

YOUNG OPERATORS FOR THE REPRESENTATIONS
[$n-1,1$] and [$2\ 1^{n-1}$] IN STANDARD
ORTHOGONAL FORM

By

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Abstract :

Explicit expressions in term of ket and bra tableau operators formed from symmetrisers and antisymmetrisers and tableau permutations are given for the Young operators for the representations [$n-1,1$] and [$2\ 1^{n-1}$] of the symmetric group S_n in standard orthogonal form.

1. *Introduction :*

Jahn (1960) gave the young operators of S_n as linear combinations of two-sided products of Young operators of S_{n-1} with the particular transposition $P_{n,n-1}$. To reach an explicit expression a long chain of calculations is required. It is the aim of the present paper to simplify the young operator expansions for the two representations [$n-1,1$] and [$2\ 1^{n-1}$] of S_n in standard orthogonal form. A complete set of young operators for the two particular representations is explicitly constructed and tabulated.

2. Young operators: notation and properties

The $(n-1)^2$ Young operators O_{ab}^n ($a, b = 2, 3, \dots, n$) for the representation $[\bar{n-1}, 1]$ of S_n in standard orthogonal form are required to satisfy

$$O_{ab}^n O_{cd}^n = \delta_{bc} O_{ad}^n, \quad \dots (2.1)$$

$$P_{a, a+1} O_{ab}^n = (1/a) O_{ab}^n + \{(a^2-1)^{1/2}/a\} O_{a+1, b}^n, \quad \dots (2.2)$$

$$O_{ab}^n P_{b, b+1} = (1/b) O_{ab}^n + \{(b^2-1)^{1/2}/b\} O_{a, b+1}^n, \quad \dots (2.3)$$

Here the numerical labels a, b are abbreviations for the standard tableau labels

$$A_n^a = \begin{matrix} 1 & \dots & \overset{\cdot}{a} & \dots & n \\ a & & & & \end{matrix} = \begin{matrix} 1, 2 & \dots & a-1 & a+1 & \dots & n-1 & n \\ a & & & & & & \end{matrix}, \quad \dots (2.4)$$

$$B_n^b = \begin{matrix} 1 & \dots & \overset{\cdot}{b} & \dots & n \\ b & & & & \end{matrix} = \begin{matrix} 1, 2 & \dots & b-1 & b+1 & \dots & n-1 & n \\ b & & & & & & \end{matrix}, \quad \dots (2.5)$$

where, following the author's notation (El-Sharkaway 1975), $\overset{\cdot}{a}$ or $\overset{\cdot}{b}$ is used to denote the omission of a or b from $2 \dots n$. The coefficient $1/a$ occurs in (2.2) and $1/b$ in (2.3) because the Young axial distance from $a+1$ to a in A_n^a is $+a$ and from $b+1$ to b in B_n^b is $+b$.

The $(n-1)^2$ Young operators $O_{a^*b^*}^n$ ($a, b = 2, 3, \dots, n$) for the representation $[21^{n-2}]$ of S_n taken in standard orthogonal form are required to satisfy

$$P_{a, a+1} O_{a^*b^*}^n = -(1/a) O_{a^*b^*}^n + \{(a^2-1)^{1/2}/a\} O_{a+1^*b^*}^n, \quad \dots (2.6)$$

$$O_{a^*b^*}^n P_{b, b+1} = -(1/b) O_{a^*b^*}^n + \{(b^2-1)^{1/2}/b\} O_{a^*b+1^*}^n, \quad \dots (2.7)$$

Here the starred numerical labels a^* , b^* are abbreviations for the standard tableau labels.

$$A_n^* = [a_1 \dots a_{a-1} \dots a_{a+1} \dots a_n] = [a_1^* 2 \dots a-1 \ a+1 \dots n-1 \ n], \quad \dots (2.8)$$

$$B_n^* = [b_1 \dots b_{b-1} \dots b_{b+1} \dots b_n] = [b_1^* 2 \dots b-1 \ b+1 \dots n-1 \ n], \quad \dots (2.9)$$

where, using again the author's notation (El-Sharkaway 1975) the abbreviations

$$[a_1 \dots a_{a-1} \dots a_{a+1} \dots a_n] = \begin{matrix} |a \\ \vdots \\ a-1 \\ a+1 \\ \vdots \\ n \end{matrix}, \quad [b_1 \dots b_{b-1} \dots b_{b+1} \dots b_n] = \begin{matrix} |b \\ \vdots \\ b-1 \\ b+1 \\ \vdots \\ n \end{matrix}, \quad \dots (2.10)$$

are employed to simplify the printing. The coefficients $-(1/a)$, and $-(1/b)$ occur in (2.6) and (2.7) because the Young axial distance from $a+1$ to a in (2.10) is $-a$ and the distance from $b+1$ to b is $-b$.

3. Symmetrisers and antisymmetrisers: properties

S and A are used to denote symmetrisers and antisymmetrisers defined by

$$S_{1 \dots n} = (1/n!) \sum_{\text{all } n! \ P \text{ in } S_n} P \quad \dots (3.1)$$

$$A_{1 \dots n} = (1/n!) \sum_{\text{all } n! \ P \text{ in } S_n} (-1)^P P, \quad (-1)^P \text{ parity of } P. \quad \dots (3.2)$$

$S_{1 \dots n}$ is the Young operator for the totally symmetric representation $[n]$ of S_n and is totally symmetric i.e.

$$P S_{1 \dots n} = S_{1 \dots n} P = S_{1 \dots n}, \quad P \text{ in } S_n, \quad \dots (3.3)$$

$A_{1\dots n}$ is the Young operator for $[1^n]$ and satisfies

$$(-1)^P P A_{1\dots n} = A_{1\dots n} (-1)^P P = A_{1\dots n}, \quad (-1)^P \text{ parity of } P, P \text{ in } S_n. \quad \dots (3.4)$$

It follows from (3.3) and (3.4) that

$$S_{1\dots a} S_{1\dots n} = S_{1\dots n} S_{1\dots a} = S_{1\dots n}, \quad a \leq n, \quad \dots (3.5)$$

$$A_{1\dots a} A_{1\dots n} = A_{1\dots n} A_{1\dots a} = A_{1\dots n}, \quad a \leq n, \quad \dots (3.6)$$

these being special cases of a general relation (Jahn 1960 (3.14)).

