Appendix A
Orthogonality and Signal Representation

We have two types of signals; periodic and aperiodic. For periodic signal $x(t+T)=x(t)$, where T is the period. We have also another measure for classifying signals; namely, energy signals and power signals. For energy signals,

$$E=\int_{t_1}^{t_2} x^2(t)dt \quad <\infty \tag{A-1}$$

Specific examples of energy signals are decaying exponentials and damped sinusoids (in the semi-infinite, $t>0$), rectangular pulses, or any signal that is nonzero in a finite time interval only and finite within that interval.

However, there are many interesting signals that do not satisfy (A-1). Examples include all periodic signals and many aperiodic signals as well. In these cases, it is often more appropriate to consider the average power of the signal. The average power associated with eqn. (A-1), for example, is simply

$$P=\frac{1}{t_2-t_1}\int_{t_1}^{t_2} x^2(t)dt \tag{A-2}$$

If this remains greater than zero when the time interval becomes infinite, then the signal has finite average power, and will be called a power signal. More specifically, a power signal satisfies the condition.

$$0<\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)dt \quad <\infty \tag{A-3}$$

Upon comparing eqns. (A-1) and (A-3), it is clear that an energy signal has zero average power, and a power signal has infinite energy. Thus, a signal may be classified as one or the other, but not both. There are some signals that may not be classified as either, since both energy and average power may be infinite. The signal $x(t)=e^{-at}$, $-\infty<t<\infty$ is an example of this.

Given $x(t)$ as a periodic function, it may be represented in terms of time functions which are well defined. Such time functions are called basis functions, which may be designated as a set $\{\phi\}$, where $\{\phi\}$ denote $\phi_0(t)$, $\phi_1(t)$,.. $\phi_N(t)$, and N may be infinity in some cases. The signal function $x(t)$ may then be expressed as a linear combination in terms of the set $\{\phi\}$.

$$x(t)=\sum_{n=0}^{N} a_n\phi_n(t) \tag{A-4}$$

These basis functions have some important properties. One property that is desired for a set of basis functions is finality of coefficients. This property allows one to determine any given coefficient without the need for knowing any other coefficient. More terms can be added to the representation (to obtain greater accuracy) without making any changes in the earlier coefficients. In order to achieve finality of coefficients, it is necessary that the basis functions be orthogonal over the time interval for which the representation is to be valid. The condition of orthogonality for real basis functions requires that

$$\int_{t_1}^{t_2} \phi_n(t)\phi_k(t)dt = 0 \qquad k \neq n \tag{A-5a}$$

$$= \lambda_k \qquad k = n$$

for all k and n. If the basis functions are complex functions of time and ϕ_k^* is the complex conjugate of $\phi_k(t)$, then the condition for orthogonality is

$$\int_{t_1}^{t_2} \phi_n(t)\phi_k^*(t)dt = 0 \quad , \quad \begin{cases} 0 & k \neq n \\ \lambda_k & k = n \end{cases} \tag{A-5b}$$

where the λ_k are real

If $\lambda_k = 1$ for all k, the basis functions are orthonormal. The limits of integration in eqn. (A-5) can define a finite interval or an infinite (or semi-infinite) interval, depending upon the nature of the problem.

In order to demonstrate how the coefficients can be determined, multiply both sides of eqn. (A-4) by $\phi_j(t)$ for any j, and integrate over the specified interval. This gives

$$\int_{t_1}^{t_2} \phi_j(t)x(t)dt = \int_{t_1}^{t_2} \phi_j(t)\left|\sum_{n=0}^{N} a_n\phi_n(t)\right|dt$$

$$= \sum_{n=0}^{N} a_n \int_{t_1}^{t_2} \phi_j(t)\phi_n(t)dt \quad , \tag{A-6}$$

which form the orthogonality condition of eqn. (A-5). This equation may be written as

$$\int_{t_1}^{t_2} \phi_j(t)x(t)dt = a_j\lambda_j \quad , \tag{A-7}$$

since all of the terms on the right side of eqn. (A-6) will be zero, except for $n = j$. Thus, the coefficient a_j may be expressed quite generally as

$$a_j = \frac{1}{\lambda_j}\int_{t_1}^{t_2} \phi_j(t)x(t)dt \quad , \tag{A-8a}$$

when the basis functions are orthogonal and real.
For complex basis functions, this becomes

$$a_k = \frac{1}{\lambda_j}\int_{t_1}^{t_2} \phi_k^*(t)x(t)dt \quad , \tag{A-8b}$$

and a_k may be complex.

It is certainly possible to use sines and cosines as basis functions. But this calls for the use of two summations; one for sine terms and the other for cosine terms. A more convenient method is to use complex exponentials and let the index of summation be negative as well as positive .The resulting series can be converted easily into sines and cosines, if desired, by the familiar relation.

$$e^{\pm jn\omega_0 t} = \cos n\omega_0 t \pm j \sin n\omega_0 t \qquad \text{(A-9)}$$

The resulting series is called Fourier series. The Fourier series representation that will be obtained will be valid in the time interval form t_1 to t_2 for almost any time function. If, however, $x(t)$ is periodic with period $T = t_2 - t_1$, then the representation is valid for all times and not just those within the interval. Note that $x(t)$ must satisfy certain mathematical conditions if the resulting series is to converge. These are the Dirichlet conditions. They require that within the interval t_1 to t_2, $x(t)$ be single-valued, have only a finite number of maxima and minima in a finite time, have only a finite number of finite discontinuities, and satisfy the inequality.

$$\int_{t_1}^{t_2} |x(t)| dt < \infty$$

The actual function $x(t)$ corresponding to any physical signal will satisfy these conditions. Although some common mathematical representations do not. For the exponential Fourier series, the basis functions may be defined as

$$\phi_n(t) = e^{jn\omega_0 t} \quad , \qquad n = 0, \pm1, \pm2, \ldots\ldots, \pm\infty \text{ , where } \omega_0 = \frac{2\pi}{T}$$

It is easy to show that these functions are orthogonal so that

$$\int_{t_1}^{t_1+T} e^{jn\omega_0 t} e^{-jn\omega_0 t} dt \begin{cases} = 0 & n \neq k \\ = T & n = k \end{cases} \qquad \text{(A-10)}$$

If coefficients for the Fourier series are designated as α_n, then from eqn. (A-8a), they may be expressed as

$$\alpha_n = \frac{1}{T} \int_{t_1}^{t_1+T} x(t) e^{-jn\omega_0 t} dt \text{ ,} \qquad \text{(A-11)}$$

and are usually complex. In any case, however, $\alpha_{-n} = \alpha_n{}^*$ .Then, $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t} \qquad \text{(A-12)}$$

As a specific example of the exponential Fourier series, consider the periodic sequence of rectangular pulses shown in Fig. A.1. The time function $x(t)$ may be defined during the time interval from 0 to T (one period) as

$$x(t) = \begin{cases} A & 0 < t < t_a \\ 0 & t_a < t < T \end{cases}$$

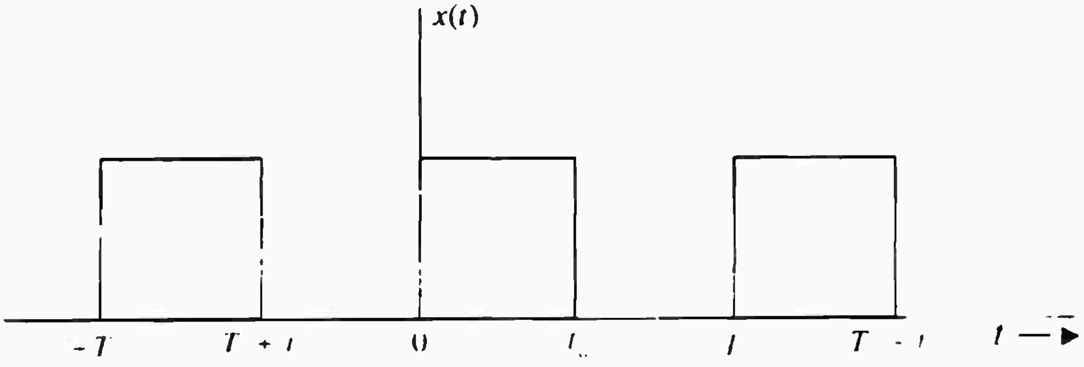

Fig. A.1 A periodic sequence of rectangular pulses

Hence, the coefficients become

$$\alpha_n = \frac{1}{T}\int_0^{t_a} e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\left[\frac{1 - e^{-jn\omega_0 t_a}}{jn\omega_0}\right]$$

$$= \frac{A}{T}\exp(-jn\omega_0 t_a/2)\left[\frac{\exp(jn\omega_0 t_a/2) - \exp(-jn\omega_0 t_a/2)}{jn\omega_0}\right]$$

$$= \frac{At_a}{T}\left[\frac{\sin(n\omega_0 t_a/2)}{n\omega_0 t_a/2}\right]\exp(-jn\omega_0 t_a/2)$$

This may be written in a slightly different form by replacing ω_0 by its equivalent $2\pi/T$. Thus,

$$\alpha_n = \frac{At_a}{T}\left[\frac{\sin(n\pi t_a/T)}{n\pi t_a/T}\right]\exp\left[-j\frac{2\pi n}{T}\left(\frac{t_a}{2}\right)\right]$$

The complete Fourier series expression for $x(t)$ now becomes

$$x(t) = \sum_{n=-\infty}^{\infty}\frac{At_a}{T}\left[\frac{\sin(n\pi t_a/T)}{n\pi t_a/T}\right]\exp\left[j\frac{2\pi n}{T}\left(t - \frac{t_a}{2}\right)\right] \qquad \text{(A-13)}$$

This representation of $x(t)$ is in terms of sinusoids having frequencies that are multiples of the fundamental frequency $1/T$. The coefficients α_n give the magnitude and phase of these sinusoids, and hence, are said to constitute a frequency-domain description of the signal. The explicit time function $x(t)$ is said to be a time-domain description of the signal.

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Appendix B
Singularity Functions

There is a class of elementary signals whose members have very simple mathematical forms, but are either discontinuous or have discontinuous derivatives. Because such signals do not have finite derivatives of all orders, they are usually referred to as singularity functions. Two of the most common singularity functions are the unit ramp function and the step function will be considered here.

Although signals such as these are mathematical idealizations and cannot really occur in any physical system, they serve several useful purposes for system analysis. In the first place, they serve as good approximations to the signals that actually do arise in systems when switching operations take place. Secondly, their simple mathematical forms make it possible to carry out system analysis much more easily than could be done with more complicated signals. Furthermore, many complicated signals can be represented as sums of these elementary ones.

The unit ramp function - designated as $r(t)$ (Fig.B.1) -is defined to start at $t=0$, and have unit slope thereafter. Hence, it may be represented mathematically as

$$r(t)=\begin{cases} t & t\geq 0 \\ 0 & t<0 \end{cases} \qquad \text{(B-1)}$$

If a slope other than unity is desired, it is necessary only to multiply by a constant. Thus, $br(t)$ is a ramp having a slope of b. An alternative way of changing the slope is to change the time scale of the argument. Since $r(t)$ has unit slope, its value must be unity whenever the argument is unity. Thus, $br(t)$ and $r(bt)$ both represent ramps with slopes of b

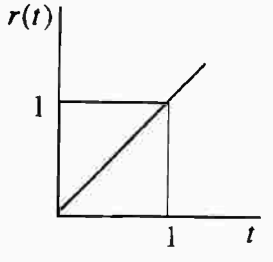

Fig. B.1 Unit ramp function

The unit step function - designated as $u(t)$ (Fig. B.2) - is defined to be zero before zero time and unity thereafter. Thus, it may be represented mathematically as

$$u(t)=\begin{cases} 1 & t\geq 0 \\ 0 & t<0 \end{cases} \qquad \text{(B-2)}$$

Fig. B.2 Unit step function

It may be noted that the unit ramp function is just the integral of the unit step function; that is,

$$r(t) = \int_{-\infty}^{t} u(\lambda)d\lambda \qquad \text{(B-3)}$$

It is also true that at all times, except $t = 0$ where a unique derivative does not exist,

$$u(t) = \frac{dr(t)}{dt} \qquad t \neq 0 \qquad \text{(B-4)}$$

A step change of value other than unity can be obtained by multiplying by a constant. Thus, $cu(t)$ is a step change of magnitude c.

All of the singularity functions just discussed were assumed to start at $t = 0$. It is often necessary to consider other starting times, and this can be done by translating the argument of the function in time. Thus, $u(t-a)$ is zero whenever $(t-a)$ is negative, and unity when it is positive, representing a step starting at $t = a$.

By using combinations of ramps and step functions, it is possible to represent many other types of functions. For example, a rectangular pulse of width a can be considered as the difference between a step function at the origin and one at $t = a$. This is illustrated in Figs. B.3, B.4. Hence, the mathematical representation of a unit rectangular pulse, $P_a(t)$ could be written as

$$P_a(t) = u(t) - u(t-a) \qquad \text{(B-5)}$$

It should also be clear that

$$cP_a(t) = c[u(t) - u(t-a)] \qquad \text{(B-6)}$$

is a rectangular pulse with magnitude c and duration a.

There is another singularity function, known as the impulse or delta function, which is of great importance in system analysis. This is not a well-behaved function in the sense that an explicit mathematical description can be written for it. Nevertheless, it has some well-defined properties that provide the best way of describing it. Before we discuss these properties, however, it may be helpful to discuss the impulse from an intuitive standpoint.

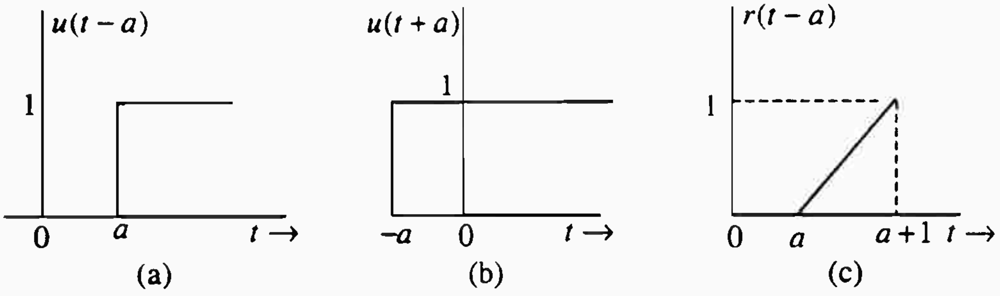

Fig. B.3 Some translated singularity functions.
a) step function translated to right b)step function translated to left
c) translated ramp function

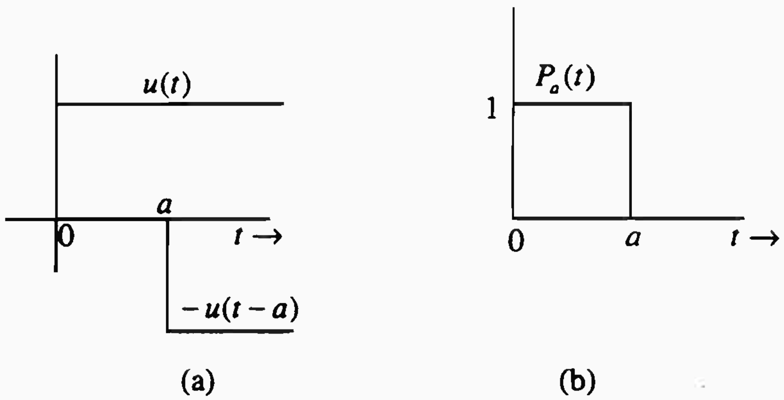

Fig. B.4 Two steps combining to form a rectangular pulse.
a) step function components b) resulting pulse function

The intuitive interpretation of an impulse is that it is an idealization of a very narrow pulse having a finite total area. For convenience, the area is usually taken to be unity. A nonmathematical approach to this interpretation which emphasizes its relation to the step function can be developed by considering the finite ramp function and its derivative. The particular forms of these functions that will be used for this discussion are shown in Fig. B.5. It is evident that the finite ramp function $f(t)$ approximates a step function when a is small. In fact, one may write

$$u(t) = \lim_{a \to 0} f(t) \qquad \text{(B-7)}$$

the derivative of the finite ramp is seen to be a rectangular pulse of duration a and magnitude $1/a$. As a becomes small, this pulse becomes narrower and taller but its area remains constant at unity. Hence, the derivative of a finite ramp approaches an impulse $\delta(t)$ as a approaches zero. Thus,

$$\delta(t) = \lim_{a \to 0} \frac{df(t)}{dt} \qquad \text{(B-8)}$$

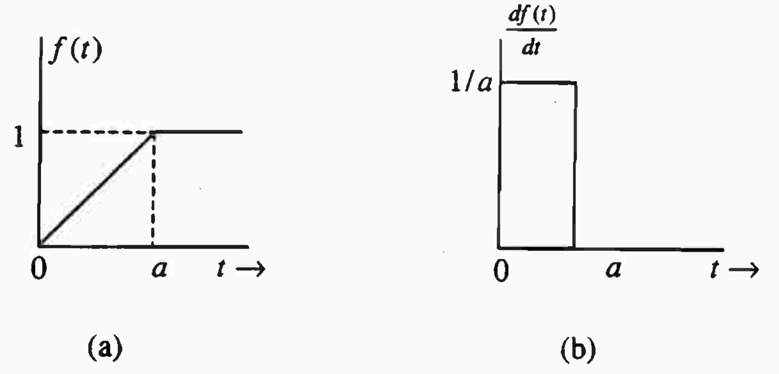

Fig. B.5 functions used to obtain the impulse function
a) the finite ramp b) the derivative of the finite ramp

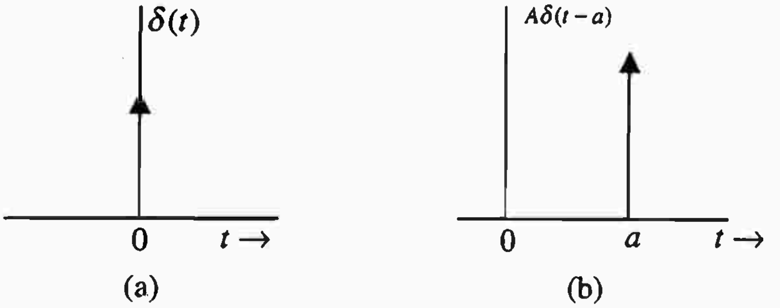

Fig. B.6 Graphical representation of an impulse.
a) the unit impulse. (b) time-shifted impulse of area *A*.

