INVESTIGATION OF THE STABILITY OF THE SOLUTION OF A PERIODIC SYSTEM OF DIFFERENTIAL EQUATIONS

By

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Consided the following system

$$\left.\begin{aligned} \frac{dy}{dt} &= Y(t, y, x), \\ \frac{dx}{dt} &= Ax + X(t, y, x). \end{aligned}\right\} \qquad (1)$$

Where :

y is a scalar,

x an n-dimensional vector,

Y a scalar function and X an n-dimensional vector function having the following properties :

$$Y(t+\omega, y, x) = Y(t, y, x),\ X(t+\omega, y, x) = X(t, y, x)\ (\omega > o)$$

$$Y(t, o, o) \equiv o, \frac{\partial Y(t, o, o)}{\partial y} \equiv o, \frac{\partial Y(t, o, o)}{\partial x} \equiv o,$$

$$X(t, o, o) \equiv o, \frac{\partial X(t, o, o)}{\partial y} \equiv o, \frac{\partial X(t, o, o)}{\partial x} \equiv o,$$

A an n X n real constant matrix, and all the roots of the characteristis equation

$$\det (A - HE) = o \qquad (2)$$

have negative real parts.

In [1] it has been proved, that there exists a function $\phi(y_0)$ which is uniquely defined on some interval

$|y_0| \leqslant h$ such that the system

$$\left.\begin{aligned} \frac{dy}{dt} &= Y(t, y, x) + \phi(y_0) \\ \frac{dx}{dt} &= Ax + X(t, y, x) \end{aligned}\right\} \qquad (3)$$

has in the neighbounhood of the origin a totality of periodic solutions depending on y_0

$$y = y(t, y_0) = y_0 + \bar{y}(t, y_0), \; x = x(t, y_0), \qquad (4)$$

for all t $(-\infty \; \infty)$ and $|y_0| \leq h$ and they satisfy the following inequalities

$$|\bar{y}(t, y_0)| \lessgtr \vartheta_1(y_0) \, |y_0| \, , \; \| x(t, y_0) \| \lessgtr \vartheta_2(y_0) \, |y_0| \qquad (5)$$

where $\vartheta_i(y_0) \geqq 0$ and $\vartheta_i(y_0) \xrightarrow[y_0 \to 0]{} 0 \; (i = 1,2)$.

Now if we let

$$y = y(t, z), \qquad x = x(t, z) + u, \qquad (6)$$

we obtain from (1) the following system of differential equations :

$$\left.\begin{aligned} \frac{dZ}{dt} &= \Phi(z) [1 + \rho(t, z)] + Z(t, z, u) \\ \frac{du}{dt} &= Au + \Phi(z) \, q(t, z) + U(t, z, u) \end{aligned}\right\} \qquad (7)$$

where the ftnctions ρ q, Z and U have the following properties :

$$\rho(t, o) \equiv o, \; q(t, o) \equiv o, \; Z(t, Z, o) \equiv o, \; U(t, Z, o) \equiv o. \qquad (8)$$

In [1] it has been discussed the case where the function $\Phi(z) = o$ for any $z \in C$ where C is an infinite set of numbers with o as limit point.

It is easy to prove from (8) that, for any $z \in C$ the system (7) has a solution $z = c$, $u = o$

Consequently from (5) and (6) the system (1) will have ω - periodic solution

$$y = y(t, c) = c + \bar{y}(t, c), \; x = x(t, c), \qquad (9)$$

which satisfies the following inequalities :

$$|\bar{y}(t, c)| \lessgtr \vartheta_1(c) \, |c| \, , \; \| x(t, c) \| \lessgtr \vartheta_2(c) \, |c| \qquad (10)$$

where $\vartheta_i(c) \geqq o$ and $\vartheta_i(c) \xrightarrow[c \to 0]{} o \; (i = 1,2)$.

In [1] it has been proved, that under certain conditions the trivial solution of the system (1) is stable. Letting

$$y = y(t, c) + \alpha, \; x = (t, c) + \psi, \tag{11}$$

in (1), we obtain the following system of differential equations :

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= Q(c, \alpha, \psi, t) \\ \frac{d\psi}{dt} &= A\psi + G(c, \alpha, \psi, t) \end{aligned}\right\} \tag{12}$$

where $Q(c, o, o, t) \equiv o, \quad G(c, o, o, t) \equiv o$

and Q and G have a period ω with respect to t.

In the following theoren we are going to discuss the stability of of the nonzero soluion in (9).