From

$$A_{1a} S_{1a} = \frac{1}{2}(I - P_{1a}) \frac{1}{2}(I + P_{1a}) = 0 = S_{1a} A_{1a} \quad \dots (3.7)$$

follows

$$A_{1a} S_{1\dots n} = \underline{A_{1a} S_{1a}} S_{1\dots n} = S_{1\dots n} \underline{S_{1a} A_{1a}} = S_{1\dots n} A_{1a} = 0 \quad \dots (3.8)$$

and more generally

$$A_{1\dots a} S_{1\dots n} = S_{1\dots n} A_{1\dots a} = A_{1\dots n} S_{1\dots a} = S_{1\dots a} A_{1\dots n} = 0, \quad a \leq n. \quad \dots (3.9)$$

A basic property of the symmetrisers and antisymmetrisers, following directly from the definitions (3.1), (3.2) is given by

$$S_{1\dots n} = \{(I + P_{1n} + P_{2n} + \dots + P_{n-1,n})/n\} S_{1\dots n}$$

$$= S_{1, \dots, n} \{ (I + P_{1n} + P_{2n} + \dots + P_{n-1, n}) / n \}, \quad \dots (3.10)$$

$$\begin{aligned} R_{1, \dots, n} &= \{ (I - P_{1n} - P_{2n} - \dots - P_{n-1, n}) / n \} R_{1, \dots, n} \\ &= R_{1, \dots, n} \{ (I - P_{1n} - P_{2n} - \dots - P_{n-1, n}) / n \}. \quad \dots (3.11) \end{aligned}$$

4. Reduction of multiple symmetriser antisymmetriser products

We show that the following four relations hold:

$$(i) \quad (R_{1a} S_{1, \dots, a}) (R_{1a} S_{1, \dots, a}) = a / \{2(1-a)\} (R_{1a} S_{1, \dots, a}) \quad (a \leq n) \quad \dots (4.1)$$

$$(ii) \quad (R_{1a} S_{1, \dots, a, \dots, b}) (R_{1a} S_{1, \dots, a, \dots, b}) = b / \{2(b-1)\} (R_{1a} S_{1, \dots, a, \dots, b}) \quad (a \leq b \leq n), \quad \dots (4.2)$$

$$(iii) \quad (S_{1, \dots, a} R_{1a}) (S_{1, \dots, a} R_{1a}) = a / \{2(a-1)\} (S_{1, \dots, a} R_{1a}) \quad (a \leq n) \quad \dots (4.3)$$

$$(iv) \quad (S_{1, \dots, a, \dots, b} R_{1a}) (S_{1, \dots, a, \dots, b} R_{1a}) = b / \{2(b-1)\} (S_{1, \dots, a, \dots, b} R_{1a}) \quad (a \leq b \leq n). \quad \dots (4.4)$$

A further set of four relations (used in connection with $[2]^{n-2}$) may be obtained from the above, interchanging R by S and S by R throughout.

Proof of (ii)

It will be sufficient to establish the second relation as

(i) follows from it ($b = a$) and (iii) and (iv) are a direct consequence of (i) and (ii). Relation (ii) may be rewritten as

$$[R_{1a} S_{1, \dots, a, \dots, b} - bI / \{2(b-1)\}] R_{1a} S_{1, \dots, a, \dots, b} = 0. \quad \dots (4.5)$$

Putting (from (3.10))

$$S_{1\dots\dot{a}\dots b} = \{(I+P_{12}+\dots+P_{1\dot{a}}+\dots+P_{1b})/(b-1)\} S_{2\dots\dot{a}\dots b}, \quad \dots (4.6)$$

commuting $S_{2\dots\dot{a}\dots b}$ with $A_{1\dot{a}}$, using (from (3.5))

$$S_{2\dots\dot{a}\dots b} S_{1\dots\dot{a}\dots b} = S_{1\dots\dot{a}\dots b}, \quad \dots (4.7)$$

writing $A_{1\dot{a}} = (I-P_{1\dot{a}})/2$, removing the common factor $2/(b-1)$

there remains

$$[(I-P_{1\dot{a}})(I+P_{12}+\dots+P_{1\dot{a}}+\dots+P_{1b}) - bI] A_{1\dot{a}} S_{1\dots\dot{a}\dots b} = 0. \quad \dots (4.8)$$

Now

$$P_{1\dot{a}} P_{1j} = P_{a1j} = P_{aj} P_{1\dot{a}} \quad (j = 2, \dots, \dot{a}, \dots, b) \quad \dots (4.9)$$

and (from (3.4))

$$-P_{1\dot{a}} A_{1\dot{a}} = +A_{1\dot{a}}. \quad \dots (4.10)$$

There remains

$$\begin{aligned} & (I+P_{12}+\dots+P_{1\dot{a}}+\dots+P_{1b} \\ & +I+P_{a2}+\dots+P_{a\dot{a}}+\dots+P_{ab} - bI) A_{1\dot{a}} S_{1\dots\dot{a}\dots b} = 0 \quad \dots (4.11) \end{aligned}$$

or, changing the overall sign,

$$\sum_{j=2, \dots, \dot{a}, \dots, b} (I-P_{1j}-P_{aj}) A_{1\dot{a}} S_{1\dots\dot{a}\dots b} = 0. \quad \dots (4.12)$$

Now (from 3.11))

$$3A_{1aj} = (I - P_{1j} - P_{aj})A_{1a} \quad (j = 2, \dots, a, \dots, b) \quad \dots (4.13)$$

and (from (3.9))

$$A_{1aj} S_{1 \dots a \dots b} = A_{1aj} \underline{A_{1j} S_{1j}} S_{1 \dots a \dots b} = 0 \quad (j = 2, \dots, a, \dots, b) \quad \dots (4.14)$$

Hence relation (ii) has been established.

Clearly the further four relations used in connection with $[21^{n-2}]$ obtained from (4.1) to (4.4) by interchanging A and S may be established in a very similar manner.

5. Bra and ket tableau operators

Bra and ket tableau operators $\langle A_n |$ and $| B_n \rangle$ are defined for $[n-1, 1]$ by

$$\langle A_n | = \langle \begin{matrix} j \dots a \dots n \\ a \end{matrix} | = 2\{(a-1)(n-1)/(an)\}^{1/2} S_{1 \dots a} A_{1a} S_{1 \dots a \dots n} \quad \dots (5.1)$$

$$| B_n \rangle = | \begin{matrix} 1 \dots b \dots n \\ b \end{matrix} \rangle = 2\{(b-1)(n-1)/(bn)\}^{1/2} S_{1 \dots b \dots n} A_{1b} S_{1 \dots b} \quad \dots (5.2)$$

and for $[21^{n-2}]$ by

$$\langle A_n^* | = \langle \left(\begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \right)^* | = \langle \begin{matrix} 1a \dots a \dots n \\ a \end{matrix} | = 2\{(a-1)(n-1)/(an)\}^{1/2} A_{1 \dots a} S_{1a} A_{1 \dots a \dots n} \quad \dots (5.3)$$

$$| B_n^* \rangle = \left| \left(\begin{matrix} 1 \dots b \dots n \\ b \end{matrix} \right)^* \right\rangle = | \begin{matrix} 1b \dots b \dots n \\ b \end{matrix} \rangle = 2\{(b-1)(n-1)/(bn)\}^{1/2} A_{1 \dots b \dots n} S_{1b} A_{1 \dots b} \quad \dots (5.4)$$

$\langle A_n |$ and $|B_n \rangle$ become equal when $a = b = n$:

$$\langle N_n | = \langle \begin{matrix} 1 \dots n \\ n \end{matrix} | = |N_n \rangle = \left| \begin{matrix} 1 \dots n \\ a \end{matrix} \right\rangle = 2\{(n-1)/n\} S_{1 \dots n} \beta_{1n} S_{1 \dots n} \dots \dots (5.5)$$

and likewise with $\langle A_n^* |$ and $|B_n^* \rangle$:

$$\langle N_n^* | = \langle \begin{matrix} 1 \dots n \\ n \end{matrix} | = |N_n^* \rangle = \left| \begin{matrix} 1 \dots n \\ a \end{matrix} \right\rangle = 2\{(n-1)/n\} \beta_{1n} S_{1 \dots n} \beta_{1n} \dots \dots (5.6)$$