From these two relationships, it is possible to write a definition of the impulse as

$$\delta(t) = \frac{du(t)}{dt} \quad , \tag{B-9}$$

even though the derivative of the step function does not exist in the strict mathematical sense. The graphical representation of $\delta(t)$ is illustrated in Fig.B.6a. A magnitude-scaled and time-shifted version is shown in Fig. B.6b. It should be emphasized that multiplying an impulse is really designating the area of the impulse and is not just scaling its magnitude. Thus, $A\delta(t-a)$ is an impulse with an area of *A* located at $t = a$.

The usefulness of the impulse or delta function in system analysis arises from the fact that the response of a linear system to a unit impulse at its input can be used to obtain the response of a linear system to any input signal. Thus, the impulse response of the system can be considered as another mathematical model for the system, since it can be used to relate the input and the output.

It can also be shown that the impulse is a good approximation to physical pulses of any shape, provided they are narrow compared to the time it takes the system to produce a significant response. The advantage of using the impulse in system analysis is that the analysis is much simpler and it is only necessary to specify the area of the physical pulse rather than its complete time function. The delta function satisfies the following properties:

$$(1)\ \delta(t-t_0)=0 \qquad t\neq t_0 \tag{B-10}$$

$$(2)\ \int_{t_1}^{t_2}\delta(t-t_0)dt=1 \qquad t_1<t_0<t_2 \tag{B-11}$$

$$(3)\ \int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt=f(t_0),\ \text{for } f(t) \text{ continuous at } t_0 \tag{B-12}$$

Conditions (B-10) and (B-11) are in complete agreement with the intuitive approach in that they define an arbitrarily narrow pulse with unit area condition. Condition (B-12 is usually referred to as the "sifting property" of the delta function and is seen to be a logical consequence of the first two: that is the integrand is zero everywhere except at $t=t_0$, where it becomes $f(t_0)\delta(t-t_0)$. Since $f(t_0)$ is a constant, and the area of $\delta(t-t\ )$ is unity, the conclusion of eqn. (B-12) follows. The sifting property is undoubtedly the most important one from the standpoint of using the delta function in system analysis.

Additional properties of the delta function are:

1- **Time scaling**

$$\delta(bt)=\frac{1}{|b|}\delta(t) \tag{B-13}$$

2- **Derivative.** It is possible to define a function that may be interpreted as the "derivative" of a delta function, even though a true derivative dose not exist in the usual sense. This derivative which is usually called a doublet has the following properties

$$\delta'(t-t_0)=0,\quad t\neq t_0 \tag{B-14}$$

$$\int_{t_1}^{t_2}\delta'(t-t_0)dt=0,\qquad t_1<t_0<t_2 \tag{B-15}$$

$$\int_{-\infty}^{\infty} f(t)\delta'(t-t_0)dt=-f'(t_0),\ \text{for } f(t) \text{ and } f'(t) \text{ continuous at } t_0 \tag{B-16}$$

3- **Multiplication by a time function**

$$f(t)\delta(t-t_0)=f(t_0)\delta(t-t_0),\ \text{for } f(t) \text{ continuous at } t_0 \tag{B-17}$$

$$f(t)\delta'(t-t_0)=-f'(t_0)\delta(t-t_0)+f(t_0)\delta'(t-t_0) \tag{B-18}$$

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Appendix C
Convolution

Consider the time function $f(t)$ shown in Fig. C.1. This function can be approximated over the time interval $T < t < T$ by a series of pulses as shown.

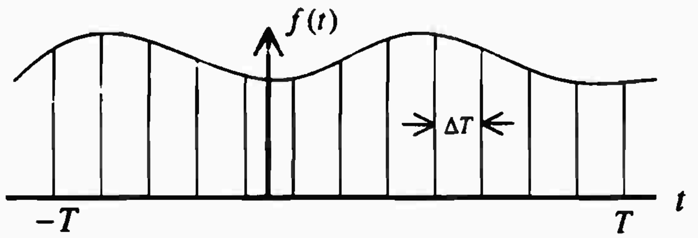

Fig. C.1. Representation of a signal by a series of pulses

The number of pulses will be $2N+1 = 2T/\Delta T$, where ΔT is the pulse width. The amplitude of the pulses can be taken as the amplitude of the function at the center of the pulse. Thus, the P_k pulse would be

$$P_k = f(k\Delta T)\left[u\left(t - k\Delta T + \frac{\Delta T}{2}\right) - u\left(t - k\Delta T - \frac{\Delta T}{2}\right)\right] \qquad \text{(C-1)}$$

The approximation to $f(t)$ is then obtained as the summation of the pulses.

$$f(t) = \sum_{k-N}^{N} f(k\Delta T)\left[u\left(t - k\Delta T + \frac{\Delta T}{2}\right) - u\left(t - k\Delta T - \frac{\Delta T}{2}\right)\right] \qquad \text{(C-2)}$$

If we multiply and divide by ΔT we obtain

$$f(t) = \sum_{k=N}^{N} f(k\Delta T)\left[\frac{u(t - k\Delta T + \frac{\Delta T}{2}) - u(t - k\Delta T - \frac{\Delta T}{2})}{\Delta T}\right]\Delta T \qquad \text{(C-3)}$$

As ΔT is made smaller, the approximation becomes better. Also, as ΔT becomes smaller, the factor within the square brackets approaches more closely a δ function located at $t = k\Delta T$. In the limiting case, as $\Delta T \to 0$, N becomes infinite. However, The product $N\Delta T$ remains constant and equal to T. The product $k\Delta T$ takes on all possible values in the interval $-T < t < T$, and can be considered to be a continuous variable λ. The increment ΔT becomes the differential $d\lambda$. In this limiting case, the summation becomes an integral with respect to λ over the range $-T$ to T. Accordingly, $f(t)$ can be written as

$$f(t) = \int_{-T}^{T} f(\lambda)\delta(t - \lambda)d\lambda \qquad -T < t < T \qquad \text{(C-4)}$$

The complete time function can be obtained by letting $T \to \infty$, giving

$$f(t) = \int_{-\infty}^{\infty} f(\lambda)\delta(t-\lambda)d\lambda \tag{C-5}$$

The function $f(t)$ is thus represented as the summation (integral) of a continuum of impulses having strengths at any time λ given by $f(\lambda)d\lambda$. The relation given in eqn. (C-4) is often used as the definition of the unit impulse. Its derivation in the foregoing manner was carried out to provide a physical basis for understanding the convolution integral.

By computing the response of a linear system to each member of this continuum of impulses and summing up these responses, the total system response will be obtained. The validity of this approach rests on the superposition theorem, which states that the response of a linear system having a number of inputs can be computed by determining the response to each input considered separately, and then summing the individual responses to obtain the total response. Superposition is only applicable to linear systems.

By the impulse response $h(t)$ of a time-invariant system is meant the output time function which results when the input signal is a unit impulse occurring at $t = 0$. It is assumed that the output was zero before the application of the impulse and would have remained zero if the impulse had not been applied.

Using the concept of impulse response $h(t)$ for a linear time-invariant system, it is now possible to determine the system output for an arbitrary input. The input $x(t)$ can be resolved into a continuum of impulses as in eqn. (C-5). Each of these impulses is of the form $x(\lambda)\delta(t-\lambda)d\lambda$. The response to each elementary impulse is $h(t)$ multiplied by the strength of the impulse $x(\lambda)d\lambda$ and is properly positioned to coincide with the time of the application of the impulse. Mathematically, this is expressed as $x(\lambda)h(t-\lambda)d\lambda$. The total responses *y(t)* is the summation of all the elementary responses and is given by

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \tag{C-6}$$

The integral relationship expressed in eqn. (C-6) is called the convolution of $x(t)$ and $h(t)$, and relates the input and output of the system by means of the system impulse response. A simple change of variables shows that convolution is commutative for time-invariant systems, and therefore an equivalent expression is

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda \tag{C-7}$$

The convolution operation is frequently denoted by a pentacle as follows

$$f_1(t)\star f_2(t)=\int_{-\infty}^{\infty} f_1(\lambda)f_2(t-\lambda)d\lambda \qquad \text{(C-8)}$$

The effective limits on the convolution integral will vary with the particular characteristics of the functions being convolved. For physically realizable systems, $h(t)=0$. for $t<0$, and this requirement establishes the upper limit in eqn. (C-6) as t. Actually, it would not be incorrect to write the upper integration limit as ∞, since the function $h(t-\lambda)$ is zero when $\lambda>t$. Similarly, if the time function starts at a time t_0, the lower limit could be t_0. Yet, it is common practice to extend the lower limit to $-\infty$. The convolution integral becomes

$$y(t)=\int_{-\infty}^{t} x(\lambda)h(t-\lambda)d\lambda \qquad \text{(C-9)}$$

When the order of convolution is changed as in eqn. (C-7), the expression for a physically realizable system becomes

$$y(t)=\int_{0}^{\infty} h(\lambda)x(t-\lambda)d\lambda \qquad \text{(C-10)}$$

In eqn. (C-10), the lower limit is determined by the physical realizability constraint on $h(t)$, and the upper limit is set to include negative as well as positive time for the excitation $x(t)$.

Consider the convolution of the two time functions $f_1(t)$ and $f_2(t)$. Formally, the convolution operation is given by

$$f_3=f_1\star f_2=\int_{-\infty}^{\infty} f_1(\lambda)f_2(t-\lambda)d\lambda \qquad \text{(C-11)}$$

The value of f_3 for any particular t is seen to be the area under the product of $f_1(\lambda)$ and $f_2(t-\lambda)$. In order for the convolution technique to be used efficiently, it is necessary to be able to sketch rapidly (or visualize mentally) the functions $f_1(\lambda)$ and $f_2(t-\lambda)$. In visualizing these functions, there is no difficulty with the function $f_1(\lambda)$, since it is identical with $f_1(t)$, except for a change in the independent variable from t to λ. The function $f_2(t-\lambda)$ as a function of λ requires a little more thought, however. It can be visualized most readily as a combination of reflection and translation of the original function $f_2(\lambda)$. This process - called folding - is most easily described by means of an example. In Fig. C.2a, where an arbitrary function $f_2(\lambda)$ is shown. The function $f_2(-\lambda)$ in Fig. C.2b is merely a reflection of $f_2(\lambda)$ about the vertical axis. In order to sketch the reflected function, it is only necessary to start at the origin with the ordinate $f_2(0)$ and sketch on the right that portion of the function which was originally on the left and sketch on the left that portion of the function which was on the right. The function $f_2(t-\lambda)$ can be thought of as $f_2[-(\lambda-t)]$ in which case it is clear that the variable λ has been replaced by $\lambda-t$, which corresponds to a delay (or translation to the right) by an amount t when the function is plotted along the λ axis.

The function $f_2(t-\lambda)$ is shown in Fig. C.2c for an arbitrary t. The amount of displacement t is measured from the position of $f_2(-\lambda)$, which corresponds to $f_2(t-\lambda)$ with $t=0$. The convolution is then given as the area under the product of $f_1(\lambda)$ and $f_2(t-\lambda)$. It is generally a function of t, since the value of t determines the relative position of $f_1(\lambda)$ and $f_2(t-\lambda)$. The convolution can be thought of as being obtained by folding or reflecting one function and then determining the area under the product as the folded function is slid along the horizontal axis to the right for positive time and to the left for negative time.

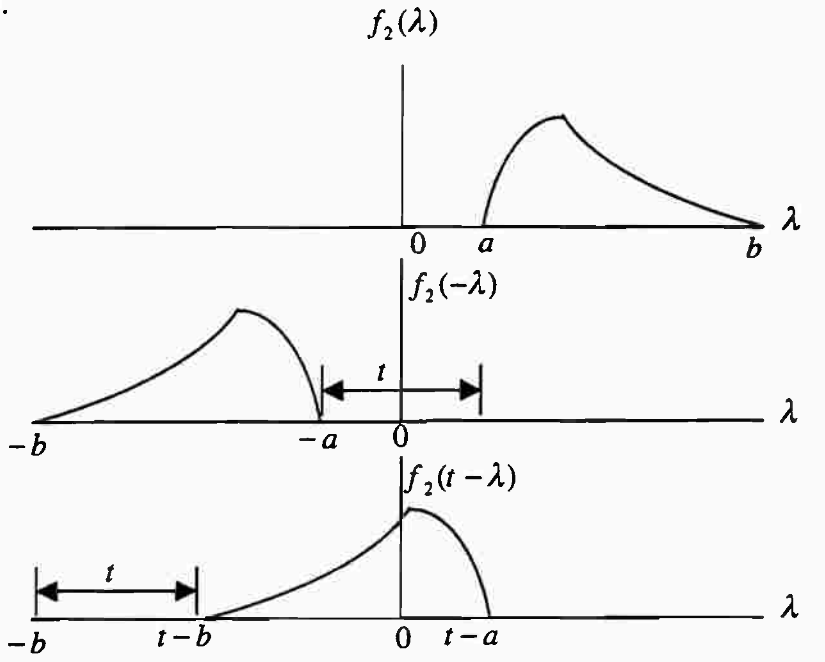

Fig. C.2 Folding and Sliding the function $f_2(\lambda)$

Consider the defining relation of eqn. (C-8). By making the change of variables $\lambda = t-\lambda_1$, we obtain

$$f_1 \star f_2 = \int_{\infty}^{-\infty} f_1(t-\lambda_1) f_2(\lambda_1)(-d\lambda_1)$$

$$= \int_{-\infty}^{\infty} f_2(\lambda_1) f_1(t-\lambda_1) d\lambda_1$$

Therefore,

$$f_1 \star f_2 = f_2 \star f_1 \tag{C-12}$$

It therefore follows that convolution is commutative and the order of the functions being convolved is immaterial.