Theorem :

i) If there exists a sequence of positive numbers $Z_i \xrightarrow[i \to \infty]{}$ such that $\phi(Z_i) = o$, $i = 1, 2, \ldots \infty$ and simultuneously a sequence of negative numbers $\overline{Z_i} \xrightarrow[i \to \infty]{} o$ such that $\phi(\overline{Z_i}) = o$, $i = 1, 2, \ldots \infty$ then the nonzero solution $y = y(t, c)$, $x = x(t, c)$, is stable.

ii) If a sequence of positive (negative) numbers $Z \xrightarrow[i \to \infty]{} o$ exists such that $\phi(Z_i) = o$, $i = 1, 2, \ldots \infty$ and $\phi(z) > o$ for $z < o$ ($\phi(z) < o$ fo = $z > o$) then the solution (9) of (1) is stable.

To prove this theorem, we need the following two lemmas which had been proved in [1].

Lemma (1) :

There exists a monotonic function $P(y)$ [$P(y) \geq o$, $P(y) \xrightarrow[y \to o]{} o$ such that, if $V(x) \geq y^2 P^2(y)$

then for sufficiently small $|y|$ and $|x|$

$$\dot{v} = \frac{dv}{dt} = \frac{\partial x}{\partial v} [Ax + X(t, y, x)] < o, \tag{15}$$

where V (x) is a quadratic form satisfying the following equation

$$\frac{\partial v}{\partial x} A x = - \| x \|^2$$

Lemma (2) :

There exist constants $E > 0$, $\lambda > 0$, $c_0 > 0$, $a_0 > 0$ and a scalar function $f(c, a, t_0)$, where c, t_0 are scalars and a is n-dimensional vector, such that

1. The function f is defined for any $c \in C$, $|c| \lessgtr c_0$, $\|a\| \lessgtr a_0$ and for all $t_0 \in (-\infty, \infty)$,
2. f is continuous with respect to a and t_0 for any fixed $c \in C$,
3. $f(c, a, t_0 + \omega) = f(c, a, t_0)$,
4. $|f(c, a, t_0) \lessgtr \|a\|$,
5. For any solution of the system (12) with initial conditions :

$$t = t_0, \ \psi = a, \ \alpha = f(c, a, t_0), \ \|a\| \lessgtr a_0$$

the following inequalities :

$$|\alpha(t)| \lessgtr \|a\| e^{-\lambda(t - t_0)}, \ \|\psi(t)\| \lessgtr E \|a\| e^{-\lambda(t - t_0}$$

hold for all $t \geq t_0$.

We turn now to the proof of the theorem.

i) For the proof of the first part, we take arbitrary

$\epsilon > 0$, $\epsilon < a_0$.

For this ϵ we can find two numbers

$$\left. \begin{array}{l} c_1 > \frac{2o}{3} c > o, \ c_1 \in C, \ c_1 \lessgtr c_0, \\ c_2 < o, \ c_2 \in C, \ |c_2| \lessgtr c_0, \end{array} \right\} \quad (13)$$

such that the following inequalities hold

$$\bar{c} = \max \{ c_1, |c_2| \} < \varepsilon/2 \quad (14)$$

$$\bar{p}(2\bar{c}) < 1/8 \quad (15)$$

$$\max \vartheta_i(c_j) < 1/8 \quad (i, j = 1,2), \quad (16)$$

where p (y) is an increasing function such that if $v(x) \lessgtr y^2 p^2(y)$ we have $\|x\| \lessgtr y \bar{p}(y)$.

For the chosen numbers c_1 and c_2 we take $\partial > o$, such that the folow-ing inqualities hold

$$\partial < {}^1/_8 \min \{ c_1 , | c_2 | \} \tag{17}$$

and

$$v (x - x (t, c)) \gtrless 4 c^{-2} P^2 (2 c) \text{ for } \| x - x (t, c) \| < \partial . \tag{18}$$

We shall try to prove now that for every solution of system (1), for which the initial data satisfy the following conditions

$$y (t_o) - y (t_o, c) | < \partial \quad , \quad \| x (t_o) - x (t_o, c) \| < \partial, \tag{19}$$

the following inqualities

$$| y (t) - y (t, c) | < 2 \bar{c} , \tag{20}$$

$$| x (t) - x (t, c) \| < 2 \bar{c}, \tag{21}$$

hold for all $t \geq t_o$.

These two inequalities combined with (14) show that the nonzero solution (9) is table.

On the contarary : suppose that, there exists a solution $y (t)$, $x(t)$ of system (1) for which the initial date satisfy (19) and a number $T > t_o$ such for any $t \in [t_o , T)$ the inequalities (20) and (21) are satisfied, save for at $t = T$ where at least ne of them turn into an equality.

We prove first that for this solution the inequality

$$v (x (t) - x (t, c)) < 4 c^{-2} p^2 (z \bar{c}) , \tag{22}$$

holds for all $t \in [t_o , T]$.