We may put $n = a$ in these expressions and obtain

$$\langle A_a | = \langle \begin{matrix} 1 \dots a \\ a \end{matrix} | = |A_a \rangle = \left| \begin{matrix} 1 \dots a \\ a \end{matrix} \right\rangle = 2\{(a-1)/a\} S_{1 \dots a} \beta_{1a} S_{1 \dots a} \dots \dots (5.7)$$

$$\langle A_a^* | = \langle \begin{matrix} 1 \dots a \\ a \end{matrix} | = |A_a^* \rangle = \left| \begin{matrix} 1 \dots a \\ a \end{matrix} \right\rangle = 2\{(a-1)/a\} \beta_{1a} S_{1 \dots a} \beta_{1a} \dots \dots (5.8)$$

these being tableau operators for the representations $[a-1, 1]$ and $[2, a-2]$ respectively of the symmetric group S_a .

6. "Diagonal" bracket and ket-bra tableau operators

Compound bracket and ket-bra tableau operators are defined as simple products of the corresponding bra and ket operators. Included in our main theorem is the statement that the "diagonal" Young operator O_{aa}^n for $[n-1, 1]$ is equal to the bracket tableau operator $\langle A_n | A_n \rangle$, and similarly for $[2, n-2]$ $O_{a^*a^*}^n$ is equal to $\langle A_n^* | A_n^* \rangle$. Thus from (5.1) - (5.4)

$$O_{aa}^n = \langle A_n | A_n \rangle = \left\langle \begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \left| \begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \right\rangle$$

$$= 4\{(a-1)(n-1)/(an)\} S_{1\dots a} R_{1a} S_{1\dots a} \dots R_{1a} S_{1\dots a} \dots \dots (6.1)$$

$$O_{a^*a^*}^n = \langle A_n^* | A_n^* \rangle = \langle 1a^* \dots a^* \dots n | 1a^* \dots a^* \dots n \rangle$$

$$= 4\{(a-1)(n-1)/(an)\} R_{1\dots a} S_{1a} R_{1\dots a} \dots R_{1a} S_{1\dots a} \dots \dots (6.2)$$

Putting $n=a$ in (6.1) and (6.2) and using the reduction formula (4.3) and the analogous one with S and A interchanged, we find

$$\begin{aligned} O_{aa}^a &= \langle A_a | A_a \rangle = \left\langle \begin{matrix} 1 \dots a \\ a \end{matrix} \middle| \begin{matrix} 1 \dots a \\ a \end{matrix} \right\rangle = 2\{(a-1)/a\} S_{1\dots a} R_{1a} S_{1\dots a} \\ &= \langle A_a | = | A_a \rangle = | A_a \rangle \langle A_a | , \end{aligned} \dots (6.3)$$

$$\begin{aligned} O_{a^*a^*}^a &= \langle A_a^* | A_a^* \rangle = \langle 1a^* \dots a^* | 1a^* \dots a^* \rangle = 2\{(a-1)/a\} R_{1\dots a} S_{1a} R_{1\dots a} \\ &= \langle A_a^* | = | A_a^* \rangle = | A_a^* \rangle \langle A_a^* | . \end{aligned} \dots (6.4)$$

Here the equality of the ket-bra tableau operator $|A_a\rangle\langle A_a|$ with the bracket tableau operator $\langle A_a | A_a \rangle$ is a consequence of $\langle A_a |$ and $|A_a\rangle$ being equal and therefore commuting. The same applies to the operators obtained by putting $a = n$ (and $A = N$):

$$\begin{aligned} O_{nn}^n &= \langle N_n | N_n \rangle = \left\langle \begin{matrix} 1 \dots n \\ n \end{matrix} \middle| \begin{matrix} 1 \dots n \\ n \end{matrix} \right\rangle = 2\{(n-1)/n\} S_{1\dots n} R_{1n} S_{1\dots n} \\ &= \langle N_n | = | N_n \rangle = | N_n \rangle \langle N_n | , \end{aligned} \dots (6.5)$$

$$\begin{aligned} O_{n^*n^*}^n &= \langle N_n^* | N_n^* \rangle = \langle 1n^* \dots n^* | 1n^* \dots n^* \rangle = 2\{(n-1)/n\} R_{1\dots n} S_{1n} R_{1\dots n} \\ &= \langle N_n^* | = | N_n^* \rangle = | N_n^* \rangle \langle N_n^* | . \end{aligned} \dots (6.6)$$

In the general case $\langle A_n |$ and $|A_n\rangle$ are neither equal nor commute, and we must distinguish the ket-bra tableau operator $|A_n\rangle\langle A_n|$ from

the bracket tableau operator $\langle A_n | A_n \rangle = O_{aa}^n$. We use a tilde \sim to make the distinction and find

$$\begin{aligned} \tilde{O}_{aa}^n &= |A_n\rangle \langle A_n| = \left| \begin{array}{c} 1 \dots \dot{a} \dots n \\ a \end{array} \right\rangle \left\langle \begin{array}{c} 1 \dots \dot{a} \dots n \\ a \end{array} \right| \\ &= 4\{(a-1)(n-1)/(an)\} S_{1 \dots \dot{a} \dots n} R_{1a} S_{1 \dots \dot{a} \dots n} R_{1a} S_{1 \dots \dot{a} \dots n} \\ &= 4\{(a-1)(n-1)/(an)\} S_{1 \dots \dot{a} \dots n} R_{1a} S_{1 \dots \dot{a} \dots n} R_{1a} S_{1 \dots \dot{a} \dots n} \\ &= 2\{(n-1)/n\} S_{1 \dots \dot{a} \dots n} R_{1a} S_{1 \dots \dot{a} \dots n} \dots (6.7) \end{aligned}$$

using (4.1). Similarly, for $[21^{n-2}]$,

$$\begin{aligned} \tilde{O}_{a^*a^*}^n &= |A_n^*\rangle \langle A_n^*| = |1a \dots \dot{a} \dots n\rangle \langle 1a \dots \dot{a} \dots n| \\ &= 2\{(n-1)/n\} R_{1a} S_{1 \dots \dot{a} \dots n} R_{1a} S_{1 \dots \dot{a} \dots n} \dots (6.8) \end{aligned}$$

We tabulate some properties of the "diagonal" operators O_{aa}^n and \tilde{O}_{aa}^n for $[n-1, \dot{1}]$. (Similarly results hold for the operators of $[21^{n-2}]$ with appropriate stars introduced).

