INVESTIGATION OF THE STABILITY
OF THE SOLUTION OF A SYSTEM
OF DIFFERENTIAL EQUATIONS

By

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Consider the following system

$$\frac{dy}{dt} = Y(y, x),$$

$$\frac{dx}{dt} = A x + X(y, x). \tag{1}$$

Where :

y   is a scalar,

x   an n -dimensional vector,

Y   a scalar function and X an n -dimensional vector function
having the following properties :

$$Y(0,0) = 0, \quad \frac{\partial Y(0,0)}{\partial y} = 0, \quad \frac{\partial Y(0,0)}{\partial x} = 0$$

$$X(0,0) = 0, \quad \frac{\partial X(0,0)}{\partial y} = 0, \quad \frac{\partial X(0,0)}{\partial x} = 0$$

A   an $n \times n$ real constant matrix, and all the roots of the
characteristic equation

$$\det(A - HE) = 0 \tag{2}$$

have negative real parts.

In [1] it was proved that the following implicit equation

$$Ax + X(y, x) = 0$$

can be solved explicitly with respect to x , the solution being in the
form

$$x = u(y).$$

Now if we let

$$x = u(y) + z \tag{3}$$

we obtain from (1) the following system of differential equations [1] :

$$\left.\begin{aligned} \frac{dy}{dt} &= \Phi(y) + Y^*(y,z) \\[2em] \frac{dz}{dt} &= A z + \Phi(y) q(y) + Z(y,z) \end{aligned}\right\} \tag{4}$$

where

$$\Phi(0) = \Phi'(0) = 0, \quad q(0) = 0. \quad Y^*(y,0) = 0. \text{ and } Z(y,0) = 0.$$

In [1], it has been discussed the case where the function $\Phi(y) = 0$ for any $y \in C$, where C is a set of numbers with o as its limit point. It has been proved that under certain conditions, the trivial solution of the system (1) is stable.

For any $y \in C$, the system (4) has a solution

$$y = c, \ z = 0 \qquad \text{, see [1].}$$

Consequently from (3) the system (1) has the following solution

$$y = c \quad, \quad x = u(c). \tag{5}$$

Letting

$$y = c + \alpha \quad, \quad x = u(c) + \Psi$$

in (1), we obtain the following system of differential equations :

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= Q(c, \alpha, \Psi) \\[2em] \frac{d\Psi}{dt} &= A\Psi + G(c, \alpha, \Psi), \end{aligned}\right\} \tag{6}$$

where

$$Q(c, \alpha, \Psi) = Y(c + \alpha, u(c) + \Psi) - Y(c, u(c)),$$

$$G(c, \alpha, \Psi) = X(c + \alpha, u(c) + \Psi) - X(c, u(c)).$$

In the following theorem we are going to discuss the stability of the nonzero solution (5).

Theorem :

i) If there exists a sequence of positive numbers $y_i \longrightarrow 0$ such that $\Phi(y_i) = 0$, $i = 1, 2, \ldots \infty$ and simultaneously a sequence of negative numbers $\overline{y_i} \longrightarrow 0$ such that $\Phi(\overline{y_i}) = 0$, $i = 1, 2, \ldots, \infty$ then the nonzero solution $y = c$, $x = u(c)$ is stable.

ii) If a sequence of positive (negative) numbers $y_i \underset{i \to \infty}{\longrightarrow} 0$ exists such that $\Phi(y_i) = 0$, $i = 1, 2, \ldots \infty$ and $\Phi(y) > 0$ for $y < 0$ ($\Phi(y) < 0$ for $y > 0$) then the solution (5) of (1) is stable.

To prove this theorem, we need the following two lemmas which had been proved in [1].

Lemma (1) :

There exists a monotonic function $P(y)$ [$P(y) \geq 0$, $P(y) \underset{y \to 0}{\longrightarrow} 0$ such that if

$$v(x) \geq y^2 \, P^2(y)$$

then for sufficiently small y and x

$$\dot{v} = \frac{dv}{dt} = \frac{\partial v}{\partial x} [Ax + X(y, x)] < 0.$$

where $v(x)$ is a quadratic form satisfying the equation

$$\frac{\partial v}{\partial x} Ax = - x^2, \quad [2].$$

Lemma (2) :