As a direct result of the superposition property of integrals (that is, the integral of a sum of terms is equivalent to the sum of the integrals of the terms taken separately), it is shown readily that convolution is distributive. Accordingly,

$$f_1 \star [f_2 + f_3] = f_1 \star f_2 + f_1 \star f_3 \tag{C-13}$$

When functions are reasonably well behaved - as they always are when they arise in physical problems - the order of integration can be changed, and it is shown readily that convolution is also associative; that is,

$$f_1 \star [f_2 \star f_3] = [f_1 \star f_2] \star f_3 \tag{C-14}$$

In view of the associative property, it is unnecessary to use brackets to separate the function being convolved, and eqn. (C-14) can be written in the equivalent forms

$$f_1 \star [f_2 \star f_3] = [f_1 \star f_2] \star f_3$$

$$[f_1 \star f_2] \star f_3 = f_1 \star [f_2 \star f_3]$$

$$= f_1 \star f_2 \star f_3 \tag{C-15}$$

Also,

$$z(t) \star \delta(t) = \int_{-\infty}^{\infty} z(\lambda)\delta(t-\lambda)\,d\lambda = z(t) \tag{C-16}$$

$$z(t) \star u(t) = \int_{-\infty}^{\infty} z(\lambda)\, u(t-\lambda)\,d\lambda = \int_{-\infty}^{t} z(\lambda)\,d\lambda \tag{C-17}$$

$$z(t) \star \delta'(t) = \int_{-\infty}^{\infty} z(\lambda)\delta'(t-\lambda)\,d\lambda = z'(t) \tag{C-18}$$

Therefore,

$$u(t) \star \delta'(t) = \int_{-\infty}^{\infty} u(\lambda)\delta'(t-\lambda)\,d\lambda = \delta(t) \tag{C-19}$$

Using these relationships may be obtained readily. Assume that

$$z(t) = x(t) \star y(t)$$

Then

$$z(t) \star \delta'(t) = x(t) \star y(t) \star \delta'(t)$$

$$z'(t) = x(t) \star y'(t) \tag{C-20}$$

$$= x'(t) \star y(t) \tag{C-21}$$

and

$$z(t)\star\delta(t)=x(t)\star y(t)\star\delta'(t)\star u(t) \qquad \text{(C-22)}$$

$$z(t)=\left[x(t)\star\delta'(t)\right]\star y(t)\star u(t)=x'(t)\star\int_{-\infty}^{t} y(\lambda)d\lambda \qquad \text{(C-23)}$$

$$=y'(t)\star\int_{-\infty}^{t} x(\lambda)d\lambda \qquad \text{(C-24)}$$

A relationship between the input and output of a system can also be obtained by using the step response of the system instead of the impulse response. The basic relationship could be obtained by a procedure analogous to that used in developing the convolution integral. However, a more direct derivation can be made using convolution algebra. The step response $w(t)$ of a system is related to the impulse response $w(t)$ as follows:

$$w(t)=u(t)\star h(t)=\int_{0}^{t} h(t-\lambda)d\lambda=\int_{0}^{t} h(\lambda)d\lambda \qquad \text{(C-25)}$$

It follows that

$$w'(t)=\frac{d}{dt}\int_{0}^{t} h(\lambda)d\lambda=h(t) \qquad \text{(C-26)}$$

Consider now the convolution of the derivative of the input, $x'(t)$, and the system step response $w(t)$: $x'(t)\star w(t)$

Using eqns. (C -20) and (C-21), we obtain

$$x'(t)\star w(t)=x(t)\star w'(t)=x(t)\star h(t)=y(t) \qquad \text{(C-27)}$$

$$y(t)=x'(t)\star w(t)=\int_{-\infty}^{t} x'(\lambda)w(t-\lambda)d\lambda \qquad \text{(C-28)}$$

In most commonly encountered applications involving the step response, $x_c(t)$ is a causal time function. It is zero for $t<0$. For this case we have

$$y(t)=\int_{-\infty}^{t} x_c'(\lambda)w(t-\lambda)d\lambda=\int_{0^-}^{t} x_c'(\lambda)w(t-\lambda)d\lambda \qquad \text{(C-29)}$$

$$=\int_{0^-}^{0^+} x_c'(\lambda)w(t-\lambda)d\lambda+\int_{0^+}^{t} x_c'(\lambda)w(t-\lambda)d\lambda \qquad \text{(C-30)}$$

$$=x_c(0^+)w(t)+\int_{0^+}^{t} x_c'(\lambda)w(t-\lambda)d\lambda \qquad \text{(C-31)}$$

The relationship given in eqn. (C-31) is known as the superposition integral or Duhamel's integral.

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Appendix D
Fourier Transform

There are several ways to obtain the analytical expression whereby a time function can be represented in terms of a continuum of elementary sinusoids. Consider a time function $f(t)$ which is to be represented in terms of complex sinusoids. This time function can be precisely represented over the interval $-T/2<t<T/2$, by means of a Fourier series having a period T. As the period T is increased, more and more of the time function will be included in the series representation. In the limit as $T\rightarrow\infty$, the entire function $f(t)$ will be included. Consider now the representation of $f(t)$ as a complex Fourier series over the interval of $-T/2<t<T/2$;

$$f(t)=\sum_{n=-\infty}^{\infty}\alpha_n \mathrm{e}^{j\frac{2\pi nt}{T}}, \qquad \text{(D -1)}$$

where

$$\alpha_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-j\frac{2\pi nt}{T}} \qquad \text{(D-2)}$$

The fundamental angular frequency is $\omega_0=2\pi/T$. In addition to being the lowest frequency component, ω_0 is also the spacing between harmonics. Using this expression for ω_0, and substituting eqn. (D-2) into eqn. (D-1) gives

$$f(t)=\sum_{n=-\infty}^{\infty}\mathrm{e}^{j\frac{2\pi nt}{T}}\left[\frac{\omega_0}{2\pi}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-jn\omega_0}dt\right] \qquad \text{(D-3)}$$

If we now let $T\rightarrow\infty$, the spacing between harmonics will become a differential, that is. $\omega_0=2\pi/T\rightarrow d\omega$; The number of components becomes infinite, that is $n\rightarrow\infty$. The angular frequency of any particular component is given by $n\omega_0$ and the summation formally passes into an integral. Eqn. (D-3) may then be written in the following form:

$$f(t)=\int_{-\infty}^{\infty} e^{j\omega t}\left[\frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right] \qquad \text{(D-4)}$$

Rearranging terms gives

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right]e^{j\omega t}d\omega \qquad \text{(D-5)}$$

This is the Fourier integral relation. Its significance becomes apparent when we separate the inner and outer integrals. It is evident that the inner integral is only a function of the angular frequency, since time is integrated out. This inner integral is called the Fourier transform of $f(t)$, and is designated as

$$\mathcal{F}[f(t)] = F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad \text{(D-6)}$$

The relationship in eqn. (D-5) may then be considered as establishing the connection between $F(j\omega)$ and $f(t)$. This is called the inverse Fourier transform and is written as follows:

$$\mathcal{F}^{-1}[F(j\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega \qquad \text{(D-7)}$$

The functions $f(t)$ and $F(j\omega)$ are called Fourier transform pairs.

Using the analogy between the Fourier series and the Fourier transform, it may be concluded that the function $F(j\omega)$ analyzes $f(t)$ into a continuum of complex sinusoids having amplitudes of $(1/2\pi)|F(j\omega)|d\omega$. If $F(j\omega)$ is finite - as it is unless there are discrete frequencies present - then these amplitudes are infinitesimal. This can be interpreted as the distribution of the signal throughout a frequency band. Such a distribution is called a frequency spectrum, and in the case of the Fourier transform, $(1/2\pi)F(j\omega)d\omega$ can be thought of as the amplitude of the signal lying in the angular frequency band of ω to $\omega + d\omega$. Noting $\omega = 2\pi f$, it is also clear that $|F(j2\pi f)|df$ equals the amplitude in the frequency band of f to $f + df$ Hertz. The Fourier transform can be expressed more clearly in terms of angular frequency spectra by writing it as

$$F(j\omega) = A(\omega)e^{j\theta(\omega)} \quad , \qquad \text{(D-8)}$$

where

$$A(\omega) = |F(j\omega)| \quad , \qquad \text{(D-9)}$$

and

$$\theta(\omega) = \tan^{-1}\left[\frac{\operatorname{Im} F(j\omega)}{\operatorname{Re} F(j\omega)}\right] \qquad \text{(D-10)}$$

$A(\omega)$ is then the amplitude spectrum (often called the angular frequency spectrum) and $\theta(\omega)$ is the angular phase spectrum corresponding to the phase (at $t = 0$) of the elementary sinusoid at the angular frequency ω.

Not all time functions can be represented by the Fourier integral. However, when such a representation is possible, there is a unique one-to-one correspondence between a function and its Fourier transform. This means that there is only a single time function corresponding to a given Fourier transform. The determining factor in the Fourier representation is whether or not the integrals are convergent. One set of conditions that assures convergence is the Dirichlet conditions, which may be stated as follows:

1. $f(t)$ must be absolutely integrable: that is,

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty \tag{D-11}$$

2. $f(t)$ must have a finite number of maxima and minima in any finite interval.
3. $f(t)$ must have a finite number of finite discontinuities in any finite interval.

These conditions are sufficient to include virtually all useful finite-energy signals. However, they exclude a number of important signals, such as periodic waveforms and the unit step function, that are not absolutely integrable. By allowing the Fourier transform to include the delta function, it will be found that signals of this type can be handled using essentially the same methods as for finite energy signals.

As an example of transform computation, consider the rectangular pulse shown in Fig. D.1. This pulse may be expressed analytically as

$$P_T(t)\begin{cases} =1 & 0<t<T \\ =0 & \text{elsewhere} \end{cases} \tag{D-12}$$

The Fourier transform is found by application of eqn. (D-6)

$$P_T(j\omega) = \int_{-\infty}^{\infty} P_T(t)\, e^{-j\omega t} dt$$

$$= \int_0^T e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_0^T = \frac{1-e^{j\omega T}}{j\omega} \tag{D-13}$$

The transform $P_T(j\omega)$ can be simplified by partially converting to trigonometric functions as follows

$$P_T(j\omega) = \frac{e^{-j\omega T/2}}{\omega/2}\left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right]$$

$$= Te^{-j\omega T/2}\frac{\sin \omega T/2}{\omega T/2} \tag{D-14}$$

The reason for putting $P_T(j\omega)$ into the form of eqn. (D-14) is to make use of the function $\sin(x)/x$, which is very easy to visualize. This function occurs so frequently in Fourier analysis that it is convenient to give it a special symbol. Accordingly, we will define

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x} \tag{D-15}$$

The function $\mathrm{sinc}(x)$ is shown plotted in Fig. D.2 to sufficient accuracy that values can be read off the curves if required. It is seen from Fig. D.2 that $\mathrm{sinc}(x)$ is an even function of x , having a maximum of unity occurring at the origin with a damped oscillatory amplitude away from the origin.

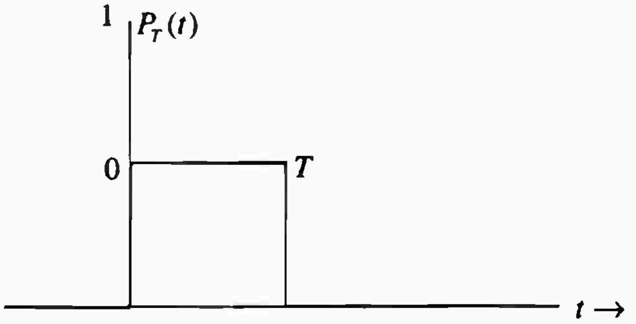

Fig.. D.1

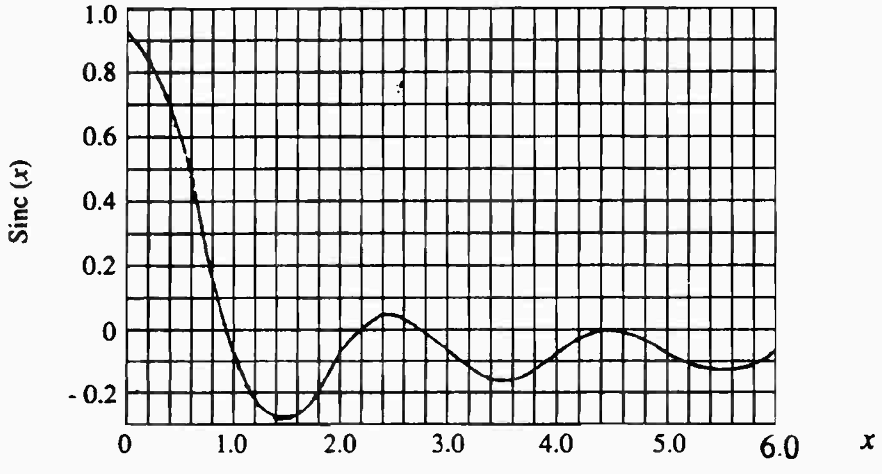

Fig. D.2 Plot of $\text{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$

Retuming now to eqn. (D-14), we see that $P_T(j\omega)$ is a complex function of angular frequency ω having an amplitude spectrum $A(\omega) = |T\text{sinc}(\omega T / 2\pi)|$. Fig. D.3 shows a plot of the amplitude and phase spectra. Note that the spectrum is concentrated over a band of frequencies in the vicinity of the origin with the first null of major lobe of the spectrum occurring at a frequency $f = 1/T$. As the pulse width is decreased, this first null moves to higher frequencies. Conversely, as the pulse width is increased, the first null moves in closer to the origin. The relative amplitudes of the various portions of the spectrum are unchanged by changes in T. It is evident that for this pulse shape there is an inverse relationship between the time duration of the signal and the frequency spread of the spectrum. This is a general property of all signals: the more compact the signal in time, the more spread out it will be in frequency and vice versa.

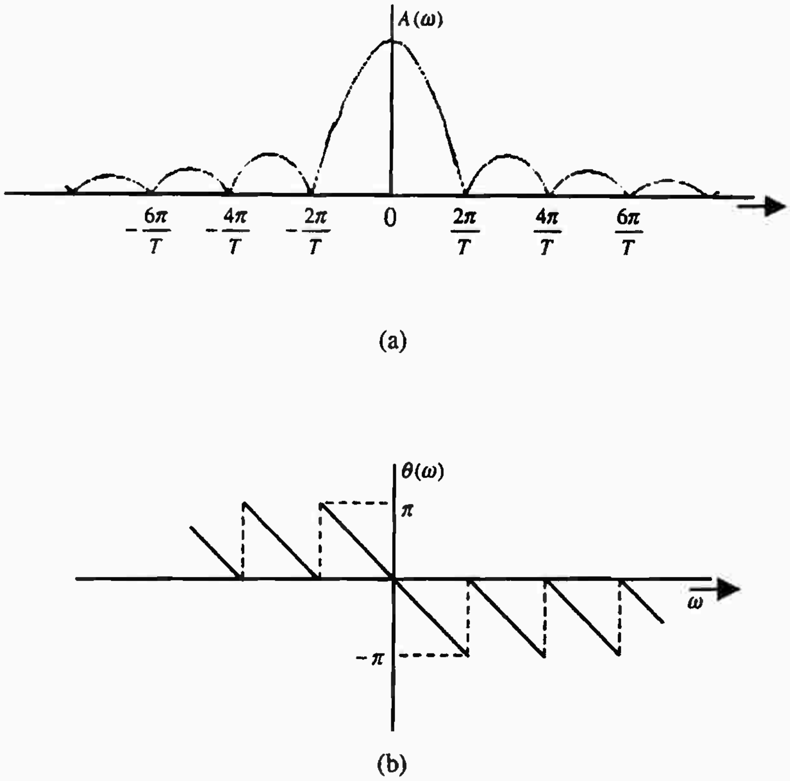

Fig. D.3 Spectra of rectangular pulse
a) amplitude b) phase

One consequence of this property is that there is a minimum value of time duration-bandwidth product that can be obtained with any signal. It is readily shown from the defining equations of the Fourier transform, eqn. (D-6), and eqn. (D-11), that

$$\int_{-\infty}^{\infty} f(t)dt = F(0) \tag{D-16}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)d\omega = f(0) \tag{D-17}$$

Multiplying these equations together and rearranging factors leads to the following relationship

$$\frac{\int_{-\infty}^{\infty} f(t)dt}{f(0)} \cdot \frac{\int_{-\infty}^{\infty} F(j\omega)d\omega}{F(0)} = 2\pi \tag{D-18}$$

The two factors can be thought of as the equivalent duration and equivalent bandwidth of the signals respectively. In each case, it is seen that the equivalent width is the area of the function divided by the ordinate at the origin. If the centroid of the time signal is at t_0 rather than at the origin, we find

$$\frac{\int_{-\infty}^{\infty} f(t)dt}{f(t_0)} \cdot \frac{\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t_0}}{F(0)} = 2\pi \tag{D-19}$$

It is seen as long as $f(t) \neq 0$ and $F(0) \neq 0$, the product (eqn. D-18) is a constant, and an increase in one must therefore always result in a compensating decrease in the other. As an example, consider the rectangular pulse, $AP_T(t)$. The centroid of the pulse is at $t = T/2$. The equivalent duration T_{eq} and the equivalent bandwidth B_{eq} are

$$T_{eq} = \frac{A\int_{-\infty}^{\infty} P_T(t)dt}{AP_T(T/2)} = \int_0^T dt = T \tag{D-20}$$

$$B_{eq} = \frac{A\int_{-\infty}^{\infty} P_T(j\omega)\, e^{j\omega T/2} d\omega}{AP_T(0)} = \frac{T\int_{-\infty}^{\infty} \mathrm{sinc}(\omega T/2\pi) d\omega}{T} = \frac{2\pi}{T} \tag{D-21}$$

From eqns. (D-20) and (D-21),

$$T_{eq}B_{eq} = 2\pi \tag{D-22}$$

The Fourier transform of the derivative of a time function can be expressed in terms of the transform of the original function.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

$$\frac{df(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(j\omega)e^{j\omega t}d\omega \tag{D-23}$$

Now since there is a unique one-to-one correspondence between a Fourier transform and its inverse, and the right-hand side of eqn. (D-23) is the inverse Fourier transform of $j\omega F(j\omega)$, it immediately follows that

$$\mathcal{F}^{-1}\{j\omega F(j\omega)\} = \frac{df(t)}{dt} \quad ,$$

and, conversely,

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(j\omega) \quad , \qquad \text{(D-24)}$$

Thus, it is seen that the transform of a derivative is just $j\omega$ times the transform of the original function.

