

# TABLES OF THE LEGENDRE POLYNOMIALS OF COMPLEX ARGUMENT

By

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## § 1. Introduction :

In this paper we produce the double entry tables of the function  $P_n(x + iy)$ . The argument  $x$ , appearing in a vertical column on each page covers the range  $x = 0, (0.1); 0.9$ , and the argument  $y$  arranged horizontally, takes the values  $y = 0.1, (0.1); 1$ .

Seven decimal places are given throughout, and the error does not exceed one unit in the last figure.

## § 2. The calculation of this table is based on the formula

$$P_n(x+iy) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r y^{2r} (1-x^2)^{-r}}{(2r)!} P_n^{2r}(x) + i \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^r y^{2r+1} (1-x^2)^{-r-1/2}}{(2r+1)!} P_n^{2r+1}(x).$$

This formula is deduced from the theorem

$$P_n(x+iy) = \sum_{m=0}^{\infty} \frac{y^m}{m!} (1-x^2)^{-\frac{m}{2}} P_n^m(x)$$

which is given by the authors in (1). To give an example, choose  $x = 0.2$ ,  $y = 0.2$  and  $n = 3$ , then

$$P_3(0.2 + 0.2i) = -0.34 - 0.2599999i$$

where

$$P_3^0(0.2) = -0.28, \quad P_3^1(0.2) = -1.17575508,$$

$$P_3^2(0.2) = 2.88, \quad P_3^3(0.2) = 14.10906092$$

$$\text{and } \sqrt{6} = 2.4494897.$$

This value in the table, which is calculated by the computer I.B.M. 1620 is read as

$P_3(0.2 + 0.2 i) = - 0.3399999 - 0.2599999 i$ ,  
and the true value is

$$P_3(0.2 + 0.2 i) = - 0.34 - 0.26 i$$

The error done by the computer does not exceed one unit in the last figure.

Interpolation in the x or y direction is performed by quadratic interpolation with a Lagrangian three-point interpolation formula [2], namely

$$f(x) = \frac{(x - a_1)(x - a_2)}{(a_0 - a_1)(a_0 - a_2)} f(a_0) + \frac{(x - a_0)(x - a_2)}{(a_1 - a_0)(a_1 - a_2)} f(a_1) \\ + \frac{(x - a_0)(x - a_1)}{(a_2 - a_0)(a_2 - a_1)} f(a_2)$$

To interpret the use of this formula, write the Lagrangian formula in the form.

$$f(x) = A_0 f(a_0) + A_1 f(a_1) + A_2 f(a_2),$$

So if we take.

$$a_0 = 0.1, a_1 = 0.2 \text{ and } a_2 = 0.3, \text{ then}$$

$$A_0 = - 0.08, A_1 = 0.96, A_2 = 0.12$$

$$f(a_0) = - 0.545 + 0.06 i,$$

$$f(a_1) = - 0.5 + 0.12 i, \text{ and}$$

$$f(a_2) = - 0.425 + 0.18 i.$$

Hence

$$P_2(0.22 + 0.2 i) = - 0.4874 + 0.1320 i$$

which is identical with the accurate value.

§ 3. The following tables give the values of  $P_n(x + iy) = u + iv$ ,

for  $n = 1, 2, \dots, 10$ .

One thousand values are calculated for  $P_n(x + iy)$ .

$P_1(x + iy)$

$y \ x$	0.1	0.2	0.3	0.4	0.5
0.0	0.000000 +0.100000i	0.000000 +0.200000i	0.000000 +0.300000i	0.000000 +0.400000i	0.000000 +0.500000i
0.1	0.100000 +0.100000i	0.100000 +0.200000i	0.100000 +0.300000i	0.100000 +0.400000i	0.100000 +0.500000i
0.2	0.200000 +0.100000i	0.200000 +0.200000i	0.200000 +0.300000i	0.200000 +0.400000i	0.200000 +0.500000i
0.3	0.300000 +0.100000i	0.300000 +0.200000i	0.300000 +0.300000i	0.300000 +0.400000i	0.300000 +0.500000i
0.4	0.400000 +0.100000i	0.400000 +0.200000i	0.400000 +0.300000i	0.400000 +0.400000i	0.400000 +0.500000i
0.5	0.500000 +0.100000i	0.500000 +0.200000i	0.500000 +0.300000i	0.500000 +0.400000i	0.500000 +0.500000i
0.6	0.600000 +0.100000i	0.600000 +0.200000i	0.600000 +0.300000i	0.600000 +0.400000i	0.600000 +0.500000i
0.7	0.700000 +0.100000i	0.700000 +0.200000i	0.700000 +0.300000i	0.700000 +0.400000i	0.700000 +0.500000i
0.8	0.800000 +0.100000i	0.800000 +0.200000i	0.800000 +0.300000i	0.800000 +0.400000i	0.800000 +0.500000i
0.9	0.900000 +0.100000i	0.900000 +0.200000i	0.900000 +0.300000i	0.900000 +0.400000i	0.900000 +0.500000i