$$(i) \quad O_{aa}^n \langle A_n | = \langle A_n | A_n \rangle \langle A_n | = \langle A_n | \tilde{O}_{aa}^n = \langle A_n | \dots (6.9)$$

$$(ii) \quad |A_n\rangle O_{aa}^n = |A_n\rangle \langle A_n | A_n \rangle = \tilde{O}_{aa}^n |A_n\rangle = |A_n\rangle \dots (6.10)$$

$$(iii) \quad |A_n\rangle O_{aa}^n \langle A_n | = |A_n\rangle \langle A_n | A_n \rangle \langle A_n | = \tilde{O}_{aa}^n \tilde{O}_{aa}^n = \tilde{O}_{aa}^n \dots (6.11)$$

$$(iv) \quad \langle A_n | \tilde{O}_{aa}^n |A_n\rangle = \langle A_n | A_n \rangle \langle A_n | A_n \rangle = O_{aa}^n O_{aa}^n = O_{aa}^n \dots (6.12)$$

Since \tilde{O}_{aa}^n has a simpler explicit expression than O_{aa}^n , we verify the last relations in (6.9) and (6.10) from which the rest follows. We find

$$\begin{aligned}
 \langle A_n | \tilde{O}_{aa}^n &= 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots a} A_{1a} S_{1\dots a} \dots R_{1a} S_{1\dots a} \dots n \\
 &\quad \times 2\{(n-1)/n\} \\
 &= 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots a} A_{1a} S_{1\dots a} \dots n = \langle A_n | ,
 \end{aligned}$$

(6.13)

making use of (4.2) with $b = n$. Similarly

$$\begin{aligned}
 \tilde{O}_{aa}^n | A_n \rangle &= 2\{(n-1)/n\} \times 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots a} \dots n A_{1a} S_{1\dots a} \dots n A_{1a} S_{1\dots a} \dots a \\
 &= 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots a} \dots n A_{1a} S_{1\dots a} \dots a = | A_n \rangle ,
 \end{aligned}$$

(6.14)

using (4.4) with $b = n$.

These relations may be summarised in the statements that the ket $|A_n$

is a left-hand eigenstate of O_{aa}^n

and a right-hand eigenstate of \tilde{O}_{aa}^n

whilst the

bra $\langle A_n |$ is a right-hand eigenstate of O_{aa}^n

and a left-hand eigenstate of \tilde{O}_{aa}^n ,

the eigenvalue being +1 in all cases. Further

O_{aa}^n and \tilde{O}_{aa}^n are idempotents.

It is clear that similar relations hold for the starred operators of $[2i^{n-2}]$.

$$7. \quad \underline{O_{nn}^n = S_{1\dots n} - S_{1\dots n}, \quad O_{n^*n^*}^n = R_{1\dots n} - R_{1\dots n}}$$

Using (3.10)

$$\begin{aligned}
 S_{1\dots a} - S_{1\dots n} &= S_{1\dots a} (I - (I + P_{1n} + P_{2n} + \dots + P_{n-1,n})/n) S_{1\dots a} \\
 &= (1/n) S_{1\dots a} \{(n-1)I - (P_{1n} + P_{2n} + \dots + P_{n-1,n})\} S_{1\dots a}
 \end{aligned}$$

$$= (2/n) S_{1\dots\dot{a}} (A_{1n} + A_{2n} + \dots + A_{n-1,n}) S_{1\dots\dot{n}} \dots (7.1)$$

Now, for $2 \leq a \leq n-1$,

$$\begin{aligned} & S_{1\dots a\dots n} A_{an} S_{1\dots a\dots n} \\ &= S_{1\dots a\dots n} P_{1a} P_{an}^{-1} S_{1\dots a\dots n} \\ &= S_{1\dots\dot{a}} A_{1n} S_{1\dots\dot{n}} \end{aligned}$$

Hence

$$\begin{aligned} S_{1\dots\dot{a}} - S_{1\dots n} &= 2(n-1)/n S_{1\dots\dot{n}} A_{1n} S_{1\dots\dot{n}} \\ &= 0_{nn}^n \dots (7.3) \end{aligned}$$

by (6.5).

Similarly

$$\begin{aligned} A_{1\dots\dot{a}} - A_{1\dots n} &= A_{1\dots\dot{a}} [I - (1 - P_{1n}^{-P} - P_{2n}^{-P} - \dots - P_{n-1,n}^{-P})/n] A_{1\dots\dot{a}} \\ &= (1/n) A_{1\dots\dot{a}} ((n-1)I + (P_{1n} + P_{2n} + \dots + P_{n-1,n})) A_{1\dots\dot{a}} \\ &= (2/n) A_{1\dots\dot{a}} (S_{1n} + S_{2n} + \dots + S_{n-1,n}) A_{1\dots\dot{a}} \\ &= (2(n-1)/n) A_{1\dots\dot{a}} S_{1n} A_{1\dots\dot{a}} = 0_{n^*n^*}^n \dots (7.4) \end{aligned}$$

by (6.6), since

$$\begin{aligned} & A_{1\dots a\dots n} S_{an} A_{1\dots a\dots n} \\ &= A_{1\dots\dot{a}} P_{1a} S_{an} P_{1a}^{-1} A_{1\dots\dot{n}} = A_{1\dots\dot{a}} S_{1n} A_{1\dots\dot{n}} \quad (a = 2, 3, \dots, n-1). \end{aligned} \dots (7.5)$$

$$8. \quad \underline{|A_n\rangle \langle B_n| = \delta_{ab} \bar{\sigma}_{aa}^n, \quad |A_n^*\rangle \langle B_n^*| = \delta_{ab} \bar{\sigma}_{a^*a^*}^n}$$

With

$$|A_n\rangle = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} A_{1a} S_{1\dots\dot{a}} \dots (8.1)$$

$$\langle B_n| = 2\{(b-1)(n-1)/(bn)\}^{\frac{1}{2}} S_{1\dots\dot{b}\dots n} A_{1b} S_{1\dots\dot{b}} \dots (8.2)$$

into the standard tableau

$$A_n = \begin{matrix} 1 & 2 & \dots & a-1 & a+1 & a+2 & \dots & n-1 & n \\ a & & & & & & & & \end{matrix} \dots (9.2)$$

by

$$(A_n | P | N_n) = P_{(a, a+1, \dots, n)} \dots (9.3)$$

and the inverse permutation, converting A_n into N_n by

$$(N_n | P | A_n) = P_{(n, n-1, \dots, a)} \dots (9.4)$$

The permutation which converts

$$B_n = \begin{matrix} 1 & \dots & a & \dots & b & \dots & n \\ b & & & & & & \end{matrix} \text{ into } A_n = \begin{matrix} 1 & \dots & a & \dots & n \\ a & & & & \end{matrix}$$

may then be written as

$$\begin{aligned} (A_n | P | B_n) &= (A_n | P | N_n) (N_n | P | B_n) \\ &= P_{(a, a+1, \dots, n)} P_{(n, n-1, \dots, b)} \dots (9.5) \end{aligned}$$

