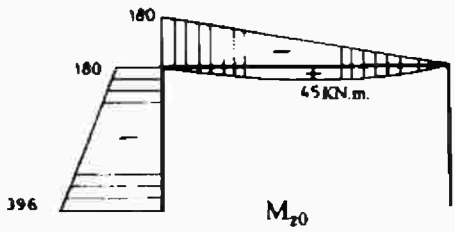


Figure 3.146

$$I_x = \frac{6}{EI} \times \frac{6^2}{3} \times 2 + \frac{12}{2EI}(6)^2 = \frac{360}{EI} \ \text{kN}^{-1}\ \text{m}$$

$$I_y = \frac{12}{2EI} \times \frac{12^2}{3} + \frac{6}{EI}(12)^2 = \frac{1152}{EI} \ \text{kN}^{-1}\ \text{m}$$

$$I_{xy} = \frac{12}{2EI}(-6)(6) + \frac{6}{EI}(-12)(3) = \frac{-432}{EI} \ \text{kN}^{-1}\ \text{m}$$

$$N_1 = \frac{2}{3} \times 45 \times \frac{12}{2EI} = \frac{180}{EI}$$

$$N_2 = \frac{-180 \times 12}{2} \times \frac{1}{2EI} = \frac{540}{EI}$$

$$N_3 = 180 \times 6 \ \frac{1}{EI} = -\frac{1080}{EI}$$

$$N_4 = \frac{216 \times 6}{2EI} = -\frac{648}{EI}$$

$$M_x = (N_1 + N_2)6 + N_3 \times 3 + N_4 \times 2 = \frac{-6696}{EI} \ \text{kN m}$$

$$M_y = N_1 \times (-6) + N_2 \times (-8) + (N_3 + N_4) \times (-12) = \frac{23976}{EI} \text{ m}$$

Temperature Effect

Neglecting axial deformation due to temperature change, one has

$$M_x = -\alpha \left(\frac{T_1 - T_2}{h} \right) \int_{BC} y \, d\ell$$

$$= -\alpha \left(\frac{30-10}{0.6} \right) (12 \times 6) = -2400 \times 10^{-5} \text{ m}$$

$$M_y = -\alpha \left(\frac{T_1 - T_2}{h} \right) \int_{BC} x \, d\ell$$

$$= -\alpha \left(\frac{30-10}{0.6} \right) \times 12 \times (-6) = 2400 \times 10^{-5} \text{ m}$$

Therefore, the total moments due to the loading and temperature effect are

$$M_x = -6696 \times 10^{-5} - 2400 \times 10^{-5} = -9096 \times 10^{-5} \text{ m}$$

$$M_y = 23976 \times 10^{-5} + 2400 \times 10^{-5} = 26376 \times 10^{-5} \text{ m}$$

To substitute into the column analogy expression given in Equation 3.63, one should calculate the following terms:

$$I_x I_y - I_{xy}^2 = 228096 \times 10^{-10} \text{ kN}^{-2} \text{ m}^2$$

$$M_x I_y - M_y I_{xy} = 915840 \times 10^{-10} \text{ kN}^{-2} \text{ m}^2$$

$$M_y I_x - M_x I_{xy} = 5565888 \times 10^{-10} \text{ kN}^{-2} \text{ m}^2$$

Equation 3.63 gives

$$M_z = M_{z0} - \left(\frac{915840}{228096} \right) y - \left(\frac{5565888}{228096} \right) x$$

$$M_A = -396 - 4.015(0) - 24.401(-12) = -103.188 \text{ kN.m}$$

$$M_B = -180 - 4.015(6) - 24.401(-12) = 88.722 \text{ kN.m}$$

$$M_C = 0 - 4.015(6) - 24.401(0) = -24.09 \text{ kN.m}$$

$$M_D = 0 - 4.015(0) - 24.401(0) = 0$$

If one wishes to include the axial deformation due to the change in temperature, one should use equation 3.67 instead of equations 3.68. In this case,

$$M_x = -\alpha\left(\frac{T_1 - T_2}{h}\right)\int_{BC} y\, d\ell + \alpha\left(\frac{T_1 + T_2}{2}\right)\int_{BC} a_{x1}\, d\ell$$

which a_{x1} is the axial force in member BC due to x_1 = 1 kN.

$$M_x = -\alpha\left(\frac{30-20}{0.6}\right)(12\times 6) + \alpha\left(\frac{30+10}{2}\right)(1\times 12) = -2160\times 10^{-5}\ \text{m}$$

Similarly, M_y is calculated as follows:

$$M_y = -\alpha\left(\frac{T_1 - T_2}{h}\right)\int_{BC} x\, d\ell + \alpha\left(\frac{T_1 + T_2}{2}\right)\int_{BC} a_{x2}\, d\ell$$

where a_{x2} is the axial force in member BC due to x_2 = 1 kN.

$$M_y = -\alpha\left(\frac{30-10}{0.6}\right)((-6)\times 12) + \alpha\left(\frac{30+10}{2}\right)(0\times 12) = 2400\times 10^{-5}\ \text{m}$$

Therefore, the moments due to loading and temperature effect, including axial deformation, are given by

$$M_x = -6696\times 10^{-5} - 2160\times 10^{-5} = -8856\times 10^{-5}\ \text{m}$$

$$M_y = 23976\times 10^{-5} + 2400\times 10^{-5} = 26376\times 10^{-5}\ \text{m}$$

Substituting into the column analogy equation one obtains

$$M_x I_y - M_y I_{xy} = 1192320\times 10^{-10}\ \text{kN}^{-2}\ \text{m}^2$$

$$M_y I_x - M_x I_{xy} = 5669568\times 10^{-10}\ \text{kN}^{-2}\ \text{m}^2$$

$$M_z = M_{z0} - \left(\frac{1192320}{228096}\right)y - \left(\frac{566958}{228096}\right)x$$

$$M_A = -396 - 5.227(0) - 24.856(-12) = -97.73\ \text{kN.m}$$

$$M_B = -180 - 5.227(6) - 24.856(-12) = 86.91\ \text{kN.m}$$

$$M_C = 0 - 5.227(6) - 24.856(0) = -31.36\ \text{kN.m}$$

$M_D = \quad 0 \quad -5.227(0) - 24.856(0) \quad = 0$

It is obvious the influence of ignoring axial deformations due to temperature change. These results can be compared with the case of including axial deformation treated in example 3.12. Since the results of example 3.12 almost the same as the case of including axial deformations due to temperature effect, it is concluded that axial deformations due to temperature should not be neglected in any structural analysis problem. The bending moment diagrams of this example for both cases of temperature effect on axial deformation are shown in Figure 3.147.

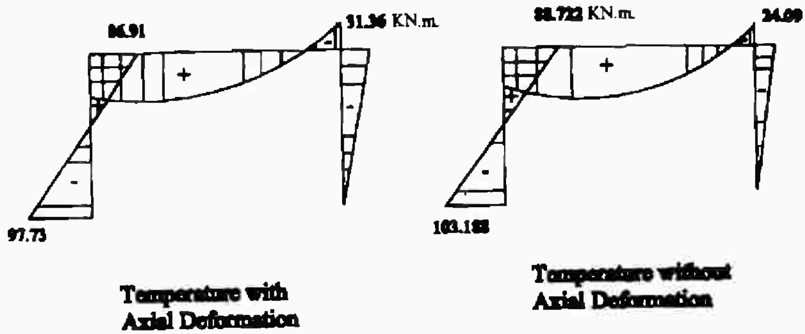


Figure 3.147

Example 3.34

Determine the bending moment diagram for the frame shown in Figure 3.148 using the column analogy method (EI = 10^3 kN.m^2).

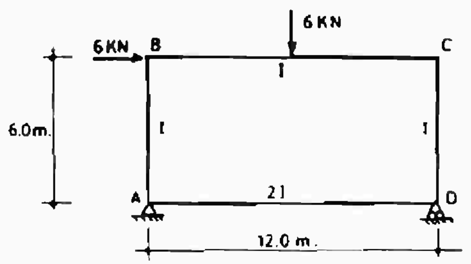
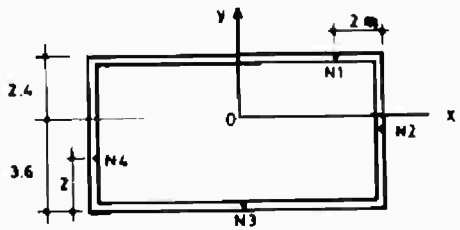


Figure 3.148

Solution

The elastic centre of this frame was determined in Example 3.27. The primary structure and the column cross section are shown in Figure 3.149.



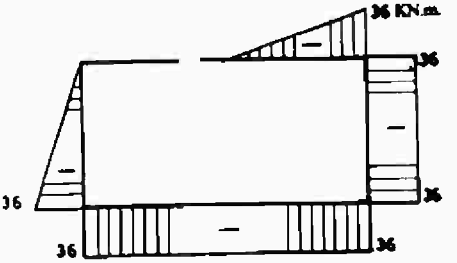


Figure 3.149

For the chosen coordinates x and y at the elastic centre properties of the column cross section are:

$$A = \frac{12}{EI} + \frac{6}{EI} + \frac{6}{EI} + \frac{12}{2EI} = \frac{30}{EI}$$

$$I_x = \frac{6}{EI} \times \frac{6^2}{12} \times 2 + \frac{12}{EI} \times (2.4)^2 + \frac{12}{2EI} \times (3.6)^2 + \frac{6}{2EI}(0.6)^2 \times 2 = \frac{187.2}{EI}$$

$$I_y = \frac{12}{EI} \times \frac{12^2}{12} + \frac{12}{2EI} \times \frac{12^2}{2EI} + \frac{6}{EI} \times 6^2 \times 2 = \frac{648}{EI}$$

$$I_{xy} = 0$$

From M_{z0} one can determine the analogous forces and moments as follows:

$$N_1 = \frac{-36 \times 6}{2EI} = \frac{-108}{EI}$$

$$N_2 = -36 \times \frac{6}{2EI} = -\frac{216}{EI}$$

$$N_3 = -36 \times \frac{12}{2EI} = -\frac{216}{EI}$$

$$N_4 = -36 \times \frac{6}{2EI} = -\frac{108}{EI}$$

$$N = N_1 + N_2 + N_3 + N_4 = -\frac{648}{EI}$$

$$M_x = 2.4\,N_1 - 0.6\,N_2 - 1.6\,N_4 - 3.6\,N_3 = \frac{820.8}{EI}$$

$$M_y = 4N_1 + 6N_2 - 6N_4 = \frac{1080}{EI}$$

The final bending moment is determined from:

$$M_z = M_{z0} - \frac{N}{A} - \frac{M_x}{I_x}y - \frac{M_y}{I_y}x$$
$$= M_{z0} + 21.6 - 4.3846y - 1.6667x$$

$$M_A = -36 + 21.6 - 4.3646(-3.6) + 1.6667(-6) = -8.615 \text{ kN.m}$$

$$M_B = 0 + 21.6 - 4.3646(2.4) + 1.6667(-6) = 1.0769 \text{ kN.m}$$

$$M_C = -36 + 21.6 - 4.3646(2.4) + 1.6667(6) = -14.875 \text{ kN.m}$$

$$M_D = -36 + 21.6 - 4.3646(-3.6) + 1.6667(6) = 11.312 \text{ kN.m}$$

The bending moment diagram is given in Figure 3.150, which is the same as in example 3.27.

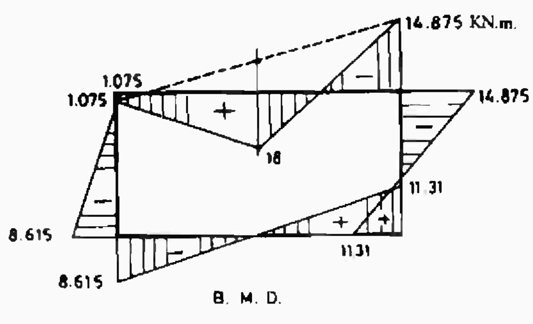


Figure 3.150

Example 3.35

Determine the bending moment diagram of the frame of example 3.14 due to a vertical settlement at D of 5 cm downward ($EI = 10^5$ kN.m^2).

Solution

The elastic centre of this frame was determined in example 3.28 and is shown in Figure 3.119. The properties of the column cross section are determined using Figure 3.151 as follows:

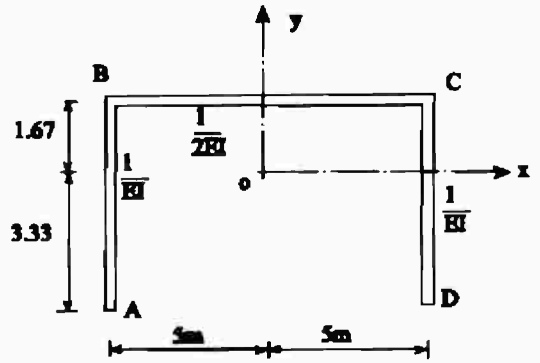


Figure 3.151

$$A = \frac{5}{EI} + \frac{10}{2EI} + \frac{5}{EI} = \frac{15}{EI}$$

$$I_x = \frac{10}{EI}(1.67)^2 + \frac{1.67}{EI}\frac{(1.67)^2}{3} \times 2 + \frac{3.33}{EI} \times \frac{(3.33)^2}{EI} \times 2 = \frac{41.67}{EI}$$

$$I_y = \frac{10}{2EI} \times \frac{10^2}{12} + \frac{5}{EI} \times 5^2 \times 2 = \frac{291.667}{EI}$$

$$I_{xy} = 0$$

The settlement at D is in opposite direction to x_2 direction for the right part connected with elastic centre, then $M_y = M_{ys} = -0.05m$, $M_x = 0$ and $N = 0$. Substituting into Equation 3.63 one has

$$M_z = M_{z0} - \frac{M_x}{I_x} y - \frac{M_y}{I_y} x - \frac{N}{A}$$
$$= 0 - 0 - \frac{(-0.05)EI}{291.667} x = \frac{0.05EI}{21.667} x$$

$$M_A = \frac{0.05 \times 10^5}{291.667} \times (-5) = -85.71 \text{ kN.m}$$

$$M_B = -85.71 \text{ kN.m}$$

$$M_C = \frac{0.05 \times 10^5}{291.667} \times (+5) = 85.71 \text{ kN.m}$$

$$M_D = 85.71 \text{ kN.m}$$

which are the same results determined in example 3.14 and 3.28.

Example 3.36

Determine the bending moment diagram for the frame of example 3.14 due to a clockwise rotation at B of 0.001 radian by the methods of elastic centre and column analogy.

Solution

a) By elastic centre method

The moment diagrams for m_{x1}, m_{x2}, and m_{x3} were given in Figure 3.120. The correlation between the support rotation and the redundants directions is given in Figure 3.152. The right hand side of the consistent deformation equations Δ_1, Δ_2, and Δ_3 are given by

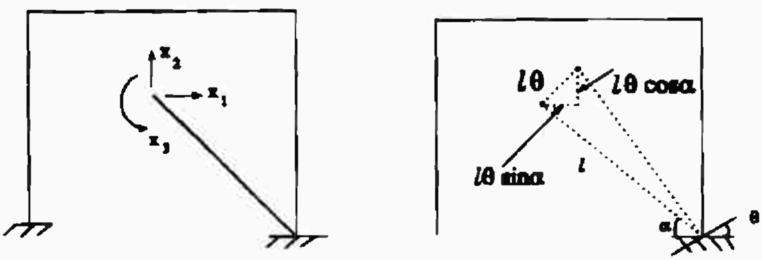


Figure 3.152

$$\Delta_1 = +\ell\theta \sin\alpha = 3.33\theta$$
$$\Delta_2 = +\ell\theta \cos\alpha = 5\theta$$
$$\Delta_3 = -\theta$$

where $\theta = 0.001$ radian and $\ell = \sqrt{3.332 + 5^2} = 6$ m.

Recalling from example 3.28 the values of f_{11}, f_{22}, and f_{33}, one obtains the values of the redundants as follows:

$$x_1 = \frac{\Delta_1}{f_{11}} = \frac{3.33\theta}{(41.67/EI)} = 7.99 \text{ kN}$$

$$x_2 = \frac{\Delta_2}{f_{22}} = \frac{5\theta}{(291.67/EI)} = 1.714 \text{ kN}$$

$$x_3 = \frac{\Delta_3}{f_{33}} = \frac{-\theta}{(15/EI)} = -6.667 \text{ kN.m.}$$

The bending moment in the frame is obtained from

$$M_A = 0 + 7.99 \times (-3.33) + 1.714 \times 5 - 6.667 \times 1 = -24.7 \text{ kN.m.}$$

$$M_B = 0 + 7.99(1.67) + 1.714(5) - 6.667 \times 1 = 15.24 \text{ kN.m.}$$

$$M_C = 0 + 7.99(1.67) + 1.714(-5) - 6.667 \times 1 = -1.89 \text{ kN.m.}$$

$$M_D = 0 + 7.99(-3.33) + 1.714(-5) - 6.667 \times 1 = -41.84 \text{ kN.m.}$$

The bending moment diagram is shown in Figure 3.153.

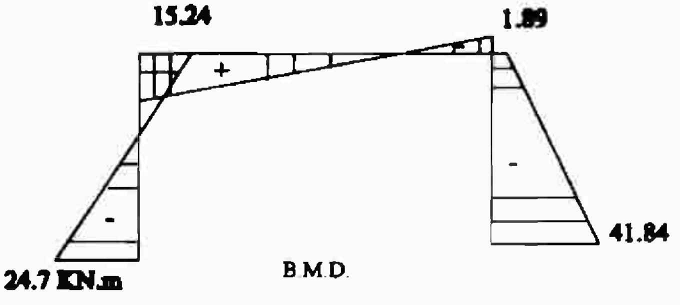


Figure 3.153

b) By column analogy

The properties of the column have been determined in example 3.35. The rotation at D is considered as axial force at D with positive sign (N = + θ) which is in agreement with the sign of x_3 in Figure 133. Therefore, one has

$$N = N_s = \theta$$

$$M_x = -3.33 \times \theta$$

$$M_y = 5 \times \theta$$

where θ = 0.001 rad.

Substituting into Equation 3.63 one obtains

$$M_z = 0 - \frac{\theta}{(15/EI)} - \frac{-3.33\theta)\,y}{(41.67/EI)} - \frac{(5\theta)\,x}{(291.667/EI)} = -\frac{100}{15} + 7.99\,y - 1.714\,x$$

$$M_A = -\frac{100}{15} + 7.99(-3.33) - 1.714(-5) = -24.7 \text{ kN.m}$$

$$M_B = -\frac{100}{15} + 7.99(+1.67) - 1.714(-5) = 15.24 \text{ kN.m}$$

$$M_C = -\frac{100}{15} + 7.99(+1.67) - 1.714(5) = -1.89 \text{ kN.m}$$

$$M_D = -\frac{100}{15} + 7.99(-3.33) - 1.714(5) = -41.84 \text{ kN.m}$$

which are the same results obtained by the method of elastic centre.

Example 3.37

Determine the fixed end moments for the beam shown in Figure 3.154 due to a unit anticlockwise rotation at A.

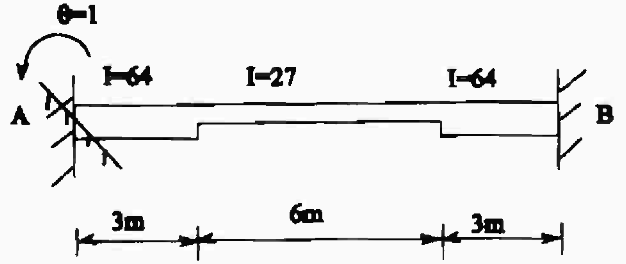


Figure 3.154

Solution

The column analogy is shown in Figure 3.155. The rotation is applied as normal force with positive direction in agreement with x_3 direction.

$$N = N_s = 1$$

$$M_y = -6 \times 1 = -6$$

$$A = \frac{1}{64E} \times 3 \times 2 + \frac{1}{27E} \times 6 = \frac{0.316}{E}$$

$$I_y = \frac{1}{27E} \times \frac{6^3}{12} + 2\left[\frac{1}{64E} \times \frac{3^3}{12} + \frac{3}{64E} \times 4.5^2\right] = \frac{2.635}{E}$$

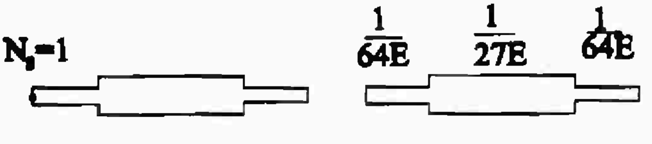


Figure 3.155

$$M = -\frac{N}{A} - \frac{M_y}{I_y} x = -\frac{E}{0.316} - \frac{-6E}{2.635} x = -3.164E + 2.277\,E\,x$$

$$M_A = -3.164E + 2.277E(-6) = -16.826\,E \ \text{kN.m.}$$

$$M_B = -3.164E + 2.277E(+6) = 10.498\,E \ \text{kN.m.}$$

The values of M_A and M_B are called stiffness coefficients and shall be used in Chapter 4. The ratio between M_B and M_A is called carry over factor. In this example, the carry over factor is 0.624.

