

**ON THE SMOOTHNESS OF THE SOLUTION OF THE
FIRST BOUNDARY VALUE PROBLEM OF QUASILINEAR
ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS**

By
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§ 1. Def :

1. A function $u(x)$ defined on the set of points A is said to satisfy Hölder's condition with exponent α ($0 < \alpha < 1$) if the ratio

$$\frac{|u(P) - u(Q)|}{PQ^\alpha}$$

is uniformly bounded above for any two points P and Q from A .

2. The space $C_{m+\alpha}(A)$ (m nonnegative integer, and $0 < \alpha < 1$) is the space of all m -times differentiable functions in A , the m^{th} derivative of which satisfy Hölder's condition with exponent α .

§ 2. Consider the convex domain G , with boundary Γ consisting of finite number of curves Γ_k , $k = 1, 2, \dots, n$. The two curves Γ_k and Γ_{k+1} intersect at (a_k, b_k) with angle θ_k , ($0 < \theta_k < \pi$). The curve Γ_k may be represented parametrically in the form $x = x(s)$ and $y = y(s)$ where s is the arc length measured from an end of Γ_k and $x(s) \in C_{m+2+\alpha}$ and $y(s) \in C_{m+2+\alpha}$ ($m \geq 0$, $0 < \alpha < 1$).

On Γ_k , a function $\Phi_k(s) \in C_{m+2+\alpha}$ is defined, such that $\Phi_k = \Phi_{k+1}$ at the common point of Γ_k and Γ_{k+1} .

Consider Dirichlet's problem

$$a_{ij}(x, y, u, u_x, u_y) u_{ij} = b(x, y, u, u_x, u_y), (x, y) \in G$$

$$u|_{\Gamma} = \Phi(s)$$

where :

$$i) \quad \Phi(s) = \Phi_k(s) \text{ on } \Gamma_k \qquad k = 1, 2, \dots, n$$

and $a_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \quad (x_1 = x, x_2 = y) \quad \text{and}$

repeated indices mean summation.

ii) The functions $a_{ij}(x, y, u, p, q)$ and $b(x, y, u, p, q)$ are defined for $(x, y) \in G$ and all finite values of u, p and q , and belong to $C_{m+\alpha}$

iii) For $(x, y) \in G$ and any positive number K , such that $|u| + p + q \leq K$, a_{ij} satisfy the inequality $m(K) (\eta_1^2 + \eta_2^2) \leq a_{ij} \eta_i \eta_j \leq M(K) (\eta_1^2 + \eta_2^2)$ for any real numbers η_1 and η_2 , where $m(K)$ and $M(K)$ depend only upon K .

It is known [1] that, the solution $u(x, y)$ of this problem, belongs to $C_{m+2+\beta}$ ($0 < \beta < \alpha$) every where in G except at the angular points

We now define the following numbers

$$1. \quad \zeta_{11}^{(k)} = a_{ij} [a_k, b_k, \Phi_{1_1}^{(k)}(a_k, b_k) \cos \alpha_k + \Phi_{1_2}^{(k)}(a_k, b_k) \cos(0_k + \alpha_k), \Phi_{1_1}^{(k)}(a_k, b_k) \sin \alpha_k + \Phi_{1_2}^{(k)}(a_k, b_k) \sin(0_k + \alpha_k)]$$

where $\Phi_{1_1}^{(k)}(a_k, b_k)$ and $\Phi_{1_2}^{(k)}(a_k, b_k)$ are the derivatives of Φ in the directions of the tangents to the two curves Γ_k and Γ_{k+1} respectively.

i.e. In the directions making with the x-axis the angles $0 = \alpha_k$ and $0 = \alpha_k + 0_k$

$$2. \quad \omega_k = \tan^{-1} \left[\frac{\zeta_{11}^{(k)} \zeta_{22}^{(k)} - \zeta_{12}^{(k)2}}{\zeta_{22}^{(k)} \cos 0_k - \zeta_{12}^{(k)}} \right]^{1/2}$$

Theorem :

In the neighbourhood of the angular point (a_k, I_k) , there exists a number $\beta \leq \alpha$ such that

i) If $\frac{\pi}{\omega_k} > m + 2 + \beta$, then $u \in C_{m+2+\beta}$

ii) If $\frac{\pi}{\omega_k} \leq m + 2 + \beta$, then $u \in C_{\frac{\pi}{\omega_k} - \epsilon}$ ($\epsilon > 0$)

iii) If $\frac{\pi}{\omega_k} < m + 2$, then there exist two numbers τ and τ_0 , ($0 < \tau, \tau_0 < 1$) such that

$$r^{\tau} u \left(\left[\frac{\pi}{\omega_k} - \epsilon \right] + 1 \right) \in C_{\tau_0}$$

where r is the distance from (x, y) to (a_k, b_k) .