Consider the general linear system shown schematically in Fig.D.4.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

Fig. D.4 A general linear system

For zero initial energy storage, the relationship between the input $x(t)$ and output $y(t)$ is given by the convolution of the input signal and the impulse response as follows:

$$y(t) = \int_{-\infty}^{\infty} h(y)x(t-\lambda)d\lambda \qquad \text{(D-25)}$$

Taking the Fourier transform of both sides of eqn. (D-25) gives

$$Y(j\omega) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda\right]e^{-j\omega t}dt$$

Inverting the order of integration on the right-hand side, changing the variable of integration and carrying out the indicated operation leads to the following:

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} h(\lambda)\left[\int_{-\infty}^{\infty} x(t-\lambda)e^{-j\omega t}dt\right]dx \\ &= \int_{-\infty}^{\infty} h(\lambda)\left[\int_{-\infty}^{\infty} x(\varepsilon)\, e^{-j\omega(\varepsilon+\lambda)}d\varepsilon\right]d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)e^{-j\omega\lambda}X(j\omega)d\lambda \\ &= X(j\omega)H(j\omega) \qquad \text{(D-27)} \end{aligned}$$

There are two important results implied in eqn. (D-27). First, the Fourier transform of the convolution of two functions is equal to the product of the Fourier transforms of the functions taken separately. Symbolically this may be stated as

$$\mathcal{F}\{f_1(t)\star f_2(t)\} = F_1(j\omega)F_2(j\omega) \qquad \text{(D-28)}$$

Second, the Fourier transform of the output of a linear system is given by the Fourier transform of the input multiplied by the Fourier transform of the system impulse response. Because of its frequent use, the Fourier transform of the impulse response, $H(j\omega) = \mathcal{F}\{h(t)\}$, is called the system function or transfer function and represents another mathematical model for the system when there is no initial stored energy. The system function is generally found as the ratio, $H(j\omega)=Y(j\omega)/X(j\omega)$, by solution of the circuit equations of the system. Use of the system function concept often greatly simplifies computation of system response and is of enormous value in the theoretical analysis of systems.

The system function $H(j\omega)$ is the ratio of the component of the output corresponding to the frequency ω to the component of the input corresponding to the same frequency. This ratio is commonly called the frequency response of a network or system. It should be noted that $H(j\omega)$ contains both amplitude and phase information. The value of $H(j\omega)$ at some particular frequency can be measured by applying a signal of known frequency and amplitude and measuring the output signal. The ratio of the phasors representing the input and output sinusoids gives the value of $H(j\omega)$. The physical explanation of why $H(j\omega)$ - the Fourier transform of the system impulse response - is the frequency response of the system is readily obtained by formally carrying out the computation of the response of a system in the frequency domain when a unit impulse is applied. The output will be the product of $H(j\omega)$ and the Fourier transform of the unit impulse is

$$X(j\omega)=\mathcal{F}\{\delta(t)\}=\int_{-\infty}^{\infty}\delta(t)\,e^{-j\omega t}dt=1 \qquad \text{(D-29)}$$

From eqn. (D-27), the output due to the impulse function input is given by

$$Y(j\omega)=H(j\omega) \qquad \text{(D-30)}$$

From eqn. (D-29), it is seen that the spectrum of the unit impulse is uniform: that is, all frequency components are present with equal amplitudes and zero initial phase. When a signal having these characteristics is applied to a system, the output is a direct measure of the transmission of each frequency component, and this in turn is just the frequency response of the system.

The energy in a signal $f(t)$ is given as

$$E=\int_{-\infty}^{\infty}[f(t)]^2dt \qquad \text{(D-31)}$$

Letting $F(j\omega)$ be the Fourier transform of $f(t)$, eqn. (D-31) can be written as

$$\int_{-\infty}^{\infty}[f(t)]^2dt=\int_{-\infty}^{\infty}f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega\right]dt \qquad \text{(D-32)}$$

Interchanging the order of integration on the right-hand side of eqn, (D-32), and rearranging gives

$$\int_{-\infty}^{\infty}[f(t)]^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\left[\int_{-\infty}^{\infty}f(t)e^{j\omega t}dt\right]d\omega \qquad \text{(D-33)}$$

The factor in this integrand that is enclosed in brackets is seen to be $F(-j\omega)$. Therefore, eqn. (D-33) can be written as

$$\int_{-\infty}^{\infty}[f(t)]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(-j\omega)d\omega \qquad \text{(D-34)}$$

This can be put into a somewhat simpler form by noting from the definition of the Fourier transform that if $f(t)$ is real (that is, it has no imaginary part), then $F(-j\omega) = F^*(j\omega)$, the complex conjugate of $F(j\omega)$.
Using this relationship, eqn. (D-34) can then be written as

$$\int_{-\infty}^{\infty}[f(t)]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F^*(j\omega)d\omega \qquad \text{(D-35)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega \qquad \text{(D-36)}$$

The relationship in eqn. (D-34), which is a fundamental property of Fourier transform, is called P Pa states that the energy in the signal $f(t)$ is equal to $1/2\pi$ times the area under the square of the magnitude of the Fourier transform of $f(t)$. The quantity $|F(j\omega)|^2$ is called the energy spectrum, or energy density spectrum of $f(t)$, since from eqn. (D-36), it can be interpreted as the distribution of energy with frequency. The units of $|F(j\omega)|^2$ are dependent on the units of $f(t)$. For example, if $f(t)$ were a voltage, then $|F(j\omega)|^2$ would have units of volts^2-seconds per hertz.

For power signals, by truncating the signal over the interval T

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt \qquad \text{(D-37)}$$

We define the power spectral density function $S_f(\omega)$ for truncated time function $f_T(t)$ over period T as

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)d\omega \qquad \text{(D-38)}$$

From eqn. (D-36)

$$\int_{-T/2}^{T/2} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega \qquad \text{(D-39)}$$

where $F_T(\omega)$ is the Fourier transform of the truncated function over period T.
Thus,

$$S_f(\omega) = \lim_{T\to\infty}\frac{|F_T(\omega)|^2}{T} \qquad \text{(D-40)}$$

For energy signals, in order to appreciate the full significance of the energy spectrum, it is necessary to understand how the system function of a linear system affects this spectrum for a signal transmitted through the system. It was previously shown that for the case of a simple system, having a system function $H(j\omega)$, the output and input are related by

$$Y(j\omega) = H(j\omega)X(j\omega)$$

The energy spectrum of the output is then found to be

$$\begin{aligned}|Y(j\omega)|^2 &= Y(j\omega)Y^*(j\omega)\\ &= \left[H(j\omega)X(j\omega)\right]\left[H^*(j\omega)X^*(j\omega)\right]\\ &= |H(j\omega)|^2|X(j\omega)|^2\end{aligned} \qquad \text{(D-41)}$$

From eqn. (D-41), it is seen that the output signal energy spectrum is related to the input signal energy spectrum by the quantity $|H(j\omega)|^2$. Because of this relationship, $|H(j\omega)|^2$ is sometimes called the energy transfer function of the system. Using the energy transfer function concept, it is possible to obtain a better appreciation of the physical significance of the energy spectrum. Suppose that a signal having an arbitrary energy spectrum is passed through an ideal band-pass filter having a narrow pass-band centered at a frequency f_0. The energy transfer function of the filter will be unity for those components lying in the filter passband and will be zero for all other components. The energy spectrum of the output will therefore be just that portion of the energy spectrum of the input corresponding to the frequency in the filter passband (Fig. D.5). The total energy of the output $V_0(t)$ for an input $V_i(t)$ will be given by

$$E_0 = \int_{-\infty}^{\infty} |V_0(f)|^2 df = 2\int_{f_0-W/2}^{f_0+W/2} |V_i(f)|^2 df \quad , \qquad \text{(D-42)}$$

where in eqn. (D-42) W is the bandwidth and the factor $1/2\pi$ was absorbed into the variable of integration. For a sufficiently narrow filter bandpass,
solving for $V_1(f_1)$ gives

$$|V_1(f_1)|^2 = \frac{E_0}{2W} \qquad \text{(D-43)}$$

From this expression it is evident that $|V_1(f_1)|^2$ can be interpreted as the energy per unit bandwidth.

As an example of the energy spectrum of a signal, consider the pulse signal, $P_T(t)$, which was discussed earlier. From eqn. (D-14), we have

$$|P_T(j\omega)|^2 = T^2\left[\frac{\sin \omega T/2}{\omega T/2}\right]^2 \qquad \text{(D-44)}$$

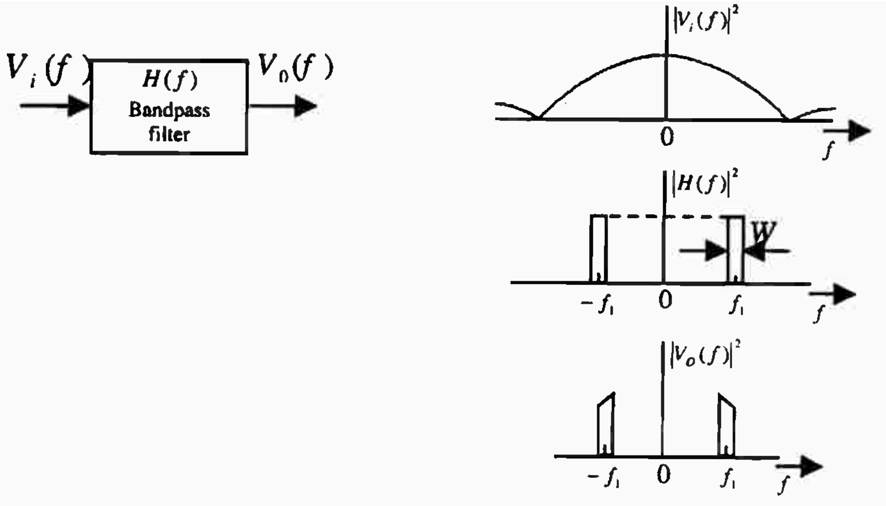

Fig. D.5 Measurement of energy spectrum

The energy spectrum $|P_T(f)|^2$ of the rectangular pulse is shown in Fig.D.6. It is seen that the energy is concentrated in the low-frequency portion of the spectrum, the extent of this concentration can be found by computing the energy in the first lobe (that is, for $|f|<1/T$) and comparing this to the total energy. The ratio, found by graphical integration, is 0.902. Thus, 90.2% of the energy in a rectangular pulse is contained in the band of frequencies below a frequency equal to reciprocal of the pulse width. As a useful rule of thumb, it is often assumed that a pulse transmission system having a bandwidth equal to the reciprocal of the pulse width will perform satisfactorily. Actually, if high-fidelity reproduction of the pulse shape is required, a much greater bandwidth will be necessary. However, it can be seen that a system with this bandwidth will transmit most of the pulse energy.

A number of mathematical operations will now be considered.

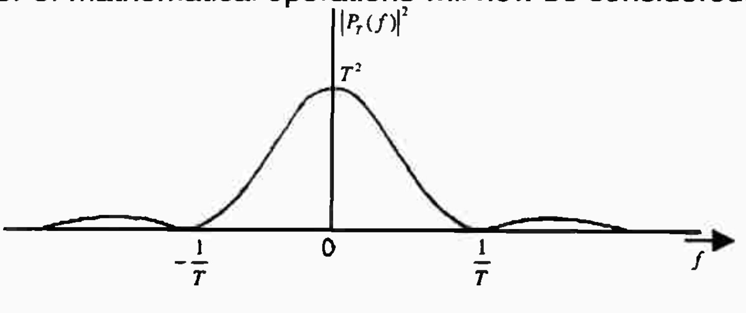

Fig. D.6 Energy spectrum of a rectangular pulse

Scaling: By scaling is meant multiplication of the variable in the time function by a constant. This has the effect of expanding or contracting the time scale depending on whether the magnitude of the constant is less than or greater than unity. If the constant is negative, the time scale is reversed. The Fourier transform of a scaled time function can be obtained as follows:

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt \qquad \text{(D-45)}$$

We must consider the case of positive and negative a separately. Consider first the positive a and change the variable of integration to $\lambda = at$.

$$\mathcal{F}\{f(at)\} = \frac{1}{a}\int_{-\infty}^{\infty} f(\lambda)e^{-j\omega\lambda/a}d\lambda$$

$$= \frac{1}{a}F\left(\frac{j\omega}{a}\right) \quad a > 0 \qquad \text{(D-46)}$$

When a is negative, the limits on the integral will be reversed when the variable of integration is changed; the final result is

$$\mathcal{F}\{f(at)\} = -\frac{1}{a}F\left(\frac{j\omega}{a}\right) \quad a < 0 \qquad \text{(D-47)}$$

These two results can be combined to give to give

$$\mathcal{F}(at) \Leftrightarrow -\frac{1}{|a|}F\left(\frac{j\omega}{a}\right) \qquad \text{(D-48)}$$

Delay: When a new variable $t - t_0$ is substituted for the original variable t in a time function, the resulting function is an exact replica of the original function delayed by an amount t_0. The Fourier transform of the modified function is

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt \qquad \text{(D-49)}$$

Changing the variable of integration of integration and carrying out the indicated operations gives

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(\lambda)\, e^{-j\omega(\lambda+t_0)}d\lambda \qquad \text{(D-50)}$$

$$= e^{j\omega t_0}\int_{-\infty}^{\infty} f(\lambda)\, e^{-j\omega\lambda}d\lambda \qquad \text{(D-51)}$$

$$= e^{-j\omega t_0}F(j\omega) \qquad \text{(D-52)}$$

$$f(t-t_0) \Leftrightarrow e^{-j\omega t_0}F(j\omega) \qquad \text{(D-53)}$$

Delay in the time domain is thus seen to correspond to introduction of a phase shift in the frequency domain that varies linearly with frequency. As an example of the use of eqn. (D-51), consider a rectangular pulse shifted by an amount $T/2$, Therefore,

$$f(t) = P_t(t + T/2).$$

The corresponding transform is then found from eqn. (D-51) to be

$$F(j\omega) = e^{j\omega T/2} P_T(j\omega) \tag{D-54}$$

$$= T e^{j\omega T/2} e^{-j\omega T/2} \frac{\sin \omega T/2}{\omega T/2}$$

$$= T \frac{\sin \omega T/2}{\omega T/2} \tag{D-55}$$

Expressed in terms of the frequency f this can be stated as

$$F(f) = T \frac{\sin \pi T f}{\pi T f} = T \operatorname{sinc}(Tf) \tag{D-56}$$

Modulation: Multiplication of a time function by the complex sinusoid causes a translation in the frequency domain.