Example 3.38

Determine the bending moment diagram for the frame shown in Figure 3.156 due to rise in temperature in members BC and CD as shown. $EI = 10^5$ kN.m^2, $\alpha = 10^{-5}/°c$, EI is constant for all members

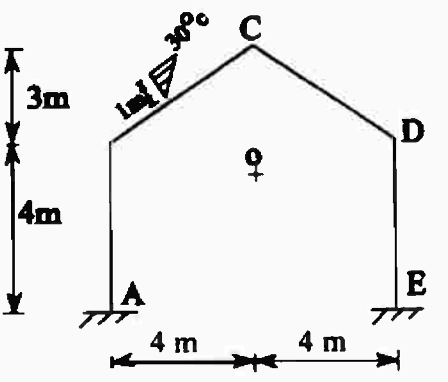


Figure 3.156

Solution

The elastic centre is determined as the centroid of $d\ell/EI$ areas. The centre is located along the axis of symmetry and at distance $\bar{y}$ from line AE as

$$\bar{y} = \frac{\frac{4}{EI} \times 2 \times 2 + \frac{5}{EI} \times 2 \times 5.5}{\frac{4}{EI} \times 2 + \frac{5}{EI} \times 2} = 3.944 \ \text{m}$$

$$I_x = \frac{2}{EI}\left[\frac{4^3}{12} + 4\,(1.944)^2\right] + \frac{2}{EI}\left[\frac{5\times 3^2}{12} + 5\,(1.5+0.056)^2\right] = \frac{72.62}{EI}$$

$$I_y = \frac{2}{EI}\left[4\times(4)^2 + 5\times\frac{4^2}{12} + 5\,(2)^2\right] = \frac{181.33}{EI}$$

$I_{xy} = 0$ (due to symmetry)

$$A = \frac{4}{EI}\times 2 + \frac{5}{EI}\times 2 = \frac{18}{EI}$$

$$M_{xT} = \alpha\left(\frac{T_1+T_2}{2}\right)\int_{\substack{BC,\\CD}} a_{x_1}\,d\ell - \alpha\left(\frac{T_1-T_2}{h}\right)\int_{\substack{BC,\\CD}} y\,d\ell$$

Diagrams of a_{x_1} and a_{x_2} for members BC and CD are shown in Figure 3.157.

$$M_{xT} = \alpha\,(15)\,(0.8\times 5\times 2) - \alpha\left(\frac{30}{1}\right)(5\times 1.556\times 2) = -346.8\alpha$$

$$M_{yT} = \alpha\left(\frac{T_1+T_2}{2}\right)\int_{\substack{BC,\\CD}} a_{x_2}\,d\ell - \alpha\left(\frac{T_1-T_2}{h}\right)\int_{\substack{BC,\\CD}} x\,d\ell = \text{zero (due to symmetry)}$$

$$N_T = -\alpha\left(\frac{T_1-T_2}{h}\right)\int_{\substack{BC,\\CD}} 1\,d\ell = -\alpha\left(\frac{30}{1}\right)(5+5) = -300\,\alpha$$

Substituting into column analogy equation one has

$$M_z = M_{z0} - \left(\frac{N}{A} + \frac{M_x}{I_x}\,y\right)$$

$$= -\left(\frac{-300\,\alpha}{18/EI} - \frac{346.8\,\alpha}{72.623/EI}\,y\right) = 16.67 + 4.775\,y$$

$$M_{ZA} = 16.67 + 4.775\,(-3.944) = -2.163 \text{ kN.m}$$

$$M_{ZB} = 16.67 + 4.775\,(+0.056) = 16.937 \text{ kN.m}$$

$$M_{ZC} = 16.67 + 4.775\,(3.056) = 31.262 \text{ kN.m}$$

The bending moment diagram is given in Figure 3.158.

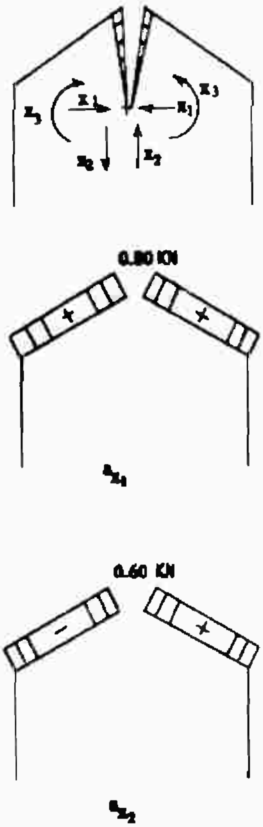


Figure 3.157

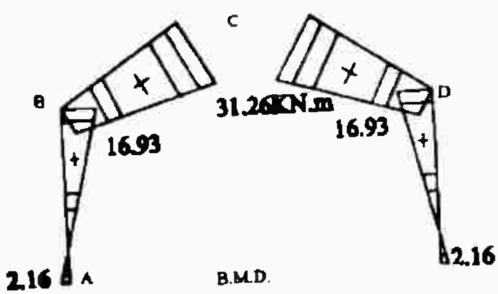


Figure 3.158

Example 3.39

Determine the bending moment diagram for the frame shown in Figure 3.159 due to the applied loads and the settlement at C of 1 cm downward. $EI = 10^5$ kN.m^2.

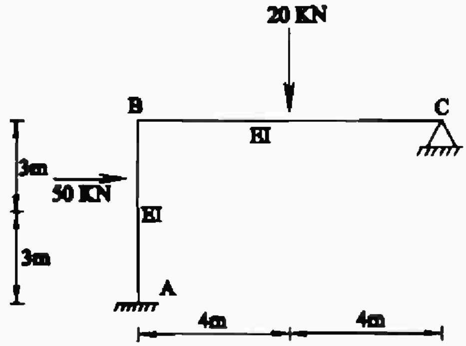


Figure 3.159

Solution

The column analogy properties and the elastic center are determined, using Figure 3.160, as follows:

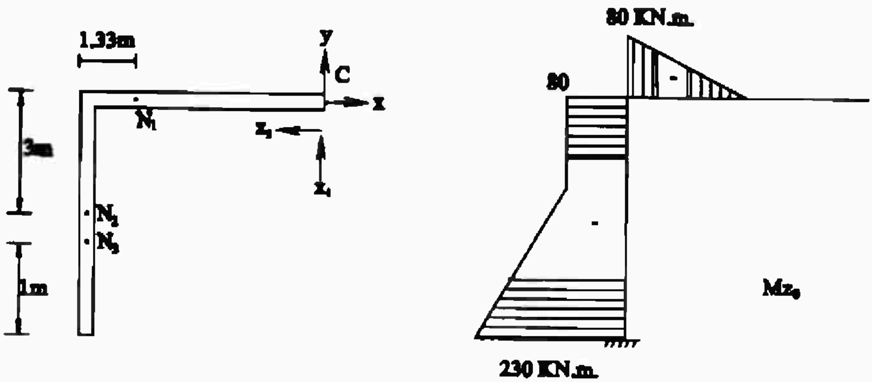


Figure 3.160

$A = \infty$

$$I_x = \frac{6^3}{3EI} = \frac{72}{EI}$$

$$I_y = \frac{8^3}{3EI} + \frac{6}{EI}\left(8^2\right) = \frac{554.67}{EI}$$

$$I_{xy} = \frac{6}{EI} \times (-8) \times (-3) = \frac{144}{EI}$$

$$N_1 = -\frac{80 \times 4}{2EI} = -\frac{160}{EI}$$

$$N_2 = -80 \times \frac{6}{EI} = -\frac{480}{EI}$$

$$N_3 = -\frac{150 \times 3}{2EI} = -\frac{225}{EI}$$

$$M_x = N_2 \times (-3) + N_3 \times (-5) = \frac{2565}{EI}$$

$$M_y = N_1 \times (1.33 - 8) + N_2 \times (-8) + N_3 \times (-8) = \frac{6706.67}{EI}$$

The settlement at C is downward with respect to y-axis, therefore, $M_{yn} = -0.01$ m.

The moment due to loading and settlement is

$$M_y = -0.01 + \frac{6706.67}{EI} = 5706.67 \times 10^{-5}$$

Substituting into Equation 3.64 one has

$$M_z = M_{z0} - \left(\frac{M_x I_y - M_y I_{xy}}{I_x I_y - I_{xy}^2} \right) y - \left(\frac{M_y I_x - M_x I_{xy}}{I_y I_x - I_{xy}^2} \right) x$$

$$= M_{z0} - 31.3\, y - 2.1625\, x$$

M_A = -230 – 31.3 × (-6) – 2.1625 × (-8) = -24.9 kN.m.

M_B = -80 – 31.3 × (-0) – 2.1625 × (-8) = -62.7 kN.m.

M_C = 0

The bending moment diagram is given in Figure 3.161 which is the same results as example 4.13.

3.8 THE FLEXIBILITY MATRIX METHOD : APPROACH I

3.8.1 Introduction

It was shown in Section 3.3.1 that the equations of consistent deformation can be expressed in a matrix form as

$$\underline{\Delta} = \underline{\Delta}_0 + [f]\,\underline{X} \qquad (3.69)$$

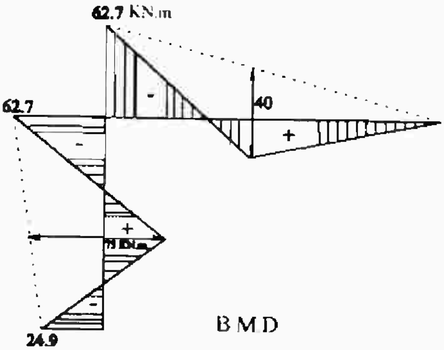


Figure 3.161

where $\underline{\Delta} = [\Delta_1 \ \Delta_2 \ \ldots\ldots \ \Delta_n]^T$, represents the actual deformations at the chosen redundants locations; $\underline{\Delta}_0 = [\Delta_{10} \ \Delta_{22} \ \ldots\ldots \ \Delta_{n0}]^T$, represents the deformations of the primary structure due to the applied actions at the released redundants locations; [f] is the structural flexibility matrix, where a general element f_{ij} represents the deformation of the primary structure at the released redundants i due to a unit load for the redundant j applied; and $\underline{X} = [x_1 \ x_2 \ \ldots\ldots \ x_n]^T$, represents the unknown redundants.

It has been shown how to determine the values of Δ_{i0} and f_{ij} by the unit load method. The objective of this section is to calculate the values of these deformations in an automatic way which assists in using the computer to solve the problem, if one provides the necessary data needed for these computations. The flexibility matrix approach-I is a direct application to Equation 3.69, which means that the analyst prepares all internal forces and moments, but the mathematical integrations are done using matrix multiplications. The multiplications are done in a systematic way in order to enable the analyst to use simple computer programs to solve the problems.

3.8.2 Transformation of Integration to Matrix Multiplication

Consider, in general, that member k in a loaded primary structure is subjected to axial force A_{x0}, bending moment at the left end $M_{z0\ell}$, and bending moment at the right end M_{z0r}, where M_{z0} diagram is linear. For simplicity, the shear deformation is neglected. The unit values of the redundants x_i and x_j induce in member k, respectively, axial forces a_{xi} and a_{xj}, and bending moments $m_{zi\ell}$, m_{zir}, $m_{zj\ell}$ and m_{zjr}, as shown in Figure 3.162.

The contribution of the internal actions in member k to the deformation Δ_{i0} and f_{ij} is given, respectively, by

$$\left(\Delta_{io}\right)_K = \left(a_{xi} \ A_{x0} \ \frac{L}{EA}\right)_K + \left[\frac{L}{6EI}\left(2M_{z0\ell} \ m_{zi\ell} + 2M_{zor} \ m_{zir} + M_{zo\ell} \ m_{zir} + M_{zor} \ m_{zi\ell}\right)\right]_K \quad (3.70)$$

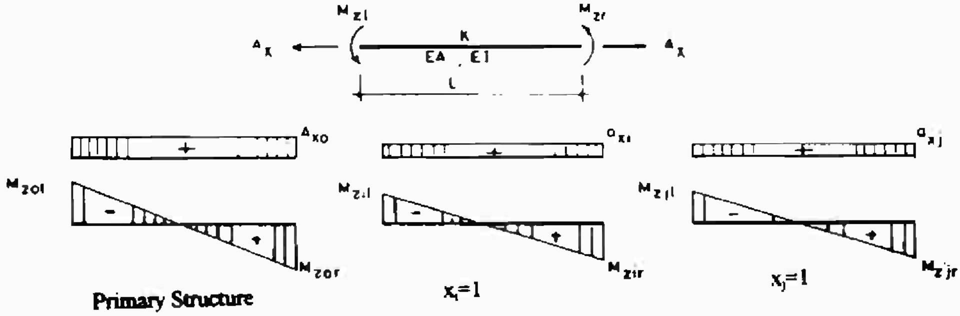


Figure 3.162

$$\left(f_{ij}\right)_K = \left(a_{xi}\, a_{xj} \frac{L}{EA}\right)_K + \left[\frac{L}{6EI}\left(2m_{zi\ell}\, m_{zj\ell} + 2m_{zir}\, m_{zjr} + m_{zi\ell}\, m_{zjr} + m_{zir}\, m_{zj\ell}\right)\right]_K \tag{3.71}$$

Equations 3.70 and 3.71 can be written in matrix form as follows:

$$\left(\Delta_{io}\right)_K = \begin{bmatrix} a_{xi} & m_{zi\ell} & m_{zir} \end{bmatrix}_K \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{L}{6EI} \\ 0 & \frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix}_K \begin{bmatrix} A_{xo} \\ M_{zo\ell} \\ M_{zor} \end{bmatrix}_K \tag{3.72}$$

$$\left(f_{ij}\right)_K = \begin{bmatrix} a_{xi} & m_{zi\ell} & m_{zir} \end{bmatrix}_K \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{L}{6EI} \\ 0 & \frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix}_K \begin{bmatrix} A_{xj} \\ Mzj\ell \\ M_{zjr} \end{bmatrix}_K \tag{3.73}$$

Equations 3.72 and 3.73 can be repeated for all structure's members, k = 1, 2, …, m to give finally

$$\Delta_{io} = \sum_{k=1}^{m} \left(\Delta_{io}\right)_k \tag{3.74}$$

$$f_{ij} = \sum_{k=1}^{m} \left(f_{ij}\right)_k \tag{3.75}$$

Let $\mathbf{a}_i$ represents an array of dimensions represent the internal actions in all structure's members k = 1, 2, …, m due to the unit action at i. Then $\mathbf{a}_i$ can be written as

$$\underline{\mathbf{a}}_i^T = \left[\left(a_{xi} \; m_{zi\ell} \; m_{zir} \right)_1 \quad \left(a_{xi} \; m_{zi\ell} \; m_{zir} \right)_2 \cdots \left(a_{xi} \; m_{zi\ell} \; m_{zir} \right)_m \right] \tag{3.76}$$

Let $\underline{\mathbf{A}}_o$ represents an array which contains the internal actions in all members, k = 1, 2, ..., m, due to the applied actions on the primary structure. It can be expressed as

$$\underline{\mathbf{A}}_o^T = \left[\left(A_{xo} \; M_{zo\ell} \; M_{zor} \right)_1 \quad \left(A_{xo} \; M_{zo\ell} \; M_{zor} \right)_2 \cdots\cdots \left(A_{xo} \; M_{zo\ell} \; M_{zor} \right)_m \right] \tag{3.77}$$

Therefore, by substituting Equations 3.76 and 3.77 into Equations 3.74 and 3.75, one obtains

$$\Delta_{io} = \underline{\mathbf{a}}_i^T \, [f_m] \, \underline{\mathbf{A}}_o \tag{3.78}$$

$$f_{ij} = \underline{\mathbf{a}}_i^T \, [f_m] \, \underline{\mathbf{a}}_j \tag{3.79}$$

where $[f_m]$ is called the members flexibility matrix which is of dimension (3m × 3m) and contains the flexibility matrices of all members k = 1, 2, ..., m arranged as follows:

$$[f_m] = \begin{bmatrix} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{L}{6EI} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{L}{6EI} & \frac{L}{3EI} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{L}{EA} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{L}{3EI} & \frac{L}{6EI} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{L}{6EI} & \frac{L}{3EI} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{L}{6EI} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{L}{3EI} & \frac{L}{6EI} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix} \tag{3.80}$$

Let the matrix $\underline{\mathbf{a}}$ contains the arrays $\underline{\mathbf{a}}_i$ for all redundants i = 1, 2, ..., n. Therefore, it is of dimension (3m × n) and obtained as follows:

$$\underline{\mathbf{a}} = \left[\underline{\mathbf{a}}_1 \;\; \underline{\mathbf{a}}_2 \cdots \underline{\mathbf{a}}_i \cdots \underline{\mathbf{a}}_n \right] \tag{3.81}$$

The terms $\underline{\Delta}$ and $\underline{f}$ in Equations 3.69 are thus given by

$$\underline{\Delta}_o = \underline{a}^T [f_m] \underline{A}_o \tag{3.82}$$

$$[f] = \underline{a}^T [f_m] \underline{a} \tag{3.83}$$

3.8.3 Effect of Member Loading

The previous formulations were based on the assumption that M_{z0} for each member k is a straight line. In practice, however, members may be subjected to direct loading. In this case, one has to transfer the load into equivalent joint loading at the member ends. This can be done using Maxwell's theorem as follows:

Consider, for example, member AB which is shown in Figure 3.163. Under the effect of the uniform distributed load, the rotations at A and B are named θ_A and θ_B. Let this case of loading and deformation be called case I. Thus, one has

$$\underline{A}_I = \begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.84}$$

$$\underline{D}_I = \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \end{bmatrix} = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} \tag{3.85}$$

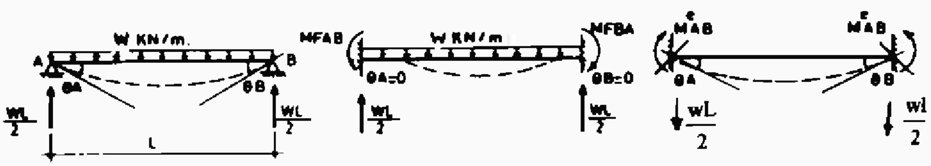


Figure 3.163

If the member AB is fixed at both ends, the rotations at the fixed supports are zero. However, at the supports, there are the fixed end moment M_{FAB} and M_{FBA}. Let this case be denoted by $\underline{A}_F$ and $\underline{D}_F$, of case of loading II.

$$\underline{A}_F = \begin{bmatrix} M_{FAB} \\ M_{FBA} \end{bmatrix} \tag{3.86}$$

$$\underline{D}_F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.87}$$

To keep the deformation consistent, the unloaded beam is rotated at the fixed supports A and B by angles of rotation θ_A and θ_B, respectively. Let the actions $\underline{A}^c$ have caused these deformations which are called $\underline{D}^c$, then

$$\underline{A}^c = \begin{bmatrix} M^c_{AB} \\ M^c_{BA} \end{bmatrix} \tag{3.88}$$

$$\underline{D}^c = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} \tag{3.89}$$

From the superposition principle it is obvious that $\underline{D}_I = \underline{D}_F + \underline{D}^c$ and $\underline{A}_I = \underline{A}_F + \underline{A}^c$. This result can also be found by Maxwell's Theorem as

$$\underline{A}_I^T\left(\underline{D}_F + \underline{D}^c\right) = \left(\underline{A}_F^T + \underline{A}_c^T\right)\underline{D}_I \tag{3.90}$$

One concludes that $\underline{A}^c = -\underline{A}_F$. This means that the loading $\underline{A}^c$, which is called the equivalent joint loading, equal in magnitude and opposite in directions to the fixed end loading $\underline{A}_f$. It is used whenever the member is subjected to direct loading. The fixed end moments for several cases of loading are given in Table 3.2.

Example 3.40

Determine the angle of rotation at A for the beam shown in Figure 3.164.