$P_1(x + iy)$

$x$	$y$	0.6	0.7	0.8	0.9	1
0.0		0.000000 +0.600000i	0.000000 +0.700000i	0.000000 +0.800000i	0.000000 +0.900000i	0.000000 +1.000000i
0.1		0.100000 +0.600000i	0.100000 +0.700000i	0.100000 +0.800000i	0.100000 +0.900000i	0.100000 +1.000000i
0.2		0.200000 +0.600000i	0.200000 +0.700000i	0.200000 +0.800000i	0.200000 +0.900000i	0.200000 +1.000000i
0.3		0.300000 +0.600000i	0.300000 +0.700000i	0.300000 +0.800000i	0.300000 +0.900000i	0.300000 +1.000000i
0.4		0.400000 +0.600000i	0.400000 +0.700000i	0.400000 +0.800000i	0.400000 +0.900000i	0.400000 +1.000000i
0.5		0.500000 +0.600000i	0.500000 +0.700000i	0.500000 +0.800000i	0.500000 +0.900000i	0.500000 +1.000000i
0.6		0.600000 +0.600000i	0.600000 +0.700000i	0.600000 +0.800000i	0.600000 +0.900000i	0.600000 +1.000000i
0.7		0.700000 +0.600000i	0.700000 +0.700000i	0.700000 +0.800000i	0.700000 +0.900000i	0.700000 +1.000000i
0.8		0.800000 +0.600000i	0.800000 +0.700000i	0.800000 +0.800000i	0.800000 +0.900000i	0.800000 +1.000000i
0.9		0.900000 +0.600000i	0.900000 +0.700000i	0.900000 +0.800000i	0.900000 +0.900000i	0.900000 +1.000000i

$P_2 ( x + iy )$

xy	0.1	0.2	0.3	0.4	0.5
0.0	-0.5150000	-0.5600000	-0.6350000	-0.7400000	-0.8750000
0.1	-0.5000000 +0.0300000i	-0.5450000 +0.0600000i	-0.6200000 +0.0900000i	-0.7250000 +0.1200000i	-0.8600000 +0.1500000i
0.2	-0.4550000 +0.0600000i	-0.5000000 +0.1200000i	-0.5750000 +0.1800000i	-0.6800000 +0.2400000i	-0.8150000 +0.3000000i
0.3	-0.3800000 +0.0900000i	-0.4250000 +0.1800000i	-0.5000000 +0.2700000i	-0.6050000 +0.3600000i	-0.7400000 +0.4500000i
0.4	-0.2750000 +0.1200000i	-0.3200000 +0.2400000i	-0.3950000 +0.3600000i	-0.5000000 +0.4800000i	-0.6250000 +0.6000000i
0.5	-0.1400000 +0.1500000i	-0.1850000 +0.3000000i	-0.2600000 +0.4500000i	-0.3650000 +0.6000000i	-0.5000000 +0.7500000i
0.6	0.0250000 +0.1800000i	-0.0200000 +0.3600000i	-0.0950000 +0.5400000i	-0.2000000 +0.7200000i	-0.3350000 +0.9000000i
0.7	0.2200000 +0.2100000i	0.1750000 +0.4200000i	0.1000000 +0.6300000i	-0.0050000 +0.8400000i	-0.1400000 +1.0500000i
0.8	0.4450000 +0.2400000i	0.4000000 +0.4860000i	0.3250000 +0.7200000i	0.2200000 +0.9600000i	0.0850000 +1.2000000i
0.9	0.7000000 +0.2700000i	0.6550000 +0.5400000i	0.5800000 +0.8100000i	0.4750000 +1.0800000i	0.3400000 +1.3500000i

$P_2(x+iy)$

$xy$	0.6	0.7	0.8	0.9	1.0
0.0	-1.0400000	-1.2350000	-1.4600000	-1.7150000	-2.0000000
0.1	-1.0250000 +0.1800000i	-1.2200000 +0.2100000i	-1.4450000 +0.2400000i	-1.7000000 +0.2700000i	-1.9850000 +0.3000000i
0.2	-0.9800000 +0.3600000i	-1.1750000 +0.4200000i	-1.4000000 +0.4800000i	-1.6550000 +0.5400000i	-1.9400000 +0.6000000i
0.3	-0.9050000 +0.5400000i	-1.1000000 +0.6300000i	-1.3250000 +0.7200000i	-1.5800000 +0.8100000i	-1.8650000 +0.9000000i
0.4	-0.8000000 +0.7200000i	-0.9950000 +0.8400000i	-1.2200000 +0.9600000i	-1.4750000 +1.0800000i	-1.7600000 +1.2000000i
0.5	-0.6650000 +0.9000000i	-0.8600000 +1.0500000i	-1.0850000 +1.2000000i	-1.3400000 +1.3500000i	-1.6250000 +1.5000000i
0.6	-0.5000000 +1.0800000i	-0.6950000 +1.2600000i	-0.9200000 +1.4400000i	-1.1750000 +1.6200000i	-1.4600000 +1.8000000i
0.7	-0.3050000 +1.2600000i	-0.5000000 +1.4700000i	-0.7250000 +1.6800000i	-0.9800000 +1.8900000i	-1.2650000 +2.1000000i
0.8	-0.0800000 +1.4400000i	-0.2750000 +1.6800000i	-0.5000000 +1.9200000i	-0.7550000 +2.1600000i	-1.0400000 +2.4000000i
0.9	0.1750000 +1.6200000i	-0.0200000 +1.8900000i	-0.2450000 +2.1600000i	-0.5000000 +2.4300000i	-0.7850000 +2.7000000i

