CARTWRIGHT'S THEOREM FOR VECTOR VALUED ENTIRE FUNCTIONS OF n-COMPLEX VARIABLES

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In this paper we deduce an interpolation formula for vector valued entire functions of n-complex variables of exponential types. This formula is used to derive a sufficient condition for these functions to have compact trajectories. The result leads to a generalization of Cartwright's theorem [3] for this class of functions.

§1. Introduction:

The vector valued functions are defined on the field of complex variables $C_z^n - z = (z_1, z_2, \dots, z_n)$, $z_k = x_k + iy_k$, $k = 1,2,3,\dots,n$ - to Banach Space X. Let $E_{\mu_1,\mu_2,\dots,\mu_n}(X)$ be the space of such vector valued functions satisfying the following two conditions :

(1) They are entire functions of exponential types $\mu_1, \mu_2, \mu_3, \dots, \mu_n$ relative to $z_1, z_2, \dots, z_n$ respectively, where $\mu_k < \pi, k = 1,2, \dots, n$.

(2) They are bounded functions on the n-dimensional Euclidean Space R^n. So if $F(z_1,z_2,\dots,z_n) \in E_{\mu_1,\mu_2,\dots,\mu_n}(X)$

then

$$(i)\ F(z_1,z_2,\dots,z_n) = \sum_{i_1=0}^{\infty} \sum_{i_1=0}^{\infty} \dots \sum_{i_n=0}^{\infty} a_{i_1,i_2,\dots,i_n} z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}$$

$$(ii)\ \| F(z_1,z_2,\dots,z_n) \|_X < A \exp \sum_{k=1}^{n} (\mu_k + \varepsilon) |z_k|$$

where ε is an arbitrary positive number, A is a constant, $\mu_k < \pi$ for all $k = 1,2,\dots,n$.

(iii) $\operatorname{Sup} \|F(x_1,x_2,\ldots,x_n)\|_X < \infty$

§2. Generalization of Cartwright's Theorem.

Theorem (1). If $F(z_1,z_2,\ldots,z_n) \in E_{\mu_1,\mu_2,\ldots,\mu_n}(X)$, then

$$F(z_1,z_2,\ldots,z_n)=\sum_{m_1=-\infty}^{+\infty}\sum_{m_2=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty} L_{m_1,m_2,\ldots,m_n}(z_1,z_2,\ldots,z_n)\, F(m_1,m_2,\ldots,m_n)$$

where

$$L_{m_1,m_2,\ldots,m_n}(z_1,z_2,\ldots,z_n) = (-1)^{m_1+m_2+\cdots+m_n} \prod_{i=1}^{n} \frac{\sin \pi z_i \sin \omega(m_i-z_i)}{\pi\ \omega\ (m_i-z_i)^2}$$

$$\omega < \pi - \max_k \mu_k , \quad k = 1,2,\ldots,n$$

Proof. This is just a generalization of the one variable expansion [4] .

Theorem (2) . Let $F(z_1,z_2,\ldots,z_n)$ be a vector valued function defined on the n-dimensional Euclidean space R^n to Banach space X with expansion,

$$F(x_1,x_2,\ldots,x_n) = \sum_{m_1=-\infty}^{+\infty}\sum_{m_2=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty} L_{m_1,m_2,\ldots,m_n}(x_1,x_2,\ldots,x_n)\, F(\alpha_{m_1},\alpha_{m_2},\ldots,\alpha_{m_n})$$

such that :

(i) The set of the numerical function $L_{m_1,m_2,\ldots,m_n}(x_1,x_2,\ldots,x_n)$ satisfies the condition

$$\sum_{m_1=-\infty}^{+\infty}\sum_{m_2=-\infty}^{+\infty}\cdots\sum_{m_n=-\infty}^{+\infty} \left| L_{m_1,m_2,\ldots,m_n}(x_1,x_2,\ldots,x_n) \right| < M$$

(M is a constant)

and ,

(ii) If the set of points

$$\Lambda = (\alpha_{m_1}, \alpha_{m_2}, \ldots , \alpha_{m_n}) \subset R^n$$

is defined for all combinations $(m_1, m_2, \ldots, m_n)$ of positive and negative integers, and if the set $F(\alpha_{m_1}, \alpha_{m_2}, \ldots, \alpha_{m_n})$ of values of the function $F(x_1, x_2, \ldots, x_n)$ at the set is compact, then the function $F(x_1, x_2, \ldots, x_n)$ is of compact trajectory.