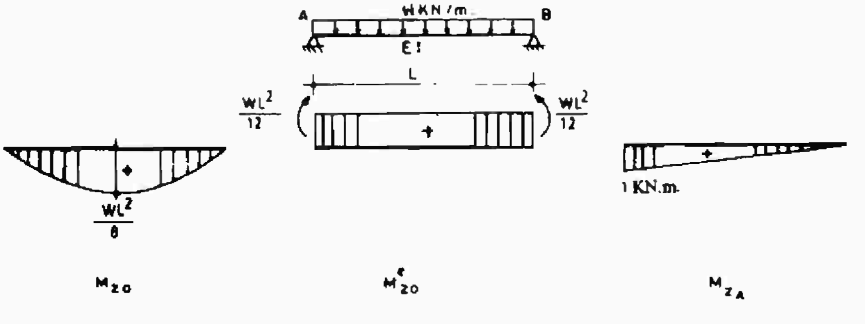
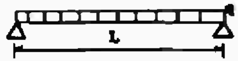
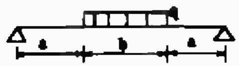
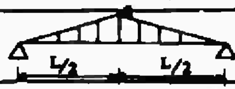
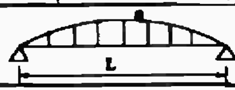
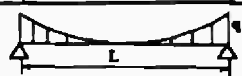
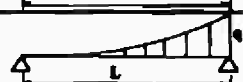
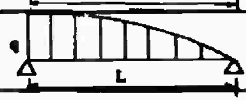


Figure 3.164

Solution

This problem can be solved using the actual loading or the equivalent joint loading. Applying a unit moment at A one obtains m_{ZA} diagram as shown in Figure 3.164. Using the moment diagrams M_{z0} and m_{ZA} one has ıle 3.2 Fixed End Moments

No.	Case of Loading	Fixed End Moments		
				
		M_A	M_B	M_A
1		$-\frac{q\times L^2}{12}$	$-\frac{q\times L^2}{12}$	$-\frac{q\times L^2}{8}$
2		$-\frac{q\times b\times L}{24}\left(3-\frac{b^2}{L^2}\right)$	$-\frac{q\times b\times L}{24}\left(3-\frac{b^2}{L^2}\right)$	$-\frac{q\times b\times L}{16}\left(3-\frac{b^2}{L^2}\right)$
3		$-\frac{q\times a^2}{6}\left(3-\frac{2a}{L}\right)$	$-\frac{q\times a^2}{6}\left(3-\frac{2a}{L}\right)$	$-\frac{q\times a^2}{4}\left(3-\frac{2a}{L}\right)$
4		$-\frac{5}{96}\times q\times L^2$	$-\frac{5}{96}\times q\times L^2$	$-\frac{5}{64}\times q\times L^2$
5		$-\frac{q}{12}\left[L^2-a^2\left(2-\frac{a}{L}\right)\right]$	$-\frac{q}{12}\left[L^2-a^2\left(2-\frac{a}{L}\right)\right]$	$-\frac{q}{8}\left[L^2-a^2\left(2-\frac{a}{L}\right)\right]$
6		$-\frac{q\times L^2}{15}$	$-\frac{q\times L^2}{15}$	$-\frac{q\times L^2}{10}$
7		$-\frac{q\times L^2}{60}$	$-\frac{q\times L^2}{60}$	$-\frac{q\times L^2}{40}$
8		$-\frac{q\times L^2}{30}$	$-\frac{q\times L^2}{60}$	$-\frac{q\times L^2}{24}$
9		$-\frac{q\times L^2}{60}$	$-\frac{q\times L^2}{30}$	$-\frac{q\times L^2}{30}$
10		$-\frac{q\times L^2}{15}$	$-\frac{q\times L^2}{20}$	$-\frac{11q\times L^3}{120}$

*Reference 9.

Table 3.2 Fixed End Moments (Cont'd…)

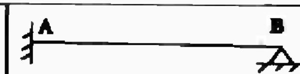
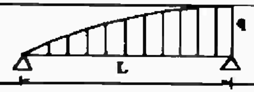
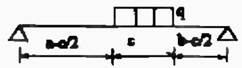
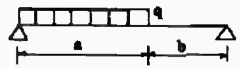
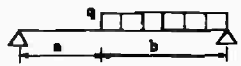
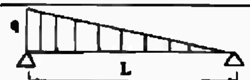
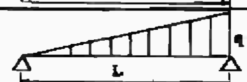
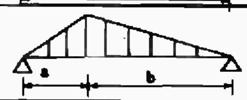
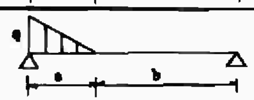
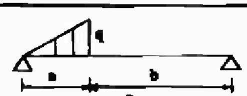
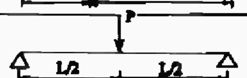
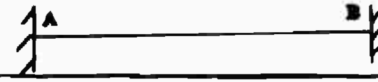
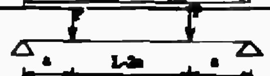
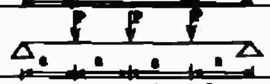
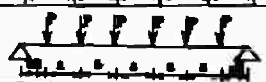
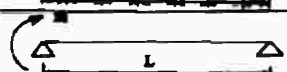
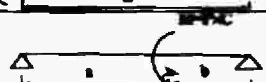
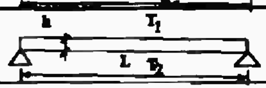
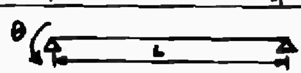
No.	Case of Loading	Fixed End Moments		
				
		M_A	M_B	M_A
11		$-\frac{q\times L^2}{20}$	$-\frac{q\times L^2}{15}$	$-\frac{q\times L^2}{12}$
12		$-\frac{q\times c}{L^2}\left[a\times b^2 - \frac{c^2}{6}\left(b-\frac{a}{2}\right)\right]$	$-\frac{q\times c}{L^2}\left[a\times b^2 - \frac{c^2}{6}\left(a-\frac{b}{2}\right)\right]$	$-\frac{q\times a\times b\times c}{2L^2}\left(L+b-\frac{c^2}{4a}\right)$
13		$-\frac{q\times a^2}{4}\left[2-\frac{a}{L}\left(\frac{8}{3}-\frac{a}{L}\right)\right]$	$-\frac{q\times a^3}{12L^2}(4L-3a)$	$-\frac{q\times a^2}{8}\left(2-\frac{a}{L}\right)^2$
14		$-\frac{q\times b^3}{12L^2}(4L-3b)$	$-\frac{q\times b^2}{4}\left[2-\frac{b}{L}\left(\frac{8}{3}-\frac{b}{L}\right)\right]$	$-\frac{q\times b^2}{8}\left(2-\frac{b^2}{L^2}\right)^2$
15		$-\frac{q\times L^2}{20}$	$-\frac{q\times L^2}{30}$	$-\frac{q\times L^2}{15}$
16		$-\frac{q\times L^2}{30}$	$-\frac{q\times L^2}{20}$	$-\frac{7\times q\times L^2}{120}$
17		$-\frac{q\times L^2}{30}\left(1+\frac{b}{L}+\frac{b^2}{L^2}-1.5\frac{b^3}{L^3}\right)$	$-\frac{q\times L^2}{30}\left(1+\frac{a}{L}+\frac{a^2}{L^2}-1.5\frac{a^3}{L^3}\right)$	$-\frac{q(L+b)}{120L}\left(7L^2-3b^2\right)$
18		$-\frac{q\times a^2}{6}\left(1-\frac{a}{L}+\frac{3}{10}\times\frac{a^2}{L^2}\right)$	$-\frac{q\times a^2}{60}\times\frac{a}{L}\left(5-3\frac{a}{L}\right)$	$-\frac{q\times a^2}{2}\left(\frac{1}{3}-\frac{a}{4L}+\frac{a^2}{20L^2}\right)$
19		$-q\times a^2\left(\frac{1}{3}-\frac{a}{2L}+\frac{a^2}{5L^2}\right)$	$-\frac{q\times a^2}{L}\left(\frac{1}{4}-\frac{a}{5L}\right)$	$-q\times a^2\left(\frac{1}{3}-\frac{3a}{8L}+\frac{a^2}{10L^2}\right)$
20		$-\frac{P\times L}{8}$	$-\frac{P\times L}{8}$	$-\frac{3P\times L}{16}$

Table 3.2 Fixed End Moments (Cont'd...)

No.	Case of Loading	Fixed End Moments		
				
		M_A	M_B	M_A
21		$-P\times a\times\dfrac{b^2}{L^2}$	$-P\times b\times\dfrac{a^2}{L^2}$	$-\dfrac{P\times a\times b}{2L^2}(L+b)$
22		$-\dfrac{P\times a}{L}(L-a)$	$-\dfrac{P\times a}{L}(L-a)$	$-\dfrac{3\times P\times a}{2L}(L-a)$
23		$-\dfrac{5\times P\times L}{16}$	$-\dfrac{5\times P\times L}{16}$	$-\dfrac{15\times P\times L}{32}$
24		$-\dfrac{P\times L(n^2-1)}{12n}$	$-\dfrac{P\times L(n^2-1)}{12n}$	$-\dfrac{P\times L(n^2-1)}{8n}$
25		$-\dfrac{P\times L}{24}\left(2n+\dfrac{1}{n}\right)$	$-\dfrac{P\times L}{24}\left(2n+\dfrac{1}{n}\right)$	$-\dfrac{P\times L}{16}\left(2n+\dfrac{1}{n}\right)$
26		-M	0	-M
27		$-P\times c\times\dfrac{a}{L}\left(4-3\dfrac{a}{L}-\dfrac{L}{a}\right)$	$-P\times c\times\dfrac{a}{L}\left(3\dfrac{a}{L}-2\right)$	$-P\times c\times\dfrac{a}{L}\left(3-1.5\dfrac{a}{L}-\dfrac{L}{a}\right)$
28		$EI\alpha\dfrac{(T_1-T_2)}{h}$	$EI\alpha\dfrac{(T_1-T_2)}{h}$	$1.5EI\alpha\dfrac{(T_1-T_2)}{h}$
29		$-6EI\dfrac{\Delta}{L^2}$	$+6EI\dfrac{\Delta}{L^2}$	$-3EI\dfrac{\Delta}{L^2}$
30		$-4\dfrac{EI}{L}\theta$	$+2\dfrac{EI}{L}\theta$	$-3\dfrac{EI}{L}\theta$

$$\theta_A = \int_0^L \frac{M_{z0}\, m_{zA}\, d\ell}{EI} = \frac{1}{EI}\left(\frac{2}{3} \times \frac{wL^2}{8} \times L \times \frac{1}{2}\right) = \frac{wL^3}{24EI} \text{ rad.}$$

Using the moment diagram of equivalent joint loading M^e_{z0} and m_{zA}, one has

$$\theta_A = \int_0^L \frac{M^e_{z0}\, m_{zA}\, d\ell}{EI} = \frac{1}{EI}\left(\frac{wL^2}{12} \times L \times \frac{1}{2}\right) = \frac{wL^3}{24EI} \text{ rad.}$$

which gives the same results.

3.8.4 Effect of Temperature Changes

It has been shown in section 2.17 that the deformation due to temperature change on any member k is given by

$$\Delta_o = \alpha\left(\frac{T_1 + T_2}{2}\right)\int a_x\, d\ell - \alpha\left(\frac{T_1 - T_2}{h}\right)\int m_z\, d\ell \tag{3.91}$$

This equation can be transformed into a matrix form, as Equation 3.78, to give

$$\Delta_o = \left[a_x\ m_{z\ell}\ m_{zr}\right]_K \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{L}{6EI} \\ 0 & \frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix}_K \begin{bmatrix} EA\alpha\left(\frac{T_1 + T_2}{2}\right) \\ -EI\alpha\left(\frac{T_1 - T_2}{h}\right) \\ -EI\alpha\left(\frac{T_1 - T_2}{h}\right) \end{bmatrix} \tag{3.92}$$

This means that the equivalent joint loading is given by the term $EA\alpha\ (T_1 + T_2)/2$ for A_{x0}, and the term $(-EI\alpha\ (T_1 - T_2)/h)$ for both $M_{z0\ell}$ and M_{z0r}.

This also indicates that the fixed end loading for a member k due to rise in temperature T_1 on top surface and T_2 on the bottom surface is given by

$$\underline{A_f} = \begin{bmatrix} -EA\alpha\left(\frac{T_1 + T_2}{2}\right) \\ EI\alpha\left(\frac{T_1 - T_2}{h}\right) \\ EI\alpha\left(\frac{T_1 - T_2}{h}\right) \end{bmatrix}$$

For a truss member subjected to change in temperature T, the fixed end loading is an axial force given by

$$\underline{A_f} = -EA\alpha T$$

3.8.5 Deformation of Statically Indeterminate Structures

By the formation of $\underline{\Delta}_0$ and [f] in Equations 3.82 and 3.83, the values of the redundants $\underline{x}$ can be found from the solution of Equation 3.69 after specifying the boundary condition for $\underline{\Delta}$. Determination of the redundants $\underline{X}$ will lead to the final internal forces in the structure by using the superposition principle. Denoting the final actions by $\underline{A}$ one has

$$\underline{A} = \underline{A}_0 + \underline{a}\,\underline{x} + \underline{A}_f \tag{3.93}$$

in which $\underline{A}_f$ is the fixed end actions due to the direct loadings and/or temperature change.

To determine the deformation at any point C one applies a unit load at this point and determine the internal actions in the primary structure. Denoting these internal actions by $\underline{a}_c$ which is of dimension (3m × 1), the deformation at point C can thus be obtained from

$$\Delta_c = \underline{a}_c^T\,[f_m]\,\underline{A} \tag{3.94}$$

If the deformations at points like a, b, c, etc..... on the structure are required, one applies, in turn, a unit load at each of these points in order to obtain $\underline{a}_a$, $\underline{a}_b$, $\underline{a}_c$, ... respectively. Let $\underline{a}_D$ be a matrix which contains all these internal actions. One can then determine the deformations at these points by

$$\underline{D} = \underline{a}_D^T\,[f_m]\,\underline{A} \tag{3.95}$$

3.8.6 Applications to Beams

If the axial deformations in the beams are neglected, the flexibility matrix of each member k, becomes in the form of

$$[f_m]_k = \begin{bmatrix} \dfrac{L}{3EI} & \dfrac{L}{6EI} \\ \dfrac{L}{6EI} & \dfrac{L}{3EI} \end{bmatrix} \tag{3.96}$$

All previous formulations are valid except that the dimension (3m) becomes (2m) in the case of beams.

Example 3.41

Draw the bending moment diagram for the beam shown in Figure 3.165 (Example 4.10) due to the applied loadings, a rotation at support A of 0.002 radian, anticlockwise, and a rise in temperature in member BC as shown. (EI = 10^5 kN.m^2, α = 10^{-5}/°c).

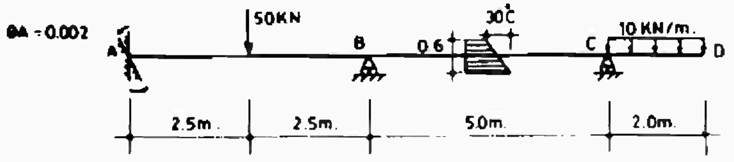


Figure 3.165

Solution

This problem was solved in example 3.10 where DSI = 2. Select the released redundants at A and C. Transform the concentrated load 50 kN into equivalent joint loading as shown in Figure 3.166.

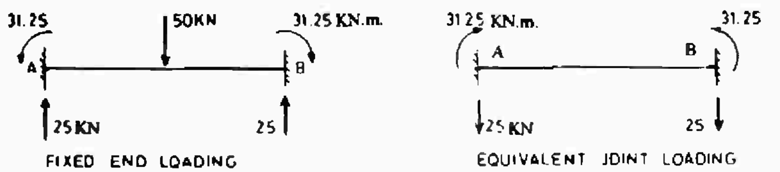


Figure 3.166

Similarly, transform the uniformly distributed load and temperature on members BC and CD into equivalent joint loads shown in Figure 3.167.

The fixed end actions are expressed for each member as follows:

$$\underline{A}_{FAB} = \begin{bmatrix} -31.25 \\ -31.25 \end{bmatrix}, \quad \underline{A}_{FBC} = \begin{bmatrix} \alpha EI\left(\dfrac{T_1 - T_2}{h}\right) \\ \alpha EI\left(\dfrac{T_1 - T_2}{h}\right) \end{bmatrix} = \begin{bmatrix} -50 \\ -50 \end{bmatrix}, \quad \underline{A}_{FCD} = \begin{bmatrix} -3.33 \\ -3.33 \end{bmatrix}$$

The reverse of these moments is applied on the primary structure to determine M_{zo}. The bending moment diagram for the primary structure due to the equivalent joint loading is shown in Figure 3.168.

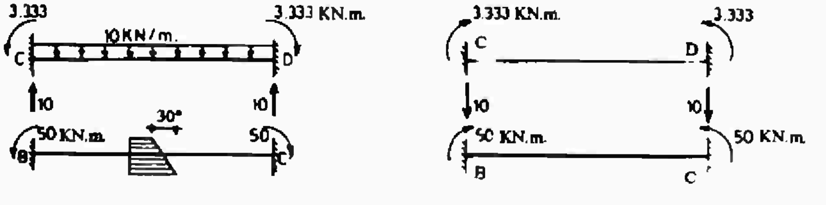


Figure 3.167

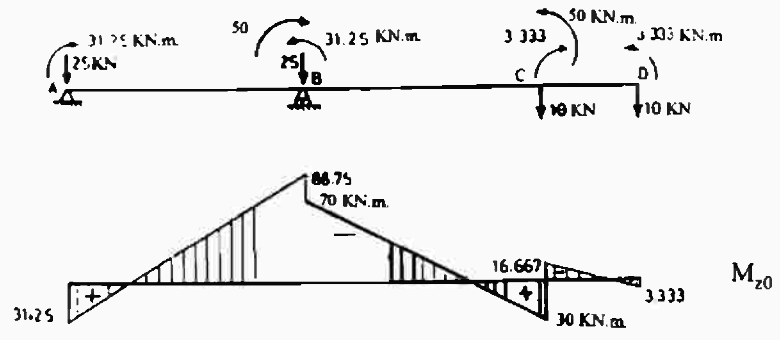


Figure 3.168

The internal action $\underline{A}_o$ due to the equivalent joint loading can thus be written as follows:

$$\underline{A}_o^T = [+31.25 \quad -88.75 \quad -70 \quad +30 \quad -16.667 \quad +3.333]$$

The internal actions due to unit values of each redundants x_1 and x_2 can be obtained using Figure 3.169 as follows:

$$\underline{a}_1^T = [-1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\underline{a}_2^T = [0 \quad +5 \quad +5 \quad 0 \quad 0 \quad 0]$$

Therefore, $\underline{a} = [\underline{a}_1 \quad \underline{a}_2]$ gives

$$\underline{a} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

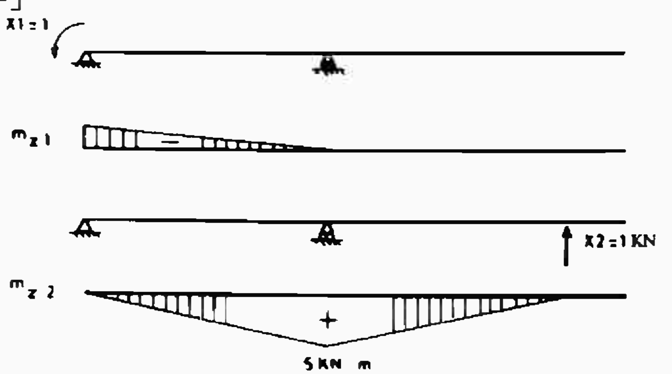


Figure 3.169

The members flexibility matrix is

$$[\mathbf{f}_m] = \frac{1}{6EI}\begin{bmatrix} 10 & 5 & 0 & 0 & 0 & 0 \\ 5 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 5 & 0 & 0 \\ 0 & 0 & 5 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

Substituting into Equations 3.82 and 3.83 one obtains

$$\underline{\Delta}_o = \underline{\mathbf{a}}_T\,[\mathbf{f}_m]\,\underline{\mathbf{A}}_o$$

$$= \frac{1}{6EI}\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 10 & 5 & 0 & 0 & 0 & 0 \\ 5 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 5 & 0 & 0 \\ 0 & 0 & 5 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 31.25 \\ -88.75 \\ -70 \\ 30 \\ -16.667 \\ 3.333 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 21.875 \\ -1067.70 \end{bmatrix}$$

$$[\mathbf{f}] = \underline{\mathbf{a}}^T\,[\mathbf{f}_m]\,\underline{\mathbf{a}} = \frac{1}{EI}\begin{bmatrix} \frac{10}{6} & \frac{-25}{6} \\ \frac{-25}{6} & \frac{500}{6} \end{bmatrix}$$

The boundary conditions are given by

$$\underline{\Delta}^T = [0.002 \quad 0]$$

Solving Equation 3.69 one obtains

$$\mathbf{X} = [\mathbf{f}]^{-1}\,[\underline{\Delta} - \underline{\Delta}_o] = [\mathbf{f}]^{-1}\begin{bmatrix} 0.002 - 21.875\times10^{-5} \\ 1067.70\times10^{-5} \end{bmatrix} = \begin{bmatrix} +158.75 \\ +20.75 \end{bmatrix} \begin{matrix} \text{kN.m} \\ \text{kN} \end{matrix}$$

The final internal actions are found using Equation 3.93, as follows:

$$\underline{\mathbf{A}} = \underline{\mathbf{A}}_o + \underline{\mathbf{a}}\,\underline{\mathbf{x}} + \underline{\mathbf{A}}_f$$

$$\underline{\mathbf{A}} = \begin{bmatrix} 31.25 \\ -88.75 \\ -70 \\ +30 \\ -16.667 \\ 3.333 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 5 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 158.75 \\ 20.75 \end{bmatrix} + \begin{bmatrix} -31.25 \\ -31.25 \\ -50 \\ -50 \\ -3.333 \\ -3.333 \end{bmatrix} = \begin{bmatrix} -158.75 \\ -16.25 \\ -16.25 \\ -20 \\ -20 \\ 0 \end{bmatrix}$$

The bending moment diagram is shown in Figure 3.170, which is the same as Example 3.10.