This may be evaluated for the two cases $a < b$ and $a > b$ as follows

$$\begin{aligned} \text{If } a < b \quad P_{(a, a+1, \dots, n)} &= P_{(a, a+1, \dots, b)} P_{(b, b+1, \dots, n)} \\ &= P_{(a, a+1, \dots, b)} P_{(n, n-1, \dots, b)}^{-1} \dots (9.6) \end{aligned}$$

so that

$$(A_n | P | B_n)_{a < b} = P_{(a, a+1, \dots, b)} \dots (9.7)$$

$$\begin{aligned} \text{If } a > b \quad P_{(n, n-1, \dots, b)} &= P_{(n, n-1, \dots, a)} P_{(a, a-1, \dots, b)} \\ &= P_{(a, a+1, \dots, n)}^{-1} P_{(a, a-1, \dots, b)} \dots (9.8) \end{aligned}$$

so that

$$(A_n | P | B_n)_{a > b} = P_{(a, a-1, \dots, b)} \dots (9.9)$$

It is easy to verify this directly from the form the tableaux A_n, B_n have in the two cases:

$$\underline{a < b} \quad B_n = \begin{matrix} 1 & 2 & \dots & a-1 & a & a+1 & \dots & b-1 & b & b+1 & \dots & n \end{matrix} \quad \dots \quad (9.10)$$

$$A_n = \begin{matrix} 1 & 2 & \dots & a-1 & a+1 & a+2 & \dots & b & b+1 & \dots & n \end{matrix} \quad \dots \quad (9.11)$$

$$(A_n | P | B_n) = P_{(a, a+1, \dots, b)} \quad \dots \quad (9.12)$$

$$\underline{a > b} \quad B_n = \begin{matrix} 1 & 2 & \dots & b-1 & b & b+1 & \dots & a-1 & a & a+1 & \dots & n \end{matrix} \quad \dots \quad (9.13)$$

$$A_n = \begin{matrix} 1 & 2 & \dots & b-1 & b & \dots & a-2 & a-1 & a+1 & \dots & n \end{matrix} \quad \dots \quad (9.14)$$

$$(A_n | P | B_n) = P_{(a, a-1, \dots, b)} \quad \dots \quad (9.15)$$

$$10. \quad \underline{O_{ab}^n = \langle A_n | (A_n | P | B_n) | B_n \rangle \text{ implies } O_{an}^n = \langle A_n | P_{(a, a+1, \dots, n)} | B_n \rangle}$$

$$\underline{O_{nb}^n = P_{(n, n-1, \dots, b)} | B_n \rangle}$$

$$O_{a^*b^*}^n = \langle A_n^* | (A_n | P | B_n) | B_n^* \rangle \text{ implies } O_{a^*n^*}^n = \langle A_n^* | P_{(a, a+1, \dots, n)} | B_n^* \rangle$$

$$\underline{O_{n^*b^*}^n = P_{(n, n-1, \dots, b)} | B_n^* \rangle}$$

Since from (6.5), (6.7)

$$|N_n\rangle \langle N_n| = 2\{(n-1)/n\} S_1 \dots S_n R_{1n} S_1 \dots S_n \quad \dots \quad (10.1)$$

$$|A_n\rangle \langle A_n| = 2\{(n-1)/n\} S_1 \dots S_n R_{1a} S_1 \dots S_n \quad \dots \quad (10.2)$$

$$|B_n\rangle \langle B_n| = 2\{(n-1)/n\} S_1 \dots S_n R_{1b} S_1 \dots S_n \quad \dots \quad (10.3)$$

with the same coefficient $2\{(n-1)/n\}$ it follows

$$P_{(a, a+1, \dots, n)} |N_n\rangle \langle N_n| P_{(a, a+1, \dots, n)}^{-1} = |A_n\rangle \langle A_n| \quad \dots \quad (10.4)$$

or, equivalently,

$$P_{(n,n-1,\dots,b)}^{-1} |N_n\rangle \langle N_n| P_{(n,n-1,\dots,b)} = |B_n\rangle \langle B_n| \dots (10.5)$$

[Note that the equality of the coefficient $2(n-1)/n$ in (10.1), (10.2), (10.3) is needed for (10.4), (10.5) to hold, since transformation by P or by P^{-1} affects only the symmetriser and antisymmetrisers: it is a pitfall to expect the transformation to change coefficients!].

It follows

$$\begin{aligned} \langle A_n | P | N_n \rangle |N_n\rangle \langle N_n| &= P_{(a,a+1,\dots,n)} |N_n\rangle \langle N_n| \\ &= P_{(a,a+1,\dots,n)} |N_n\rangle \langle N_n| P_{(a,a+1,\dots,n)}^{-1} P_{(a,a+1,\dots,n)} \\ &= |A_n\rangle \langle A_n| P_{(a,a+1,\dots,n)} \dots (10.6) \end{aligned}$$

$$\begin{aligned} |N_n\rangle \langle N_n| \langle N_n | P | B_n \rangle &= |N_n\rangle \langle N_n| P_{(n,n-1,\dots,b)} \\ &= P_{(n,n-1,\dots,b)} |P_{(n,n-1,\dots,b)}^{-1} |N_n\rangle \langle N_n| P_{(n,n-1,\dots,b)} \\ &= P_{(n,n-1,\dots,b)} |B_n\rangle \langle B_n| \dots (10.7) \end{aligned}$$

Using then the relation (from (6.5))

$$|N_n\rangle = |N_n\rangle \langle N_n| = \langle N_n| \dots (10.8)$$

it follows (using the statement of the theorem to follow)

$$O_{an}^n = \langle A_n | \langle A_n | P | N_n \rangle |N_n\rangle = \langle A_n | A_n \rangle \langle A_n | P_{(a,a+1,\dots,n)} = \langle A_n | P_{(a,a+1,\dots,n)} \dots (10.9)$$

$$O_{nb}^n = \langle N_n | \langle N_n | P | B_n \rangle |B_n\rangle = P_{(n,n-1,\dots,b)} |B_n\rangle \langle B_n| B_n \rangle = P_{(n,n-1,\dots,b)} |B_n\rangle \dots (10.10)$$

The starred relations are established in an identical manner.

11. Statement of theorem

For the representation $[\bar{n-1}, 1]$ of S_n in standard orthogonal form the Young operators are given by

$$O_{an}^n = \langle A_n | P_{(a, a+1, \dots, n)} \rangle, \quad O_{ab}^n = \langle A_n | (A_n | P | B_n) | B_n \rangle, \\ O_{nb}^n = P_{(n, n-1, \dots, b)} | B_n \rangle, \\ (a, b=2, 3, \dots, n), \quad \dots (11.1)$$

$$\langle A_n | = \left\langle \begin{matrix} 1 \dots \dot{a} \dots n \\ a \end{matrix} \right| = 2\{(a-1)(n-1)/(an)\}^{1/2} S_1 \dots \dot{a} A_a S_1 \dots \dot{a} \dots n, \\ \dots (11.2)$$

$$| B_n \rangle = \left| \begin{matrix} 1 \dots \dot{b} \dots n \\ b \end{matrix} \right\rangle = 2\{(b-1)(n-1)/(bn)\}^{1/2} S_1 \dots \dot{b} \dots n A_b S_1 \dots \dot{b}, \\ \dots (11.3)$$

$$(A_n | P | B_n) = P_{(a, a+1, \dots, n)} P_{(n, n-1, \dots, b)} = P_{(a, a+1, \dots, b), (a < b)} \\ = P_{(a, a-1, \dots, b), (a > b)} \dots (11.4)$$

