ANGLES BETWEEN SUBSPACES IN N-DIMENSIONAL VECTOR SPACE

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ABSTRACT :   In this paper, the concept of
angles between two subspaces V and W in
n-dimensional vector space R_n is introduced,
and some properties of these angles are given
to be used in the proof of one theorem which
determines in R_n the number of subspaces W of
different dimensions, inclining to a fixed
subspace V at the same fixed angle Υ . It
is of interist to note that this theorem remains
valid in 3-dimensional vector space even for
ordinary angles between vectors.

INTRODUCTION : It is clear that in 3-
dimensional vector space there exist only 2
subspaces of different dimensions, which are
inclined to a fixed subspace V at the same
angle α , $0 \leqslant \alpha \leqslant 90°$. Obviously, one of
the two integers (3-dim V) and (4-dim V)
equals 2, for dim V = 1 if V is a straight
line, and dim V = 2 if V is a plane. Thus
in 3-dimensional vector space the number of
subspaces W of different dimensions, which
are inclined to a fixed subspace V at the

same angle α is equal to one of the two integers
(3-dim V) and (4-dim V).   This fact is generalized
in the present paper, where the concept of angles
between subspaces in unitary n-dimensional vector
space R_n is introduced and a theorem is proved
to assure that if in R_n a subspace V of dimension
m is fixed, and an angle γ is given in V, then
the number of subspaces W of different dimensions,
which are inclined to subspace V at the same angle
γ equals one of the two integers $(n - m)$ and
$(n + 1 - m)$.

BASIC LEMMAS : Let V and W be two subspaces
in R_n, and let P_V and P_W be projection operators
of R_n onto subspaces V and W respectively. If
P_V^W is the restriction of P_V on W, and P_W^V is
the restriction of P_W on V, then

$$\gamma = P_V^W \, P_W^V \, , \qquad \phi = P_W^V \, P_V^W \qquad (1)$$

are two linear operators in V and W respectively.

DEFINITION : Linear operators γ and ϕ
determined by equalities (1) are called the angles
between subspaces V and W. For angles γ and ϕ
we can use the notations:

$$\gamma = \measuredangle (V,W) \, , \qquad \phi = \measuredangle (W,V).$$

(ii) $\alpha_o = 0$. Here we have

$$m = \alpha_1 + \alpha_2 + \ldots + \alpha_s + \alpha ,$$

$$\dim W = m' = \alpha'_o + m ,$$

$$\alpha'_o = \dim (W \cap V^{\perp}).$$

Since W is a proper subspace in R_n, α'_o must satisfy the condition

$$0 \leqslant \alpha'_o \leqslant n-m-1$$

and so in this case there are $(n-m)$ subspaces of dimensions $m, m+1, \ldots , n-1$.

(iii) $m > \alpha_o > 0$. In this case

$$m = \alpha_o + \alpha_1 + \alpha_2 \ldots + \alpha_s + \alpha ,$$

$$\dim W = m' = \alpha'_o + m - \alpha_o ,$$

where α'_o must satisfy the condition

$$0 \leqslant \alpha'_o \leqslant n-m$$

and so in this case there are $(n+1-m)$ subspaces W of dimensions $m- \alpha_o , m- \alpha_o +1, \ldots , n- \alpha_o ,$ and the Theorem is proved.

REMARK 1 :   In cases (ii) and (iii) it is easy to show how to construct any subspace W of given dimension m' to be inclined to V at the angle $\psi = \sphericalangle (V,W)$, where

$$m \leqslant m' \leqslant n-1 \qquad \text{in case (ii)}$$

$$m - \alpha_0 \leqslant m' \leqslant n - \alpha_0 \qquad \text{in case (iii)}$$

REMARK 2 :   Examples show that two different subspaces W_1 and W_2 of the same dimension may satisfy the condition $\sphericalangle (V,W_1) = \sphericalangle (V,W_2)$.   It is not difficult to prove that two subspaces W_1 and W_2 of the same dimension m' are inclined to subspace V at the same angle $\psi = \sphericalangle (V,W_1) = \sphericalangle (V,W_2)$ if and only if there exists in R_n a unitary operator τ, which transforms W_1 into W_2 and induces in V a unit operator, i.e. τ is such that

$$\tau W_1 = W_2 , \quad \tau v = v \quad \forall v \in V.$$

REMARK 3 :   The concept of angles between subspaces in R_n is a rich concept which has important applications (see [2] and [3]).