Proof. For any arbitrary positive number ε, there exists a finite set of points in R^n,

$$\Gamma = \left[(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \right] \quad r = 1, 2, \ldots, s;$$

The set generates a finite number of divisions of the set of points

$$\beta = \bigcup_{r=1}^{s} \beta_r$$

such that for any point $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \beta_r$ we get,

$$\| F(\alpha_1, \alpha_2, \ldots, \alpha_n) - F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \|_X < \varepsilon.$$

Consider the following equality :

$$F(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{s} \sum_{\beta_r} L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) \left\{ F(\alpha_{m_1}, \ldots, \alpha_{m_n}) - F(\gamma_1^r, \ldots, \gamma_n^r) \right\} +$$

$$+ \sum_{r=1}^{s} \sum_{\beta_r} L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) F(\gamma_1^r, \ldots, \gamma_n^r) \quad \ldots\ldots\ldots\ldots (1)$$

For the last term in (1) we get

$$\sum_{r=1}^{s} \sum_{\beta_r} L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) =$$

$$= \sum_{r=1}^{s} F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \sum_{\beta_r} L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n)$$

Let

$$\sum_{\beta_r\; m_1,m_2,\ldots,m_n} L(x_1,x_2,\ldots,x_n) = G_r(x_1,x_2,\ldots,x_n)$$

From condition (1) of the theorem we get that the numerical function $G_r(x_1,x_2,\ldots,x_n)$ is bounded, and so it has a compact trajectory.

Let

$$F_s(x_1,x_2,\ldots,x_n) = \sum_{r=1}^{s} G_r(x_1,x_2,\ldots,x_n)\, F(\gamma_1^r,\gamma_2^r,\ldots,\gamma_n^r) \qquad (2)$$

It is clear that $F_s(x_1,x_2,\ldots,x_n)$ is a polynomial in the bounded numerical functions $G_r(x_1,x_2,\ldots,x_n)$, $r = 1,2,\ldots, s$, and so the vector valued function $F_s(x_1,x_2,\ldots,x_n)$ is of compact trajectory .

From (1) and (2) we have

$$F(x_1,x_2,\ldots,x_n) - F_s(x_1,x_2,\ldots,x_n) =$$

$$= \sum_{r=1}^{s} \sum_{\beta_r\; m_1,m_2,\ldots,m_n} L(x_1,x_2,\ldots,x_n)\left\{F(\alpha_{m_1},\ldots,\alpha_{m_n}) - F(\gamma_1^r,\ldots,\gamma_n^r)\right\}$$

$$\therefore \|F(x_1,x_2,\ldots,x_n)-F_s(x_1,x_2,\ldots,x_n)\|<\varepsilon M.$$

Thus the trajectory of the vector valued function $F_s(x_1,x_2,\ldots,x_n)$ is εM net for the trajectory of the function $F(x_1,x_2,\ldots,x_n)$, and so the trajectory of the vector valued function $F(x_1,x_2,\ldots,x_n)$ is compact, Q.E.D.

Theorem (3). If $F(z_1,z_2,\ldots,z_n) \in E_{\mu_1,\mu_2,\ldots,\mu_n}(X)$, and if the set $[F(m_1,m_2,\ldots,m_n)]$ of values of the function $F(x_1,x_2,\ldots,x_n)$ at the lattice point * is compact, then the function $F(x_1,x_2,\ldots,x_n)$ is of a compact trajectory.

Proof. From theorem (1), we have the expansion

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* All combinations $(m_1,\ldots,m_n)$ of positive and negative integers

$$F(x_1,x_2,\ldots,x_n) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_n=-\infty}^{+\infty} L(x_1,x_2,\ldots,x_n)\, F(m_1,m_2,\ldots,m_n)$$

where

$$L_{m_1,m_2,\ldots,m_n}(x_1,x_2,\ldots,x_n) = (-1)^{m_1+m_2+\ldots+m_n} \prod_{i=1}^{n} \frac{\sin \pi z_i \ \sin\omega(m_i-z_i)}{\pi^n\, \omega^n \ (m_i-z_i)^2}$$

So we have

$$\sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_n=-\infty}^{+\infty} L_{m_1,m_2,\ldots,m_n}(x_1,x_2,\ldots,x_n) =$$

$$= \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_n=-\infty}^{+\infty} \prod_{i=1}^{n} \left| \frac{\sin z_i \ \sin \omega(m_i-z_i)}{\pi^n\ \omega^n \ (m_i-z_i)^2} \right|$$

$$= \sum_{m_1=-\infty}^{+\infty} \left| \frac{\sin \pi z_1 \ \sin \omega(m_1-z_1)}{\pi\omega \ (m_1-z_1)} \right| \cdots \sum_{m_n=-\infty}^{+\infty} \left| \frac{\sin z_n \ \sin \omega(m_n-z_n)}{\pi\ \omega(m_n-z_n)} \right| \leqslant \left(\frac{\pi}{\omega}\right)^n$$

From theorem (2) we get the required result .

From the above we can generalize the Cartwright's theorem to be :

If $F(z_1,z_2,\ldots,z_n)$ is a vector valued entire function of n-variables $z_1,z_2,\ldots,z_n$ of exponential types $\mu_1,\mu_2,\ldots,\mu_n$ repectively , $\mu_i < k$, $i = 1,2,\ldots,n$, and if the set of its values at the lattice points is compact, then the function $F(z_1,z_2,\ldots,z_n)$ is of compact trajectory.

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