MOTION OF A CONDUCTING FLUID DUE TO AN INFINITE ROTATING DISK

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ABSTRACT

In the present paper, the unsteady motion of a weakly conducting, incompressible and viscous fluid due to non-uniform rotation of an infinite flat disk, is considered. The motion is subjected to a uniform external magnetic field normal to the disc and parallel to the axis of rotation. The equations of motion are written assuming low magnetic Reynold's number i.e. The external magnetic field is undisturbed during motion. The general solution is determined by the aid of Green's function, and is obtained in the form of a system of integral equations which can be solved by successive approximation. A special case of the solution, which corresponds to uniform rotation of the disk, is deduced and discussed.

INTRODUCTION

The flow of viscous, electrically conducting fluids in the neighbourhood of a rotating disk is of great practical importance, particularly in connection with rotary magnetohydrodynamic machines. Similar problem in classical hydrodynamics is studied by many authors[1]. The exact solution of the équation of motion of a viscous fluid around a flat infinite disk which rotates about an axis perpendicular to its plane with a uniform angular velocity was studied by Cochran[2], and the steady state solution was obtained by similarity. The motion due to a rotating disk in a fluid at rest, was also examined to study the transition of a three-dimensional boundary layer to turbulence[1], and it was proved that it becomes turbulent at large Reynold's numbers, $Re > 3x10^5$.

In the presence of an external axial magnetic field, the steady flow of an electrically conducting fluid was studied due to the rotation of an infinite cylinder.[3] Other problems of steady motion of conducting fluids due to rotating disks are described in many references[4,5]. Such problems are related to magnetohydrodynamic (MHD) generators and MHD vortex flow, and the solution was obtained mainly by numerical methods.

In the present work, is considered, the non-uniform rotation of an infinite thin disk through the origin, about the z-axis, in an incompressible, viscous and electrically conducting fluid. The fluid is assumed initially at rest, and is subjected to a uniform external magnetic field normal to the disk and parallel to the z-axis. Following the cylindrical-coordinates (r,θ,z), the case considered is a fully three dimensional flow i.e. there exists three components of the velocity of flow v_r , v_θ and v_z. The fluid layer near the disk is carried by it through friction and is thrown outwards owing to the action of centrifugal forces. This is compensated by particles which flow in an axial direction towards the disk to be in turn carried and ejected centrifugally.

FUNDAMENTAL EQUATIONS

Denoting the three components of velocity by $v_r = u(r,z,t)$, $v_\theta = v(r,z,t)$ and $v_z = w(r,z,t)$, and assuming low magnetic Reynold's number $Re_m < 1$, the equations of motion in cylindrical coordinates are written in the form(3,5);

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u - \frac{u}{r}\right) - m^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\Delta v - \frac{v}{r^2}\right) - m^2 v \quad .. (1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta w$$

and

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

where,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$m^2 = \text{const} = \dfrac{\sigma H_o^2}{\rho c^2}$, and H_o = constant external magnetic field.

The angular velocity of the rotation of the disk is a given function of time $w(t)$, so that the inital and boundary conditions can be represented in the form;

$$u = v = w = 0, \quad \text{at} \quad t = 0$$

$$u = 0 \ , \ v = r\ w(t), \quad w = 0 \quad \text{at} \quad z = 0 \ , \quad \ldots(2)$$

and $\quad u = v = 0. \quad \text{as} \quad z \longrightarrow \infty$

It is required to obtain the solution of system of equations (1), which satisfies the boundary and initial conditions (2). The solution is assumed in the form;

$$u = rf(z,t) \ , \ v = r\ g(z,t) \ , \ w = 2\psi(z,t) \ ,$$

$$P = P(z,t) \qquad \ldots \quad (3)$$

Substituting from (3), the system of equations (1) is reduced to the form;

$$\nu \frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial t} - m^2 f = 2\psi \frac{\partial f}{\partial z} + f^2 - g^2 ,$$

$$\nu \frac{\partial^2 g}{\partial z^2} - \frac{\partial g}{\partial t} - m^2 g = 2\psi \frac{\partial g}{\partial z} + 2 f g , \qquad \ldots \quad (4)$$