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PROOF : Here also the proof is given only for $\curlyvee$. Let $\curlyvee(x) = \lambda x$. Since the length of any vector cannot be increased by projection, therefore

$$\| x \| \geqslant \| P_W(x) \| \geqslant \| P_V P_W(x) \| = \| \curlyvee(x) \| = \lambda \| x \|$$

Corollary 1 and relation $\| x \| \geqslant \lambda \| x \|$ imply that $0 \leqslant \lambda \leqslant 1$. To find the multiplicity of eigenvalue 1 for operator $\curlyvee$ let, on one hand, $x \in (V \cap W)$; therefore

$$\curlyvee(x) = P_V P_W(x) = P_V(x) = x .$$

On the other hand, if $\curlyvee(x) = x$, then

$$\| x \| \geqslant \| P_W(x) \| \geqslant \| \curlyvee(x) \| = \| x \|$$

i.e. $\| x \| = \| P_W(x) \|$, which can be valid only when $P_W(x) = x$, and consequently $x \in (V \cap W)$. Actually we obtained that $\curlyvee(x) = x$ if and only if $x \in (V \cap W)$, and the Lemma is proved.

LEMMA 3 : Operators $\curlyvee$ and $\emptyset$ have the same eigenvalues, different from zero. Moreover, each of these eigenvalues has the same multiplicity for $\curlyvee$ and $\emptyset$.

PROOF : Let $\psi(x) = \lambda \, x$, where $x \in$
V, $\lambda \neq 0$. Then we obtain

$$\phi(P_W(x)) = P_W \, P_V \, (P_W(x))$$

$$= P_W \, (P_V \, P_W(x))$$

$$= P_W \, (\psi(x))$$

$$= P_W \, (\lambda \, x)$$

$$= \lambda \, (P_W(x))$$

i.e. λ is also an eigenvalue for operator ϕ,
where corresponding eigenvector is $P_W(x)$.
Analogically, if $\phi(y) = \mu \, y$, $y \in W$, $\mu \neq 0$,
then we obtain $\psi(P_V(y)) = \mu \, (P_V(y))$. Thus
we proved that ψ and ϕ have the same eigenvalues,
different from zero. Now, suppose that $\lambda \neq 0$
is an eigenvalue for ψ and ϕ with multiplicities
β and γ rrespectively. Let x_1, x_2, ..., x_β be
complete orthonormal system of eigenvectors for
operator ψ, which corresponds to eigenvalue λ.
From equality (2) we obtain

$$(P_W(x_i) \cdot P_W(x_j)) = (\psi(x_i) \cdot x_j)$$

$$= \lambda (x_i \cdot x_j)$$

$$= \lambda \delta_{ij}$$

where δ_{ij} is the Kronecker sympol, i.e. δ_{ij} = 1 if $i = j$, δ_{ij} = 0 if $i \neq j$; $i,j = 1,2,$ ..., β. Consequently, $P_w(x_1),\ldots, P_w(x_\beta)$ is an orthogenal system of eigenvectors for operator $\emptyset$, corresponding to eigenvalue λ . Actually we proved that $\beta \leqslant \gamma$. Analogically we can prove that $\gamma \leqslant \beta$ and consequently $\beta = \gamma$. Proof is accomplished

From lemmas 1-3 we deduce that if $0, \lambda_1, \lambda_2,\ldots,\ldots, \lambda_s, 1$, $(0 < \lambda_1 < \lambda_2 \ldots < \lambda_s < 1)$, are the eigenvalues of the operator γ with multiplicities α_0 , α_1 , $\alpha_2, \ldots ,$ α_s , α respectively, then $0,$ $\lambda_1, \lambda_2, \ldots , \lambda_s , 1$ are also the eigenvalues of the operator $\emptyset$ with multiplicities $\alpha_0' $, α_1 , $\alpha_2, \ldots ,$ α_s , α where

$$\alpha_0 = \dim (V \cap W^\perp), \quad \alpha_0' = \dim (W \cap V^\perp), \quad \alpha = \dim (V \cap W).$$

It is also clear that

$$\dim V = m = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_s + \alpha ,$$

$$\dim W = m' = \alpha_0' + \alpha_1 + \alpha_2 + \ldots + \alpha_s + \alpha .$$

and, consequently, $m = m'$ if and only if $\alpha_0 = \alpha_0'$.