Example 3.42

Determine the bending moment diagram and the deflection at B for the beam shown in Figure 3.171 (Example 3.11). ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}/°c$, K = 10 kN/cm).

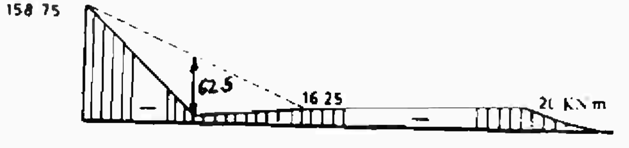


Figure 3.170

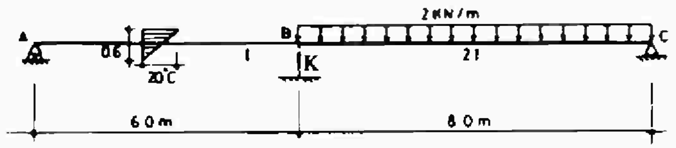


Figure 3.171

Solution

This beam is one degree statically indeterminate as determined in Example 3.11. The equivalent joint actions are determined as shown in Figure 3.172.

The fixed end actions in each member are expressed as follows:

$$\underline{A}_{FAB} = \begin{bmatrix} \alpha EI\left(\dfrac{T_1 - T_2}{h}\right) \\ \alpha EI\left(\dfrac{T_1 - T_2}{h}\right) \end{bmatrix} = \begin{bmatrix} 33.33 \\ 33.33 \end{bmatrix} \quad , \quad \underline{A}_{FBC} = \begin{bmatrix} -10.667 \\ -10.667 \end{bmatrix}$$

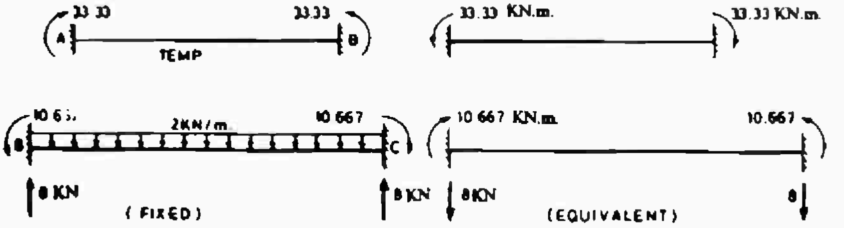


Figure 3.172

The reverse of the fixed end actions are applied on the primary structure. The internal actions for the primary structure due to the equivalent joint loading are shown in Figure 3.173.

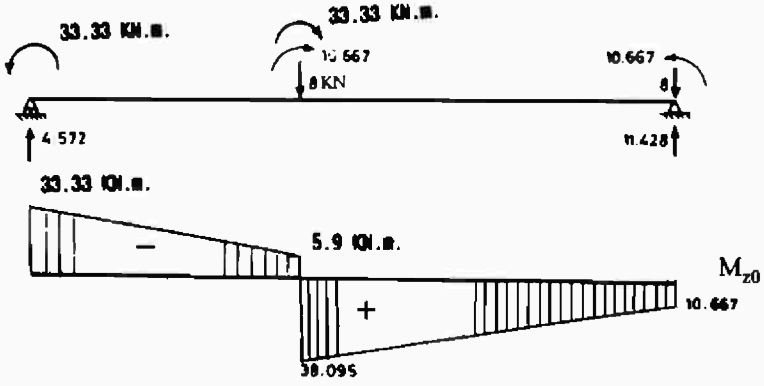


Figure 3.173

One can form $\underline{A}_o$ as follows:

$$\underline{A}_o^T = [-33.3 \quad -5.90 \quad 38.095 \quad 10.667]$$

The internal actions due to the redundant x_1 = 1 kN are determined as shown in Figure 3.174.

$$\underline{a}^T = \underline{a}_1^T = [0 \quad -3.428 \quad -3.428 \quad 0]$$

The flexibility matrix $[f_m]$ is formed as

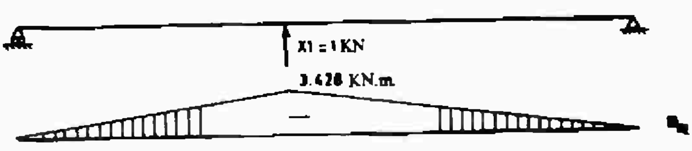


Figure 3.174

$$[\mathbf{f_m}] = \begin{bmatrix} \frac{6}{3EI} & \frac{6}{6EI} & 0 & 0 \\ \frac{6}{6EI} & \frac{6}{3EI} & 0 & 0 \\ 0 & 0 & \frac{8}{6EI} & \frac{8}{12EI} \\ 0 & 0 & \frac{8}{12EI} & \frac{8}{6EI} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1.333 & 0.667 \\ 0 & 0 & 0.667 & 1.333 \end{bmatrix}$$

Substituting into Equations 3.83 and 3.84 one obtains

$$\underline{\Delta}_o = \underline{\mathbf{a}}^T [\mathbf{f_m}] \underline{\mathbf{A}}_o$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -3.428 & -3.428 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1.333 & 0.667 \\ 0 & 0 & 0.667 & 1.333 \end{bmatrix} \begin{bmatrix} -33.33 \\ -5.905 \\ 38.095 \\ 10.667 \end{bmatrix} = \begin{bmatrix} \frac{-43.715}{EI} \end{bmatrix}$$

$$[\mathbf{f}] = \underline{\mathbf{a}}^T [\mathbf{f_m}] \underline{\mathbf{a}} = \begin{bmatrix} \frac{39.166}{EI} \end{bmatrix}$$

Substituting into Equation 3.70, where $\Delta = -0.001\ x_1$ one obtains

$$\frac{-43.715}{EI} + \frac{39.166}{EI} x_1 = -0.001 x_1 \quad ; \quad x_1 = \frac{43.175}{139.166} = 0.314\ kN$$

The final actions are determined from Equation 3.94 as follows:

$$\underline{\mathbf{A}} = \begin{bmatrix} -33.33 \\ -5.905 \\ 38.095 \\ 10.667 \end{bmatrix} + \begin{bmatrix} 0 \\ -3.428 \\ -3.428 \\ 0 \end{bmatrix} \times 0.314 + \begin{bmatrix} 33.33 \\ 33.33 \\ -10.667 \\ -10.667 \end{bmatrix} = \begin{bmatrix} 0 \\ 26.35 \\ 26.35 \\ 0 \end{bmatrix} \begin{matrix} \\ kN.m \\ kN.m \\ \\ \end{matrix}$$

The deformation at B can be determined from $\Delta_B = x_1/K$ or by applying a unit load at B and integrating m_{zB} with M_{zo} using Equation 3.95, where $\underline{\mathbf{a}}_B = \underline{\mathbf{a}}$,

$$\Delta_B = \underline{\mathbf{a}}_B^T [\mathbf{f_m}] \underline{\mathbf{A}}$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -3.428 & -3.428 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1.333 & 0.667 \\ 0 & 0 & 0.667 & 1.333 \end{bmatrix} \begin{bmatrix} 0 \\ 26.35 \\ 26.33 \\ 0 \end{bmatrix} = 0.003\ cm$$

The bending moment diagram is shown in Figure 3.175, which is the same as example 3.11.

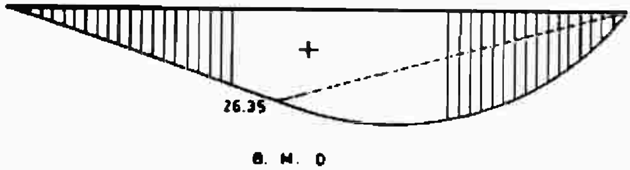


Figure 3.175

3.8.7 Applications to Frames

Example 4.43

Determine the bending moment and axial force diagrams for the frame shown in Figure 3.176 (example 3.12) due to the applied loads, and a rise in temperature in member BC. ($EI = 10^5$ kN.m^2, $EA = 50 \times 10^5$ kN, $\alpha = 10^{-5}/°c$).

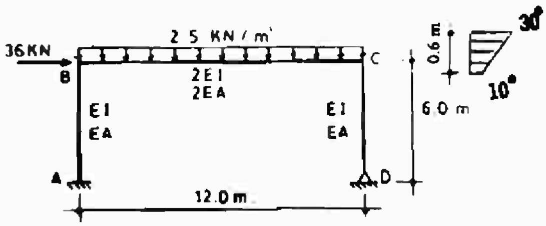


Figure 3.176

Solution

The fixed and equivalent joint actions are determined as shown in Figure 3.177. Then, M_{zo} and A_{xo} for the primary structure are constructed as shown in Figure 3.178. The m_{z1}, a_{x1}, m_{z2}, and a_{x2} diagrams due to unit values of the redundants are obtained as shown in Figure 3.179.

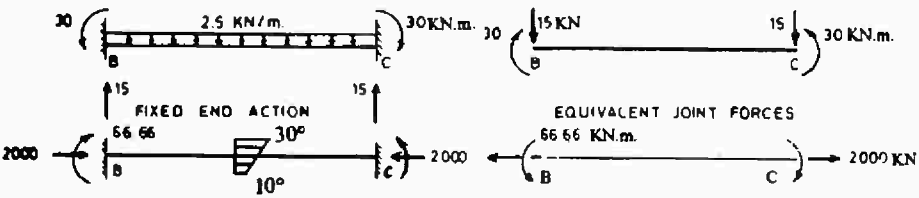


Figure 3.177

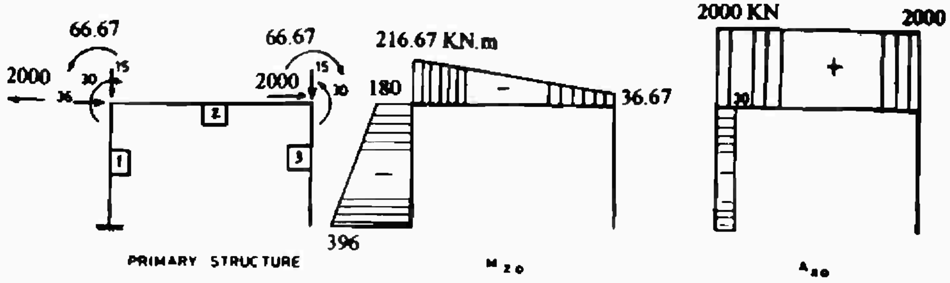


Figure 3.178

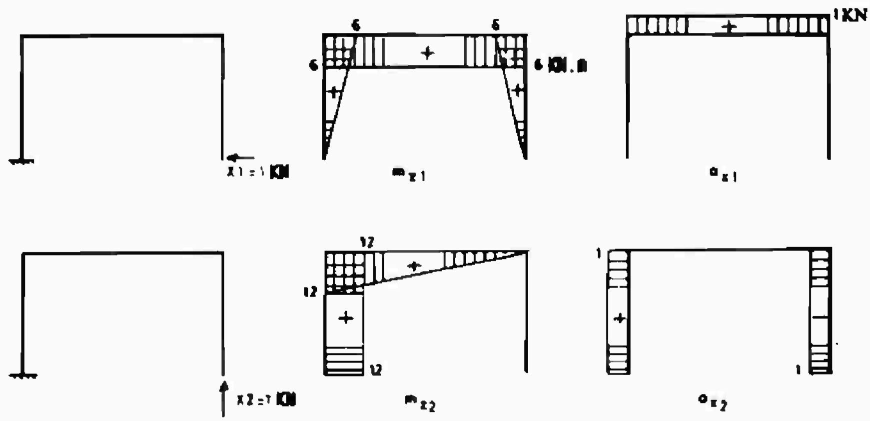


Figure 3.179

The fixed end actions for member BC are

$$\underline{\mathbf{A}}_{FBC} = \begin{bmatrix} 0 \\ -30 \\ -30 \end{bmatrix} + \begin{bmatrix} -\alpha\, EI(T_1 + T_2)/2 \\ +\alpha\, EI(T_1 - T_2)/h \\ +\alpha\, EI(T_1 - T_2)/h \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ -30 \end{bmatrix} + \begin{bmatrix} -2000 \\ 66.66 \\ 66.66 \end{bmatrix} = \begin{bmatrix} -2000 \\ 36.66 \\ 36.66 \end{bmatrix}$$

From A_{xo} and M_{zo} one can form $\underline{\mathbf{A}}_o$ as follows:

$$\underline{\mathbf{A}}_o^T = [-30 \quad -396 \quad -180 \quad 2000 \quad -216.666 \quad -36.666 \quad 0 \quad 0 \quad 0]$$

$$\underline{\mathbf{a}}^T = \begin{bmatrix} 0 & 0 & 6 & +1 & 6 & 6 & 0 & 6 & 0 \\ 1 & 12 & 12 & 0 & 12 & 0 & -1 & 0 & 0 \end{bmatrix}$$

The members flexibility matrix $[\mathbf{f_m}]$ is given by

$$[f_m]=\begin{bmatrix} \frac{6}{EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{3EI} & \frac{6}{6EI} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{6EI} & \frac{6}{3EI} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{12}{2EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{12}{3\times 2EI} & \frac{12}{6\times 2EI} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{12}{6\times 2EI} & \frac{12}{3\times 2EI} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{EA} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{3EI} & \frac{6}{6EI} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{6EI} & \frac{6}{3EI} \end{bmatrix}$$

$$=10^{-5}\begin{bmatrix} 0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Substituting into Equation 3.82 and 3.83 one obtains

$\underline{\Delta}_o = \underline{a}^T\,[f_m]\,\underline{A}_o$

$$=10^{-5}\begin{bmatrix} 0 & 6 & 12 & 0.12 & 18 & 18 & 0 & 12 & 6 \\ 0.12 & 36 & 36 & 0 & 24 & 12 & 0.12 & 0 & 0 \end{bmatrix}\begin{bmatrix} -30 \\ -396 \\ -180 \\ 2000 \\ -216.666 \\ -36.666 \\ 0 \\ 0 \\ 0 \end{bmatrix}=10^{-5}\begin{bmatrix} -8856 \\ -26379.6 \end{bmatrix}$$

$[\mathbf{f}] = \underline{\mathbf{a}}^T\,[\mathbf{f}_m]\,\underline{\mathbf{a}}$

$$= 10^{-5}\begin{bmatrix} 0 & 6 & 12 & 0.12 & 18 & 18 & 0 & 12 & 6 \\ 0.12 & 36 & 36 & 0 & 24 & 12 & 0.12 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 12 \\ 6 & 2 \\ 1 & 0 \\ 6 & 12 \\ 6 & 0 \\ 0 & -1 \\ 6 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 10^{-5}\begin{bmatrix} 360.12 & 432 \\ 432 & 1152.2 \end{bmatrix}$$

Substituting into Equation 3.69 where $\underline{\Delta} = \underline{0}$, one obtains

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\mathbf{f}]^{-1}\,[\underline{\Delta} - \underline{\Delta}_o] = \begin{bmatrix} -5.219 \\ 24.85 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \end{matrix}$$

The end actions are determined from Equation 3.93 as follows:

$\underline{\mathbf{A}} = \underline{\mathbf{A}}_o + \underline{\mathbf{a}}\,\underline{x} + \underline{\mathbf{A}}_f$

$$\underline{\mathbf{A}} = \begin{bmatrix} -30 \\ -396 \\ -180 \\ 2000 \\ -216.667 \\ -36.667 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 12 \\ 6 & 12 \\ 1 & 0 \\ 6 & 12 \\ 6 & 0 \\ 0 & -1 \\ 6 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -5.219 \\ -24.85 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2000 \\ +36.667 \\ +36.667 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5.15 \\ -97.8 \\ 86.886 \\ -5.219 \\ 86.886 \\ -31.314 \\ -24.85 \\ -31.314 \\ 0 \end{bmatrix}$$

The bending moment and axial force diagrams are shown in Figure 3.180.

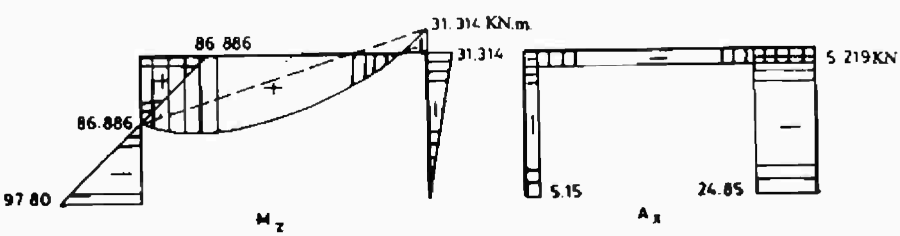


Figure 3.180

3.8.8 Application to Trusses

In trusses, the member flexibility matrix for any member is $\left[\dfrac{L}{EA}\right]$. All other steps are the same, where the dimension (3m) is replaced by just (m) for trusses.

Example 3.44

Determine the member forces and the horizontal deflection of support C for the truss shown in Figure 3.181 (example 3.15) due to the applied loads and a rise in temperature for member ED and DC of 20°c. (EA = 20 × 10^5 kN, $\alpha = 10^{-5}$/°c).

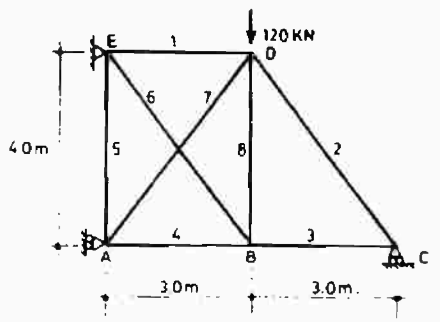


Figure 3.181

Solution

The truss is two degree statically indeterminate, one external redundant and one internal redundant as was determined in example 3.15. The member forces due to the applied loads in the primary structure are determined as shown in Figure 3.182. Members have to be numbered as shown in Figure 3.181.

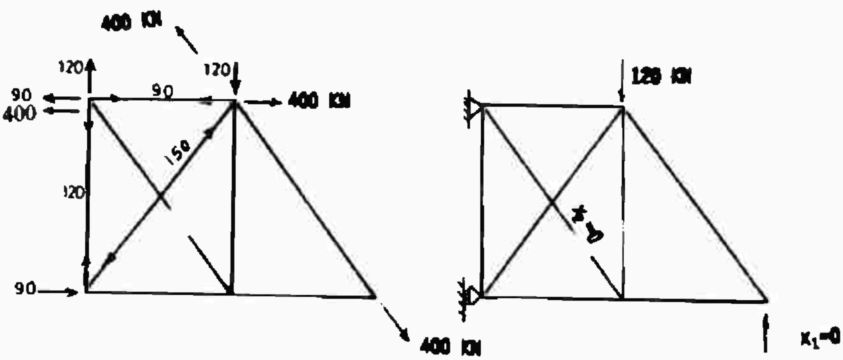


Figure 3.182

Since members ED and DC are subjected to temperature change, the fixed end action in these members are as follows:

$$\underline{A}_{FED} = [-\alpha EA\,T] = [-400] \qquad , \qquad \underline{A}_{FDC} = [-\alpha EA\,T] = [-400]$$

The reverse of these forces result in the equivalent joint loading as shown in Figure 3.182. Therefore, the member forces due to loading and temperature are

$$\underline{A}_0^T = [490 \quad 400 \quad 0 \quad 0 \quad 120 \quad 0 \quad -150 \quad 0]$$

Applying unit load for each of x_1 and x_2, one can form $\underline{a}_1$ and $\underline{a}_2$ from the member forces shown in Figures 3.183.