Thus,

$$\mathcal{F}\{e^{j\omega_0 t} f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j(\omega-\omega_0)t} dt$$

$$= F[j(\omega - \omega_0)] \tag{D-57}$$

$$e^{j\omega_0 t} f(t) \Leftrightarrow F[j(\omega - \omega_0)]$$

$$\Leftrightarrow F(f - f_0) \tag{D-58}$$

The relationship in eqn. (D-57) can be thought of as a process in which the complex sinusoid is modulated by the time function $f(t)$. If instead of $e^{j\omega_0 t}$, we consider the real function, $\cos \omega_0 t$, we obtain the following relationship:

$$\mathcal{F}\{f(t)\cos \omega_0 t\} = \int_{-\infty}^{\infty} f(t) \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-j\omega t} dt$$

$$f(t)\cos \omega_0 t \Leftrightarrow \frac{1}{2} F[j(\omega - \omega_0)] + \frac{1}{2} F[j(\omega + \omega_0)] \tag{D-59}$$

$$\Leftrightarrow \frac{1}{2} F(f - f_0) + \frac{1}{2} F(f + f_0) \tag{D-60}$$

Thus, modulation of a cosine wave by a time function $f(t)$ leads to a new function having a spectrum consisting of half the original spectrum translated along the positive frequency axis by an amount f_0 and half the original spectrum translated along the negative frequency axis by the amount $-f_0$. An example of this process is shown in Fig. D.7.

Reversal: When a time function is reflected about the origin, the corresponding spectrum is also reflected about the origin. In equation form,

$$f(-t) \Leftrightarrow F(-j\omega) \tag{D-61}$$

This relationship follows immediately taking the scale factor to be -1.

Symmetry: Because of the similarity between the integrals defining the Fourier transform and its inverse, there is a very close relationship between the transform of a particular function of t and the inverse transform of that same function of $j\omega$. The precise relationship is

$$\mathcal{F}(jt) \Leftrightarrow 2\pi f\,(-\omega) \tag{D-62}$$

$$\frac{1}{2\pi}\mathcal{F}(-t) \Leftrightarrow f\,(j\omega) \tag{D-63}$$

As an example, consider the time function corresponding to a transform that is constant over a specified frequency band and zero elsewhere. This situation is illustrated in Fig. D.8a, in which the shape of the transform is assumed to be a rectangular pulse of width $4\pi W$ radians per second and unity amplitude. The phase function is assumed to be zero. Such a transform corresponds to the transfer function, $H(j\omega)$ of an ideal low-pass filter.

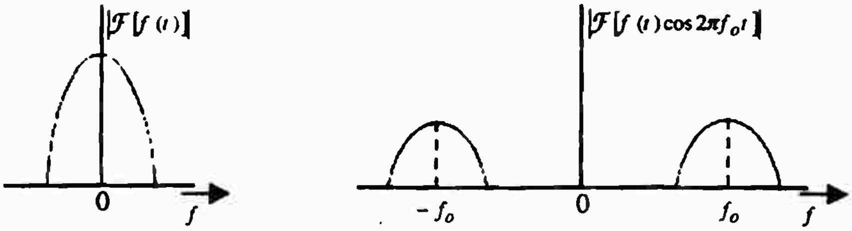

Fig. D.7. Spectrum of modulated cosine wave

(a) (b)

Fig. D.8. System function and impulse response of an ideal low-pass filter.

Accordingly, the inverse transform will be the impulse response $h(t)$ of such a filter. The inverse transform can be found directly from eqn. (D-63) by using the previously established relationship for a pulse signal; that is,

$$P_T(t) \Leftrightarrow T\,\frac{\sin \omega T/2}{\omega T/2}e^{-j\omega T/2} \tag{D-64}$$

In the present instance, we can write the frequency function as

$$H(j\omega) = P_{4\pi W}(\omega + 2\pi W) \tag{D-65}$$

Therefore, the corresponding transform is

$$h(t)=\frac{1}{2\pi}\left[4\pi W\frac{\sin 4\pi Wt/2}{4\pi Wt/2}\right]=2W\mathrm{sinc}(2Wt) \qquad \text{(D-66)}$$

This impulse response is shown in Fig. D.8b. It is clear from the figure that this is not a physically realizable system because the output occurs prior to application of the input. One way of approximating the ideal filter response is to employ a system having a response of form Fig. D.8b, but delayed in time. The greater the delay the more nearly the shape of Fig. D.8b can be reproduced by a physically realizable filter. The effect of the delay in the frequency domain is to produce a phase shift which varies linearly with frequency.

Fourier transforms of power signals. The ordinary Fourier transform is limited to the transformation of functions that are absolutely integrable-that is, functions that obey the inequality

$$\int_{-\infty}^{\infty}|f(t)|dt<\infty \qquad \text{(D-67)}$$

A number of functions having great usefulness do not meet this requirement; for example, a sine wave or a step function does not satisfy eqn. (D-67). Many such functions, can nevertheless, be handled by allowing the Fourier transform to contain impulses, or - in some cases higher order singularity function. This procedure can be put on a rigorous mathematical basis by means of the theory of generalized functions and by showing that correct results are obtained when this method is used.

Consider the function $\mathrm{sgn}(t)$, called signum *t*, which is defined as

$$\mathrm{sgn}(t)=\begin{cases}-1 & t<0\\ 0 & t=0\\ +1 & t>0\end{cases} \qquad \text{(D-68)}$$

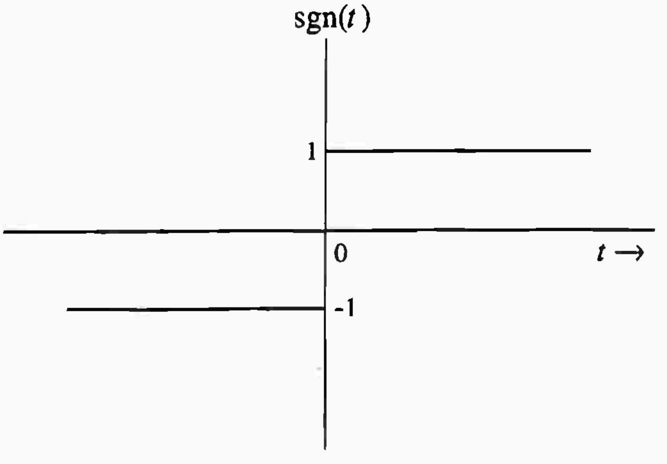

Fig. D.9 Signum function, sgn(t)

This function is shown in Fig. D.9. It is evident that it has a zero average value and is not absolutely integrable. The Fourier transform of this function cannot be computed in the formal manner since it leads to a divergent integral. Consider instead a sequence of functions that approaches $\operatorname{sgn}(t)$ as a limit. Such a sequence can be obtained by introducing a suitable convergence factor multiplying $\operatorname{sgn}(t)$. A suitable function is $e^{-a|t|}\operatorname{sgn}(t)$. The transform may now be computed as

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \mathcal{F}\left\{\lim_{a\to 0} e^{-a|t|}\operatorname{sgn}(t)\right\} \tag{D-69}$$

Interchanging the limiting and integration operations gives

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-a|t|}\operatorname{sgn}(t)e^{-j\omega t}\,dt \tag{D-70}$$

$$= \lim_{a\to 0}\left[\int_{-\infty}^{0} -e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt\right]$$

$$= \lim_{a\to 0}\left[\left.\frac{-e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}\right]$$

$$= \lim_{a\to 0}\left[\frac{-1}{a-j\omega} + \frac{1}{a+j\omega}\right] = \frac{2}{j\omega} \tag{D-71}$$

$$\operatorname{sgn}(t) \Leftrightarrow \frac{2}{j\omega} \tag{D-72}$$

The amplitude of the spectrum is shown plotted in Fig. D.10.

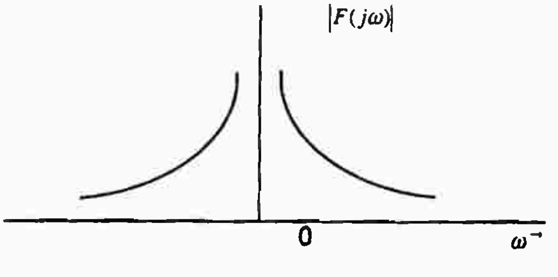

Fig. D.10 Amplitude spectrum of sgn(t)

The spectrum of a constant can be found in a similar fashion using the same type of convergence factor:

$$\mathcal{F}\{1\} = \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}\,dt \tag{D-73}$$

$$= \lim_{a\to 0}\left[\int_{-\infty}^{0} -e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt\right]$$

$$= \lim_{a\to 0}\left[\frac{e^{(a-j\omega)t}}{a-j\omega}\bigg|_{-\infty}^{0} - \frac{e^{-(a+j\omega)t}}{a+j\omega}\bigg|_{0}^{\infty}\right]$$

$$= \lim_{a\to 0}\left[\frac{1}{a-j\omega} + \frac{1}{a+j\omega}\right]$$

$$= \lim_{a\to 0}\left[\frac{2a}{a^2+\omega^2}\right] \qquad \text{(D-74)}$$

It is seen that the limit as $a \to 0$ in eqn. (D-74) is zero except when $\omega = 0$ for which case an indeterminate form is obtained - the indeterminate form can be evaluated by L'Hospital's rule to give

$$\lim_{a\to 0}\left[\frac{2}{2a}\right] = \infty$$

The area under the function is found to be

$$Area = \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2}d\omega = 2\tan^{-1}\left(\frac{\omega}{a}\right)\bigg|_{-\infty}^{\infty}$$

$$= 2\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 2\pi \qquad \text{(D-75)}$$

Therefore, we can write the Fourier transform of a constant of unit magnitude as

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \qquad \text{(D-76)}$$

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

$$\Leftrightarrow \delta(f) \qquad \text{(D-77)}$$

This result can be extended to include sinusoidal functions in the following manner. From eqn. (D-76) we have

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-j\omega t}dt = 2\pi\delta(\omega) \qquad \text{(D-78)}$$

By use of eqn. (D-78) and the modulation theorem of the Fourier transform, we can obtain the transform of the complex exponential as

$$\mathcal{F}\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} e^{j\omega_0 t - j\omega t}dt = 2\pi\delta(\omega-\omega_0) \qquad \text{(D-79)}$$

The transform of the sine and cosine functions are found to be

$$\mathcal{F}\{\cos\omega_0 t\} = \mathcal{F}\left\{\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right\} \qquad \text{(D-80)}$$

$$\cos\omega_0 t \Leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \qquad \text{(D-81)}$$

$$\Leftrightarrow \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)] \tag{D-82}$$

$$\sin\omega_0 t \Leftrightarrow -j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] \tag{D-83}$$

$$\Leftrightarrow -j\frac{1}{2}[\delta(f-f_0)-\delta(f+f_0)] \tag{D-84}$$

From these relationships and the various transform operations, it is possible to derive most of the transforms needed. For example, the transform of the step function can be found using the equality $u(t)=1/2+1/2\,\mathrm{sgn}(t)$ as follows.

$$\mathcal{F}\{u(t)\}=\mathcal{F}\left\{\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t)\right\}=\pi\delta(\omega)+\frac{1}{j\omega} \tag{D-85}$$

$$u(t) \Leftrightarrow \pi\delta(\omega)+\frac{1}{j\omega} \tag{D-86}$$

A useful expression for the Fourier transform of an indefinite integral can be obtained by considering the integral as the convolution of a function $f_1(t)$ and the unit step. Thus,

$$f_2(t)=\int_{-\infty}^{t} f_1(\lambda)d\lambda=\int_{-\infty}^{\infty} f_1(\lambda)u(t-\lambda)d\lambda \tag{D-87}$$

$$=f_1(t)\star u(t) \tag{D-88}$$

From eqn. (D-28), which states that the transform of the convolution of two functions is equal to the product of the transforms of the functions taken separately, we have

$$F_2(j\omega)=F_1(j\omega)\left[\frac{1}{j\omega}+\pi\delta(\omega)\right] \tag{D-89}$$

Therefore,

$$\int_{-\infty}^{t} f_1(\lambda)d\lambda \Leftrightarrow \frac{1}{j\omega}F_1(j\omega)+\pi F_1(0)\delta(\omega) \tag{D-90}$$

$$\int_{-\infty}^{t} f_1(\lambda)d\lambda \Leftrightarrow \frac{1}{j2\pi f}F_1(f)+\frac{1}{2}F_1(0)\delta(f) \tag{D-91}$$

As an example of this expression, we will compute the Fourier transform of $\sin\omega_0 t$.

$$\mathcal{F}\{\sin\omega_0 t\}=\mathcal{F}\{\omega_0\int_0^t\cos\omega_0\lambda d\lambda\}=\omega_0\mathcal{F}\int_{-\infty}^{t}\cos(\omega_0\lambda)u(\lambda)d\lambda \tag{D-92}$$

$$=\omega_0\frac{1}{j\omega}[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]+\pi\sin(0)\delta(\omega) \tag{D-93}$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] \tag{D-94}$$

Periodic functions:

Since a periodic function can be expressed as a sum of complex exponentials, and complex exponentials can be Fourier transformed, it should be possible to take the Fourier transform of a periodic function by taking the transform of each term in the expansion. Suppose we are given a periodic function $f(t)$ with period T. We can proceed formally to obtain the Fourier transform of $f(t)$ by first writing the Fourier series for $f(t)$ as

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \tag{D-95}$$

The Fourier transform then becomes

$$F(j\omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right\} \tag{D-96}$$

$$F(j\omega) = \sum_{n=-\infty}^{\infty} a_n \mathcal{F}\{e^{jn\omega_0 t}\} \tag{D-97}$$

$$= 2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0) \tag{D-98}$$

Now a_n can be expressed in terms of the Fourier transform of $f_T(t)$, the waveform over one period, as follows:

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt \tag{D-99}$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} f_T(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}F_T(jn\omega_0) \tag{D-100}$$

Therefore,

$$F(j\omega) = \omega_0 \sum_{n=-\infty}^{\infty} F_T(jn\omega_0)\delta(\omega - n\omega_0) \tag{D-101}$$

$$f(t) \Leftrightarrow \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} F_T\left(j\frac{2\pi n}{T}\right)\delta\left(\omega - \frac{2\pi n}{T}\right) \tag{D-102}$$

Converting to the variable f instead of ω, the relationship becomes

$$F(f) = \sum_{n=-\infty}^{\infty} a_n \delta(f - nf_0) \tag{D-103}$$

$$f(t) \Leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} F_T\left(j\frac{2\pi n}{T}\right)\delta\left(f-\frac{n}{T}\right) \tag{D-104}$$

The Fourier transform is thus seen to be a series of impulses at the harmonics of the repetition period with strengths determined by the shape of the waveform in one period.

As an example of the Fourier transform of a periodic function, consider the sampling function $f_s(t)$ shown in Fig. D.11.

The Fourier transform of the truncated function $f_{sT}(t)$ is

$$f_{sT}(t) = \begin{cases} f_s(t) & -T/2 < t < T/2 \\ 0 & \text{elsewhere} \end{cases} \tag{D-105}$$

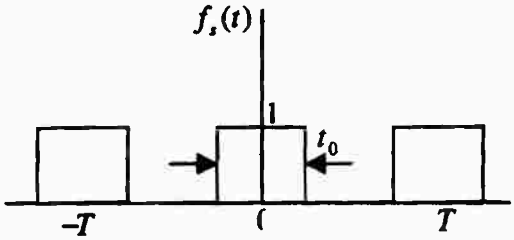

Fig. D.11 The sampling function $f_s(t)$ ·

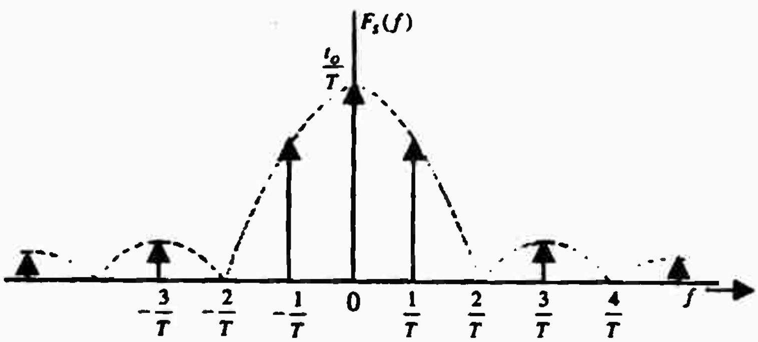

Fig. D.12 Amplitude spectrum of the sampling function.

$$F_{sT}(j\omega) = t_0 \frac{\sin \omega t_0/2}{\omega t_0/2} \tag{D-106}$$

Therefore, the Fourier transform of $f_s(t)$ is

$$F_s(j\omega) = \frac{2\pi t_0}{T}\sum_{n=-\infty}^{\infty} \frac{\sin \pi n t_0/T}{\pi n t_0/T}\delta\left(\omega-\frac{2\pi n}{T}\right) \tag{D-107}$$

$$F_s(f) = \frac{t_0}{T}\sum_{n=-\infty}^{\infty} \frac{\sin \pi n t_0/T}{\pi n t_0/T}\delta\left(f-\frac{n}{T}\right) \tag{D-108}$$

The amplitude spectrum of $f_s(t)$ is shown plotted in Fig. D.12. for $t_0/T = 1/2$

Another important signal is the unit impulse train. Consider a function $f(t)$ composed of an infinite train of unit impulses having a repetition period T. Such a function is shown in Fig. D.13. The Fourier transform of this function is very important in sampling theory and can be determined by first expanding $f(t)$ in a Fourier series and then taking the Fourier transform of the terms in the expansion.