Now we are ready to prove the following theorem which is the main aim of the present paper.

THEOREM : Let in R_n a proper subspace V of dimension m is fixed, and let φ be a given angle in V. Then the number of proper subspaces W of different dimensions, which are inclined to V at the same angle $\varphi = \angle (V,W)$ is equal to $(n+1-m)$ or $(n-m)$ according as α_0 satisfies or does not satisfy the condition $m > \alpha_0 > 0$ respectively.

PROOF : Let the eigenvalues of operator φ be $0, \lambda_1, \lambda_2, \ldots, \lambda_s, 1, (0 < \lambda_1 < \lambda_2 \ldots < \lambda_s < 1)$, with multiplicities $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_s, \varkappa$ respectively. Here α_0 plays the leading part. There are three possible cases.

(i) $\alpha_0 = m$. In this case V and W are multally orthogonal, and so W coincides with any subspace in $V^\perp$. Since dim $V^\perp = n-m$, we can construct in $V^\perp$ subspaces W of dimensions $1, 2, \ldots, n-m$, where subspace W with dimension $(n-m)$ coincides with $V^\perp$.

LEMMA 1 : Υ and $\emptyset$ are positive-semidefinite operators in V and W respectively. Moreover, Υ and $\emptyset$ will be positive-definite if and only if $(V \wedge W^{\perp}) = 0$ and $(W \wedge V^{\perp}) = 0$ respectively.

PROOF : We prove the lemma only for the operator Υ . For any two vectors x and y from V we obtain

$$(\Upsilon (x).y) = (P_V^W \; P_W^V \; (x).y)$$

$$= (P_V \; P_W \; (x).y)$$

$$= ((I - P_V^{\perp}) \; P_W \; (x).y)$$

$$= (P_W \; (x).y)$$

$$= (P_W(x).(P_W + P_W^{\perp}) \; (y))$$

$$\therefore \quad (\emptyset (x).y) = (P_W(x).P_W(y)) \qquad (2)$$

$$= ((I - P_W^{\perp}) (x).P_W(y))$$

$$= (x.P_W(y))$$

$$= (x.(P_V + P_V^{\perp}) \; P_W(y))$$

$$= (x.P_V P_W \; (y))$$

$$= (x. \; P_V^W \; P_W^V \; (y))$$

$$\therefore \quad (\psi(x).y) = (x.\psi(y)) \qquad (3)$$

Equality (3) proves that ψ is hermitian operator in V.

From (2) we obtain for every vector $x \in V$

$$(\psi(x).x) = (P_W(x).P_W(x)) = \left\| P_W(x) \right\|^2 \geqslant 0$$

i.e. ψ is positive-semidefinite operator in V. It is clear that ψ will be positive-definite if and only if $P_W(x) \neq 0 \ \forall \ x \in V$, i.e. if and only if $(V \cap W^{\perp}) = 0$, and the proof is finished.

COROLLARY 1 : All eigenvalues of operators ψ and $\emptyset$ are non-negative (see [1], P. 274).

COROLLARY 2: Zero is an eigenvalue for ψ and $\emptyset$ with multiplicities α_0 and α_0' respectively, where

$$\alpha_0 = \dim(V \cap W^{\perp}), \quad \alpha_0' = \dim(W \cap V^{\perp}).$$

LEMMA 2 : All eigenvalues of operators ψ and $\emptyset$ are included between 0 and 1, moreover, eigenvalue 1 has the same multiplicity α for ψ and $\emptyset$, where $\alpha = \dim(V \cap W)$.

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