$P_3(x + iy)$

$x \ y$	0.1	0.2	0.3	0.4	0.5
0.0	0.0000000 -0.1524999i	0.0000000 -0.3199999i	0.0000000 -0.5174999i	0.0000000 -0.7599999i	0.0000000 -1.0624999i
0.1	-0.1549999 -0.1449999i	-0.1774999 -0.3049999i	-0.2149999 -0.4949999i	-0.2674999 -0.7299999i	-0.3349999 -1.0249999i
0.2	-0.2949999 -0.1224999i	-0.3399999 -0.2599999i	-0.4149999 -0.4274999i	-0.5199999 -0.6399999i	-0.6549999 -0.9124999i
0.3	-0.4049999 -0.0849999i	-0.4724999 -0.1849999i	-0.5849999 -0.3149999i	-0.7424999 -0.4899999i	-0.9449999 -0.7249999i
0.4	-0.4699999 -0.0325000i	-0.5599999 -0.0800000i	-0.7099999 -0.1575000i	-0.9199999 -0.2499999i	-1.1899999 -0.4624999i
0.5	-0.4749999 -0.0350000i	-0.5874999 +0.0549999i	-0.7749999 +0.0450000i	-1.0374999 -0.0100000i	-1.3749999 -0.1250000
0.6	-0.4049999 +0.1174999i	-0.5399999 +0.2199999i	-0.7649999 +0.2924999i	-1.0799999 +0.3199999i	-1.4849999 +0.2874999i
0.7	-0.2450000 +0.2149999i	-0.4025000 +0.4149999i	-0.6649999 +0.5849999i	-1.0324999 +0.7099999i	-1.5049999 +0.7749999i
0.8	0.0199999 +0.3274999i	-0.1600000 +0.6399999i	-0.4599999 +0.9225000i	-0.8799999 +1.1600000i	-1.4199999 +1.3374999i
0.9	0.4050000 +0.4549999i	0.2024999 +0.8949999i	-0.1350000 +1.3050000i	-0.6074999 +1.6699999i	-1.2149999 +1.9749999i

$P_3(x + iy)$

x y	0.6	0.7	0.8	0.9	1.—
0.0	0.0000000 -1.4399999i	0.0000000 -1.9074999i	0.0000000 -2.4799999i	0.0000000 -3.1724999i	0.0000000 -3.9999999i
0.1	-0.4174999 -1.3949999i	-0.5149999 -1.8549999i	-0.6274999 -2.4199999i	-0.7549999 -3.1049999i	-0.8974999 -3.9249999i
0.2	-0.8199999 -1.2599999i	-1.0149999 -1.6974999i	-1.2399999 -2.2399999i	-1.4949999 -2.9024999i	-1.7799999 -3.6999999i
0.3	-1.1924999 -1.0349999i	-1.4849999 -1.4349999i	-1.8224999 -1.9399999i	-2.2049999 -2.5649999i	-2.6324999 -3.3249999i
0.4	-1.5199999 -0.7199999i	-1.9099999 -1.0674999i	-2.3599999 -1.5199999i	-2.8699999 -2.0924999i	-3.4399999 -2.7999999i
0.5	-1.7874999 -0.3150000i	-2.2749999 -0.5949999i	-2.8374999 -0.9799999i	-3.4749999 -1.4849999i	-4.1874999 -2.1249999i
0.6	-1.9799999 +0.1799999i	-2.5649999 -0.0175000i	-3.2399999 -0.3200000i	-4.0049999 -0.7424999i	-4.8599999 -1.2999999i
0.7	-2.0824999 +0.7650000i	-2.7649999 +0.6650000i	-3.5524999 +0.4599999i	-4.4449999 +0.1349999i	-5.4424999 -0.3250000i
0.8	-2.0799999 +1.4400000i	-2.8599999 +1.4524999i	-3.7599999 +1.3599999i	-4.7799999 +1.1475000i	-5.9199999 +0.8000000i
0.9	-1.9574999 +2.2049999i	-2.8349999 +2.3449999i	-3.8474999 +2.3799999i	-4.9949999 +2.3799999i	-6.2774999 +2.0749999i

$P_i(x+iy)$

xy	0.1	0.2	0.3	0.4	0.5
0.0	0.4129374	0.5319999	0.7479374	1.0869999	1.5859374
0.1	0.3732500 -0.0749999i	0.4844374 -0.1604999i	0.6872500 -0.2669999i	1.0079374 -0.4049999i	1.4832499 -0.5849999i
0.2	0.2594375 -0.1394999i	0.3470000 -0.2999999i	0.5104375 -0.5024999i	0.7759999 -0.7679999i	1.1804374 -1.1174999i
0.3	0.0872500 -0.1829999i	0.1354375 -0.3974999i	0.2332500 -0.6749999i	0.4069375 -1.0469999i	0.6932500 -1.5449999i
0.4	-0.1170624 -0.1949999i	-0.1239999 -0.4319999i	-0.1180624 -0.7529999i	-0.0729999 -1.1999999i	0.0479375 -1.8149999i
0.5	-0.3167499 -0.1649999i	-0.3945624 -0.3825000i	-0.5067499 -0.7049999i	-0.6270624 -1.1849999i	-0.7187499 -1.8749999i
0.6	-0.4645624 -0.0825000i	-0.6289999 -0.2280000i	-0.8855624 -0.4995000i	-1.2079999 -0.9599999i	-1.5595624 -1.6724999i
0.7	-0.5027499 +0.0629999i	-0.7695624 +0.0524999i	-1.1967499 -0.1050000i	-1.7580624 -0.4829999i	-2.4167499 -1.1549999i
0.8	-0.3630625 +0.2819999i	-0.7479999 +0.4799999i	-1.3720624 +0.5100000i	-2.2089999 +0.2880000i	-3.2220624 -0.2700000i
0.9	0.0332500 +0.5849999i	-0.4855625 +1.0754999i	-1.3327500 +1.3770000i	-2.4020624 +1.3950000i	-3.8967499 +1.0350000i