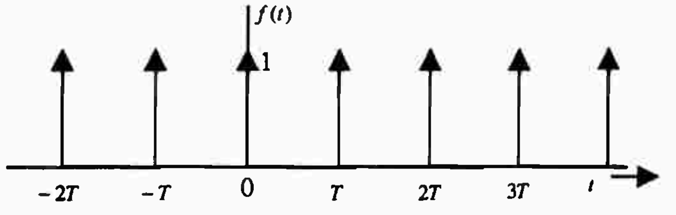

Fig. D.13 Unit impulse train.

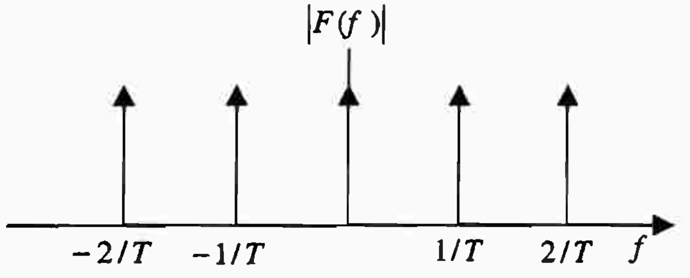

Fig. D.14. Amplitude spectrum of a unit impulse train.

Following this procedure we obtain

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{D-109}$$

$$= \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \quad , \qquad \omega_0 = \frac{2\pi}{T} \tag{D-110}$$

The coefficient a_n is given by

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \tag{D-111}$$

Therefore, $f(t)$ can be written as

$$f(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{D-112}$$

Taking the Fourier transform gives

$$F(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathcal{F}\{e^{jn\omega_0 t}\} \tag{D-113}$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0) \tag{D-114}$$

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) \Leftrightarrow \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - \frac{2\pi n}{T}) \tag{D-115}$$

In terms of the frequency f, eqn. (D-115) can be written as

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) \Leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(f - \frac{n}{T}) \tag{D-116}$$

Thus, it is seen that an impulse train in the time domain has its Fourier transform an impulse train in the frequency domain. The amplitude spectrum is shown in Fig. D.14.

The property of the Fourier transform that convolution in the time domain corresponds to multiplication in the frequency domain also has a converse; that is, the transform of the product of two time functions is the convolution of their transforms. The exact relationship is

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\xi)F_2\{j(\omega-\xi)\}d\xi = \frac{1}{2\pi}F_1(j\omega)\star F_2(j\omega) \tag{D-117}$$

$$f_1(t)f_2(t) \Leftrightarrow F_1(f)\star F_2(f) \tag{D-118}$$

These expressions can be verified by formally inverting the right-hand side.

As an example of the application of eqn. (D-117), consider the determination of the spectrum of an amplitude-modulated signal. Such a signal may be represented as follows

$$f(t) = A[1+f_m(t)]\cos(\omega_0 t + \phi) \tag{D-119}$$

In eqn. (D-119), the modulating function f_m must be normalized to have a maximum amplitude less than or equal to unity in order to prevent over-modulation. The function f_m(t) can have either a continuous or a discrete spectrum. The spectrum of the modulation signal is given by the following convolution:

$$F(j\omega) = A\left[\mathcal{F}\{\cos(\omega_0 t+\phi)\} + \frac{1}{2\pi}\mathcal{F}\{f_m(t)\}\star\mathcal{F}\{\cos(\omega_0 t+\phi)\}\right]$$

$$= A\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]e^{j\phi\omega/\omega_0} + \frac{A\pi}{2\pi}F_m(j\omega)\star[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]e^{j\phi\omega/\omega_0}$$

$$= A\pi\{\delta(\omega-\omega_0)e^{j\phi}+\delta(\omega+\omega_0)e^{-j\phi}+\frac{1}{2\pi}F_m\left[j(\omega-\omega)_0\right]e^{j\phi}+\frac{1}{2\pi}F_m\left[j(\omega+\omega_0)\right]e^{-j\phi}\} \tag{D-120}$$

It is seen that the spectrum of $F(j\omega)$ consists of a discrete component at the carrier frequency ω_0 and a reproduction of $F_m(j\omega)$ of $F(j\omega)$ around $\pm\omega_0$. Fig. D.15 shows the spectrum involved in an amplitude modulation system.

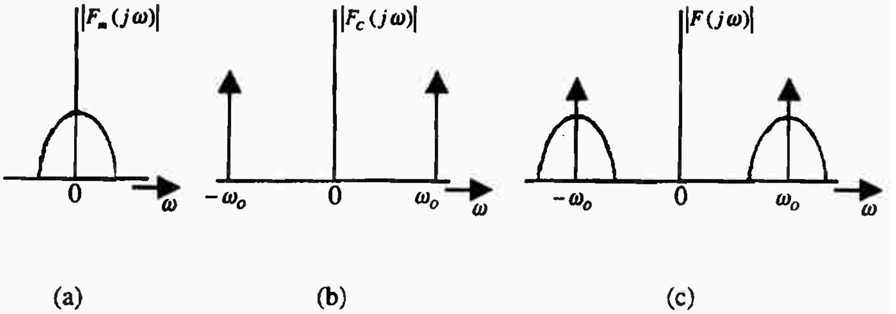

Fig. D.15. Amplitude modulation
a) modulating signal b) carriar c) AM signal

References

1- "Methods of Signal and System Analysis", G. Cooper and C. McGillem, Holt Rinehart, Winston, 1967.

2- "Fourier Integrals for practical Applications", G. Campbell and R. Foster, Van Nostrand, Princeton, N. Y., 1948.

3- "Fourier Integral and Its Applications", A. Papoulis, McGraw Hill, N. Y., 1962.

4- "Fourier Integral and its Applications", R. Bracewell, McGraw Hill, N. Y., 1965.

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Appendix E
Sampling Theorem

The sampling theorem for low passband limited signals states that: A low passband limited function with frequency components within the interval $-W$ to W Hz may be described by a set of sample values taken at time instants separated by $1/2W$ seconds or less. Sampling is accomplished by a switch, which periodically opens and closes. The switch operation may be mechanical, but it is more likely to be electronic. If the switch is in series with the signal source Fig. E.1, then the interval during which the switch is closed is normally very short compared to its open interval. The mathematical representation of this type of sampling operation simply involves multiplying the signal function by a sampling function $f_s(t)$, which is zero whenever the switch is open and unity when the switch is closed. This is indicated in Fig. E.2.

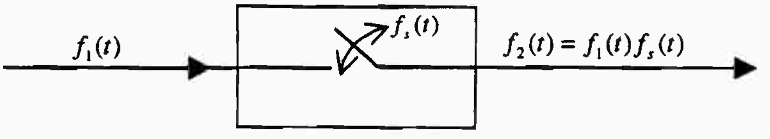

Fig. E.1 Signal samples

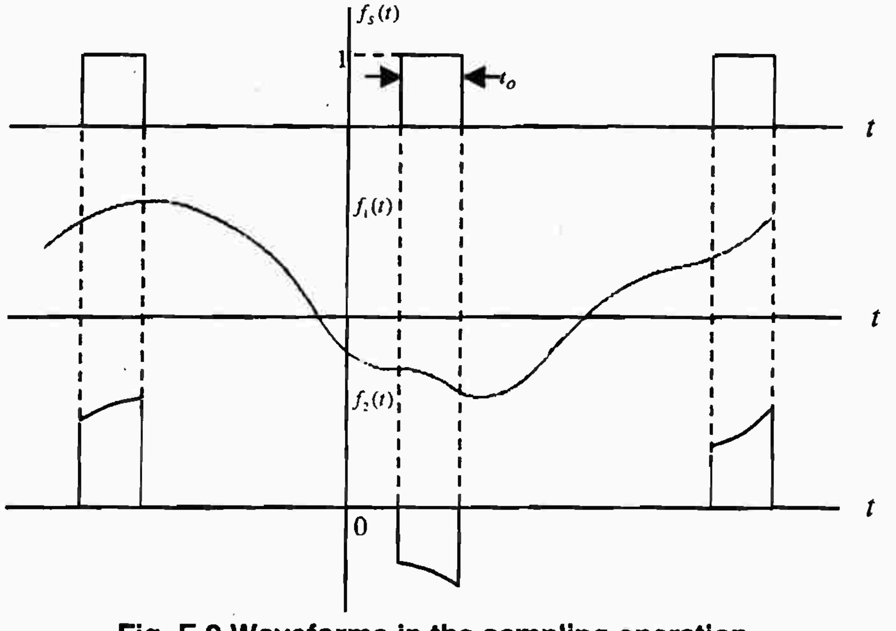

Fig. E.2 Waveforms in the sampling operation

The pulse width assumed to be infinitesmally small. Since the output of the sampler is the product of two time functions, it is now possible to use convolution in the frequency domain to show why, and under what circumstances, the sampling theorem is valid. In order to do this, let $f_1(t)$ be a low-passband-limited function as shown

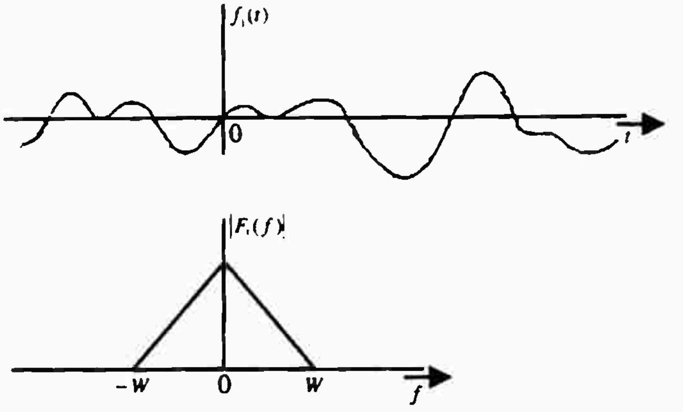

Fig. E.3 Continuous Function to be sampled

The sampler output, $f_2(t)$, has a Fourier transform of

$$F_2(j\omega) = \mathcal{F}\{f_1(t)f_s(t)\} = \frac{1}{2\pi} F_1(\omega) \star F_2(j\omega) \qquad \text{(E-1)}$$

Since $f_s(t)$ is a periodic function, it follows that its Fourier spectrum is a set of delta functions occurring at zero frequency, the sampling frequency and harmonics of the sampling frequency. The strengths of these delta functions vary in accordance with the magnitude of the Fourier spectrum of the sampling pulse shape.

The two spectra and their convolution are shown in Fig. E.4. The convolution is particularly easy to visualize in this case because one of the functions is a series of impulses, and when an impulse is convolved with another function it reproduces that function at the place where the impulse was located. In this case, it is easiest to imagine the function $F_1(f)$ along the impulses being reproduced with an amplitude proportional to the strength of each impulse.

The original signal can be recovered from the sampled signal by passing $f_2(t)$ through a low pass filter of bandwidth W. This filter will pass the portion of the spectrum $f_2(t)$ located at the origin; this is identical to the original signal spectrum $f_1(t)$ except for a scale factor. The mathematical relationships can be derived in the following manner. Referring to Fig. E.5, the output signal can be related to the input by means of the spectrum of the signals.

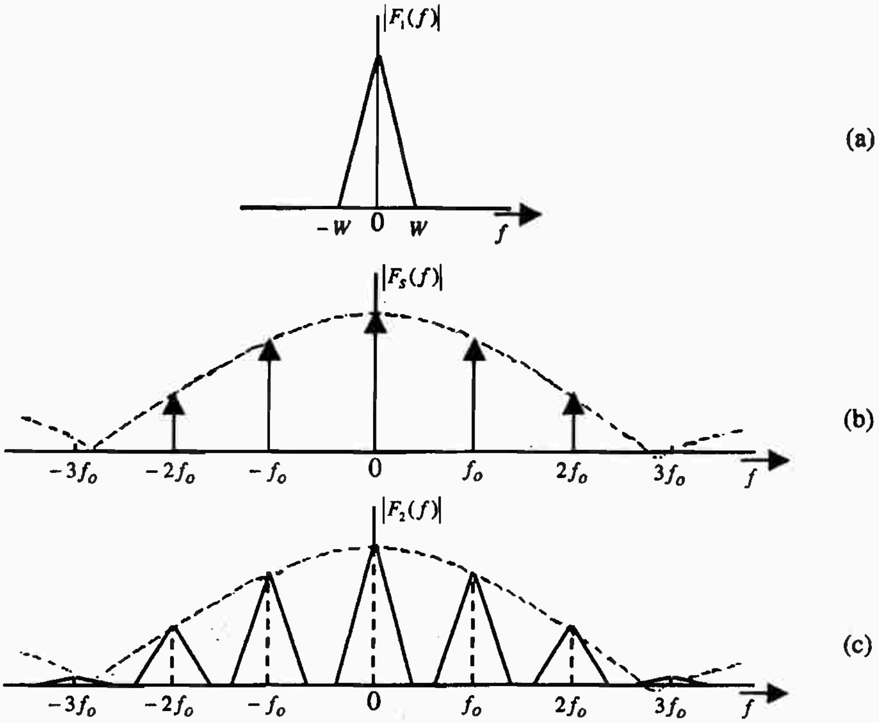

Fig. E.4 Amplitude spectrum of the sampled function
a) band limited signal b) sampling c) sampled function spectrum

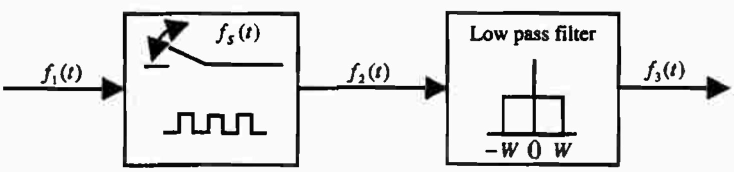

Fig. E.5 Block diagram of the signal recovery system.

$$f_2(t) = f_1(t) f_s(t) \tag{E-2}$$

$$F_2(f) = F_1(f) \star F_s(f) \tag{E-3}$$

$$= F_1(f) \star \frac{t_0}{T} \sum_{n=-\infty}^{\infty} \frac{\sin \pi n t_0 / T}{\pi n t_0 / T} \delta\left(f - \frac{n}{T}\right) \tag{E-4}$$

$$= \frac{t_0}{T} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n t_0}{T}\right) \int_{-\infty}^{\infty} F_1(f - \xi) \delta\left(\xi - \frac{n}{T}\right) d\xi \tag{E-5}$$

$$= \frac{t_0}{T} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n t_0}{T}\right) F_1\left(f - \frac{n}{T}\right) \tag{E-6}$$

$$F_3(f) = \mathrm{LPF}\{F_2(f)\} = \frac{t_0}{T} F_1(f) \tag{E-7}$$

$$F_1(f) = \frac{T}{t_0} F_3(f) \tag{E-8}$$

Hence,

$$f_1(t) = \frac{T}{t_0} f_3(t) \tag{E-9}$$

The original signal can thus be recovered from the sampled signal by passing the sampled signal through an ideal low-pass filter having a gain of T/t_0. Proper sampling requires that the samples be taken at a rate at least twice the highest frequency present in $f(t)$. The necessity for this may be seen by reference to Fig. E.4. Since the spectrum of the original signal is reproduced symmetrically around each harmonic of the sampling frequency, it is necessary that the harmonic be separated by at least twice the width of the spectrum.