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial \psi}{\partial t} = 2\psi \frac{\partial \psi}{\partial z} + \frac{1}{2\rho}\frac{\partial P}{\partial z} ,$$

and

$$\frac{\partial \psi}{\partial z} = - f$$

The system of equations (4) is solved with the following boundary and initial conditions;

$$\text{at} \quad t = 0 , \quad f = g = \psi = 0 ,$$
$$\text{at} \quad z = 0 , \quad f = 0 , \quad g = w(t) , \psi = 0 \qquad \ldots \qquad (5)$$
$$\text{and as} \quad z \rightarrow \infty , \quad f = g = 0.$$

Introducing the new function $F = f + i\,g$, ... (6)
the two first equations of the system (4) are written in the form;

$$\nu \frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial t} - m^2 F = 2\psi \frac{F}{z} + F^2 \qquad \ldots \qquad (7)$$

The function ψ will be deduced from the relation;

$$\psi = - \int_0^z f \, dz = - \int_0^z \text{Re } (F) \, dz, \ldots \qquad (8)$$

where,

Re(F) is the real part of the function F.

Boundary and initial conditions of the new function F are;

$$F = 0 \quad \text{at} \quad t = 0 , \quad \text{and} \quad F = i\,w(t) \quad \text{at} \quad z = 0 ,$$
$$\text{while as} \quad z \rightarrow \infty , \quad F \rightarrow \text{zero} \qquad \ldots \qquad (9)$$

The method of solution followed here, is described by Sharikadze and Megahed[6]. The unsteady solution is obtained by superposition, as the sum of two parts. The first part is the solution of the homogeneous differential equation which satisfies the given boundary conditions and zero initial conditions, while the second part is the solution of the given differential equation which corresponds to homogeneous boundary conditions.

SOLUTION OF THE PROBLEM

The solution of the left-hand side of equation (7), which satisfie the boundary conditions (9) can be proved to be;

$$F_o(z,t) = \frac{1}{2\sqrt{\pi \nu}} \int_0^t w(\tau) \exp -\left[\frac{z^2}{4\nu(t-\tau)} - m^2(t-\tau)\right] \frac{z \, d\tau}{(t-\tau)^{3/2}} \quad \ldots \quad (10)$$

Introducing Green's function[6,7]

$$G(z,\xi,t) = -\frac{1}{2\sqrt{\pi \nu t}} \exp\left[-\frac{(z-\xi)}{4\nu t} - m^2 t\right] + \\ + \int_0^t \exp\left(-\frac{\xi^2}{4\nu\tau} - m^2\tau\right) \exp\left[-\frac{z^2}{4\nu(t-\tau)}\right] \frac{z \, d\tau}{4\pi\nu\sqrt{\tau(t-\tau)^3}}, \quad \ldots \quad (11)$$

which satisfies at $z \neq \xi$ equation (7) without the right-hand side and tends to zero at $t = 0$, and at $z = 0, \infty$, the final required solution $F(z,t)$ can be represented in the form;

$$F(z,t) = F_o(z,t) + \int_0^t d\tau \int_0^\infty \left(2\psi \frac{\partial F}{\partial \xi} + F^2\right) G(z,\xi,t-\tau) d\xi \quad \ldots \quad (12)$$

Differentiating equation (12) with respect to z, under the sign of integration, we get;

$$\frac{\partial F}{\partial z} = \frac{\partial F_o}{\partial z} + \int_0^t d\tau \int_0^\infty \left(2\psi \frac{\partial F}{\partial \xi} + F^2\right) \frac{\partial G}{\partial z} d\xi \quad \ldots \quad (13)$$

The system of equations (12) and (13) and be solved to obtain the unknowns F, $\frac{\partial F}{\partial z}$, hence the two functions f and g can be determined, Equation (8) enables to obtain ψ while the third equation of (4) is applied to determine P. The components of velocity are deduced from f , g and ψ as defined before.

The method of successive approximation is applied to solve the system (12) and (13). Denoting $\frac{\partial F}{\partial z}$ = W, the required functions are represented in the form of the following series;

$$F = \sum_{n=0}^{\infty} \lambda^n F_n, \quad W = \sum_{n=0}^{\infty} \lambda^n W_n, \quad \psi = \sum_{n=0}^{\infty} \lambda^n \psi_n \qquad \ldots(14)$$

where,

λ is a parameter. Substitute from (14) into equations (8) , (12) and (13) , starting initially with;

$$F_o \,, \quad W_o = \frac{\partial F_o}{\partial z} \quad \text{and} \quad \psi_o = - \int_0^z \mathrm{Re}\,(F_o)\, dz = 0,$$