There exists constants $E > 0$, $\lambda > 0$, $c_0 > 0$, $a_0 > 0$ and a scalar function $f(c, a, t_0)$, where c, t_0 are scalars and a is an n-dimensional vector, such that :

1. the function f is defined for any $c \varepsilon C$, $|c| \leq c_0$, $|a| \leq a_0$ and for all $t_0 \varepsilon (0, \infty)$,

2. f is continuous with respect to a and t, for any fixed $c \varepsilon C$,

3. $|f(c, a, t_0)| \leq |a|$,

4. for any solution of the system (6) with initial conditions :

$$t = t_0, \quad \Psi = a, \quad \alpha = f(c, a, t_0), \quad \|a\| \leq a_0$$

the following inequalities

$$|x(t)| \leqslant |a| e^{-\lambda(t-t_0)},$$

$$\|\Psi(t)\| \leqslant E\|a\| e^{\lambda(t-t_0)},$$

hold for all $t \geq t_0$.

We turn now to the proof of the Theorem.

i) For the proof of the first part, we take arbitrary

$$\varepsilon > 0 \quad, \quad \varepsilon < u_0 \ .$$

for this ε we can find two numbers

$$\left.\begin{array}{ll} c_1 > 2c > 0 \quad, \quad c_1 \varepsilon C \quad, \quad c_1 \leqq c_0 \ , \\ c_2 < 0 \qquad \quad, \quad c_2 \varepsilon C \quad, \quad |c_2| \leqq c_0 \ , \end{array}\right\} \tag{7}$$

such that the following inequalities hold

$$\bar{c} = \max \{ c_1, |c_2| \} < \varepsilon/2 , \tag{8}$$

$$\bar{P}(2\bar{c}) < 1/8 \tag{9}$$

where $\bar{P}(y)$ is an increasing function such that if $v(x) \leqq y^2 P^2(y)$ we have $\|x\| < y\bar{P}(y)$.

For the chosen numbers c_1, and c_2 we take $\delta > 0$ such that the following inequalities hold

$$\delta < 1/8 \ \min \{ c_1, |c_2| \} , \tag{10}$$

and

$$v(x(t) - u(c)) < 4\bar{c}^2 P^2(2\bar{c}) \text{ for } \|x(t) - u(c)\| < \delta . \tag{11}$$

We shall try to prove now that for every solution of system (1), for which the initial data satisfy conditions :

$$|y(t_0) - c| < \delta , \|x(t_0) - u(c)\| < \delta \tag{12}$$

the following inequalities

$$|y(t) - c| < 2\bar{c}, \ \cdots \tag{13}$$

$$\|x(t) - u(c)\| < 2\bar{c}, \ \cdots \tag{14}$$

hold for all $t \geq t_0$

These two inequalities combined with (8) show that the nonzero solution (5) is stable.

On the contrary, suppose that, there exists a solution $y(t)$, $x(t)$ of system (1) for which the initial data satisfy conditions (12), and a number $T > t_o$, such that for any $t \in [t_o, T)$ inequalities (13) and (14) satisfied, save for at $t = T$ where at least one of them turn into an equality.

We prove first that for this solution the inequality

$$v(x(t) - u(c)) < 4\bar{c}^2 P^2(2\bar{c}), \cdots \qquad (15)$$

holds for all $t \in [t_o, T]$.

According to (11) inequality (15) takes place for $t = t_o$.

Suppose, that inequality (15) is broken first at

$$t = t^* \varepsilon (t_o, T].$$

From inequality (13) and in virtue of the monotonic behaviour of the function $P(y)$, it follows that

$$v(x(t^*) - u(c)) \geqq (y - c)^2 P^2(y - c),$$

and then from Lemma (1), it follows that

$$\dot{v}(x(t^*) - u(c)) < 0.$$

This contradicts the definition of t^*.

Therefore

$$v(x(t) - u(c)) < 4\bar{c}^2 P^2(2\bar{c})$$

for all $t \in [t_o, T]$, and consequently

$$\|x(t) - u(c)\| < 2\bar{c}\bar{P}(2\bar{c}) < 2\bar{c} \text{ for all } t \in [t_o, T]. \qquad (16)$$

Thus from the definition of T, it follows that

$$|y(t) - c| = 2\bar{c},$$

which means that

either $y(T) - c = 2\bar{c}$,

or $\quad y(T) - c = -2\bar{c}$.