Proof : Without loss of generality, we can suppose that there exists only a single angle on the boundary Γ of the region, and that the x axis ($x \geq 0$) touches one branch of Γ

$$(i. e. n = 1, \alpha_1 = 0, \theta_1 = 0).$$

Substituting the value of $u(x, y) \in C_{1+\alpha}$ in the functions a_{ij} and b , we obtain

$$a_{ij}(x, y, u(x, y), u_x(x, y), u_y(x, y)) = A_{ij}(x, y)$$

$$b(x, y, u(x, y), u_x(x, y), u_y(x, y)) = B(x, y).$$

The functions a_{ij} and b belong to $C_{m+\alpha}$ thus the functions A_{ij} and B belong to C_β , where $\beta = \alpha$ if $m > 0$ and $\beta = \alpha^2$ if $m = 0$.

Consider now Dirichlet's problem for the linear elliptic equation

$$A_{ij}(x, y) u_{ij} = B(x, y) \quad (x, y) \in G$$

$$u|_{\Gamma} = \Phi \in C_{m+2+\alpha}$$

For this problem, it was proved [1] that if

$$A_{ij}(x, y) \in C_{m+\beta}, \quad B(x, y) \in C_{m+\beta}$$

$$\text{and } \omega = \tan^{-1} \frac{[A_{11}(0,0)A_{22}(0,0) - A_{12}^2(0,0)]^{1/2}}{A_{22}(0,0) \cot \theta - A_{12}(0,0)}$$

then the following results are valid

- i) If $\frac{\pi}{\omega} > m + 2 + \beta$, then $u(x, y) \in C_{m+2+\beta}(\bar{G})$
- ii) If $\frac{\pi}{\omega} \leq m + 2 + \beta$, then $u(x, y) \in C_{\frac{\pi}{\omega} - \epsilon}(\bar{G})$
- iii) If $\frac{\pi}{\omega} < m + 2$, then there exist two constants τ and

τ_0 ; $0 < \tau, \tau_0 < 1$, such that

$$r^{\tau} u \left(\left[\frac{\pi}{\omega} - \epsilon \right] + 1 \right) \in C_{\tau_0}(\bar{G})$$

where r is the distance from (x, y) to the origin.

Thus $u(x,y) \in C_{2+\beta}(\bar{G})$ if $\frac{\pi}{\omega} > 2 + \beta$ otherwise $u \in C_{\frac{\pi}{\omega}-\varepsilon}(\bar{G})$.

If $\frac{\pi}{\omega} > 2 + \beta$, we substitute the value of $u(x,y) \in C_{2+\beta}(\bar{G})$ in a_{ij} and b to obtain A_{ij} and B which belong to $C_{1+\beta}(\bar{G})$ and using the same result of the linear equations we conclude that $u \in C_{3+\beta}(\bar{G})$ if $\frac{\pi}{\omega} > 3 + \beta$ otherwise $u \in C_{\frac{\pi}{\omega}-\varepsilon}(\bar{G})$.

Repeating this process, we obtain that if $\frac{\pi}{\omega} > m + 2 + \beta$, then $u \in C_{m+2+\beta}(\bar{G})$ otherwise $u \in C_{\frac{\pi}{\omega}-\varepsilon}(\bar{G})$,

Similarly if $\frac{\pi}{\omega} < m + 2$, then we can find τ and τ_0 , ($0 < \tau, \tau_0 < 1$), such that

$$r^\tau u \left(\left[\frac{\pi}{\omega} - \varepsilon \right] + 1 \right) \in C_{\tau_0}(\bar{G})$$

[1] A. H. Azzam ; Ph.D. thesis. Moscow State University 1967, Unpublished.