$P_4(x+iy)$

xy	0.6	0.7	0.8	0.9	1.—
0.0	2.2919999	3.2629374	4.5669999	6.2829374	8.4999999
0.1	2.1604374 -0.8174999i	3.0972499 -1.1129999i	4.3619374 -1.4819999i	6.0332499 -1.9319999i	8.2004374 -2.4824999i
0.2	1.7709999 -1.5719999i	2.6054374 -2.1524999i	3.7519999 -2.8799999i	5.2894374 -3.7754999i	7.3069999 -4.8599999i
0.3	1.1394374 -2.2004999i	1.8032500 -3.0449999i	2.7529375 -4.1099999i	4.0672499 -5.4269999i	5.8354374 -7.0274999i
0.4	0.2920000 -2.6399999i	0.7169375 -3.7169999i	1.3910000 -5.0879999i	2.3929374 -6.7949999i	3.8119399 -8.8799999i
0.5	-0.7345624 -2.8274999i	-0.6167500 -4.0949999i	-0.2970625 -5.7299999i	0.3032500 -7.7849999i	1.2784274 -10.3124999i
0.6	-1.8929999 -2.6999999i	-2.1505624 -4.1054999i	-2.2639999 -5.9519999i	-2.1545624 -8.3024999i	1.7329999 -11.2199999i
0.7	-3.1255624 -2.1944999i	-3.8267499 -3.6749999i	-4.4520524 -5.6699999i	-4.9227499 -8.2529999i	-5.1495624 -11.4974999i
0.8	-4.3639999 -1.2479999i	-5.5770624 -2.7299999i	-6.7929999 -4.7999999i	-7.9330624 -7.5419999i	-8.9079999 -11.0399999i
0.9	-5.5295624 +0.2025000i	-7.3227499 -1.1969999i	-9.2060624 -3.2579999i	-11.1067499 -6.0749999i	-12.9295624 -9.7424999i

$P_5(x + iy)$

x y	0.1	0.2	0.3	0.4	0.5
0.0	0.0000000 +0.1963287i	0.0000000 +0.4475199i	0.0000000 +0.8178862i	0.0000000 +1.3206899i	0.0000000 +2.2773437i
0.1	0.2046849 +0.1696849i	0.2869787 +0.3895074i	0.4363849 +0.7199549i	0.6879387 +1.2336149i	1.0614849 +2.0496249i
0.2	0.3545074 +0.0944787i	0.5049199 +0.2249200i	0.7271974 +0.4837362i	1.0488199 +0.7942399i	1.9347074 +1.3209937i
0.3	0.4040549 -0.0151149i	0.5942362 -0.0740937i	0.9364549 +0.0134580i	1.5945862 +0.1194150i	2.5268549 +0.3696850i
0.4	-0.3268150 +0.1354712i	-0.5162399 +0.2916799i	0.8299149 -0.249127i	1.5244899 -0.6925599i	2.6200149 -0.8936562i
0.5	0.1246249 -0.2335149i	0.2745937 -0.530292i	0.7246249 -0.941449i	1.4783437 -1.5095849i	3.1407249 -2.231749i
0.6	-0.1628774 -0.2667212i	-0.1852359 -0.6486799i	-0.0746774 -1.2516637i	0.258540 -2.1625599i	1.8085225 -3.4399062i
0.7	-0.4488049 -0.1831150i	-0.6665187 -0.5428925i	-0.9192049 -1.2468449i	-1.9413987 -2.4428649i	-0.8016049 -4.2503749i
0.8	-0.5895699 +0.0707287i	-1.1219199 -0.6900800i	-1.8231699 -0.7445237i	-2.6843199 -2.1037599i	-3.2607699 -4.3596562i
0.9	-0.3754349 +0.5896049i	-1.3357912 +0.2515074i	-2.7946350 +0.4670550i	-4.5393412 -0.8637849i	-6.2722049 -3.4103749i