we obtain the recurrent formulae;

$$F_{n+1} = \int_0^t d\tau \int_0^{\infty} \sum_{m=0}^{n} (2\psi_{n-m} W_m + F_{n-m} F_m)\, G\, d\zeta,$$

$$\psi_{n+1} = - \int_0^z \mathrm{Re}\,(F_{n+1})\, dz, \qquad \ldots(15)$$

and $$W_{n+1} = \int_0^t d\tau \int_0^{\infty} \sum_{m=0}^{\infty} (2\Psi_{n-m} W_m + F_{n-m} F_m) \frac{\partial G}{\partial z} d\zeta$$

The function F_o could be calculated from (10) as a pure imaginary function with a real part equals zero. The solution deduced in (14) and (15) can be proved to form a system of convergent series within the conditions under consideration.

The first two approximations of such solution are determined. Consider as a first approximation;

$$F_o = 0 \quad , \quad \Psi_o = 0 \text{ i.e.} \qquad f_o = \Psi_o = 0,$$

Therefore from (4) and (10), we get;

$$u = 0 \quad , \quad w = 0 \quad , \quad P = P_o(t)$$

$$v_\theta = v = \frac{r \ z}{2\sqrt{\pi\nu}} \int_0^t w(\tau) \exp\left[\frac{-z^2}{4\nu(t-\tau)} - m^2(t-\tau)\right] \frac{d\tau}{(t-\tau)^{3/2}}$$

...(16)

The last equations indicates that the first approximation effects only the rotating motion of the fluid, while the radial and axial components of the velocity are zero. If the disc rotates with uniform angular velocity i.e. $w(\tau) = \omega$ = const., we get after integrating (16);

$$v = \frac{r\omega}{2}\left[\exp(-mz/\sqrt{\nu}) \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - m\sqrt{t}\right) + \right.$$

$$\left. + \exp(mz/\sqrt{\nu}) \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + m\sqrt{t}\right)\right], \qquad \text{...(17)}$$

where, $\operatorname{erfc} x = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-n^2} dn$

Equation (17) represents the distribution of the transverse component of the velocity of flow due to a uniformly rotating infinite disk in the presence of external magnetic field. As m tends to zero the corresponding solution in classical hydrodynamics is obtained.

$$v = r\omega \left(\operatorname{erfc} \left(z/2\sqrt{\nu t} \right) \right) \quad \ldots \quad (18)$$

For the second approximation the following results are obtained;

$$F_1 = \int_0^t d\tau \int_0^\infty F_o^2 \, G d\zeta \, , \quad W_1 = \int_0^t d\tau \int_0^\infty F_o^2 \frac{\partial G}{\partial z} d\zeta \, ,$$

and the following expressions are deduced;

$$f_1 = \int_0^t d\tau \int_0^\infty F_o^2 \, G \, d\zeta \, , \quad \Psi_1 = - \int_0^t d\tau \int_0^\infty F_o^2 \, d\zeta \int_0^z G \, dz .$$

All the results obtained in this work for the case of infinite disk, can be applied as well to the case of circular rotating disk of finite radius R, provided that the radius is large compared to δ the thickness of the viscous boundary layer i.e. $R \gg \delta$.

Equation (16) which represents the solution for the velocity corresponding to the first approximation, allows to calculate the moment of the force of friction allover a rotating disc of finite radius R. The moment of the force of friction is defined and calculated as follows;

$$M = -2\pi\mu \int_0^R r^2 \frac{\partial v_\theta}{\partial z}\Big|_{z=0} dr =$$

$$= \rho R^4 \sqrt{\pi\nu} \int_0^t \left(\frac{dw}{d\tau} + m^2 w \right) \frac{e^{-m^2(t-\tau)}}{\sqrt{t-\tau}} d\tau \quad \ldots (19)$$

It is evident from (16) and (19) that even in the first approximation the magnetic field has an effect on the velocity distribution and the frictional moment.

Substituting m = 0 in the obtained results, the corresponding results of classical hydrodynamics are obtained, without taking into consideration the conductivity of the fluid or the presence of external magnetic field.