According to (18) inequality (22) takes place for $t = t_o$. Suppose, that inequality (22) is broken first at $t = t^* \in (t_o ,T]$. From inequality (20) and in virtue of the monotomic behavior of the function $P (y)$ it follows that

$$v (x (t^*) - x (t^*, c)) \gtrless [y (t^*) - y (t^*,)]^2 P^2 (y (t^* c) - y (t^*,c)),$$

and then frm lemma (1), it follows that

$$\dot{v} (x (t^*) - x (t^*, c)) < o$$

This contradicts the definition of t^*

Therefore $v (x (t) - x (t,c)) < 4 \bar{c}^{-2} P^2 (z \bar{c})$ for all $t \in [t_o T]$, and consequantly

$$| x(t) - x(t, c) | < 2\bar{c}\,\bar{P}(2\bar{c}) < 2\bar{c} \text{ fos all } t \in [t_0, T] . \tag{23}$$

Thus from the definition of T it follows that

$$| y(T) - y(T, c) | = 2\bar{c} ,$$

which means that :

either $y(T) - y(T, c) = 2\bar{c}$,

or $y(T) - y(T, c) = -2\bar{c}$

We shall prove that these two posibilities are not true. Consider the following function

$$\gamma(t) = y(t, c_1) - y(t, c) + (c, x(t) - x(t, c_1), t) - (y(t) - y(t,c)) \quad (24$$

where y (t), x (t) is the chosen solution and the functions $y(t,c_1)$, $x(t,c_1)$ and f are defined above.

From inequalities (10), (14), (15), (16) and (23) we have

$$\| x(t) - x(t, c_1) \| < \frac{\bar{c}}{2} < a_0 \quad \text{for all } t \in [t_0, T] .$$

Thus the function γ (t) is defined and continuous for all $t \in [t_0, T]$. From inequalities (9), (10), (13), (14), (16), (17) and from lemma (2), it follows that

$$y(t_0, c_1) - y(t_0, c) + f(c_1, x(t_0) - x(t_0, c_1), t_0)$$

$$= c_1 + \bar{y}(t_0, c_1) - c - \bar{y}(t_0, c) + f(c_1, x(t_0) - x(t_0, c_1), t_0) \geq$$

$$\geq c_1 + | \bar{y}(t_0, c_1) - c - \bar{y}(t_0, c) | - | f | > \frac{5}{16} c_1$$

Now from the inequality $y(t_0) - y(t_0, c) < \delta < \frac{1}{8} c_1$

if follows that

$$\gamma(t_0) > 0 \tag{25}$$

By the same way, using (10), (13), (16), (23) and from lemma (2) we can prve, that

$$y(T, c_1) - y(T, c) + f(c_1, x(T) - x(T, c_1), T) < \frac{27}{16} c$$

and since $y(T) - y(T, c) = 2\bar{c}$ it follows that,

$$\gamma(T) < 0 \tag{26}$$

From (25) and (26) it follows that, there exists an instance $t^1 \in (t_0, T)$ where $\gamma(t^1) = 0$, i.e.

i. e. $y(t^1) - y(t^1, c) = y(t^1, c_1) - y(t^1, c) + f(c_1, x(t^1) - x(t^1, c_1), t^1)$. (27)

Considyering now the variables α (t) and ψ (t), we have

$$\alpha(t) = y(t) - y(t, c_1), \tag{28}$$

$$\psi(t) = x(t) - (t, c_1). \tag{29}$$

From (27) it follows that for $t = t^1$,

$$\psi = x(t^1) - x(t' c_1) = \bar{a}, \text{ say}. \tag{30}$$

and $$\alpha = f(c_1, \bar{a}, t^1). \tag{31}$$

As we have prved above, the following inequality

$| x(t) - x(t, c_1) | < \frac{\bar{c}}{2}$ holds for all $t \in [t_0 T]$.

There from (30) it follows that $\| \bar{a} \| < a_0$ and from lemma (2), we have

$| \alpha(t) \lesseqgtr \| x(t) - x(t, c_1) \| < \frac{27}{16} \bar{c}$ for all $t \geq t^1$.

Now from (10) and (28) we have $| y(t) - y(t, c) | < \frac{27}{16} \bar{c}$

for all $t \geq t^1$.

This inequality contradicts our assumption that

$$y(t) - y(t, c) = 2 \quad c \qquad \text{for } t = T > t^1 .$$

This proves that

$$y(T) - y(T, c) < 2 \bar{c} \tag{32}$$

Similarly we can prove, that

$$y(T) - y(T, c) > -2 \bar{c} \tag{33}$$

From (32) and (33) it follows that $| y(T) - y(T, c) | < 2 \bar{c}$, which proves the first part of the theorem.

ii) The second part of the theorm now follows by applying Krasovesky theorem [2], and the first part of this theorem.

Refferences

1. Mounir N. Bishia «About the stability of motion in critical cases». Candidate thesis, Leningrad University, 1966.

2. Krasvesky N. N. «The stability of motion in the case of one zero root». (Mat. Shornik, vol. 37 1955 (Russian).