$P_5(x + iy)$ 

$x \ y$	0.6	0.7	0.8	0.9	1.
0.0	0.0000000 +3.6273599i	0.0000000 +5.6373012i	0.0000000 +8.5604799i	0.0000000 +12.7163587i	0.0000000 +18.4999999i
0.1	1.6057787 +3.3021224i	2.3718849 +5.1861949i	3.4212287 +7.9504299i	4.8246849 +11.9095649i	6.6625787 +17.4539374i
0.2	2.9913199 +2.3547599i	4.4021074 +3.8659512i	6.4899199 +6.1580799i	9.2165074 +9.5317087i	12.8025199 +14.3629999i
0.3	3.9458362 +0.8703224i	5.9984549 +1.7757950i	8.8629862 +3.2268300i	12.7520549 +5.7103649i	17.9066362 +9.3689374i
0.4	4.2774399 -1.0094399i	6.7276149 -0.9188987i	10.2162399 -0.4443199i	15.0268149 +0.6581587i	21.4806399 +02.7079999i
0.5	3.8225937 -3.0860774i	6.4246249 -3.9866049i	10.2538437 -4.8007699i	15.6646249 -5.3272349i	23.0585937 -5.2890624i
0.6	2.4555600 -5.1044399i	4.9023224 -7.1296487i	8.7177599 -9.4323199i	14.3271224 -11.8630912i	22.2123599 -14.1969999i
0.7	0.0978512 -6.7526774i	2.0207949 -9.9842049i	5.3972012 -13.9231699i	10.7231949 -18.4816349i	18.5610512 -23.4960624i
0.8	-3.2723200 -7.6622399i	-2.3031699 -12.1202987i	0.1380800 -17.7819199i	4.6184299 -24.6300412i	11.7804799 -32.5719999i
0.9	-7.6105912 -7.4078774i	-8.0866350 -13.0418049i	-7.1475412 -20.4415699i	-4.1554349 -29.6704349i	1.6126087 -40.7160624i

$P_6(x + iy)$

x y	0.1	0.2	0.3	0.4	0.5
0.0	-0.3801081	-0.6074239	-1.0731186	-1.9256359	-3.4091796
0.1	-0.3046250 +0.1311344i	-0.4939045 +0.3103852i	-0.8875429 +0.5962634i	-1.6209903 +1.0680704i	-2.9202589 +1.8362924i
0.2	-0.1035329 +0.2158852i	-0.1865000 +0.5176079i	-0.3769634 +1.0117957i	-0.7713919 +1.8466559i	-1.5412244 +3.2327662i
0.3	0.1522929 +0.2182634i	0.2205259 +0.5392957i	0.3253749 +1.0970504i	0.4353401 +2.0878514i	0.4699389 +3.8015774i
0.4	0.3620528 +0.1230705i	0.5873919 +0.3346560i	1.0147223 +0.7648515i	1.7034999 +1.6269119i	2.7005813 +3.2588324i
0.5	0.4210089 -0.0537075i	0.7547869 -0.0747337i	1.4408109 +0.0215775i	2.6554811 +0.4238325i	4.6093749 +1.4765625i
0.6	0.2568690 -0.2545042i	0.5802519 -0.5841359i	1.3442385 -1.0016527i	2.8681599 -1.3950719i	5.5626975 -1.4653012i
0.7	-0.1234369 -0.3697784i	-0.0150415 -0.9848632i	0.5032449 -2.0177954i	1.9196726 -3.4955864i	4.8814089 -5.2269524i
0.8	-0.5765841 -0.2276190i	-0.9891199 -0.9434879i	-1.2091146 -2.5526969i	-0.5534119 -5.2940159i	-1.8980243 -9.1567349
0.9	-0.7552849 +0.4166504i	-2.1116395 +0.0189472i	-3.7574030 -1.9139084i	-4.7885253 -5.9156054i	-3.9757189 -12.2391674i

$P_6(x + iy)$

$x \ y$	0.6	0.7	0.8	0.9	1.—
0.0	-5.9000959	-9.9536511	-16.3612039	-26.2177731	-40.9999999
0.1	-5.1383065 +3.0529957i	-8.8018129 +4.9222214i	-14.6683533 + 7.7103309i	-23.7939649 +11.7566504i	-37.6111105 +17.4833662i
0.2	-2.9692519 +5.4638639i	-5.4963959 +8.9376367i	- 9.7788799 +14.1765119i	-16.7557979 +21.8484472i	-27.7270759 +32.7877199i
0.3	0.2633239 +6.6528472i	-0.4824950 +11.2122044i	- 2.2548228 +18.2373029i	- 5.7978470 +28.7040914i	-12.1906000 +43.8379987i
0.4	4.0033399 +6.1649279i	5.5153893 +11.0574134i	6.9344119 +18.8979839i	7.9395878 +30.9393944i	7.6111359 +48.7670399i
0.5	7.5108649 +3.7321987i	11.5193409 +8.0030474i	16.6880381 +15.4132649i	22.8964689 +27.4508324i	29.7724609 +46.0195312i
0.6	9.8934999 -0.6637679i	16.3344060 +1.8699502i	25.3102719 + 7.3704959i	37.1294040 +17.5091017i	51.9055959 +34.4559599i
0.7	10.1581564 - 6.7295497i	8.5953749 -7.1572094i	31.0571096 - 5.2271729i	48.3584229 +00.8526735i	71.1874324 +13.4565637i
0.8	7.2657279 -13.7975039i	16.8239528 -18.4571729i	31.9434999 -21.8776319i	54.0241508 -22.2186509i	84.4167039 -16.9747199i
0.9	0.1991584 -20.7633982i	9.4963269 -30.8993264i	25.8609116 -41.4966509i	51.3553749 -50.7501854i	88.0815544 -56.1063037i