For the representation $[\bar{2}^{n-2}]$ of S_n in standard orthogonal form the Young operators are given by

$$O_{a^*n^*}^n = \langle A_n^* | P_{(a, a+1, \dots, n)} \rangle, \quad O_{a^*b^*}^n = \langle A_n^* | (A_n^* | P | B_n^*) | B_n^* \rangle, \\ O_{n^*b^*}^n = P_{(n, n-1, \dots, b)} | B_n^* \rangle, \\ (a, b=2, 3, \dots, n), \quad \dots (11.5)$$

$$\langle A_n^* | = \left\langle \left(\begin{matrix} 1 \dots \dot{a} \dots n \\ a \end{matrix} \right)^* \right| = \langle 1 a^* \dots \dot{a} \dots n | = 2\{(a-1)(n-1)/(an)\} \\ A_1 \dots \dot{a} S_a A_1 \dots \dot{a} \dots n, \\ \dots (11.6) \\ (3)$$

$$|B_n^k\rangle = \left| \left(\begin{matrix} 1 \dots \dot{b} \dots n \\ b \end{matrix} \right)^k \right\rangle = |1b \dots \dot{b} \dots n\rangle = 2\{(b-1)(n-1)/(bn)\} \\ \mathcal{A}_{1 \dots \dot{b} \dots n} \mathcal{S}_{1b} \mathcal{A}_{1 \dots \dot{b} \dots n} \dots (11.7)$$

These expressions satisfy the relations

$$O_{ab}^n O_{cd}^n = \delta_{bc} O_{ad}^n, \dots (11.8)$$

$$O_{a^*b^*}^n O_{c^*d^*}^n = \delta_{bc} O_{a^*d^*}^n, \dots (11.9)$$

$$P_{a,a+1} O_{ab}^n = (1/a) O_{ab}^n + \{(a^2-1)^{1/2}/a\} O_{a+1,b}^n, \dots (11.10)$$

$$O_{ab}^n P_{b,b+1} = (1/b) O_{ab}^n + \{(b^2-1)^{1/2}/b\} O_{a,b+1}^n, \dots (11.11)$$

$$P_{a,a+1} O_{a^*b^*}^n = -(1/a) O_{a^*b^*}^n + \{(a^2-1)^{1/2}/a\} O_{a+1^*b^*}^n, \dots (11.12)$$

$$O_{a^*b^*}^n P_{b,b+1} = -(1/b) O_{a^*b^*}^n + \{(b^2-1)^{1/2}/b\} O_{a^*b+1}^n, \dots (11.13)$$

12. Proof that $O_{ab}^n O_{cd}^n = \delta_{bc} O_{ad}^n$ (and $O_{a^*b^*}^n O_{c^*d^*}^n = \delta_{bc} O_{a^*d^*}^n$)

We have already shown (8.8) that

$$|B_n\rangle \langle C_n| = \delta_{bc} |B_n\rangle \langle B_n| \dots (12.1)$$

It remains to show

$$O_{ab}^n O_{bc}^n = O_{bc}^n, \dots (12.2)$$

$$\text{L.H.S.} = \langle A_n | (A_n | P | B_n) | B_n \rangle \langle B_n | (B_n | P | C_n) | C_n \rangle \dots (12.3)$$

$$\text{R.H.S.} = \langle A_n | (A_n | P | C_n) | C_n \rangle \dots (12.4)$$

We have shown (6.7) that

$$|B_n\rangle\langle B_n| = 2\{(n-1)/n\} S_{1\dots\dot{b}\dots n} \mathcal{A}_{1b} S_{1\dots\dot{b}\dots n} \cdot \dots \quad (12.5)$$

Hence with

$$\langle A_n| = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} \mathcal{A}_{1a} S_{1\dots\dot{a}\dots n} \cdot \dots \quad (12.6)$$

$$|C_n\rangle = 2\{(c-1)(n-1)/(cn)\}^{\frac{1}{2}} S_{1\dots\dot{c}\dots n} \mathcal{A}_{1c} S_{1\dots\dot{c}\dots n} \cdot \dots \quad (12.7)$$

and taking, for simplicity, $a < b < c$ we have

$$\begin{aligned} \text{L.H.S.} &= 8\{(n-1)/n\}^2\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} \mathcal{A}_{1a} S_{1\dots\dot{a}\dots n} \\ &\quad \times P_{(a,a+1,\dots,b)} S_{1\dots\dot{b}\dots n} \mathcal{A}_{1b} S_{1\dots\dot{b}\dots n} P_{(b,b+1,\dots,c)} \\ &\quad \quad \quad S_{1\dots\dot{c}\dots n} \mathcal{A}_{1c} S_{1\dots\dot{c}\dots n} \\ &\quad \quad \quad \dots \quad (12.8) \end{aligned}$$

$$\begin{aligned} &= 8\{(n-1)/n\}^2\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} P_{(a,a+1,\dots,b)} \\ &\quad \times \mathcal{A}_{1b} S_{1\dots\dot{b}\dots n} \mathcal{A}_{1b} S_{1\dots\dot{b}\dots n} \mathcal{A}_{1b} P_{(b,b-1,\dots,c)} S_{1\dots\dot{c}\dots n} \\ &\quad \quad \quad \dots \quad (12.9) \end{aligned}$$

$$\begin{aligned} &= 4\{(n-1)/n\}\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} P_{(a,a+1,\dots,b)} \\ &\quad \times \mathcal{A}_{1b} S_{1\dots\dot{b}\dots n} \mathcal{A}_{1b} P_{(b,b+1,\dots,c)} S_{1\dots\dot{c}\dots n} \cdot \dots \quad (12.10) \end{aligned}$$

Using (4.2) with a replaced by b and b by n ,

$$\begin{aligned} \text{R.H.S.} &= 4\{(n-1)/n\}\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} \mathcal{A}_{1a} S_{1\dots\dot{a}\dots n} \\ &\quad \times P_{(a,a+1,\dots,b)} P_{(b,b+1,\dots,c)} S_{1\dots\dot{c}\dots n} \mathcal{A}_{1c} S_{1\dots\dot{c}\dots n} \\ &\quad \quad \quad \dots \quad (12.11) \end{aligned}$$

$$\begin{aligned} &= 4\{(n-1)/n\}\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}} S_{1\dots\dot{a}\dots n} P_{(a,a+1,\dots,b)} \\ &\quad \mathcal{A}_{1b} S_{1\dots\dot{b}\dots n} \mathcal{A}_{1b} P_{(b,b+1,\dots,c)} S_{1\dots\dot{c}\dots n} = \text{L.H.S.} \\ &\quad \quad \quad \dots \quad (12.12) \end{aligned}$$

The proof is similar with the restriction $a < b < c$ removed, thus (11.8) is proved and (11.9) by interchanging S and \mathcal{H} throughout.