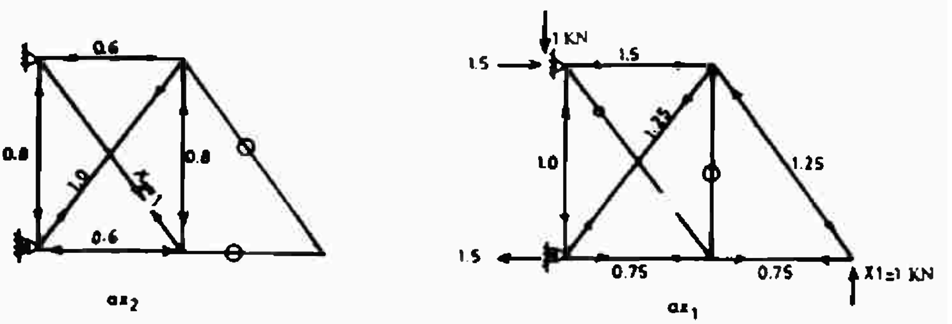


Figure 3.183

$$\underline{a}^T = \begin{bmatrix} -1.5 & -1.25 & 0.75 & 0.75 & -1.0 & 0 & 1.25 & 0 \\ -0.6 & 0 & 0 & 0.6 & -0.8 & 1.0 & 1.0 & -0.8 \end{bmatrix}$$

$$[r_m] = \begin{bmatrix} \frac{3}{EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{EA} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{EA} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{EA} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{EA} & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & \frac{5}{EA} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{EA} \end{bmatrix} = \frac{1}{EA}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Substituting into Equations 3.82 and 8.83 one obtains

$$\underline{\Delta} = \underline{a}^T [f_m] \underline{A}_o = \frac{1}{EA}\begin{bmatrix} -6122.5 \\ -2016 \end{bmatrix}$$

$$[f] = \underline{a}^T [f_m] \underline{a} = \frac{1}{EA}\begin{bmatrix} 29.75 & 10.8 \\ 10.8 & 17.28 \end{bmatrix}$$

Substituting into Equation 3.69 where $\underline{\Delta} = \underline{0}$ one obtains

$$\underline{x} = [f]^{-1} [\underline{\Delta} - \underline{\Delta}_o] = \begin{bmatrix} 211.42 \\ -15.49 \end{bmatrix} \text{ kN}$$

The final end forces are obtained from Equation 3.93 as

$$\underline{A} = \underline{A}_o + \underline{a}\,\underline{x} + \underline{A}_f$$

$$\underline{A} = \begin{bmatrix} 490 \\ 400 \\ 0 \\ 0 \\ 120 \\ 0 \\ -150 \\ 0 \end{bmatrix} + \begin{bmatrix} -1.5 & -0.6 \\ -1.25 & 0 \\ 0.75 & 0 \\ 0.75 & -0.6 \\ -1.0 & -0.8 \\ 0 & 1.0 \\ 1.25 & 1.0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} 211.42 \\ 15.49 \end{bmatrix} + \begin{bmatrix} -400 \\ -400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -217.836 \\ -264.275 \\ 158.565 \\ 167.859 \\ -79.03 \\ -15.49 \\ 98.785 \\ 12.392 \end{bmatrix}$$

which are the same results obtained before.

To determine the horizontal deflection at support C, one applies a unit load at C and determine $\underline{a}_c$ from the analysis of the primary structure as shown in Figure 3.184.

$$\underline{a}_c^T = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

Substituting into Equation 3.95 one obtains

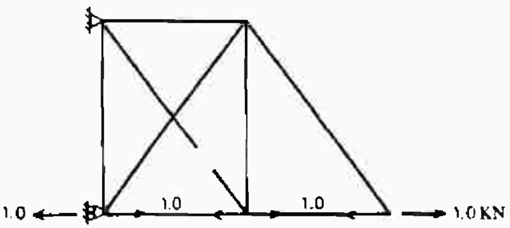


Figure 3.184

$$\Delta_c = \underline{a}_c^T [f_m] \underline{A}$$

$$= \frac{1}{EA}[0 \quad 0 \quad 3 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0] \underline{A} = \frac{979.27}{EA} = 0.0489 \text{ cm}$$

which indicates it is in the same direction of the unit load of Figure 3.181.

3.8.9 Application to Frame-Truss Structures

In these types of structures, either to consider the truss members as frame members connected together by hinges, or use the flexibility matrix of truss members combined with the frame members. Both approaches lead to the same results.

Example 3.45

Determine the bending moment and axial force diagrams for the structure shown in Figure 3.185 (example 3.17) (EI = 10^5 kN m^2, EA = 0.5×10^5).

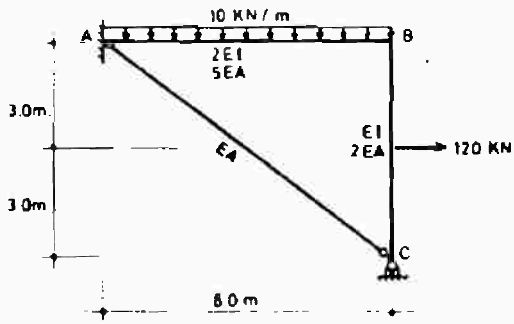


Figure 3.185

Solution

The structure is two degree statically indeterminate, one external redundant and one internal redundants as was determined in example 3.17. The fixed and equivalent joint actions are determined as demonstrated in Figure 3.186.

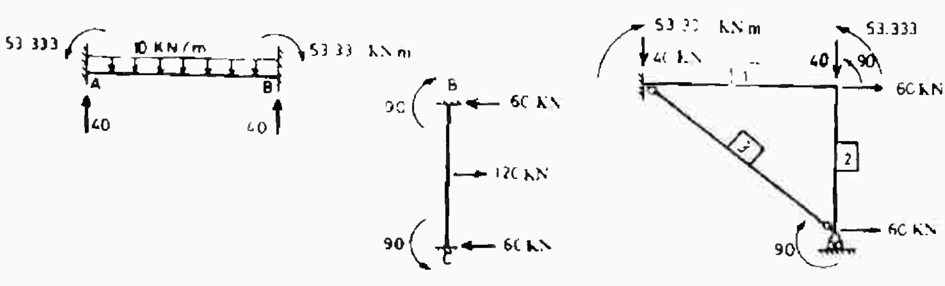


Figure 3.186

The fixed end actions for the members are formed as follows:

$$\underline{A}_{FAB} = \begin{bmatrix} 0 \\ -53.33 \\ -53.33 \end{bmatrix} \quad ; \quad \underline{A}_{FBC} = \begin{bmatrix} 0 \\ +90 \\ +90 \end{bmatrix}$$

One constructs A_{xo} and M_{zo} for the chosen primary structure as shown in Figure 3.187. The action vector $\underline{A}_o$ is formed considering the order of the members shown in Figure 3.186 as

$$\underline{A}_o^T = [120 \;\; 93.333 \;\; 413.333 \;\; 0 \;\; 270 \;\; -90 \;\; 0]$$

where member AC is considered as truss member since both ends are hinges.

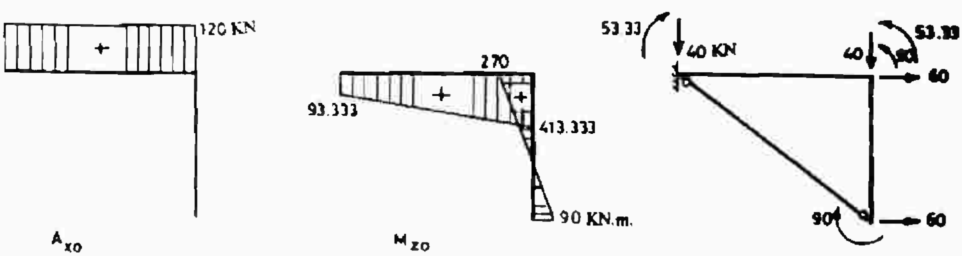


Figure 3.187

To determine $\underline{a}$, let x_1 and x_2 be unity to find a_{x1}, a_{x2}, m_{z1} and m_{z2}. The results are given in Figure 3.188.

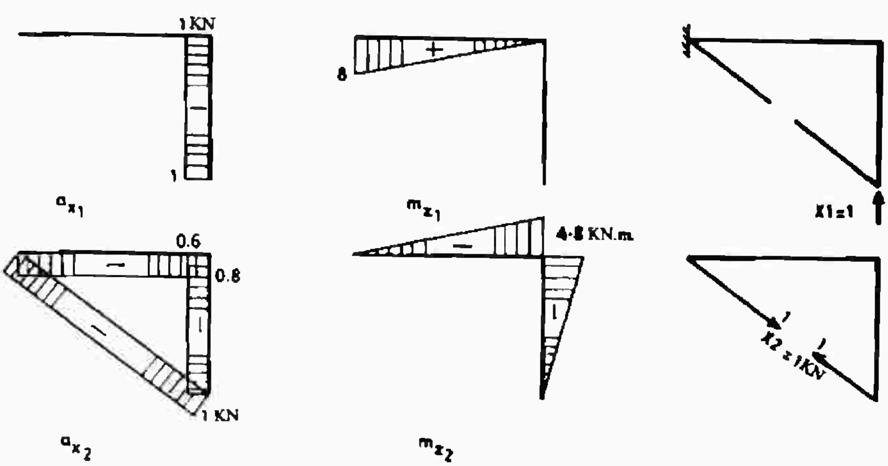


Figure 3.188

$$\underline{a}^T = \begin{bmatrix} 0 & 8 & 0 & -1 & 0 & 0 & 0 \\ -0.8 & 0 & -4.8 & -0.6 & -4.8 & 0 & 1 \end{bmatrix}$$

The members flexibility matrix $[f_m]$ is formed according to the numbering of the members as follows:

$$[f_m] = \begin{bmatrix} \frac{8}{5EA} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{6EI} & \frac{8}{12EI} & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{12EI} & \frac{8}{6EI} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{2EA} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{6}{3EI} & \frac{6}{3EI} & 0 \\ 0 & 0 & 0 & 0 & \frac{6}{3EI} & \frac{6}{3EI} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{EA} \end{bmatrix}$$

$$= 10^{-5} \begin{bmatrix} 3.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.333 & 0.667 & 0 & 0 & 0 & 0 \\ 0 & 0.667 & 1.333 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix}$$

Substituting into Equations 3.82 and 3.83 one obtains

$$\underline{\Delta}_o = \underline{a}^T [f_m] \underline{A}_o = 10^{-5} \begin{bmatrix} 3199.89 \\ -5411.19 \end{bmatrix}$$

$$[f] = \underline{a}^T [f_m] \underline{a} = 10^{-5} \begin{bmatrix} 91.336 & -22 \\ -22 & 101.008 \end{bmatrix}$$

Substituting into Equation 3.69, where $\underline{\Delta} = \underline{0}$, one obtains

$$\underline{x} = [f]^{-1} [\underline{\Delta} - \underline{\Delta}_o] = \begin{bmatrix} -23.358 \\ 48.48 \end{bmatrix} \begin{matrix} kN \\ kN \end{matrix}$$

The final forces are obtained using Equation 3.93 as

$$\underline{A} = \underline{A}_o + \underline{a}\,\underline{x} + \underline{A}_r$$

$$\underline{A} = \begin{bmatrix} 120 \\ 93.333 \\ 413.333 \\ 0 \\ 270 \\ -90 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -0.8 \\ 8 & 0 \\ 0 & -4.8 \\ -1 & -0.6 \\ 0 & -4.8 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -23.358 \\ 48.4890 \end{bmatrix} + \begin{bmatrix} 0 \\ -53.33 \\ -53.33 \\ 0 \\ 90 \\ 90 \\ 0 \end{bmatrix} = \begin{bmatrix} 81.216 \\ -146.86 \\ 127.296 \\ -5.73 \\ 127.296 \\ 0 \\ 48.48 \end{bmatrix}$$

The bending moment and axial force diagrams are shown in Figure 3.189.

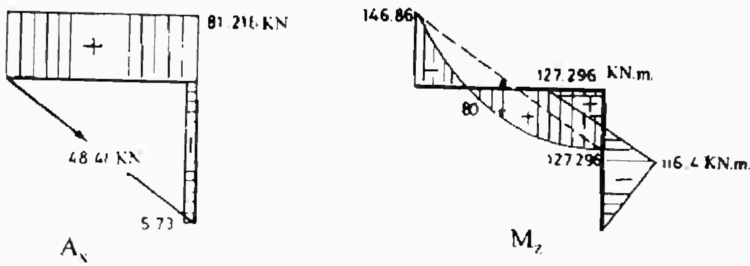


Figure 3.189

3.9 THE FLEXIBILITY MATRIX METHOD : APPROACH – II

It was shown in the previous section how the mathematical integrations were calculated through matrix multiplication, in a way that the analyst can use computer programming to solve the structural analysis problem. The analyst needs only to introduce the data to form the flexibility matrix $[f_m]$, the internal actions in the primary structure $\underline{A}_o$, and the internal actions due to the unit redundants $\underline{a}$. One then obtains Equation 3.69 which provides the values of the redundants.

In this section, a more general approach is introduced. This approach deals with all joint loadings and the desired free joints deformations, so that one can obtain, as well, the deformations at these joints.

3.9.1 Mathematical Formulations

In this approach, all external loadings and redundants on the structure are considered to be the structural coordinates. If the structure is subjected to joints loading $\underline{A}_j$, the redundants $\underline{x}$ can be augmented to these joints loading to form the structure loading vector $\underline{A}_s$ as

$$\underline{A}_s = [\underline{A}_j \; \vdots \; \underline{x}]^T \qquad (3.97)$$

where $\underline{\mathbf{A}}_j$ contains the loading at the free joints; and $\underline{\mathbf{x}}$ contains the chosen redundants for the structure.

Denoting $[\underline{\mathbf{A}}_m]_k$ is the internal actions in member k and $[\mathbf{f}_m]_k$ is the flexibility matrix of member k, the internal work (strain energy) stored in the structure is given by

$$U = W = \frac{1}{2} \sum_{k=1}^{m} [\underline{\mathbf{A}}_m]_k^T \, [\mathbf{f}_m]_k \, [\underline{\mathbf{A}}_m]_k^T \tag{3.98}$$

where m is the number of members in the structure.

This equation can be written in a shorter form as

$$U = W = \frac{1}{2} \underline{\mathbf{A}}_m^T [\mathbf{f}_m] \underline{\mathbf{A}}_m \tag{3.99}$$

where $[\mathbf{f}_m]$ is the members flexibility matrix, and $[\underline{\mathbf{A}}_m]$ contains the internal actions of all members, i.e.

$$\underline{\mathbf{A}}_m = \left[[\underline{\mathbf{A}}_m]_1 \quad [\underline{\mathbf{A}}_m]_2 \cdots [\underline{\mathbf{A}}_m]_k \cdots [\underline{\mathbf{A}}_m]_m \right] \tag{3.100}$$

From the law of conservation of energy, the strain energy in the structure must equal the external work done by the loads $\underline{\mathbf{A}}_s$. Therefore, one has

$$U = W = \frac{1}{2} \underline{\mathbf{A}}_s^T [\mathbf{f}_s] \underline{\mathbf{A}}_s \tag{3.101}$$

where $[\mathbf{f}_s]$ is the structural flexibility matrix

From Equations 3.99 and 3.101 one obtains

$$\underline{\mathbf{A}}_s^T [\mathbf{f}_s] \underline{\mathbf{A}}_s = \underline{\mathbf{A}}_m^T [\mathbf{f}_m] \underline{\mathbf{A}}_m \tag{3.102}$$

The static equilibrium conditions enable one to make a relationship between the members internal actions $\underline{\mathbf{A}}_m$ and the applied external actions $\underline{\mathbf{A}}_s$. Let this relationship be expressed as

$$\underline{\mathbf{A}}_m = \underline{\mathbf{E}} \, \underline{\mathbf{A}}_s = [\underline{\mathbf{E}}_j \; \vdots \; \underline{\mathbf{a}}] \begin{bmatrix} \underline{\mathbf{A}}_j \\ --- \\ \underline{\mathbf{x}} \end{bmatrix} \tag{3.103}$$

where $\underline{\mathbf{E}}$ is called the equilibrium matrix. It can be obtained by assuming a unit value for each of $\underline{\mathbf{A}}_j$ and $\underline{\mathbf{x}}$ to determine the members internal actions $\underline{\mathbf{A}}_m$. Substituting Equations 3.103 into Equation 3.102, one obtains

$$\underline{\mathbf{A}}_s^T [\mathbf{f}_s] \underline{\mathbf{A}}_s = \underline{\mathbf{A}}_s^T \, \underline{\mathbf{E}}^T [\mathbf{f}_m] \underline{\mathbf{E}} \, \underline{\mathbf{A}}_s \tag{3.104}$$

From Equation 3.104, it is obvious that $[\mathbf{f}_s]$ can be determined from

$$[\mathbf{f}_s] = \underline{\mathbf{E}}^T\,[\mathbf{f}_m]\,\underline{\mathbf{E}} \tag{3.105}$$

Determination of $[\mathbf{f}_s]$ leads us to write the deformations-actions relations for the structure as

$$\underline{\mathbf{D}}_s = [\mathbf{f}_s]\,\underline{\mathbf{A}}_s \tag{3.106}$$

which enables the determination of the deformation at the structure joints in the directions of $\underline{\mathbf{A}}_s$. Since the loads $\underline{\mathbf{A}}_s$ constitute known joint loads, $\underline{\mathbf{A}}_j$, and the unknown redundants $\underline{\mathbf{x}}$, the deformations are also decomposed into unknown deformations at the joint of known loads, $\underline{\mathbf{D}}_j$, and known deformations $\underline{\Delta}$ at the redundants locations. Equation 3.106 can thus be written as follows:

$$\begin{bmatrix} \underline{\mathbf{D}}_j \\ --- \\ \underline{\Delta} \end{bmatrix} = \left[\mathbf{f}_s\right] \begin{bmatrix} \underline{\mathbf{A}}_j \\ --- \\ \underline{\mathbf{x}} \end{bmatrix} \tag{3.107}$$

By partitioning the flexibility matrix $[\mathbf{f}_s]$, Equation 3.107 can be written as

$$\begin{bmatrix} \underline{\mathbf{D}}_j \\ --- \\ \underline{\Delta} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{f}_{jj} & \mathbf{f}_{jx} \\ \hline \mathbf{f}_{xj} & \mathbf{f} \end{array}\right] \begin{bmatrix} \underline{\mathbf{A}}_j \\ --- \\ \underline{\mathbf{x}} \end{bmatrix} \tag{3.108}$$

From Equations 3.108 one has

$$\underline{\mathbf{D}}_j = \left[\mathbf{f}_{jj}\right]\underline{\mathbf{A}}_j + \left[\mathbf{f}_{jx}\right]\underline{\mathbf{x}} \tag{3.109}$$

$$\underline{\Delta} = \left[\mathbf{f}_{xj}\right]\underline{\mathbf{A}}_j + \mathbf{f}\,\underline{\mathbf{x}} \tag{3.110}$$

Equation 3.110 is exactly the same as Equation 3.69, which we have dealt with in approach I, where $[\mathbf{f}_{xj}]\,\underline{\mathbf{A}}_j$ provides the deformations, $\underline{\Delta}_o$, due to the applied loading at the redundants locations.

The redundants $\underline{\mathbf{x}}$ can thus be determined from Equation 3.110 as

$$\underline{\mathbf{x}} = [\mathbf{f}]^{-1}\left[\underline{\Delta} - [\mathbf{f}_{xj}]\,\underline{\mathbf{A}}_j\right] \tag{3.111}$$

The deformations at the free joints can also be determined from Equation 3.109 as

$$\underline{\mathbf{D}}_j = [\mathbf{f}_{jj}]\,\underline{\mathbf{A}}_j + [\mathbf{f}_{jx}]\,[\mathbf{f}]^{-1}\left[\underline{\Delta} - [\mathbf{f}_{xj}]\,\underline{\mathbf{A}}_j\right] \tag{3.112}$$

The type of the member flexibility matrix should match the number and order of the member internal actions $\left[\underline{\mathbf{A}}_m\right]_k$. For example, in section 3.8, the member flexibility matrix for a plane frame member was given as

$$[f_m]_k = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{L}{6EI} \\ 0 & \frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix}_k \tag{3.113}$$

which is ordered according to the member internal actions $\lfloor A_m \rfloor_k^T = \lfloor A_x \; M_{z\ell} \; M_{zr} \rfloor_k$.