$P_7(x+iy)$

xy	0.1	0.2	0.3	0.4	0.5
0.0	0.0000000 -0.2388733i	0.0000000 -0.6092031i	0.0000000 -1.2989257i	0.0000000 -2.6224495i	0.0000000 -5.1176757i
0.1	-0.2563710 -0.1776639i	-0.4529928 -0.4596708i	-0.6740776 -1.0900864i	-1.6683466 -2.0659354i	-3.1573068 -4.1274647i
0.2	-0.3824116 -0.0200233i	-0.6943143 -0.0628056i	-1.0785937 -0.2025579i	-2.7283071 -0.5679104i	-5.2497727 -1.4219011i
0.3	-0.3020185 +0.1628420i	-0.5867967 -0.3977106i	-1.2514563 +0.7009613i	-2.6520740 +1.2816599i	-5.4289459 +2.2375722i
0.4	-0.0340295 +0.2747803i	-0.1237296 +0.7167871i	-0.4348619 +1.5428783i	-1.2694832 +3.0876131i	-3.2655897 +5.6945207i
0.5	0.2940697 +0.2284718i	0.5121229 +0.6654902i	0.8500527 +1.6320409i	1.1910760 +3.6604171i	1.9742187 +7.5604062i
0.6	0.4730998 -0.0972679i	1.0051655 +0.1200392i	2.1199387 +0.7656602i	4.0175231 +2.4297471i	6.6942205 +6.4810883i
0.7	0.2928950 -0.3466203i	0.9175798 -0.8205726i	2.5063088 -1.2742876i	5.9819233 -1.0247470i	11.8865204 +1.3808772i
0.8	-0.3090215 -0.5127385i	-0.1574655 -1.6919295i	1.2399636 -3.8280733i	5.4113311 -6.4732783i	14.1175500 -8.1058497i
0.9	-1.0179814 +0.0406900i	-2.3915283 -1.4825178i	-2.8185088 -5.6927044i	0.2816357 -12.7043294i	10.0960123 -21.2254547i

$P_7(x+iy)$

$x y$	0.6	0.7	0.8	0.9	1.—
0.0	0.0000000 -9.6835583i	0.0000000 -17.7713190i	0.0000000 -31.6456287i	0.0000000 -54.7202712i	0.0000000 -91.9999999i
0.1	-5.7325482 -7.9889473i	-10.0665546 -14.9735400i	-17.1120277 -27.1757084i	-28.2047250 -47.7947335i	-45.1648110 -81.5628123i
0.2	-9.7551591 -3.2975255i	-17.4851098 -7.1392935i	-30.2571903 -14.5414143i	-50.6415646 -28.0610179i	-82.1636967 -51.6259879i
0.3	-10.6486089 +3.2540137i	-19.9862173 +4.0974461i	-35.9485253 +3.9849063i	-62.1374005 +1.4069938i	-103.5538773 -6.2461758i
0.4	-7.5615871 +9.9063167i	-16.0440179 +16.1717128i	-31.6453615 +24.7961887i	-58.6951355 +35.6898680i	-103.3253503 +48.0407679i
0.5	-0.4608700 +14.4800719i	-5.2142522 +25.8236345i	-16.1926014 +43.2209199i	-38.0479202 +68.3260726i	-77.5834960 +102.5576171i
0.6	9.6590471i +14.6597927i	11.5682692 +29.4295142i	9.7800575 +53.9015167i	-0.1728391 +91.7376526i	-25.1911416 +146.9587479i
0.7	20.6203720 +8.3586832i	31.7462498 +23.4275051i	43.5141556 +51.2808123i	52.2494570 +97.7775268i	51.6434284 +169.8382466i
0.8	28.9740031 -5.8354176i	50.9640576 +4.8378805i	79.844703 +30.1966495i	113.4426864 +78.3984059i	146.8460287 +159.4731519i
0.9	29.9926010 -28.1330083i	62.9729581 -28.1221020i	111.0032978 -13.4154164i	1742239525 +26.4433324i	250.0372118 +104.7018326i

$P_9(x - iy)$

$x - y$	0.1	0.2	0.3	0.4	0.5
0.0	0.3773834	0.7599471	1.6696245	3.6517686	7.7808532
0.1	0.2517892 -0.1961242i	0.5196069 -0.5276475i	1.1752391 -1.2069154i	2.6521905 -2.5844315i	5.8327297 -5.3406307i
0.2	-0.0490586 -0.2691165i	-0.0716032 -0.7392519i	-0.0697406 -1.7388767i	0.6777256 -3.8750207i	0.7129417 -3.2835464i
0.3	-0.3336748 -0.1560100i	-0.6721715 -0.4682179i	-1.4279437 -1.2245722i	-2.9107893 -3.0373693i	-5.5627027 -7.1573827i
0.4	-0.3938397 +0.0920180i	-0.8793163 +0.1964927i	-2.0230231 +0.2447998i	-4.7434072 -0.0749280i	-10.1508516 -1.6421251i
0.5	-0.1355122 +0.3163282i	-0.4298821 +0.2813352i	-1.3733707 +1.5943752i	-3.9514508 +3.9540026i	-10.1196289 +6.8085937i
0.6	0.3088396 +0.3032209i	0.5778510 +1.0237751i	0.8117885 +2.8627794i	0.1877631 +6.9672959i	-3.4123154 +14.8492515i
0.7	0.5574346 +0.0765439i	1.5251996 +0.1288461i	3.6447277 +1.5366172i	6.9440586 +6.2023502i	10.0352527 +17.5295977i
0.8	0.1371173 +0.6278826i	1.2637552 +1.7713919i	5.0712120 +2.8110206i	13.4561910 +1.0191551i	27.1147879 +9.0885716i
0.9	-1.0664737 -0.4699012i	-1.6320752 -3.4151508i	1.7336401 -9.5171800i	14.1938419 -16.0520180i	40.4146387 -15.6436717i