Therefore, the first harmonic or fundamental of the sampler must be at least $f_0 = 2W$. If the signal sampled at a rate lower than twice the bandwidth of the original signal, there will be overlapping of the spectra as shown in Fig. E.6. When this occurs, it is not possible to recover the original function without distortion (aliasing). The amount of distortion will depend on the amount of under sampling and on the shape of the spectrum of the signal being sampled. It is important to note that the duration and shape of the pulses in $f_s(t)$ do not affect the ability to recover $f_1(t)$ from its samples. As can be seen from Fig. E.4, it is necessary only that $f(t)$ be periodic with a fundamental frequency f_0, that is at least twice the highest frequency contained in $f_1(t)$.

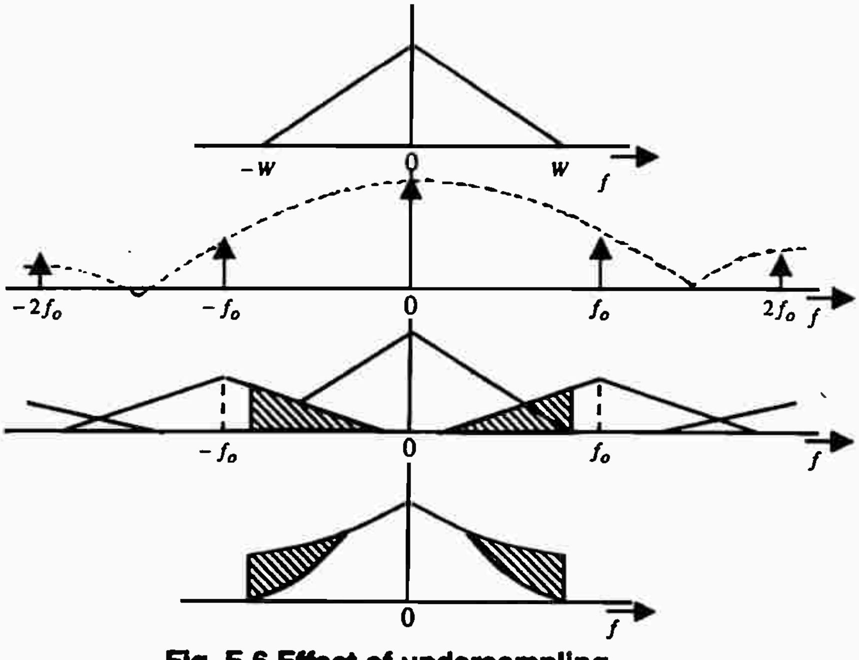

Fig. E.6 Effect of undersampling

This same operation can be carried out in the time domain by performing on the samples an operation that is equivalent to low pass filtering. Since low pass filtering in the frequency domain corresponds to multiplying the spectrum by the filter transfer function, it follows that the equivalent time domain operation will be convolution of the inverse transform of the filter function with the sampled signal $f_s(t)$. The precise relationship in the time domain can be established by carrying out the indicated convolution. However, a simpler demonstration of the desired relationship can be obtained by use of the Fourier series in the following manner. Let the time function $f(t)$ have a Fourier spectrum that is limited to the band $\pm W$, as shown in Fig. E.3. Assume that $f(t)$ is known at discrete times $t_n = n/2W$. The value of $f(t_n)$ can be found in terms of the Fourier transform $F(j\omega)$ as follows:

$$f(t_n) = f\left(\frac{n}{2W}\right) = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} F(j\omega)\exp\left(j\omega\frac{n}{2W}\right)d\omega \qquad \text{(E-10)}$$

This can be written as

$$f(t_n) = 2W\left\{\frac{1}{4\pi W}\int_{-2\pi W}^{2\pi W} F(j\omega)\exp\left[-j\left(\frac{n\omega}{2W}\right)\right]d\omega\right\} \quad , \qquad \text{(E-11)}$$

which is just $2W$ times the n^{th} coefficient of the exponential Fourier series expansion for $F(j\omega)$ over the interval $-2\pi W < \omega < 2\pi W$. In view of this, $F(j\omega)$ can be expressed in terms of $f(t_n)$ as

$$F(j\omega) = \sum_{n=-\infty}^{\infty}\frac{f(t_n)}{2W}\exp\left(-j\frac{n\omega}{2W}\right) \qquad \text{(E-12)}$$

Taking the inverse transform gives

$$f(t) = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W}\sum_{n=-\infty}^{\infty}\frac{f(t_n)}{2W}\exp\left(-j\frac{n\omega}{2W}\right)\exp(j\omega t)d\omega \qquad \text{(E-13)}$$

$$= \sum_{n=-\infty}^{\infty}\frac{f(t_n)}{4\pi W}\int_{-2\pi W}^{2\pi W}\exp\left[j\omega\left(t-\frac{n}{2W}\right)\right]d\omega \qquad \text{(E-14)}$$

$$= \sum_{n=-\infty}^{\infty}\frac{f(t_n)}{4\pi W}\left.\frac{\exp\left[\int_{-2\pi W}^{2\pi W} j\omega\left(t-\frac{n}{2W}\right)\right]}{j\left(t-\frac{n}{2W}\right)}\right|_{-2\pi W}^{2\pi W} \qquad \text{(E-15)}$$

$$= \sum_{n=-\infty}^{\infty}\frac{f(t_n)}{4\pi W}\frac{\sin 2\pi W\left(t-\frac{n}{2W}\right)}{\left(t-\frac{n}{2W}\right)} \qquad \text{(E-16)}$$

$$= \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wt - n) \qquad \text{(E-17)}$$

From eqn. (E-17), it is seen that $f(t)$ can be recovered from $f(t_n)$ by summing the properly weighted values of $f(t_n)$. At a particular time t_1, the value of $f(t_1)$ is obtained by multiplying each of the samples $f(t_n)$ by the factor $\operatorname{sinc}(2Wt - n)$ and summing the resulting terms. The influence of samples away from t_1 on the value of $f(t)$ at $t = t_1$ is determined by how rapidly $\operatorname{sinc}(2Wt - n)$ decreases with n. Depending on the sampling interval and the spectrum of the particular function being reconstructed, the number of samples required to obtain a given accuracy may vary.

References

1- "Methods of Signal and System Analysis", G. Cooper and C. McGillem, Holt, Rinehart and Winston, N. Y., 1968.

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4- "Digital and Analog Communication Systems", L. Couch 6th ed., Prentice Hall, Englewood Cliffs, N. J., 2001.

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Appendix F
Power and Voltage Spectral Densities

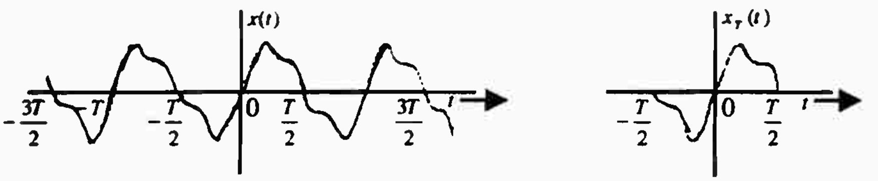
To obtain the value of the average power P_{av} of a power signal consider first the periodic signal $x(t)$ shown in Fig. F.1a. Assume that a sample of this signal is extracted over the period $-T/2 < t < +T/2$ to produce the pulse-like signal $x_T(t)$ shown in Fig. F.1b. If $F_T(\omega)$ is the Fourier transform of $x_T(t)$ then

$$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_T(\omega)\, e^{j\omega t} d\omega \qquad \text{(F-1)}$$

Also, we have

$$\frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)dt = \frac{1}{T}\int_{-T/2}^{+T/2} x_T(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} F_T(\omega) e^{j\omega t} d\omega\right\}dt \quad , \qquad \text{(F-2)}$$

where the second quantity $x_T(t)$ is replaced by the previous expression eqn. (F-1)



(a) (b)

Fig. F.1 A periodic Signal

a) $x(t)$ over all time   b) truncated function $x_T(t)$

Hence, rearranging the order of integration yields

$$\frac{1}{T}\int_{-\infty}^{\infty} x_T^2(t)dt = \frac{1}{2\pi T}\int_{-\infty}^{+\infty} F_T(\omega)d\omega\left[\int_{-T/2}^{+T/2} x_T(t)e^{j\omega t}dt\right] \qquad \text{(F-3)}$$

$$\frac{1}{T}\int_{-T/2}^{+T/2} x_T^2(t)dt = \frac{1}{2\pi T}\int_{-\infty}^{+\infty} F_T(\omega)d\omega\left[\int_{-\infty}^{\infty} x_T(t)e^{j\omega t}dt\right] \qquad \text{(F-4)}$$

We note that $x_T(t)$ is zero over the intervals $-\infty < t < -T/2$ and $+T/2 < t < +\infty$. Also, we have

$$\int_{-\infty}^{+\infty} x_T(t)\, e^{j\omega t} dt = F_T^*(\omega) \qquad \text{(F-5)}$$

where $F_T^*(\omega)$ $F(\omega)$ is the conjugate of $F_T(\omega)$ such that $F_T(\omega)F_T^*(\omega) = |F_T(\omega)|^2$. Hence,

$$\frac{1}{T}\int_{-\infty}^{\infty} x_T^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|F_T(\omega)|^2}{T}d\omega \qquad \text{(F-6)}$$

As more samples of $x(t)$ are removed over time intervals of T and added to $x_T(t)$, the signal $x_T(t)$ will eventually resemble the periodic signal $x(t)$, and so in the limit when $T \to \infty$ we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x_T^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)dt = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|F_T(\omega)|^2}{T}d\omega \qquad \text{(F-7)}$$

The quantity in the center is the average power of the periodic signal $x(t)$, and so we obtain

$$P_{av} = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|F_T(\omega)|^2}{T}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)d\omega \qquad \text{(F-8)}$$

Hence, by inspection, we obtain the power spectral density $S(\omega)$.

$$S(\omega) = \lim_{T\to\infty}\frac{|F_T(\omega)|^2}{T} \qquad \text{(F-9)}$$

We may also obtain the voltage spectral density $S_V(f)$ defined by

$$\int_{-\infty}^{+\infty} S_V(f)df = \overline{v_n^{\,2}} \qquad \text{(F-10)}$$

Where $\overline{v_n^2}$ is the mean-square noise voltage and $\overline{\overline{V_n}}$ is the rms noise voltage. In the case of thermal noise, the noise spectrum is constant over a finite bandwidth *B*. Hence,

$$\int_{-B}^{+B} S_V(f)df = 4kTBR \qquad \text{(F11)}$$

$$S_V(f)\int_{-B}^{+B} df = 4kTBR \qquad \text{(F-12)}$$

Hence

$$S_V(f)2B = 4kTBR \qquad \text{(F-13)}$$

$$S_V(f) = 2kTR \qquad \text{(F-14)}$$

For noise power spectral density S_n

$$\int_{-B}^{B} S_n(f)df = kTB \qquad \text{(F-15)}$$

$$S_n(f) = \frac{kT}{2} \qquad \text{(F-16)}$$

References

1- "Noise", F. Conner, Arnold, London, 1982.

2- "Detection of Signals in Noise", A. Whalen, Academic Press, N. Y., 1971.

3- "Introduction to Mechanical Statistics", P. Hool, J. Wiley, N. Y., 1971.

4- "An Introduction to the Theory of Random Signals and Noise", W. Davenport and W. Root, McGraw Hill, N. Y., 1958.

5- "Communication Systems Design", P. Panter, McGraw Hill, N. Y., 1972.

Appendix G
Wiener Khintchine Theorem

For a sample function $x_T(t)$ which is a single pulse signal

$$S(\omega) = \lim_{T\to\infty} \frac{|F_T(\omega)|^2}{T} \tag{G-1}$$

where $F_T(\omega)$ is the Fourier transform of $x_T(t)$

Now, If $x_T(t)$ is real, we have from transform properties that

$$F_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t} dt \tag{G-2}$$

$$F_T(\omega) = F_T(-\omega) \tag{G-3}$$

also,

$$|F_T(\omega)|^2 = F_T(\omega) F_T(-\omega)$$

or

$$|F_T(\omega)|^2 = \int_{-\infty}^{+\infty} x_T(t_1) e^{-j\omega t_1} dt_1 \int_{-\infty}^{+\infty} x_T(t_2) e^{j\omega t_2} dt_2 \tag{G-4}$$

Hence,

$$S(\omega) = \lim_{T\to\infty} \frac{\int_{-T/2}^{+T/2} x(t_1) e^{-j\omega t} dt_1 \int_{-T/2}^{+T/2} x(t_2) e^{j\omega t_2} dt_2}{T} \tag{G-5}$$

where $x(t)$ replaces $x_T(t)$ since only a sample of $x(t)$ in the interval $-T/2 < t < +T/2$ is being considered.

Interchanging the order of integration, and substituting $t_2 = t_1 - \tau$ yields

$$S(\omega) = \lim_{T\to\infty} \frac{\int_{-T/2}^{+T/2} \int_{-T/2}^{+T/2} x(t_1) x(t_1 - \tau) dt_1 e^{-j\omega\tau} d\tau}{T} \tag{G-6}$$

Now

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t_1) x(t_1 - \tau) dt_1 \tag{G-7}$$

Hence,

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t_1) x(t_1 - \tau) dt_1 = R(\tau) \tag{G-8}$$

This is called Wiener-Khintchine theorem where $R(\tau)$ is the autocorrelation function.

Hence,

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau) e^{-j\omega\tau} d\tau \tag{G-9}$$

By taking the inverse transform we obtain

$$R(\tau) = \frac{1}{2\pi} \int S(\omega) e^{j\omega\tau} d\omega \tag{G-10}$$

Thus, $S(\omega)$ and $R(\tau)$ form a Fourier transform pair.

References

1- "Noise", F. Conner, Arnold, London, 1982.

2- "Detection of Signals in Noise", A. Whalen, Academic Press, N.Y., 1971.

3- "Introduction to Mechanical Statistics", P. Hool, J. Wiley, N. Y., 1971.

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Appendix H
Narrowband Noise

If Gaussian white noise is passed through a narrow bandpass filter with a bandwidth B which is very much less than the center frequency f_c of the filter, the output from the filter is known as narrowband noise. It has the spectrum shown in Fig. H.1a. which can be approximated by a finite number of noise components spaced Δf apart, where $\Delta f \to 0$, and this is shown in Fig. H.1b.