We shall prove that, these two possibilities are not true.

Consider the following function

$$\gamma(t) = c_1 - c + f(c_1, x(t) - u(c_1), t) - (y(t) - c)$$
$$\qquad (17)$$

where $y(t)$, $x(t)$ is the chosen solution and the function f is defined in Lemma (2).

Since from Lemma (1).

$$\| u(c_j) \| < \mid c_j \mid \overline{P}(c_j), \ (j = 1, 2), \tag{18}$$

then we have from inequalities (7), (8), (9) and (16)

$$\| x(t) - u(c_1) \| \leq \| x(t) - u(c) \| + \| u(c) \| + \| u(c_1) \| <$$

$$< 2\overline{c}\,\overline{P}(2\overline{c}) + c\,\overline{P}(c) + c_1\overline{P}(c_1) < \ \frac{7}{16}\overline{c} < a_0,$$

for all $t \in [t_o, T]$.

Thus, the function $\gamma(t)$ is defined and continuous for all $t \in [t_o, T]$.

From inequalities (7) and (10) and from lemma (2) it follows that,

$$c_1 - c + f(c_1, x(t_o) - u(c_1), t_o) > c_1 - c - |f| > \frac{3}{16} c_1.$$

$$\tag{19}$$

From the inequality $|y(t_o) - c| < \delta < \ ^1\!/_8 \ c_1$

it then follows that

$$\gamma(t_o) > 0 \tag{20}$$

By the same way, using (7), (9), (16), (18) and lemma (2) we can prove, that

$$c_1 - c + f(c_1, x(T) - u(c_1), T) < \frac{29}{16}c,$$

and since $y(T) - c = 2\,\overline{c}$ then, it follows, that

$$\gamma(T) < 0 \tag{21}$$

From (20) and (21) it follows that, there exists an instance $t' \in (t_o, T)$, where $\gamma(t') = 0$,

i.e. $y(t') - c = c_1 - c + f(c_1, x(t') - u(c_1), t')$ $\tag{22}$

Considering now the variables $\alpha(t)$ and $\Psi(t)$, we have

$$\alpha(t) = y(t) - c_1, \tag{23}$$

$$\psi(t) = x(t) - u(c_1) \tag{24}$$

From (22), it follows that, for $t = t'$

$$\psi = x\,(t') - u\,(c_I) = \bar{a} \qquad\qquad \text{say,} \qquad\qquad (25)\;.$$

and

$$\alpha = f\,(\,c_1,\ \bar{a},\ t')\cdot \qquad\qquad (26)$$

As we have proved above, the following inequality

$$\|\,x\,(t) - u\,(c_1)\,\| \ < \frac{7}{16}\,\bar{c} < a_0$$

holds for all $t \in [\,t_o\,,\,T\,]$.

Then from (25) it follows that $\|\,\bar{a}\,\| < a_0$, and from lemma (2), we have

$$|\,\alpha\,(t)\,| \leq \|\,x\,(t) - u\,(c_1)\,\| < \frac{7}{16}\,\bar{c}\ ,$$

for all $t \geq t'$.

Now from (7), (8) and (23) we have

$$|\,y\,(t) - c\,| \leq |\,y\,(t) - c_1\,| + c_1 + c < \frac{7}{16}\,\bar{c} + c_1 + \tfrac{1}{2}\,c_1 < 2\,\bar{c}$$

for all $t \geq t'$.

This inequality contradicts our assumption that

$$y\,(t) - c = 2\,\bar{c} \quad \text{for} \quad t = T > t'\;.$$

This proves that

$$y\,(T) - c < 2\,\bar{c}. \qquad\qquad (27)$$

Similarly we can prove, that

$$y\,(T) - c > -\,2\,\bar{c} \qquad\qquad (28)$$

From (27) and (28) it follows that

$$|\,y\,(T) - c\,| < 2\,\bar{c}\ ,$$

which proves the first part of the theorem.

ii)   The second part can be proved on similar lines as the proofs of Krasovesky's theorem [3] and the first part of this theorem.

REFERENCES

[1]   Mounir N. Bishai « Investigation of stability of Autonomous system in the case of one Zero root ».   Journal differential equations, vol. II No. 8. (Russian).

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[3]   Krasovesky N. N. « The stability of motion in the case of one Zero root » (Mat. Sbornik), vol. 37. 1955 (Russian).