$P_8(x + iy)$

$x \cdot y$	0.6	0.7	0.8	0.9	1
0.0	16.0565871	32.0348260	61.7844966	115.2820217	208.3749999
0.1	12.4087312 -10.6184157i	25.4668786 -20.3268355i	50.3898666 -37.5100703i	96.1849462 -66.8440552i	177.3715927 -115.2749935i
0.2	2.6496271 -16.9920071i	7.6227530 -33.4468739i	19.0221952 -63.2432639i	43.0237042 -115.0979133i	90.2485326 -202.1059319i
0.3	9.8810164 -15.9705437i	-16.1979622 -33.7377757i	-24.2254350 -67.6389183i	-32.2534738 -129.1815094i	-35.8707013 -236.0352428i
0.4	-20.3191567 -6.4713600i	-38.0843526 -18.6042257i	-67.0396648 -45.4066367i	-111.1951433 -99.3526104i	-174.2301967 -200.3756159i
0.5	-23.2941534 +9.7909146i	-48.8613052 +10.3632247i	-94.6140522 +2.7439833i	-171.0045832 -24.1696507i	-291.0809631 -89.5683789i
0.6	-11.2826512 +27.9394919i	-39.8045397 +46.6553504i	-91.9961983 +68.8601087i	-187.4699614 +87.5927292i	-349.3050332 +87.9462359i
0.7	8.7723327 +40.0570462i	-5.3526007 +78.6781117i	-46.9838601 +137.1510761i	-138.7357243 +215.7578734i	-312.9537161 +307.9696337i
0.8	43.6683375 +35.9154431i	55.3754095 +90.2971728i	46.5211686 +184.2046079i	-9.4044123 +328.4734619i	-152.6077327 +529.3989119i
0.9	82.0878851 +4.4351980i	134.8678396 +62.2326710i	184.8128919 +180.1760010i	204.4430432 +383.0324557i	148.5504986 +694.5971307i

$P_9(x + iy)$

x y	0.1	0.2	0.3	0.4	0.5
0.0	0.0000000 +0.2836153i	0.0000000 +0.8286051i	0.0000000 -2.1007212i	0.0920090 -5.0001004i	0.0900000 -11.0076287i
0.1	0.3124912 +0.1684380i	0.7001518 +0.5052254i	1.6794590 +1.3381031i	3.9950018 -3.8510234i	8.9521618 +8.1687609i
0.2	0.3720312 -0.0927542i	0.8696930 -0.2441225i	2.1825551 -0.5570200i	5.8023411 -0.9003101i	12.7791476 -1.1920910i
0.3	0.1089471 -0.2961821i	0.5175759 -0.8727757i	0.9971741 -2.1972836i	3.0028532 -5.1507701i	8.4332219 -11.2921670i
0.4	-0.2847004 -0.2491164i	-0.6261721 -0.8208668i	-1.210129 -2.3000223i	-2.3982659 -6.5673139i	-3.2158454 -15.8894254i
0.5	-0.4491477 +0.0790716i	-1.1941690 +0.0784253i	-3.1500297 -0.3430098i	-5.7797022 -2.5042045i	-10.9426219 -9.8527232i
0.6	-0.1277899 +0.4083472i	-0.6253421 +1.271343i	-2.504047 +3.0712433i	-3.6225126 +5.8783590i	-23.2420557 +7.8154415i
0.7	0.4911596 +0.3122454i	1.1523511 +1.4759476i	1.6072268 +5.2297233i	-0.8245634 +14.3380926i	-13.2526929 +31.4283159i
0.8	0.6004853 -0.4671329i	2.7188364 -0.6954140i	8.1530267 +2.0516522i	16.2937560 +14.3002664i	19.8408780 +46.5474522i
0.9	-0.8193737 -1.0453349i	0.6414431 -5.1045245i	10.8455982 -10.1356282i	36.0073799 -5.2710128i	74.5052316 +30.4422099i