$$13. \frac{P_{a,a+1} O_{an}^n = (1/a)O_{an}^n + ((a^2-1)^{1/2}/a)O_{a+1,n}^n, P_{a,a+1} O_{a^*n^*}^n = -(1/a)O_{a^*b^*}^n + \frac{((a^2-1)^{1/2}/a)O_{a+1^*n^*}^n}{((a^2-1)^{1/2}/a)O_{n^*a+1^*}^n}}$$

$$\frac{O_{na}^n P_{a,a+1} = (1/a)O_{na}^n + ((a^2-1)^{1/2}/a)O_{n,a+1}^n, O_{n^*a^*}^n P_{a,a+1} = -(1/a)O_{n^*a^*}^n + \frac{((a^2-1)^{1/2}/a)O_{n^*a+1^*}^n}{((a^2-1)^{1/2}/a)O_{n^*a+1^*}^n}}$$

From (10.9) we have

$$O_{an}^n = \langle A_n | P_{(a,a+1,a+2,\dots,n)} = \langle A_n | P_{a,a+1} P_{(a+1,a+2,\dots,n)} \dots (13.1)$$

$$O_{a+1,n}^n = \langle (A+1)_n | P_{(a+1,a+2,\dots,n)} \dots (13.2)$$

Hence

$$P_{a,a+1} O_{an}^n = (1/a) O_{an}^n + ((a^2-1)^{1/2}/a) O_{a+1,n}^n \dots (13.3)$$

requires (cancelling the common term $P_{(a+1,a+2,\dots,n)}$ on the right)

$$P_{a,a+1} \langle A_n | P_{a,a+1} = (1/a) \langle A_n | P_{a,a+1} + ((a^2-1)^{1/2}/a) \langle (A+1)_n | \dots (13.4)$$

With, from (5.1),

$$\langle A_n | = 2((a-1)(n-1)/(an))^{1/2} S_{1,\dots,a} \mathcal{H}_{a+1} S_{1,\dots,a,\dots,n} \dots (13.5)$$

we deduce, by some cancellation,

$$((a^2-1)^{1/2}/a) \langle (A+1)_n | = 2((a-1)(n-1)/(an))^{1/2} S_{1,\dots,a+1} \mathcal{H}_{a+1} S_{1,\dots,a+1,\dots,n} \dots (13.6)$$

and also, noting that $S_{1\dots a} = S_{1\dots a-1}$ is independent of a and $a+1$

$$P_{a,a+1} \langle A_n \rangle_{a,a+1}^P = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\dots a} A_{1a+1} S_{1\dots a+1\dots n} \dots (13.7)$$

Hence, removing the common factor $2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}}$ equation (13.4) requires

$$S_{1\dots a} A_{1a+1} S_{1\dots a+1\dots n} = (1/a) S_{1\dots a} A_a S_{1\dots a\dots n} P_{a,a+1} + S_{1\dots a+1} A_{1a+1} S_{1\dots a+1\dots n} \dots (13.8)$$

Now from (7.3) with $n = a$ we have

$$S_{1\dots a} - S_{1\dots a+1} = 2\{(a-1)/a\} S_{1\dots a} A_{1a} S_{1\dots a} \dots (13.9)$$

Hence it remains to show (multiplying through by a)

$$2(a-1) S_{1\dots a} A_{1a} S_{1\dots a} A_{1a+1} S_{1\dots a+1\dots n} = S_{1\dots a} A_a S_{1\dots a\dots n} P_{a,a+1} \dots (13.10)$$

Cancelling the common factor $S_{1\dots a}$ on the left, putting

$$2A_{1a} = I - P_{1a}, \quad (a-1) S_{1\dots a} = (I + P_{12} + P_{13} + \dots + P_{1,a-1}) S_{2\dots a} \dots (13.10)$$

and writing

$$A_{1a} S_{1\dots a\dots n} P_{a,a+1} = P_{a,a+1} A_{1a+1} S_{1\dots a+1\dots n} \dots (13.12)$$

it remains to show

$$(I - P_{1a})(I + P_{12} + P_{13} + \dots + P_{1,a-1}) S_{2\dots a} A_{1a+1} S_{1\dots a+1\dots n} = P_{a,a+1} A_{1a+1} S_{1\dots a+1\dots n} \dots (13.13)$$

Now, by (3.1),

$$S_{2\dots a} A_{1a+1} S_{1\dots a+1\dots n} = A_{1a+1} S_{2\dots a} S_{1\dots a+1\dots n} = A_{1a+1} S_{1\dots a+1\dots n} \dots (13.14)$$

Further

$$\begin{aligned}
 (I-P_{1a})^p A_{1a} S_{1 \dots a+1 \dots n} &= (I-P_{1a})^p A_{1a} S_{1 \dots a+1 \dots n} \\
 &= (I-P_{1a})^p A_{\tau a+1} S_{1 \dots a+1 \dots n} \quad \text{for } \tau = 2, 3, \dots, a-1.
 \end{aligned}
 \tag{13.15}$$

The since

$$\begin{aligned}
 P_{1a} A_{\tau a+1} S_{1 \dots a+1 \dots n} &= A_{\tau a+1} P_{1a} S_{1 \dots a+1 \dots n} \\
 &= A_{\tau a+1} S_{1 \dots a+1 \dots n} \quad \dots \tag{13.16}
 \end{aligned}$$

it follows

$$(I-P_{1a})^{(P_{12}+P_{13}+\dots+P_{1,a-1})} A_{1a} S_{1 \dots a+1 \dots n} = 0$$

and it remains to show that

$$(I-P_{1a}^{-P_{a,a+1}}) A_{1a} S_{1 \dots a+1 \dots n} = 0 \quad \dots \tag{13.17}$$

or $A_{1,a,a+1} S_{1 \dots a+1 \dots n} = 0$

or $A_{1,a,a+1} A_{1a} S_{1a} S_{1 \dots a+1 \dots n} = 0, \quad \dots \tag{13.18}$

which is true and establishes the required result.

The relation

$$O_{na}^n P_{a,a+1} = (1/a) O_{na}^n + \{(a^2-1)^{1/2}/a\} O_{n,a+1}^n \quad \dots \tag{13.19}$$

is established in the same way with multiplication from the right.

The starred relations

$$P_{a,a+1} O_{a^*n^*}^n = -(1/a) O_{a^*n^*}^n + \{(a^2-1)^{1/2}/a\} O_{a+1^*n^*}^n \quad \dots \tag{13.20}$$

$$O_{n^*a^*}^n P_{a,a+1} = -(1/a) O_{n^*a^*}^n + \{(a^2-1)^{1/2}/a\} O_{n^*a+1^*}^n \quad \dots \tag{13.21}$$

are also established in a similar manner (with : and $\bar{\cdot}$ interchanged throughout, with the minus sign in front of $1/a$ being required because the final relation is

$$(I+P_{1a}) S_{1a+1} R_{1\dots a+1\dots n} = - P_{a,a+1} S_{1a+1} R_{1\dots a+1\dots n} \quad \dots (13.22)$$

reducing to

$$S_{1,a,a+1} \underline{S_{1a} R_{1a}} A_{1\dots a+1\dots n} = 0 \quad \dots (13.23)$$

which is true as before.

Multiplication of (13.3) on the right by O_{nb}^n leads to the general relation (11.10). The other general relations (11.11) - (11.13) may be established in the same way from the corresponding special cases established above.