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دراسة حركة مائع موصل
ناشئة عن دوران قرص لا نهائي

عادل عبد الرحمن مجاهد

قسم الرياضيات والفيزيقا الهندسية ـ كلية الهندسة ـ جامعة القاهرة

يتناول البحث دراسة حركة مائع لزج وموصل ضعيف للكهربية الناشئة عن دوران قرص رقيق ذى قطر لا نهائي بينما يتعرض المائع لمجال مغناطيسي خارجي عمودي على مستوى القرص . تم الحصول على الحل العام لمركبات سرعة المائع عندما يتحرك القرص بسرعة دورانية تتغير كدالة معطاه للزمن .
كما تم استنتاج بعض الحالات الخاصة للحل .

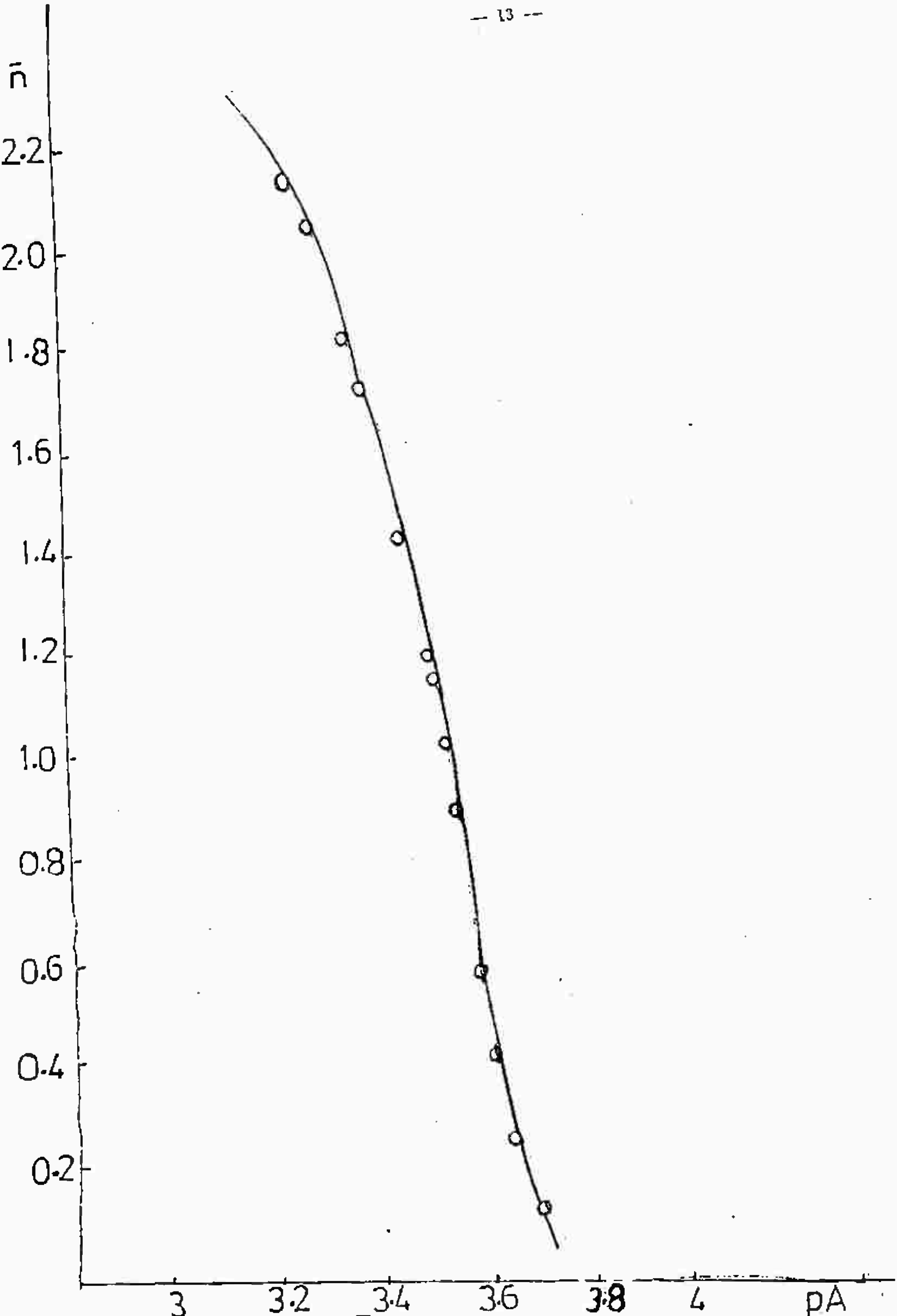

Fig 1