$P_9(x+iy)$

$x y$	0.6..	.....0.7	0.8	0.9	1
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
	+26.8050729i	+58.1542424i	-121.4926872i	-244.6202114i	+475.3749999i
0.1	19.4736744	40.6350526	81.4106612	156.8740282	291.3916762
	-19.1588145i	+43.1431729i	+93.2156371i	+193.3725169i	+385.7613429i
0.2	28.9298310	62.6462265	129.6490419	256.9345762	488.8839366
	-0.4851581i	+3.7895265i	+17.5784524i	+54.6019902i	+140.0080879i
0.3	21.9661370	53.1955181	120.4364193	256.5648425	517.5655078
	-23.1409166i	-44.1775528i	-78.4781835i	-129.2844220i	-195.9569171i
0.4	-1.2966329	10.0854257	46.0914924	137.0587832	338.6914374a
	-36.7234646i	-78.7873594i	-157.6526759i	-295.8224876i	-523.1992975i
0.5	-32.6867414	-55.2144627	-79.1106851	-86.5955497	-36.7241973
	-30.0242406i	-77.7722986i	-178.7994012i	-374.2689706i	-725.5931701i
0.6	-56.4375821	-117.1967923	-217.0110182	-261.2426278	-539.6676365
	+2.4468258i	-26.0461735i	-108.8874803i	-301.0057548i	-690.7560880i
0.7	-52.1282320	-139.3261641	-308.0528685	-596.6718837	-1041.4200470
	+55.4763532i	+76.1281711i	+64.7634232i	-37.4843356i	-334.8978339i
0.8	-0.4711283	-81.0159165	-279.0278375	-673.4539407	-1361.1127667
	+108.9500457i	+205.3673200i	+321.8108184i	+410.6849251i	+369.9676143i
0.9	107.8627570	91.0136796	-56.7536388	-458.4663767	-1281.7362542
	+125.5798916i	+309.1185595i	+597.4968310i	+975.2045239i	+1368.3422127i

$P_{10}(x + iy)$

$x \ y$	0.1	0.2	0.3	0.4	0.5
0.0	-0.3935319	-0.9988224	-2.7000732	-7.1551289	-18.3055152
0.1	-0.1992402 +0.2678883i	-0.5266049 +0.8369296i	-1.4956545 +2.2904620i	-4.1840698 +5.9648594i	-11.3088690 +14.8631860i
0.2	0.2031479 +0.2767388i	0.4871530 +0.9032237i	1.1840786 +2.6158181i	2.6595228 +7.2359778i	5.3393150 +19.1233874i
0.3	0.4186247 -0.0077338i	1.1176280 +0.0445932i	3.1059902 +0.4180579i	8.2459068 +2.0798733i	20.5469467 +8.0131360i
0.4	0.1854155 -0.3262378i	0.6289379 -1.0378877i	2.2170770 -2.7763155i	7.2850870 -6.5948615i	21.7866809 -13.6531061i
0.5	-0.3180429 -0.3034654i	-0.7773682 -1.1724818i	-1.5983852 -3.9418069i	-1.9310316 -11.8507430i	2.3723144 -31.5833740i
0.6	-0.5012412 +0.1684508i	-1.7160663 +0.2903033i	-5.4333677 -0.5428087i	-14.4662041 -6.1223467i	-31.5620290 -27.0704760i
0.7	0.0922245 +0.5774963i	-0.4009127 +2.2849423i	-4.0438266 +6.5229916i	-18.2587013 +12.8878800i	-57.3127092 +12.8629640i
0.8	0.8780874 -0.0308555i	3.2595091 +1.5703812i	6.6734840 +10.2611884i	1.7264772 +35.1594143i	-38.4652618 +81.4212423i
0.9	-0.2426891 -1.5202927i	4.5054548 -5.4113528i	22.7635748 -2.5861812i	52.8041316 +32.7089930i	62.1106716 +136.9154537i

$P_{10}(x + iy)$ .

$x + y$	0.6	0.7	0.8	0.9	1
0.0	— 45.0087116	— 106.1764859	— 240.2749316	— 522.0543812	— 1090.7499999
0.1	— 29.3099086 + 35.2967377i	— 72.5799508 + 80.5359749i	— 171.5706227 + 175.2142395i	— 387.4273903 + 365.1550162i	— 837.2165665 + 730.6863342i
0.2	— 9.1617516 + 48.0284527i	— 11.9050180 + 114.8616878i	— 5.1234124 + 260.7412932i	— 34.4555981 + 563.6950037i	— 161.4531505 + 1163.9778917i
0.3	— 47.7942579 — 26.22945631i	— 103.6557566 + 75.9328321i	— 209.7304895 + 199.2050193i	— 396.3464463 + 481.2971186i	— 699.6141132 + 1084.1107406i
0.4	— 59.1665497 — 28.2687707i	— 146.5280290 — 29.7169737i	— 334.9973000 — 3.8909920i	— 710.0967581 + 98.9627732i	— 1403.2913350 + 426.2203193i
0.5	— 89.1399430 — 74.5277370i	— 94.9585922 — 156.6458754i	— 281.7725860 — 292.5772575i	— 711.6382933 — 481.8812247i	— 1605.7119026 — 678.4599456i
0.6	— 51.2186859 — 43.6950049i	— 63.0388465 — 227.5541871i	— 0.9129678 — 515.9625732i	— 271.6462102 — 1039.7049104i	— 1011.5584367 — 1891.8680622i
0.7	— 140.4636907 — 21.6939763i	— 281.7369252 — 154.8536312i	— 465.8652442 — 505.5499757i	— 604.6131855 — 1264.3451736i	— 467.1244335 — 2701.2848789i
0.8	— 164.2206710 + 132.7430843i	— 446.1205973 + 123.1907018i	— 955.1438039 — 100.7540161i	— 1717.4572407 — 822.9912681i	— 2634.4929132 — 2500.2250361i
0.9	— 32.5948585 + 333.7134756i	— 376.8753477 + 593.6813467i	— 1171.5755083 + 773.2956490i	— 2635.5766992 + 538.8930215i	— 4925.3146478 — 720.5711168i

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