If one uses the flexibility matrix defined in Equation 2.60, section 2.21, which is given by

$$[f]_m = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \tag{3.114}$$

then, the member internal actions should be arranged as axial force, shear force, and bending moment at the right end of the member as follows:

$$\lfloor A_m \rfloor_k^T = \lfloor A_x \;\; A_y \;\; M_{zr} \rfloor_k \tag{3.115}$$

3.9.2 Applications to Beams

In the case of beams, if the member end actions are defined as the bending moments $M_{z\ell}$ and M_{zr}, then the flexibility matrix of the member is the same as Equation 3.96. However, if the internal actions are defined as the shear force and bending moment at the right end of the member, then the flexibility matrix becomes

$$[f_m] = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \tag{3.116}$$

In the following example we use the members flexibility matrices as were used in approach I.

Example 3.46

Determine the bending moment diagram for the beam shown in Figure 3.190 (example 3.41) due to the applied loading, a rotation at A of 0.002 rad, anticlockwise, and a rise in temperature in member BC. (EI = 10^5 kN.m^2, $\alpha = 10^{-5}/$ °c).

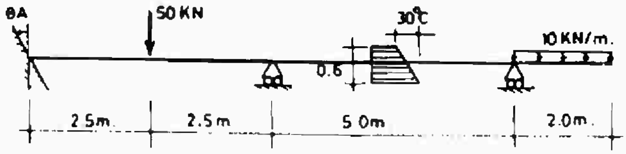


Figure 3.190

Solution

One defines the structural coordinates to be related to the members coordinates as shown in Figure 3.191, where the last two numbers 5 and 6 in the structural coordinates are used for the redundants x_1 at A and x_2 at C. The members and their coordinates are numbered as shown in the Figure.

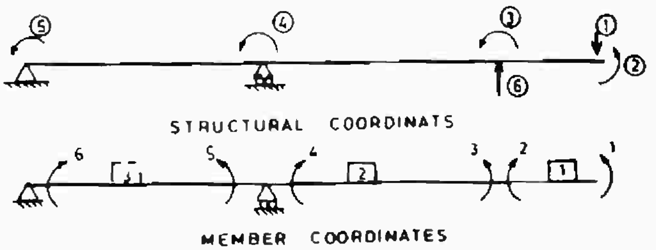


Figure 3.191

By applying a unit load or moment for each structural coordinate, one determines the member internal actions in the directions of members coordinates from which we form the equilibrium matrices $\underline{E}$, $\underline{E}_j$ and $\underline{a}$. From Figure 3.192 one has:

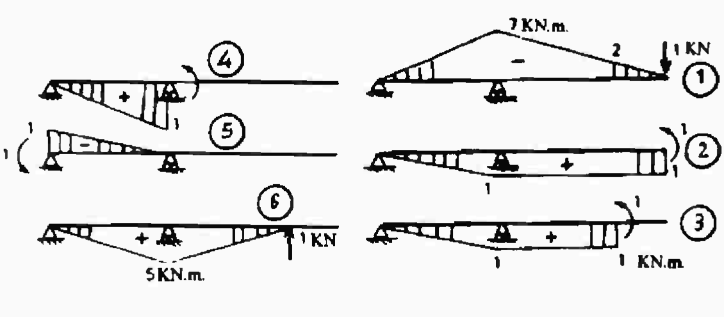


Figure 3.192

$$\underline{E} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 \\ -7 & 1 & 1 & 0 & 0 & 5 \\ -7 & 1 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad \underline{E}_j = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -7 & 1 & 1 & 0 \\ -7 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 5 \\ 0 & 5 \\ -1 & 0 \end{bmatrix}$$

The member flexibility matrix $[f_m]$ is then determined in accordance with the numbering of the members such that along the diagonal we put $[f_m]_1$, $[f_m]_2$, and $[f_m]_3$, respectively, as shown below

$$[f_m] = \frac{1}{EI}\begin{bmatrix} \frac{2}{3} & \frac{2}{6} & 0 & 0 & 0 & 0 \\ \frac{2}{6} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{3} & \frac{5}{6} & 0 & 0 \\ 0 & 0 & \frac{5}{6} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{3} & \frac{5}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{5}{3} \end{bmatrix}$$

The fixed-end actions and equivalent joint actions are determined as shown in Figure 3.193.

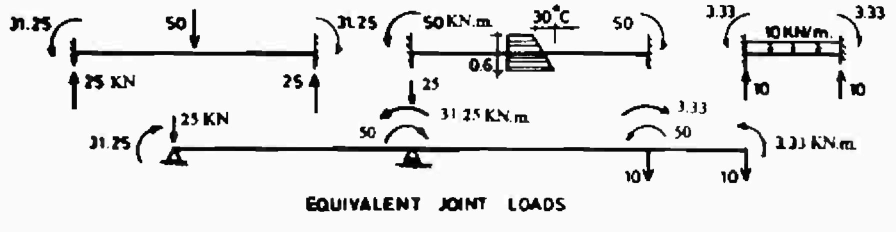


Figure 3.193

The values of $\underline{A}_j$ are then determined from the equivalent joint loading according to the structural coordinates, and the fixed end actions are expressed according to the member coordinates shown in Figure 3.191 as follows:

$$\underline{A}_s^T = [10 \quad 3.33 \quad 46.67 \quad -18.75 \quad (-31.25 + x_1) \quad (-10 + x_2)]$$

$$A_j^T = [10 \quad 3.33 \quad 46.67 \quad -18.7]$$

$\underline{A}_{fm}^{T} = [-3.33 \;\; -3.33 \;\; -50 \;\; -50 \;\; -31.25 \;\; -31.25]$

The structural flexibility matrix is then determined from

$$[f_s] = \underline{E}^T [f_m] \underline{E}$$

$$= \begin{bmatrix} \underline{E}_j^T \\ \\ \underline{a}^T \end{bmatrix} [f_m] [\underline{E}_j \;\vdots\; \underline{a}] = \begin{bmatrix} f_{jj} & f_{jx} \\ \\ f_{xj} & f \end{bmatrix}$$

from which we have

$$[f] = \underline{a}^T [f_m] \underline{a}$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 5 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0.67 & 0.33 & 0 & 0 & 0 & 0 \\ 0.33 & 0.67 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.67 & 0.835 & 0 & 0 \\ 0 & 0 & 0.835 & 1.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.67 & 0.835 \\ 0 & 0 & 0 & 0 & 0.835 & 1.67 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 5 \\ 0 & 5 \\ -1 & 0 \end{bmatrix}$$

$$= 10^{-5} \begin{bmatrix} 1.67 & -4.167 \\ -4.167 & 83.33 \end{bmatrix}$$

Similarly, $[f_{xj}]$ is determined from

$$[f_{xj}] = \underline{a}^T [f_m] \underline{E}_j$$

$$= 10^{-5} \begin{bmatrix} 5.845 & -0.835 & -0.835 & -0.835 \\ -125.03 & 20.835 & 20.835 & 8.35 \end{bmatrix}$$

Substituting into Equation 3.110, one obtains

$$\underline{\Delta} = [f_{xj}] \underline{A}_j + [f] \underline{x}$$

$$\begin{bmatrix} 0.002 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 5.484 & -0.835 & -0.835 & -0.835 \\ \\ -125.03 & 20.835 & 20.835 & 8.35 \end{bmatrix} \begin{bmatrix} 10 \\ 3.33 \\ 46.67 \\ -18.75 \end{bmatrix} + \begin{bmatrix} 1.67 & -4.167 \\ \\ -4.167 & 83.33 \end{bmatrix} \begin{bmatrix} (x_1 - 31.25) \\ \\ (x_2 - 10) \end{bmatrix}$$

The values of x_1 and x_2 are obtained as

$x_1 = 158.75$ kN.m ; $x_2 = 20.75$ kN

The deformations at the free joints are obtained from

$$\underline{D}_j = [F_{jj}]\,\underline{A}_j + [f_{jx}]\,\underline{X}$$

$$[D_1 \;\; D_2 \;\; D_3 \;\; D_4] = 10^{-5}\,[-136.24 \;\; 64.785 \;\; 78.125 \;\; -81.25]$$

To determine the member internal actions one has

$$\underline{A}_m = \underline{E}\,\underline{A}_s + \underline{A}_{fm} = \begin{bmatrix} 3.33 \\ -16.67 \\ 30 \\ 33.75 \\ 15 \\ 127.5 \end{bmatrix} + \begin{bmatrix} -3.33 \\ -3.33 \\ -50 \\ -50 \\ -31.25 \\ -31.25 \end{bmatrix} = \begin{bmatrix} 0 \\ -20 \\ -20 \\ -16.25 \\ -16.25 \\ -158.75 \end{bmatrix}$$

which results in the values of the end moments according to the numbering of the member coordinates. The bending moment diagram can now be plotted according to the conventions of Figure 3.191 as shown in Figure 3.194.

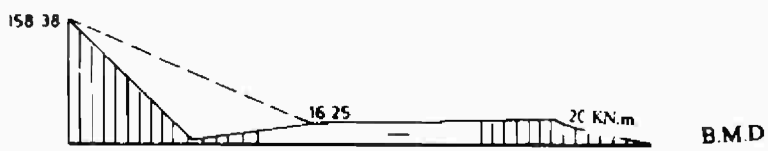


Figure 3.194

3.9.3 Applications to Frames

In plane frames, when considering axial deformations, the flexibility matrices are the same as given in Equations 3.113 or Equation 3.114.

Example 3.47

Determine the bending moment and axial force diagrams for the frame shown in Figure 3.195 (Example 3.42) due to the applied loads and a rise in temperature in member BC. ($EI = 10^5$ kN m^2, $EA = 50 \times 10^5$ kN, $\alpha = 10^{-5}/°c$).

Solution

The structural and member coordinates are selected as shown in Figure 3.196. The redundants x_1 and x_2 are in coordinates number 7 and 8.

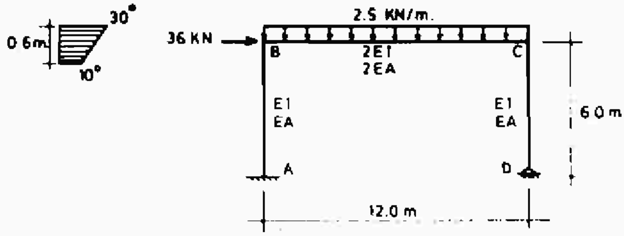


Figure 3.195

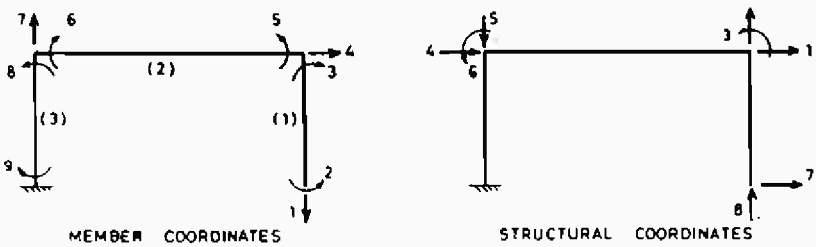


Figure 3.196

Applying a unit load in each structural coordinate, one can determine the equilibrium matrix **E** as described in Figure 3.197.

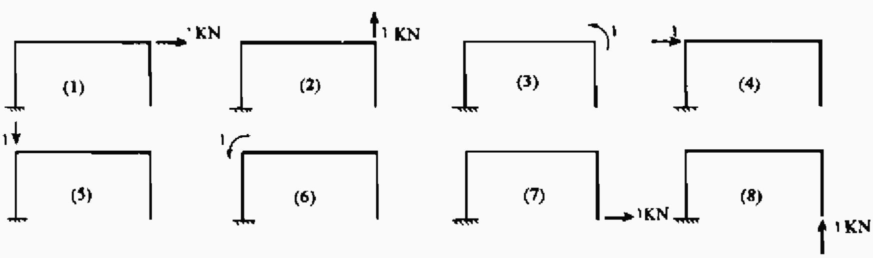


Figure 3.197

$$
\underline{\mathbf{E}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 6 & 0 \\
0 & 12 & 1 & 0 & 0 & 0 & 6 & 12 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 12 & 1 & 0 & 0 & 1 & 6 & 12 \\
-6 & 12 & 1 & -6 & 0 & 1 & 0 & 12
\end{bmatrix} = [\underline{\mathbf{E}}_l \vdots \underline{\mathbf{a}}] \quad ; \quad \underline{\mathbf{a}} = \begin{bmatrix}
0 & -1 \\
0 & 0 \\
6 & 0 \\
1 & 0 \\
6 & 0 \\
6 & 12 \\
0 & 1 \\
6 & 12 \\
0 & 12
\end{bmatrix}
$$

The members flexibility matrix according to the numbering of the members is

$$
[\mathbf{f}_m] = 10^{-5} \begin{bmatrix}
0.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
$$

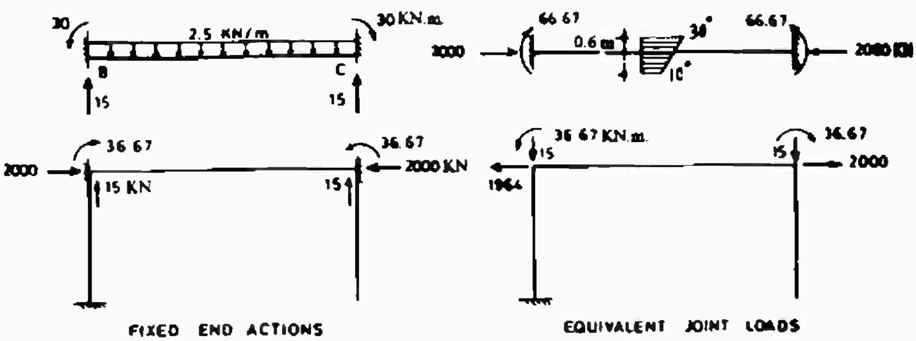


Figure 3.198

The fixed end actions and the equivalent joint loading are determined as in Figure 3.198. They can be arranged in matrix forms as

$$\underline{\mathbf{A}}_j^T = [2000 \quad -15 \quad -36.67 \quad -1964 \quad 15 \quad 36.67]$$

$$\underline{\mathbf{A}}_{fm}^T = [0 \quad 0 \quad 0 \quad -2000 \quad 36.67 \quad 36.67 \quad 0 \quad 0 \quad 0]$$

$$\underline{x}^T = [x_1 \ \ x_2]$$

The structural flexibility matrix $[f_s]$ is determined from $[f_s] = \underline{E}^T\,[f_m]\,\underline{E}$. This matrix is partitioned into $[f_{jj}]$, $[f_{jx}]$, $[f_{xj}]$ and $[f]$. The matrix $[f]$ is determined from

$$[f] = \underline{a}^T [f_m]\,\underline{a} = 10^{-5}\begin{bmatrix} 360.12 & 432 \\ 432 & 1152.24 \end{bmatrix}$$

The matrix $[f_{xj}]$ is determined from

$$[f_{xj}] = \underline{a}^T\,[f_m]\,\underline{E}_j = 10^{-5}\begin{bmatrix} -35.88 & 432 & 54 & -36 & 0 & 18 \\ -216 & 1152.12 & 108 & -216 & -0.12 & 72 \end{bmatrix}$$

Substituting into Equation 3.110, one has

$$\underline{\Delta} = [f_{xj}]\,\underline{A}_j + [f]\,\underline{x}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 10^{-5}\begin{bmatrix} -8856.12 \\ -26379.72 \end{bmatrix} + 10^{-5}\begin{bmatrix} 360.12 & 432 \\ 432 & 1152.24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

from which we obtains $x_1 = -5.22$ kN and $x_2 = 24.85$ kN.

The member internal actions are determined from

$\underline{A}_m = \underline{E}\,\underline{A}_s + \underline{A}_{fm}$, or from $\underline{A}_m = \underline{E}_j\,\underline{A}_j + \underline{a}\,\underline{x} + \underline{A}_{fm}$

$$\underline{A}_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2000 \\ -36.67 \\ -216.67 \\ -30 \\ -180 \\ -396 \end{bmatrix} + \begin{bmatrix} -24.85 \\ 0 \\ -31.32 \\ -5.22 \\ -31.32 \\ 266.68 \\ 24.85 \\ 266.88 \\ 298.2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2000 \\ 36.67 \\ 36.67 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -24.85 \\ 0 \\ -31.32 \\ -5.22 \\ -31.32 \\ 86.88 \\ -5.15 \\ 86.88 \\ -97.8 \end{bmatrix}$$

The deformation at the free joints are obtained from

$$\underline{D}_j = [f_{jj}]\,\underline{A}_j + [f_{jx}]\,\underline{X}$$

This gives

$$\underline{D}_j^T = 10^{-5}\,[891.428 \ \ -2.982 \ \ -85.94 \ \ 652.05 \ \ 0.618 \ \ -32.667]$$

The bending moment and axial force diagrams can be drawn as shown in Figure 3.199. The results are the same as obtained before.

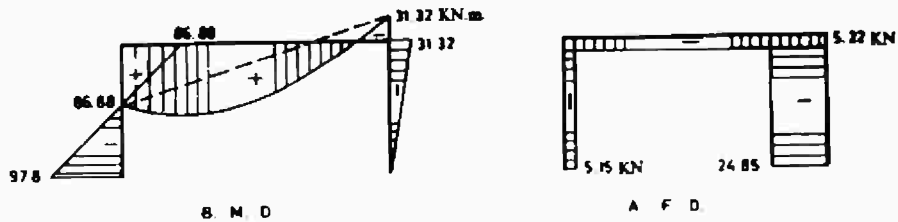


Figure 3.199

3.9.4 Applications to Trusses

Example 3.48

Determine the member forces and the horizontal deflection at C for the truss shown in Figure 3.200 (example 3.44) due to the applied loads and a rise in temperature for members ED and DC of 20°c ($EA = 20 \times 10^5$ kN, $\alpha = 10^{-5}/°c$).

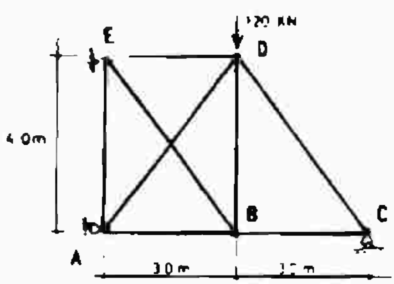


Figure 3.200

Solution

The structural coordinates and member coordinates are selected as shown in Figure 3.201.

Applying a unit load in each structural coordinate, the equilibrium matrix **E** can be determined using the members forces shown in Figure 3.202.

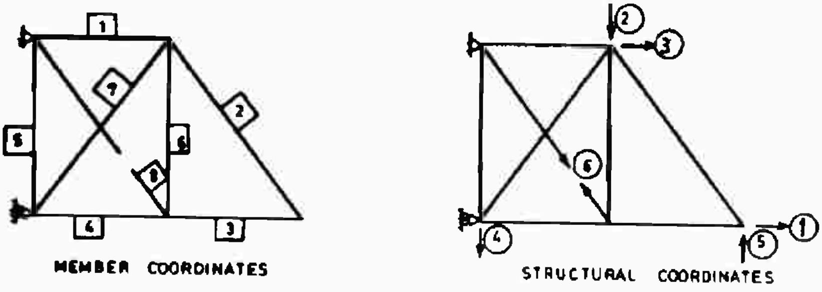


Figure 3.201

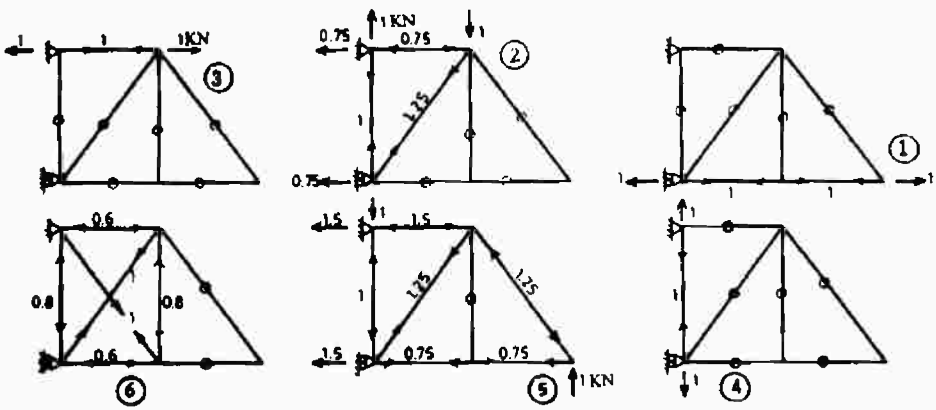


Figure 3.202

$$\underline{\mathbf{E}}_j = \left[\begin{array}{cccc:cc} 0 & 0.75 & 1 & 0 & -1.5 & -0.6 \\ 0 & 0 & 0 & 0 & -1.25 & 0 \\ 1 & 0 & 0 & 0 & 0.75 & 0 \\ 1 & 0 & 0 & 0 & 0.75 & -0.6 \\ 0 & 1 & 0 & 1 & -1 & -0.8 \\ 0 & 0 & 0 & 0 & 0 & -0.8 \\ 0 & -1.25 & 0 & 0 & 1.25 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{array}\right] \quad ; \quad \underline{\mathbf{a}} = \begin{bmatrix} -1.5 & -0.6 \\ -1.25 & 0 \\ 0.75 & 0 \\ 0.75 & -0.6 \\ -1 & -0.8 \\ 0 & -0.8 \\ 1.25 & 1.0 \\ 0 & 1.0 \end{bmatrix}$$

$$\underline{\mathbf{E}} = [\mathbf{E}_j \ \vdots \ \mathbf{a}]$$

The member flexibility matrix $[f_m]$ is determined as

$$[f_m] = \frac{10^{-5}}{20}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

The fixed end forces due to temperature change and the equivalent joint loads are determined as shown in Figure 3.203.