14. Results in tabular form

Table 1. Young operators $\left\{ \begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \middle| 0 \right\} \begin{matrix} 1 \dots b \dots n \\ b \end{matrix}$ for $[n-1, 1]$

$a < n$	$2\sqrt{\left\{ \begin{matrix} (a-1)(n-1) \\ an \end{matrix} \right\}} S_{1 \dots a} A_{1a} S_{1 \dots a \dots n} \left(\begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \right)$	$\left\{ \begin{matrix} 1 \dots b \dots n \\ b \end{matrix} \right\} R_{1b} S_{1 \dots b} = 2\sqrt{\left\{ \begin{matrix} (b-1)(n-1) \\ bn \end{matrix} \right\}}$	$b < n$
$a = n$	$\sqrt{\left\{ \begin{matrix} 2(n-1) \\ n \end{matrix} \right\}} S_{1 \dots n} A_{1n} \left(\begin{matrix} 1 \dots n \\ n \end{matrix} \right)$	$\left\{ \begin{matrix} 1 \dots n \\ n \end{matrix} \right\}$	$b = n$
$a < n$	$\sqrt{\left\{ \begin{matrix} 2(n-1) \\ n \end{matrix} \right\}} S_{1 \dots n} A_{1n} \left(\begin{matrix} 1 \dots n \\ n \end{matrix} \right)$	$\left\{ \begin{matrix} 1 \dots b \dots n \\ b \end{matrix} \right\} S_{1 \dots b} \sqrt{\left\{ \begin{matrix} 2(b-1) \\ b \end{matrix} \right\}}$	$b < n$
$a = n$	$\sqrt{\left\{ \begin{matrix} 2(n-1) \\ n \end{matrix} \right\}} S_{1 \dots n} \sqrt{\left\{ \begin{matrix} 2(n-1) \\ n \end{matrix} \right\}}$	$\left\{ \begin{matrix} 1 \dots n \\ n \end{matrix} \right\} S_{1 \dots n} \sqrt{\left\{ \begin{matrix} 2(n-1) \\ n \end{matrix} \right\}}$	$b = n$
e.g. $O_{ab}^n = \left(\begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \middle 0 \right) \begin{matrix} 1 \dots b \dots n \\ b \end{matrix}$	$= 4 \left(\frac{n-1}{n} \right) \sqrt{\left\{ \begin{matrix} (a-1)(b-1) \\ ab \end{matrix} \right\}} S_{1 \dots a} A_{1a} S_{1 \dots a \dots n} \left(\begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \right) \begin{matrix} 1 \dots b \dots n \\ b \end{matrix}$	$R_{1b} S_{1 \dots b}$	
$O_{cn}^n = \left(\begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \middle 0 \right) \begin{matrix} 1 \dots n \\ n \end{matrix}$	$= 2 \sqrt{\left\{ \begin{matrix} (a-1)(n-1) \\ an \end{matrix} \right\}} S_{1 \dots a} A_{1a} S_{1 \dots a \dots n} \left(\begin{matrix} 1 \dots a \dots n \\ a \end{matrix} \right) \begin{matrix} 1 \dots n \\ n \end{matrix}$		
$O_{nb}^n = \left(\begin{matrix} 1 \dots n \\ n \end{matrix} \middle 0 \right) \begin{matrix} 1 \dots b \dots n \\ b \end{matrix}$	$= 2 \sqrt{\left\{ \begin{matrix} (b-1)(n-1) \\ bn \end{matrix} \right\}} S_{1 \dots n} A_{1n} \left(\begin{matrix} 1 \dots n \\ n \end{matrix} \right) \begin{matrix} 1 \dots b \dots n \\ b \end{matrix}$	$S_{1 \dots b} = 2 \sqrt{\left\{ \begin{matrix} (b-1)(n-1) \\ bn \end{matrix} \right\}} \left(\begin{matrix} 1 \dots n \\ n \end{matrix} \right) \begin{matrix} 1 \dots b \dots n \\ b \end{matrix}$	
$O_{nn}^n = \left(\begin{matrix} 1 \dots n \\ n \end{matrix} \middle 0 \right) \begin{matrix} 1 \dots n \\ n \end{matrix}$	$= \frac{2(n-1)}{n} S_{1 \dots n} A_{1n} S_{1 \dots n}$	$\times S_{1 \dots b} \dots n A_{1b} S_{1 \dots b}$	

Table 2. Young operators

$$\begin{bmatrix} 1a \\ \vdots \\ 0 \\ \vdots \\ 1b \\ \vdots \\ n \end{bmatrix}$$

for $[2]^{n-2}$

$a < b$	$2 \sqrt{\frac{(a-1)(a-1)}{an}} \{ A_{1...a} S_{1a} A_{1...b} \}$	$\begin{bmatrix} 1a \\ \vdots \\ 0 \\ \vdots \\ 1b \\ \vdots \\ n \end{bmatrix}$	$\begin{bmatrix} 1b \\ \vdots \\ 2 \sqrt{\frac{(b-1)(n-1)}{bn}} \\ \vdots \\ 1n \end{bmatrix}$	$b < n$
$a = n$	$\sqrt{\frac{2(a-1)}{n}} \{ A_{1...a} S_{1a} \}$	$\begin{bmatrix} 1a \\ \vdots \\ 1 \\ \vdots \\ n \end{bmatrix}$	$\begin{bmatrix} 1n \\ \vdots \\ 2 \sqrt{\frac{2(b-1)}{b}} \\ \vdots \\ n \end{bmatrix}$	$b = n$
$a < b$	$4 \sqrt{\frac{(a-1)}{n}} \sqrt{\frac{(a-1)(b-1)}{ab}} \{ A_{1...a} S_{1a} A_{1...b} \}$	$\begin{bmatrix} 1a \\ \vdots \\ 0 \\ \vdots \\ 1b \\ \vdots \\ n \end{bmatrix}$	$\begin{bmatrix} 1b \\ \vdots \\ 2 \sqrt{\frac{2(b-1)}{b}} \\ \vdots \\ n \end{bmatrix}$	$b < n$
$a < b$	$2 \sqrt{\frac{(a-1)(n-1)}{2n}} \{ A_{1...a} S_{1a} A_{1...b} \}$	$\begin{bmatrix} 1a \\ \vdots \\ 0 \\ \vdots \\ 1b \\ \vdots \\ n \end{bmatrix}$	$\begin{bmatrix} 1n \\ \vdots \\ 2 \sqrt{\frac{2(n-1)}{n}} \\ \vdots \\ n \end{bmatrix}$	$b = n$

$$\begin{aligned} O_{a^2 b^2}^n &= (1a_1 \dots a_1 | 0 | 1b_1 \dots b_1) \{ A_{1...a} S_{1a} A_{1...b} \} \\ O_{a^2 n}^n &= (1a_1 \dots a_1 | 0 | 1n) \{ A_{1...a} S_{1a} \} \\ O_{n^2 a^2}^n &= (1a_1 \dots a_1 | 0 | 1b_1 \dots b_1) \{ A_{1...a} S_{1a} \} \\ O_{n^2 a^2 b^2}^n &= (1a_1 \dots a_1 | 0 | 1n) \{ A_{1...a} S_{1a} \} \end{aligned}$$