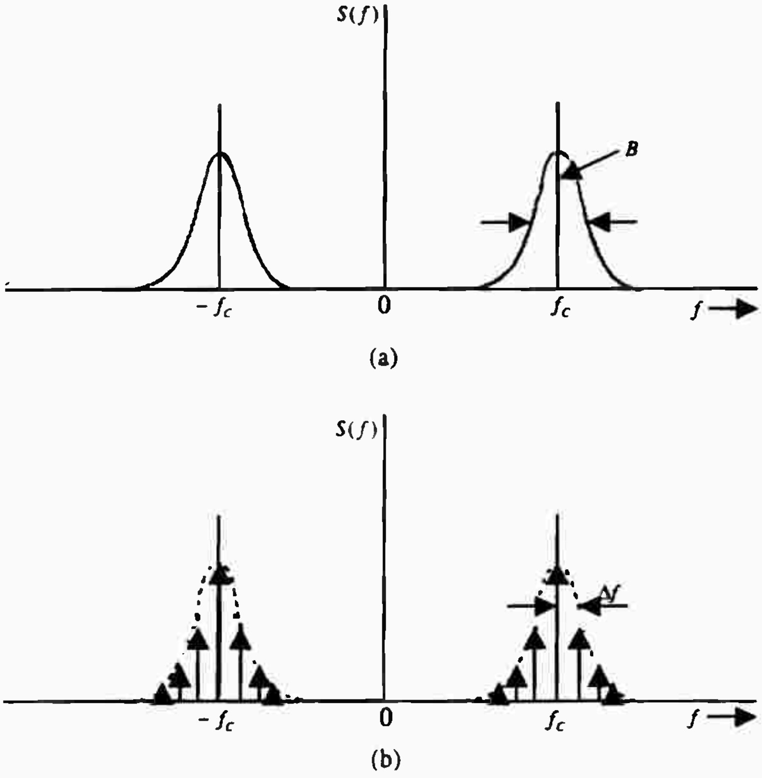

Fig. H.1 Narrowband filter

a) spectrum at the output of filter with center frequency $f_c > B$

b) approximation by noise components in the from of delta functions spaced at $\Delta f\,(\Delta f \to 0)$

Each pair of delta functions, such as $\delta(f + f_c)$ and $\delta(f - f_c)$ can be represented by a sine wave function of arbitrary phase, and the sum of all such spectral components yields the time function $n(t)$ of narrowband noise. Hence, we obtain

$$n(t) = \lim_{\Delta f \to 0} \sum_{0}^{n} a_n \{\sin[(\omega_c + 2\pi n \Delta f)t + \theta_n]\} \quad , \qquad \text{(H-1)}$$

where a_n is the amplitude of the n^{th} frequency component and θ_n is its arbitrary phase. Expanding this expression then yields

$$n(t) = \lim_{\Delta f \to 0} \sum_0^n a_n \{\sin \omega_c t \cos(2\pi n \Delta f t + \theta_n) + \cos \omega_c t \sin(2\pi n \Delta f t + \theta_n)\} \quad \text{(H-2)}$$

or

$$n(t) = x(t) \sin \omega_c t + y(t) \cos \omega_c t \quad , \quad \text{(H-3)}$$

where

$$x(t) = \lim_{\Delta f \to 0} \sum_0^n a_n \cos(2\pi n \Delta f t + \theta_n) \quad \text{(H-4)}$$

and

$$y(t) = \lim_{\Delta f \to 0} \sum_0^n a_n \sin(2\pi n \Delta f t + \theta_n) \quad \text{(H-5)}$$

The components $x(t)$ and $y(t)$ are called the in-phase and quadrature phase components if a sine function is used as a reference for the center frequency component f_c. Both $x(t)$ and $y(t)$ are Gaussian distributions with the same mean and variance as $n(t)$

The expression for $n(t)$ can be written in polar form by the substitution

$$x(t) = R(t) \cos \phi(t)$$

$$y(t) = R(t) \sin \phi(t) \quad \text{(H-7)}$$

where

$$R(t) = \sqrt{x^2(t) + y^2(t)} \quad \text{(H-8)}$$

and

$$\phi(t) = \tan^{-1}[y(t)/x(t)] \quad , \quad \text{(H-9)}$$

where the amplitude $R(t)$ varies with a Rayleigh distribution and the phase of $\phi(t)$ varies uniformly over the interval 0 to 2π. Hence, $n(t)$ resembles a sine wave which is amplitude-modulated and phase-modulated randomly as shown in Fig. H.2.

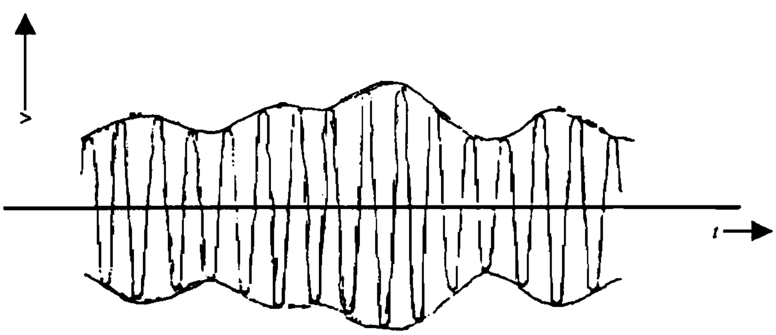

Fig. H.2. Noise at the output of narrowband filter

References

1- "Noise", F. Conner, Arnold, London, 1982.

2- "Detection of Signals in Noise", A. Whalen, Academic Press, N. Y., 1971.

3- "Introduction to Mechanical Statistics", P. Hool, J. Wiley, N. Y., 1971.

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5- "Communication Systems Design", P. Panter, McGraw Hill, N. Y., 1972.

Appendix I
Thermal noise

Nyquist's derivation of the noise power associated with a resistor is based on the consideration of a lossless transmission line terminated at either end in its characteristic impedance $Z_0 = R$, as shown in Fig. I.1a. In thermal equilibrium at temperature T, the average noise power N from each resistor travels along the line as an electromagnetic wave and is completely absorbed at the other end.

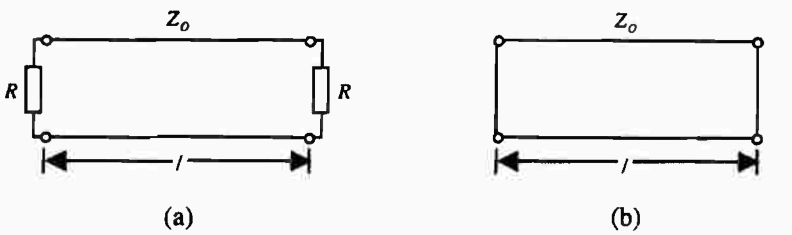

Fig. I.1 Lossless transmission line
a) terminated by R at both ends b) short circuited at both ends

If the line is suddenly short-circuited at either end, as in Fig. I.1b, the noise power from *each* resistor traveling along the line is reflected from each end to produce standing waves. The energy 'trapped' is stored in oscillating modes as the system becomes a resonator or one-dimensional harmonic oscillator. For such a resonator, the m^{th} mode has a wavelength λ is given by

$$m\lambda / 2 = 1 \qquad \text{(I-1)}$$

or

$$m = 2l/\lambda = 2lf/c \quad , \qquad \text{(I-2)}$$

where l is the length of the line, f is the mode frequency, and c is the velocity of propagation of the wave. If the number of oscillating modes Δm occupy a bandwidth $\Delta f = B$, by using increments we obtain

$$\Delta m = 2(l/c)B \qquad \text{(I-3)}$$

or

$$2l/c = \Delta m / B \qquad \text{(I-4)}$$

The average noise power *N* from each resistor travels for a time $t = l/c$ before reflection and is stored as energy ΔW where

$$\Delta W = 2Nt = 2Nl/c = N\Delta m / B \qquad \text{(I-5)}$$

From classical theory, it is known that the one- dimensional harmonic oscillator in thermal equilibrium at temperature T is associated with the oscillating modes, and the energy in each mode is kT, where *k* is Boltzmann's constant. Hence, for Δm modes we have

$$kT\Delta m = N\Delta m / B \tag{I-6}$$

$$N = kTB \quad \text{watts} \tag{I-7}$$

From quantum theory, the average energy associated with each oscilla-
, and we obtain

$$\Delta W = \frac{hf}{e^{hf/kT} - 1} , \tag{I-8}$$

where f is the frequency of oscillation and h is Planck's constant. For frequencies up to about 10^{13}Hz, $\Delta W \cong kT$, and is constant which is the classical value.

References

1- "Noise", F. Conner, Arnold, London, 1982.

2- "Detection of Signals in Noise", A. Whalen, Academic Press, N.Y., 1971.

3- "Introduction to Mechanical Statistics", P. Hool, J. Wiley, N. Y., 1971.

4- "An Introduction to the Theory of Random Signals and Noise", W. Davenport and W. Root, McGraw Hill, N. Y., 1958.

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Appendix J
Shot Noise

The shot noise rms current I_s in a diode is due to the random emission of electrons from the cathode. Each electron arriving at the anode carries a discrete electronic charge q which gives rise to a current pulse $i(t)$ in the anode during the transit time τ, as shown in Fig. J.1a. The actual shape of the current pulse is immaterial if the time-average interval chosen is such that $\tau << T$. Each pulse can be regarded as a Dirac delta function $\delta(t)$ and approximated by a short rectangular pulse, as illustrated in Fig. J.1b. Hence, we have

$$\int_{-\infty}^{+\infty} \delta(t) dt = q \tag{J-1}$$

i.e., the area of the rectangular pulse is such that q/τ x $\tau = q$

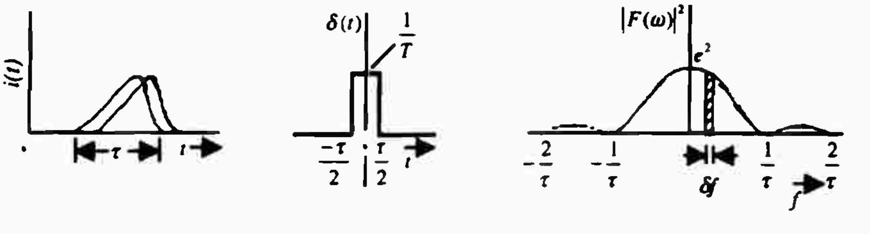

Fig. J.1. Shot noise

a) actual pulseb) approximate pulse shape c) Fourier transform

If $F(\omega)$ is the Fourier transform of $\delta(t)$ then

$$F(\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = q \frac{\sin \omega\tau/2}{\omega\tau/2} \tag{J-2}$$

and

$$|F(\omega)|^2 = q^2 \left[\frac{\sin \omega\tau/2}{\omega\tau/2} \right]^2 , \tag{J-3}$$

where $|F(\omega)|^2$ is the energy spectral density and is shown in Fig. J.1c.

From Fig. J.1c, we observe that if the transit time τ is very small (about $10^{-9} s$) then $1/\tau = 10^9$ Hz, and the spectral density over a bandwidth $\Delta f = B$ is fairly constant, especially at lower frequencies. Hence, the total energy E in a bandwidth B is given by

$$E = \int_{-\infty}^{\infty} |F(\omega)|^2 df = 2\int_{0}^{\infty} |F(\omega)|^2 df \tag{J-4}$$

Using eqn. (J-3) for very small τ,

$$E = 2q^2B \tag{J-5}$$

If n electrons arrive at the anode in time T - where T is sufficiently large - the average shot noise power in a 1Ω load becomes

$$I_s^2 = nE/T = 2nq^2B/T \tag{J-6}$$

and substituting for the average anode current $I_a = nq/T$ yields

$$I_s^2 = 2qI_aB \tag{J-7}$$

$$I_s = \sqrt{2qI_aB} \tag{J-8}$$

References

1- "Noise", F. Conner, Arnold, London, 1982.

2- "Detection of Signals in Noise", A. Whalen, Academic Press, N.Y., 1971.

3- "Introduction to Mechanical Statistics", P. Hool, J. Wiley, N. Y., 1971.

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Appendix K
Rules for Root Locus construction

A root locus plot is a drawing of the loci of the poles of a rational function as some system parameter is varied. The basic root locus problem applies directly to the simple feedback system of Fig. K.1, for which the transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \qquad \text{(K-1)}$$

where the constant gain K is the parameter of interest. The poles of the transfer function are the roots of

$$1 + KG(s)H(s) = 0 \quad , \qquad \text{(K-2)}$$

which depend upon the parameter K. The product of the forward transmittance $KG(s)$ and the feedback transmittance $H(s)$ is termed the open-loop transmittance (or gain) of the system. The poles and zeros of $G(s)H(s)$ are called the open-loop poles and zeros, while the poles and zeros of $T(s)$ are closed-loop poles and zeros. The open-loop transfer function can also be visualized as being the transfer function $Y_1(s)/R(s)$ when the loop in Fig. K.1. is broken at the point denoted by x.

We consider now K in the range $0 \le K < \infty$. From eqn. (K-2),

$$G(s)H(s) = -\frac{1}{K} \qquad \text{(K-3)}$$

and for positive K, this means that a point s which is a pole of $T(s)$ makes

$$|G(s)H(s)| = \frac{1}{K} \qquad \text{(K-4)}$$

and

$$\angle G(s)H(s) = \text{odd multiple of } 180^{\circ} \qquad \text{(K-5)}$$

Suppose that there is a point s for which the second of these conditions is satisfied. Then whatever the magnitude of $G(s)H(s)$ for this value of s, there is a corresponding value of K.

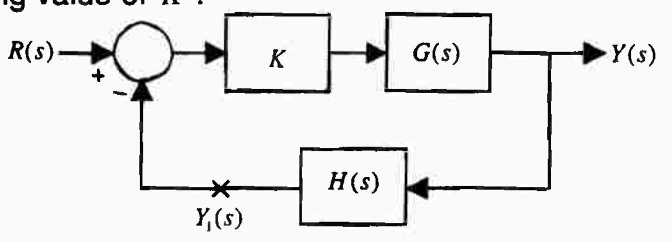

Fig. K.1. A simple feedback system with adjustable gain K

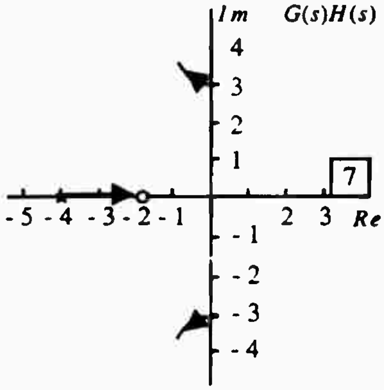

Fig. K.2. A root locus plot

Thus, any point s for which $\angle G(s)H(s) = 180^0$ is a point of the root locus for some positive value of K.

Fig. K.2. shows an example root locus plot for a feedback system with open loop transmittance

$$KG(s)H(s) = \frac{K(s+2)(s^2 - 2s + 17)}{(s+4)(s^2+9)}$$

It consists of a pole-zero plot for $G(s)H(s)$, which is generally easy to construct because $G(s)$ and $H(s)$, being components of the system, are usually known in factored or partially factored form. Superimposed upon the pole-zero plot for $G(s)H(s)$ are the curves that are loci of the poles of $T(s)$ as K varies from zero to infinity. The locus segments are symmetrical about the real axis, and the sense of increasing K usually is indicated on each segment. As $K \to 0$, $|G(s)H(s)| \to \infty$, and as $K \to \infty$, $|G(s)H(s)| \to 0$. This is to say that the poles of $T(s)$ are near the poles of $G(s)H(s)$ for small K and are near the zeros of $G(s)H(s)$ for large K. The loci being on the poles of GH and end on the zeros of GH.

To determine if a given point s_o is a point on the root locus for some value of K between zero and + ∞, it is only necessary to determine whether or not the angle of $G(s)H(s)$ is 180°. This determination is easily made graphically, using directed line segments.

$\angle G(s_o)H(s_o) =$ sum of zero angle to s_o

– sum of pole angles to s_o

+ 180° if the multiplying constant is negative

1. The branches of the locus are continuous curves that start at each of the n poles of GH, for $K=0$. As K approaches $+\infty$, the locus branches approach the m zeros of GH. Locus branches for excess poles extend infinitely far from the origin; for excess zeros, locus segments extend from infinity.
2. The locus includes all points along the real axis to the left of an odd number of poles plus zeros of GH.
3. As K approaches $+\infty$, the branches of the locus become asymptotic to straight lines with angles.
$$\theta = \frac{180^0 + i360^0}{n-m}$$
for $i = 0,\pm 1,\pm 2,\ldots$ until all $n-m$ or $m-n$ angles are obtained, where n is the number of poles and m is the number of zeros of GH.
4. The starting point of the asymptotes, the centroid of the pole-zero plot, is on the real axis at
$$\sigma = \frac{\sum \textit{pole values of GH} - \sum \textit{zero values of GH}}{n-m}$$
5. Loci leave the real axis at a gain K that is the maximum K in that region of the real axis. Loci enter the real axis at the minimum value of K in that region of the real axis. These points are termed breakaway points and entry points, respectively. A pair of locus segments leave or enter real axis at angles of $\pm$ 90°.
6. The angle of departure ϕ of a locus branch from a complex pole is given by

   (sum of the other GH pole angles to the pole under consideration) +
   ϕ (sum of the GH zero angles to the pole) = 180°.

   The angle of approach ϕ' of a locus branch to a complex zero is given by

   (sum of the GH pole angles to the zero under consideration)
   (sum of the other GH zero angles to the zero) ϕ' = 180°.

   Multiple angles of departure and approach at repeated complex poles of GH are found similarly, using multiple contributions of ϕ or ϕ' and equating to 180° i 360°.

References

1- "Design of Feedback Control Systems", G. Hostetter, C. Savant and R. Stefani, 2nd ed., Holt, Rinehart and Winston, N. Y., 1989.

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3- "Control System Dynamics", W. Evans, McGraw Hill, N. Y., 1954.

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