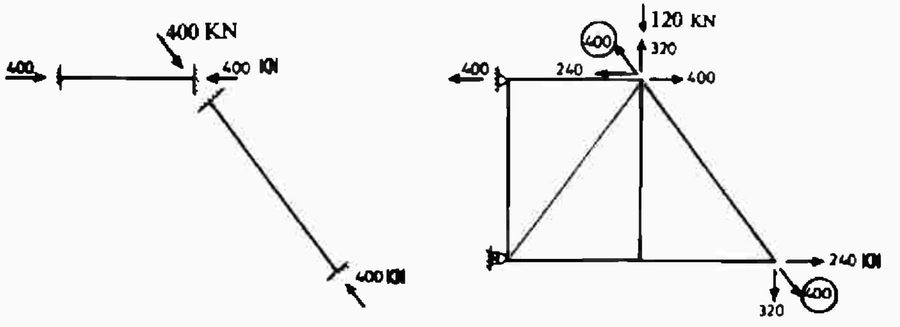


Figure 3.203

The values of $\underline{A}_j$ and $\underline{A}_{fm}$ can be arranged as follows:

$$\underline{A}_j^T = [240 \quad -200 \quad 160 \quad 0 \quad -320 \quad 0]$$

$$\underline{A}_{fm}^T = [-400 \quad -400 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\underline{x}^T = [x_1 \quad x_2]$$

The structural flexibility matrix $[f_s]$ is determined from $[f_s] = \underline{E}^T\,[f_m]\,\underline{E}$, and $[f]$ and $[f_{xj}]$ are determined as follows:

$$[f] = \underline{a}^T\,[f_m]\,\underline{a} = 10^{-5}\begin{bmatrix} 1.4875 & 0.54 \\ 0.54 & 0.864 \end{bmatrix}$$

$$[f_{xj}] = \underline{a}^T\,[f_m]\,\underline{E}_j = 10^{-5}\begin{bmatrix} 0.225 & -0.7594 & -0.225 & -0.2 \\ -0.09 & -0.54 & -0.09 & -0.16 \end{bmatrix}$$

Substituting into Equation 3.110, one obtains

$$\underline{\Delta} = [\mathbf{f}_{xj}]\,\underline{\mathbf{A}}_j + [\mathbf{f}]\,\underline{\mathbf{x}}$$

The solution is $x_1 = 211.413$ kN and $x_2 = -15.466$ kN

The member internal forces are determined from

$$\underline{\mathbf{A}}_m = \underline{\mathbf{E}}_j\,\underline{\mathbf{A}}_j + \underline{\mathbf{a}}\,\underline{\mathbf{x}} + \underline{\mathbf{A}}_{fm} \quad , \quad \text{or} \quad \text{from } \underline{\mathbf{A}}_m = \underline{\mathbf{E}}\,\underline{\mathbf{A}}_s + \underline{\mathbf{A}}_{fm}$$

$$\underline{\mathbf{A}}_m = \begin{bmatrix} 590 \\ 400 \\ 0 \\ 0 \\ 120 \\ 0 \\ -150 \\ 0 \end{bmatrix} + \begin{bmatrix} -307.84 \\ -264.26 \\ 158.55 \\ 167.83 \\ -199.04 \\ 12.368 \\ 248.8 \\ -15.46 \end{bmatrix} + \begin{bmatrix} -400 \\ -400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -217.84 \\ -264.26 \\ 158.55 \\ 167.83 \\ -79.04 \\ 12.368 \\ 98.8 \\ -15.46 \end{bmatrix} \text{kN}$$

which are the same results as example 3.44.

To determine the free joints displacements, Equation 3.109 is used as follows:

$$\underline{\mathbf{D}}_j = [\mathbf{f}_{jj}]\,\underline{\mathbf{A}}_j + [\mathbf{f}_{jx}]\,\underline{\mathbf{x}}$$

where $[\mathbf{f}_{jx}] = [\mathbf{f}_{xj}]^T$ and $[\mathbf{f}_{jj}]$ is given by

$$[\mathbf{f}_{jj}] = \begin{bmatrix} 6 & 0 & 0 & 0 & 4.5 & -1.8 \\ 0 & 13.5 & 2.25 & 4 & -15.1875 & -10.8 \\ 0 & 2.25 & 3 & 0 & -4.5 & -1.8 \\ 0 & 4 & 0 & 4 & -4 & -3.2 \\ 4.5 & -15.1875 & -4.5 & -4 & 29.75 & 10.8 \\ -1.8 & -10.8 & -1.8 & -3.2 & 10.8 & 17.28 \end{bmatrix}$$

Substituting, one obtains

$$\underline{\mathbf{D}}_j^T = 10^{-5}\,[48.96 \quad -26.19 \quad 27.32 \quad -15.8]\ \text{m}$$

The horizontal displacement of coordinate number 1 equals 0.04896 cm which is the same result as Example 3.44.

3.9.5 Application to Frame-Truss Structures

As shown in Section 3.8.8, one may consider the truss members possess only axial force and therefore their flexibility matrix is just [L/EA], or they can be considered as frame members but the end moments are zero.

Example 3.49

Determine the bending moment and axial force diagrams for the structure shown in Figure 3.204. ($EI = 10^5$ kN.m^2, $EA = 0.5 \times 10^5$ kN).

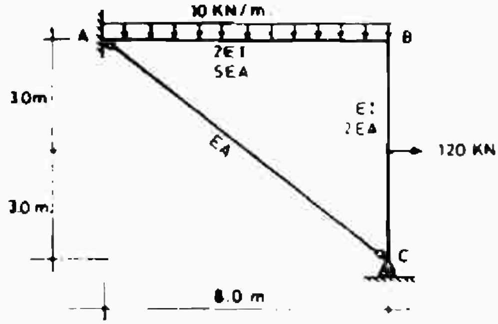


Figure 3.204

Solution

The structural coordinates and members coordinates are selected as shown in Figure 3.205 where the redundants x_1 and x_2 are in coordinates 6 and 7.

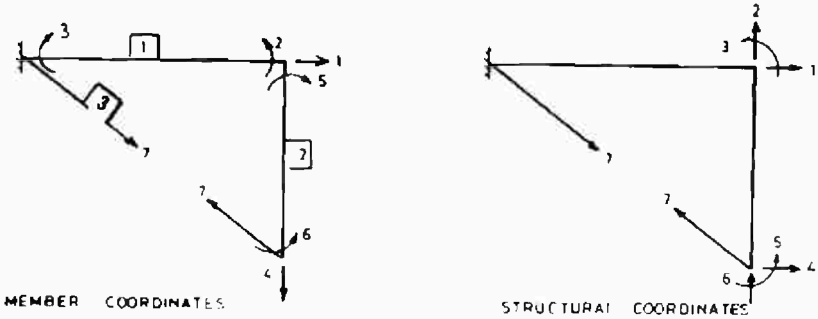


Figure 3.205

The equilibrium matrix **E** is determined by applying a unit load in direction of each structural coordinate and find the members forces as shown in Figure 3.206.

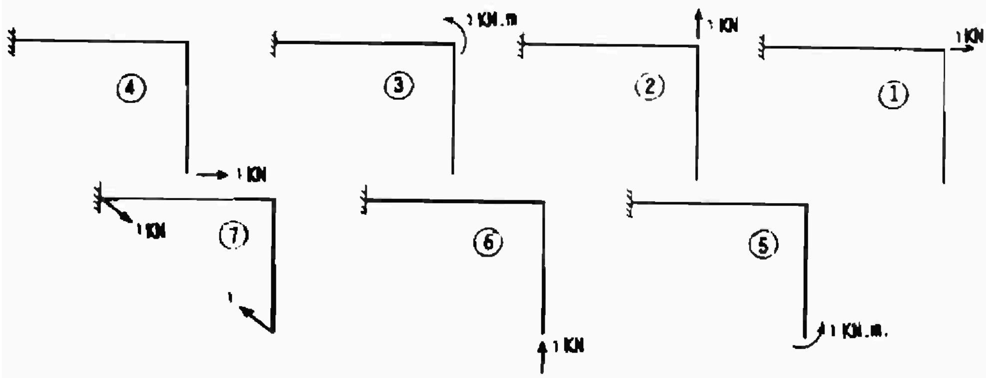


Figure 3.206

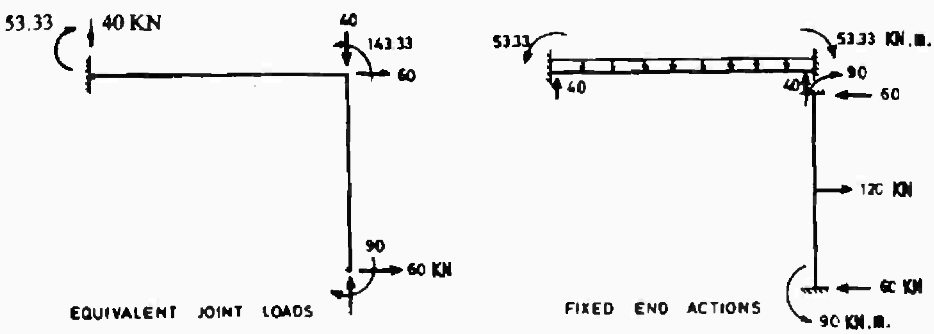


Figure 3.207

$$\underline{\mathbf{E}} = \left[\begin{array}{ccccc:cc} 1 & 0 & 0 & 1 & 0 & 0 & -0.8 \\ 0 & 0 & 1 & 6 & 1 & 0 & -4.8 \\ 0 & 8 & 1 & 6 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -0.6 \\ 0 & 0 & 0 & 6 & 1 & 0 & -4.8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\underline{\mathbf{E}}_j \ \vdots \ \underline{\mathbf{a}}\right]$$

The fixed end actions and the equivalent joint actions are determined as shown in Figure 3.207. The loading $\underline{\mathbf{A}}_j$ and $\underline{\mathbf{A}}_{fm}$ are given by

$$\underline{\mathbf{A}}_j^T = [60 \;\; -40 \;\; +143.33 \;\; 60 \;\; -90]$$

$$\underline{\mathbf{A}}_{fm}^T = [0 \;\; -53.33 \;\; -53.33 \;\; 0 \;\; 90 \;\; 90 \;\; 0]$$

The member flexibility matrix $[\mathbf{f}_m]$ is assembled as

$$[f_m]=10^{-5}\begin{bmatrix}3.2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1.333 & 0.667 & 0 & 0 & 0 & 0\\ 0 & 0.667 & 1.333 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 6 & 0 & 0 & 0\\ 0 & 0 & 0 & 6 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 20\end{bmatrix}$$

The structural flexibility matrix, $[f_s]$, is determined using Equation 3.105 as follows:

$$[f]=\underline{a}^T[f_m]\underline{a} = 10^{-5}\begin{bmatrix}91.334 & -22\\ -22 & 101\end{bmatrix}$$

$$[f_{xj}]=\underline{a}^T[f_m]\underline{E}_j = 10^{-5}\begin{bmatrix}0 & 85.336 & 16 & 96 & 16\\ -2.56 & -25.6 & -9.6 & -117.76 & -24\end{bmatrix}$$

Substituting into Equation 3.110 one obtains

$$\begin{bmatrix}0\\0\end{bmatrix}=10^{-5}\begin{bmatrix}3199.88\\-5411.2\end{bmatrix}+10^{-5}\begin{bmatrix}91.334 & -22\\ -22 & 101\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

which gives x_1 = -23.35 kN and x_2 = 48.49 kN.

The member internal forces are computed from

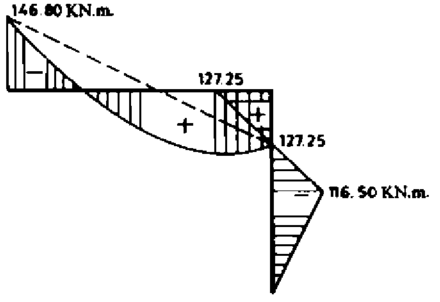
$\underline{A}_m=\underline{E}_j\,\underline{A}_j+\underline{a}\,\underline{x}+\underline{A}_{fm}$ or from $\underline{A}_m=\underline{E}\,\underline{A}_s+\underline{A}_{fm}$

$$=\begin{bmatrix}120\\413.333\\93.333\\0\\270\\-90\\0\end{bmatrix}+\begin{bmatrix}-38.79\\-232.75\\-186.8\\-5.744\\-232.75\\0\\48.49\end{bmatrix}+\begin{bmatrix}0\\-53.33\\-53.33\\0\\90\\90\\0\end{bmatrix}=\begin{bmatrix}81.21\\127.25\\-146.8\\-5.744\\127.25\\0\\48.49\end{bmatrix}$$

The deformations at the free joints are calculated from $\underline{D}_j=[f_{jj}]\,\underline{A}_j+[f_{jx}]\,\underline{x}$ which gives

$$\underline{D}_j^T=10^{-5}\begin{bmatrix}26 & -3.32 & 17.42 & 121.21 & 1.6\end{bmatrix}$$

The bending moment and axial force diagrams are plotted as shown in Figure 3.208.



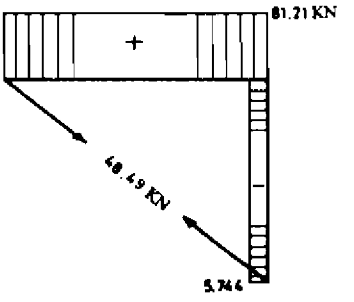


Figure 3.208

Exercises

1. Determine the flexibility matrix associated with the actions A_1, A_2 and A_3 for the truss shown in Figure 1. (EA = constant).

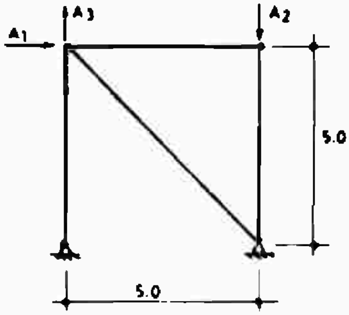


Figure 1

2. Use the column analogy method to determine the bending moment diagram for the frame shown in Figure 2.

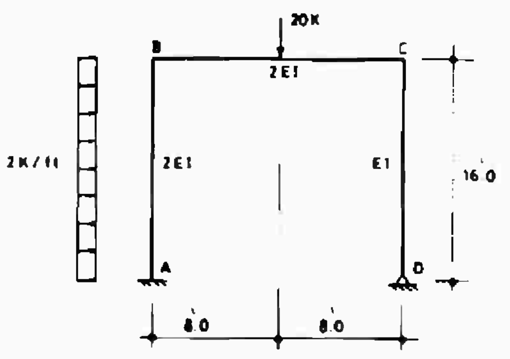


Figure 2

3. Use the consistent deformations method to solve problem No. 2.

4. Use Castigliano's second theorem to solve problem No. 2.

5. Use any of the force methods to determine bending moment diagram for the beam shown due to the applied loading and rise in temperature as shown in Figure 3. ($EI = 10^6$ kN.m^2, $\alpha = 10^{-5}$/°c, K = 10 kN/m).

6. Use the three moment equation to determine the bending moment diagram for the beams shown in Figure 4.

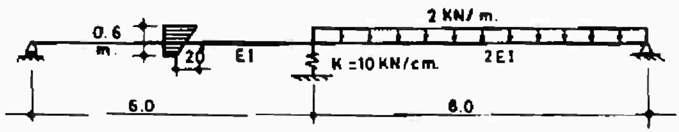


Figure 3

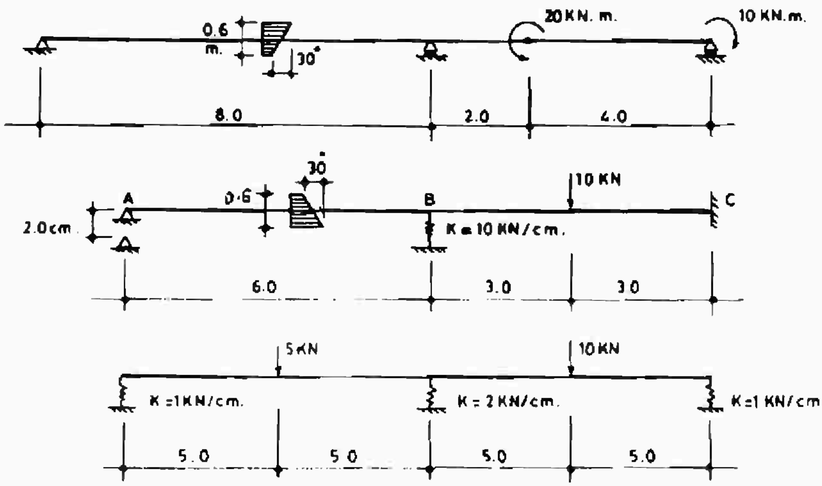


Figure 4

7. Determine the bending moment diagram and the horizontal deflection at C for the frame shown in Figure 5 using any of the force methods. ($EI = 2\times10^6$ $kN.m^2$, $EA = 8\times10^6$ kN).

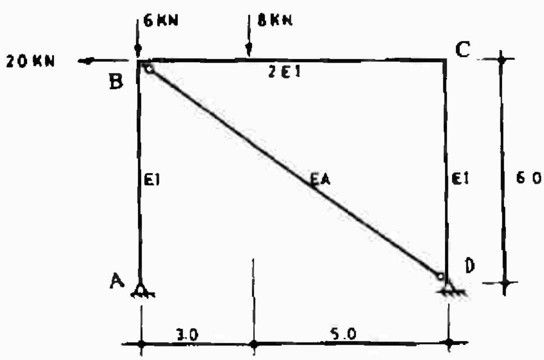


Figure 5

8. For the truss shown in Figure 6, if support A has moves towards the right 1 inch, determine the member forces due to the settlements and loading given (EA = 1000 Kips).

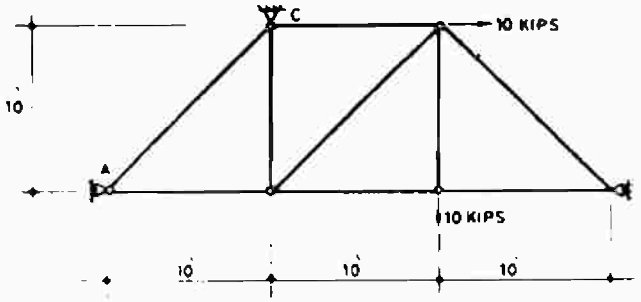


Figure 6

9. Solve problem No. 8 if support A is an elastic with spring constant of 2 Kip/in.

10. Use three moment equation to draw being moment and shear force diagrams for the continuous beam shown in Figure 7 due to the loads shown and settlement at B of 2 cm down. (EI = 1.5×10^6 kN.m^2).

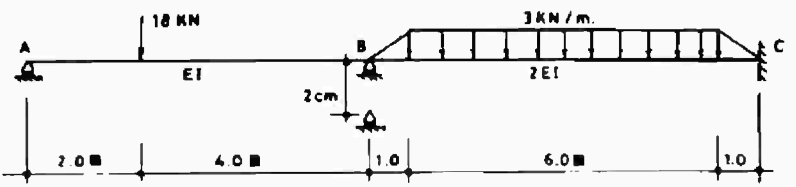


Figure 7

11. Use any of the force methods to determine the member forces in the truss shown in Figure 8. (EA = 10^5 kN for all members).

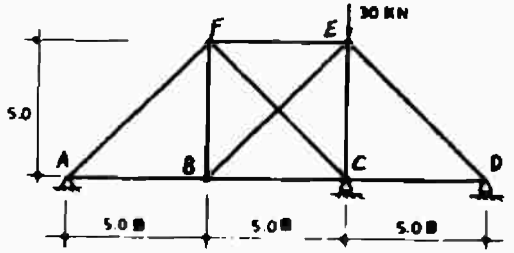


Figure 8

12. Use the consistent deformation method to determine the bending moment diagram for the frame shown in Figure 9. ($EI = 10^5$ kN m^2).

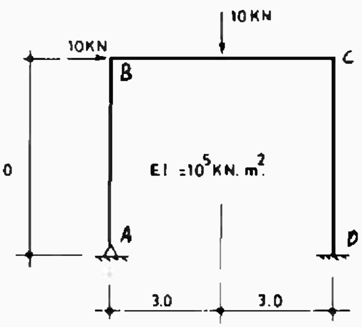


Figure 9

13. Use any of the force methods to determine the bending moment diagram for the beams shown in Figure 10.

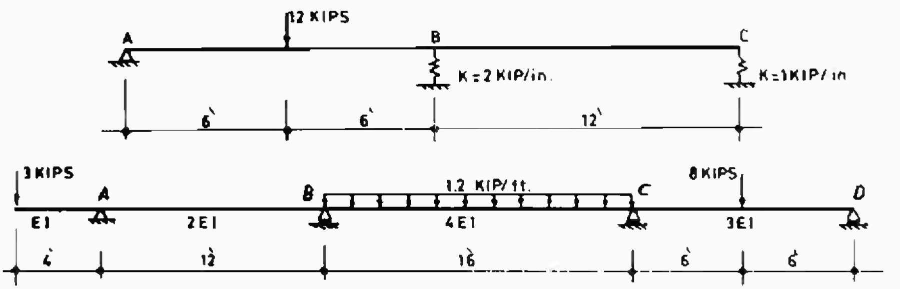


Figure 10

14. Use the force method. Matrix approach I, to analyze the truss shown in Figure 11. (EA = 2500 Kips for all members).

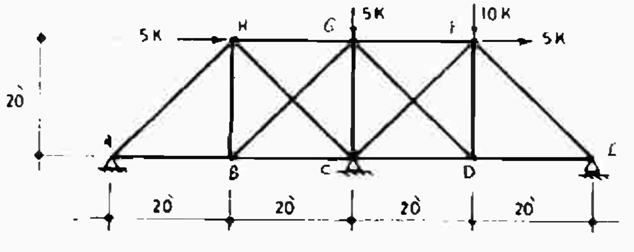


Figure 11

15. Use the force method, matrix approach I, to analyze the truss shown in Figure 12. (EA = 2000 Kips for all members).

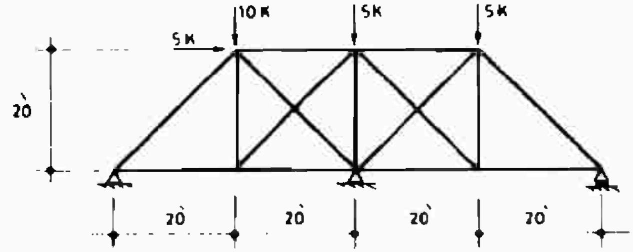


Figure 12

16 Use the force method, matrix approach I, to analyze the frame shown in Figure 13, considering axial deformation (EA = 3000 K, EI = 30000 K.ft^2).

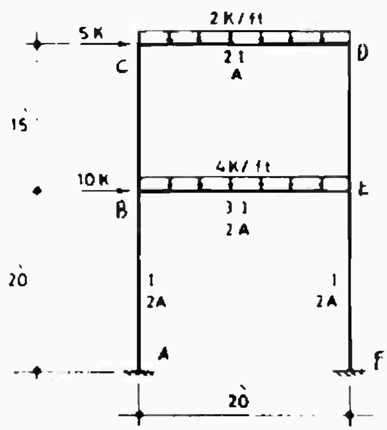


Figure 13

17 Use any of the matrix approaches of the force method to determine the bending moment diagram for the structures shown in Figure 14.

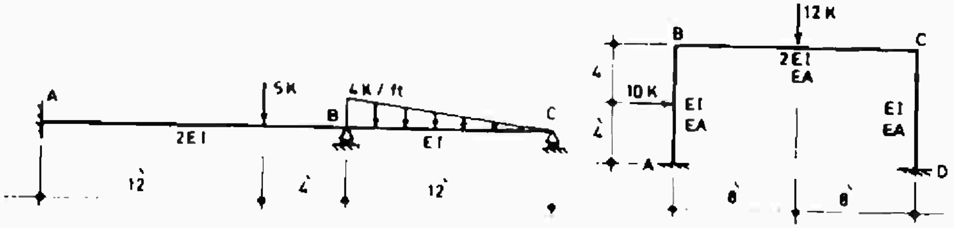


Figure 14

18. Use the elastic centre or column analogy methods to determine the bending moment diagram for the frame shown in Figure 15.

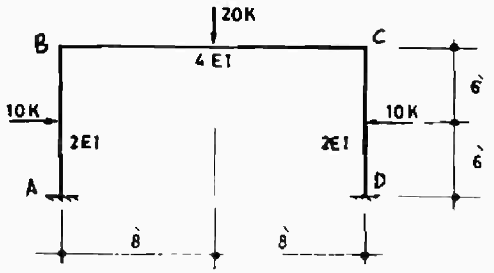


Figure 15

19. Use the three moment equation to determine the bending moment diagram for the beam shown in Figure 16 due to the applied loads and a settlement at A of 0.4 inch downward. (EI = 10^5 K.ft^2).

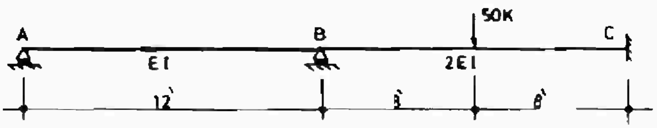


Figure 16

20. Use any of the matrix approaches of the force method to determine the member forces and horizontal displacements at B and D for the truss shown in Figure 17. (EA = 3000 Kips for all members).

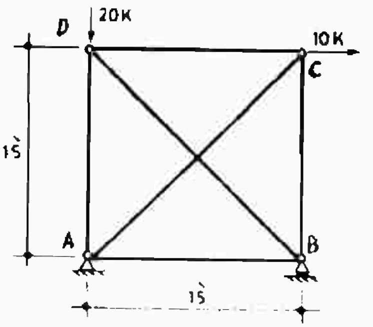


Figure 17

21. If member DE was subjected to rise in temperature as shown in Figure 18. Determine the member forces in the truss using any of the force methods. ($\alpha = 6.5 \times 10^{-6}$/°F, T = 50°F).

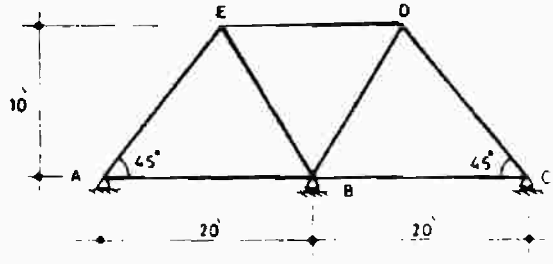


Figure 18

22. Solve problem No. 21 if support B yields down of ½ inch. (EA = 3000 Kips).

23. Use any of the force methods to analyze the beam shown in Figure 19 due to the rise in temperature. (EA = 2000 kN, EI = 20000 kN.m^2, $\alpha = 10^{-5}$/°c).

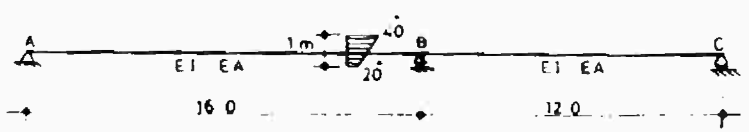


Figure 19

24. Use any of the force methods to analyze the truss shown in Figure 20 and determine the horizontal deflection at D. (EA = constant).

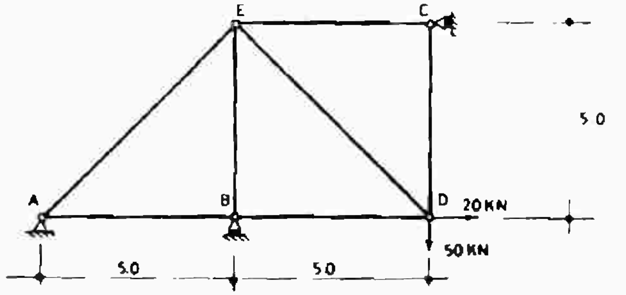


Figure 20

25. Determine the bending moment diagram for the frame shown in Figure 21 due to the loads and settlement at B of 2 inches downward, using any of the force methods. (EI = 30000 K.ft^2).

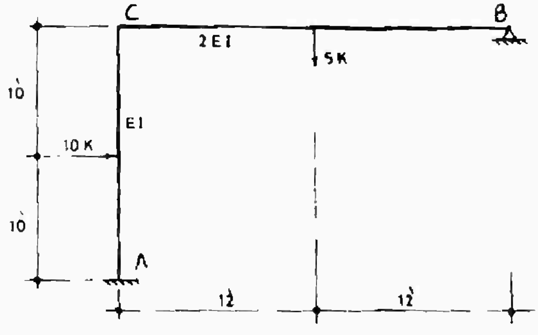


Figure 21

26. For the truss shown in Figure 22, determine the member forces, and the displacements of joint B, using the matrix formulation of the force method. Consider the length of each member is 12 ft, and EA = 1000 Kips for all members.

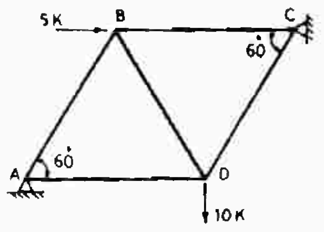


Figure 22

27. Use the three moment equation to determine the bending moment for the beam shown in Figure 23, if support A settles 15 mm downward, support B settles 25 mm downward, and support C rotates 0.0005 radian counterclockwise. (EI = 6 × 10^5 kN.m^2).

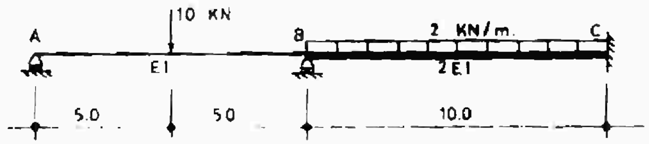


Figure 23

28 If the fixed support at C has rotated 0.0005 radian, clockwise, determine the bending moment diagram for the frame shown in Figure 24 and determine the horizontal deflection at B. (Neglect the axial deformation, $EI = 6 \times 10^5$ $kN.m^2$)

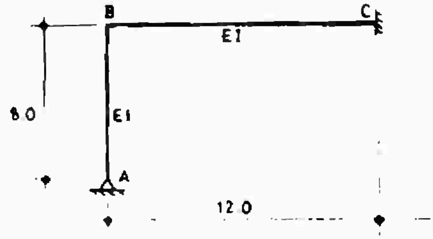


Figure 24

29 Determine the bending moment for the frame shown in Figure 25 using the method of elastic centre.

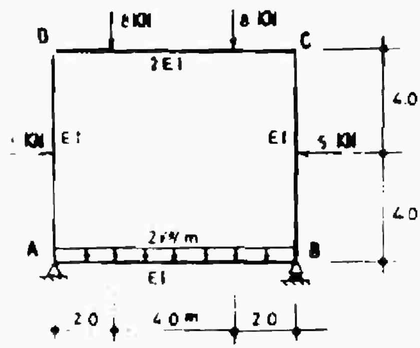


Figure 25

30. Determine the reactions and member forces of the truss shown in Figure 26 using the consistent deformation method based on the matrix approach (EA = constant for all members).

31. Determine the bending moment and shear force diagrams for the frame shown in Figure 27 using matrix approach I of the flexibility matrix method. (EA = 1000 Kips, $EI = 10000\ K.ft^2$)

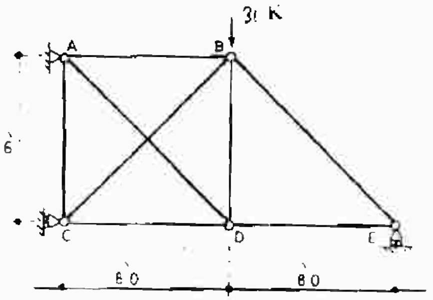


Figure 26

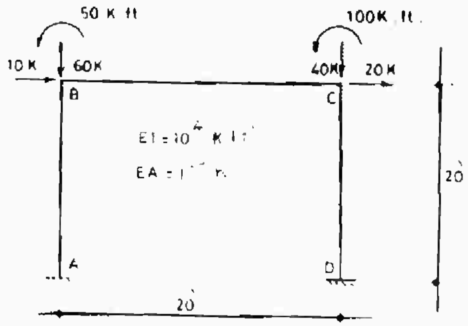


Figure 27

32. Use the matrix approach II of the flexibility matrix method to determine the displacements and reactions of the frame shown in Figure 28. (EA = 1000 Kips, EI = 1000 K ft^2).

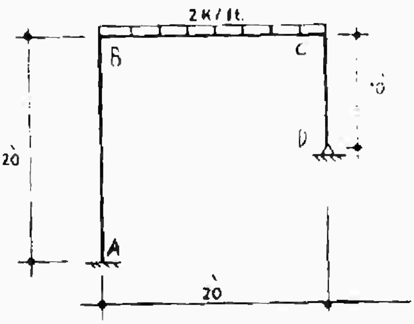


Figure 28

33. Use any of the force methods to determine the bending moment diagram for the beam shown in Figure 29.

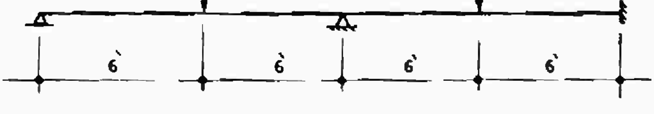


Figure 29

34. Use Castigliano's second theorem to determine the reactions in the beam shown in Figure 30. (EI = constant).

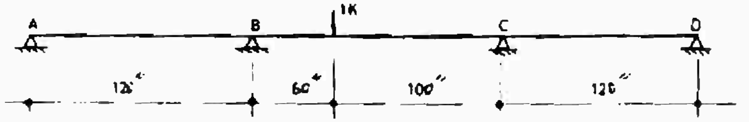


Figure 30

35. Use any of the force methods to analyze the truss shown in Figure 31 (E = 30000 Ksi, and A as indicated).

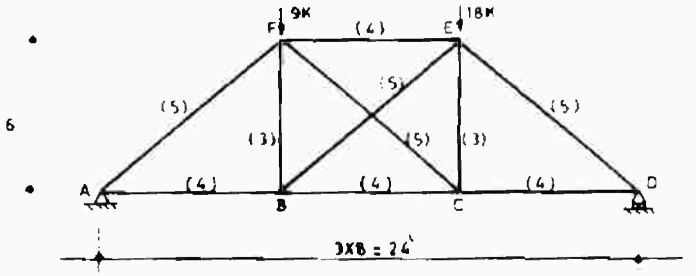


Figure 31

36. For the beam shown in Figure 32 find:

(a) Stiffness factors and carry over factors at A and B respectively.

(b) Fixed end moment at A and B.

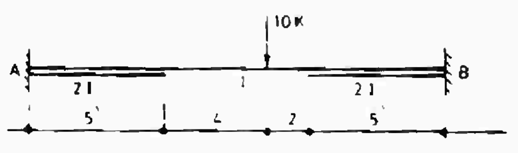


Figure 32

37. The structure shown in Figure 33 is used as a culvert for irrigation and drainage purposes. Determine the bending moment diagram for this structure using the method of column analogy. (EI = constant).

38. The frame shown in Figure 34 is being designed for an industrial building. The wind force is estimated as 8 kips and the roof load is 0.5 kip/ft. If member BC is subjected to a rise in temperature as shown, use the consistent deformation method to determine the moment diagram for the frame. (EI = 10000 K.ft^2, EA = 20000 kips, $\alpha = 6.5 \times 10^{-6}/^{\circ}$ F).

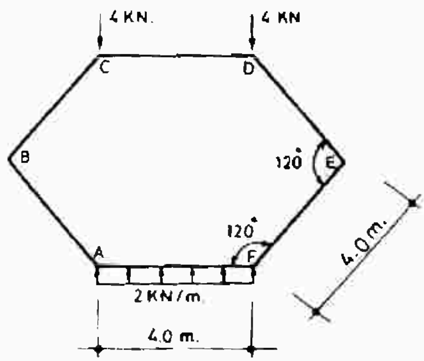


Figure 33

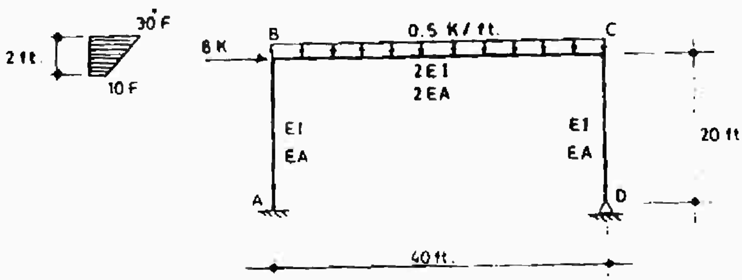


Figure 34

39. Use the matrix formulation of the flexibility matrix method to determine the bending moment diagram and the rotation at B for the frame shown in Figure 35. (Neglect axial deformation, EI = 10,000 Kft^2).

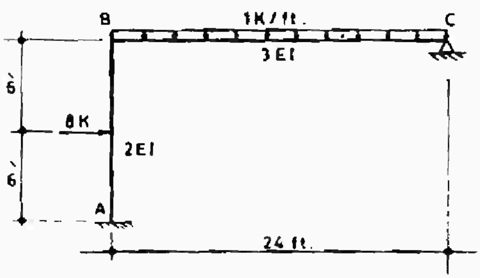


Figure 35

40. Determine the forces in AD and CB for the truss shown in Figure 36. Consider EA is constant for all members. Determine also the horizontal displacement of joint C.

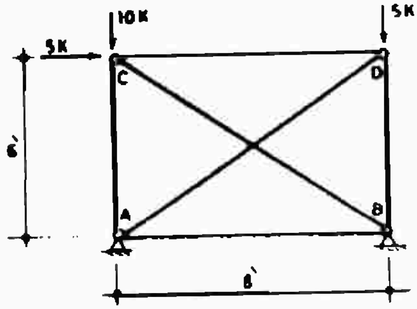


Figure 36

41. The beam shown in Figure 37 is fixed at A and simply supported at B and C. If support C settles down a value of 2 inches,

(a) Determine the bending moment diagram for the beam.

(b) Prove that $\theta_A = 0$.

(c) Find the reaction at B.

(EI = 3000 K.ft^2).

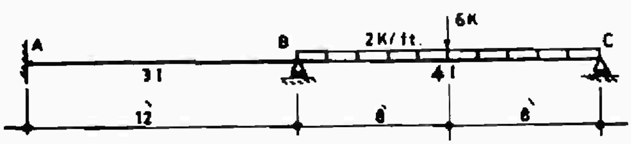


Figure 37

42. For the frame shown in Figure 38, EI = 3000 K.ft^2, EA = 10000 Kips

(a) Determine the forces in the bracings AC and BD.

(b) Draw bending moment diagrams for the frame.

43. Use the three moment equation to determine the bending moment diagram for the beam shown in Figure 39. (EI = constant).

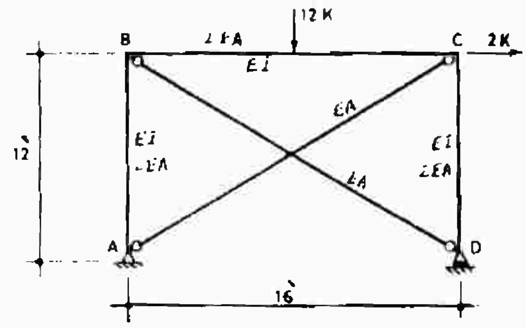


Figure 38

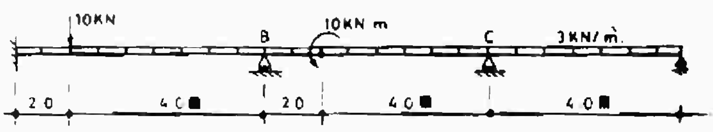


Figure 39

44. (a) Determine the bending moment diagram for the frame shown in Figure 40 using the method of column analogy.

(b) If EI = 20,000 kN.m^2, determine how much the sway in the frame.

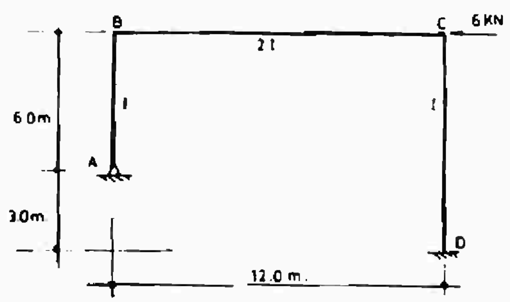


Figure 40