CHAPTER 7
Noise Power

7.1 Spectral Density Functions:

For signals of finite energy, Parseval's theorem gives a relation between a time signal $f(t)$ and its Fourier transform $F(\omega)$ - see Appendix D,

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega, \qquad (7-1)$$

where E_f is the energy across a one ohm resistance for $f(t)$ being voltage or current.

Thus, $|F(\omega)|$ is the normalized energy per unit frequency in one ohm resistance, hence, called energy spectral density or energy density. The total area under $|F(\omega)|$ is the energy of the signal.

If we apply a signal $f(t)$ to the input of a linear time invariant system whose transfer function is $H(\omega)$, the output $G(\omega)$ is given by

$$G(\omega) = F(\omega)\, H(\omega) \qquad (7-2)$$

$$|G(\omega)|^2 = |F(\omega)|^2\, |H(\omega)|^2 \qquad (7-3)$$

The normalized energy in the output signal is

$$E_o = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 |H(\omega)|^2 d\omega \qquad (7-4)$$

Not all signals in nature have finite energy. Some signals have infinite energy because they are extended in time, but they may have a finite time average of energy. This time average is called average power. Such signals are called power signals. They may or may not be periodic.

The normalized time averaged power of such signals is given by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt \qquad (7-5)$$

Thus, the time averaged power of a signal is its mean-square value $\overline{f^2(t)}$

Let us define the power spectral density function $S(\omega)$. This function has units of power per Hz. Its integral gives the total power in $f(t)$ - see Appendix F,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega \qquad (7-6)$$

This function is also valid for random signals. Suppose we have a power signal $f(t)$ as shown in Fig. 7.1a. Assume a truncated portion in the interval $(-T/2, T/2)$. The truncated function is $f(t)p_T(t/T)$.

Thus,

$$F_T(\omega) = \mathcal{F}[f(t)p_T(t/T)] \qquad (7-7)$$

From Parsevals theorem,

$$\int_{-T/2}^{T/2} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega$$

$$P = \overline{f^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega \qquad (7-8)$$

From eqn. (7-6),

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega) d\omega = \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega \qquad (7-9)$$

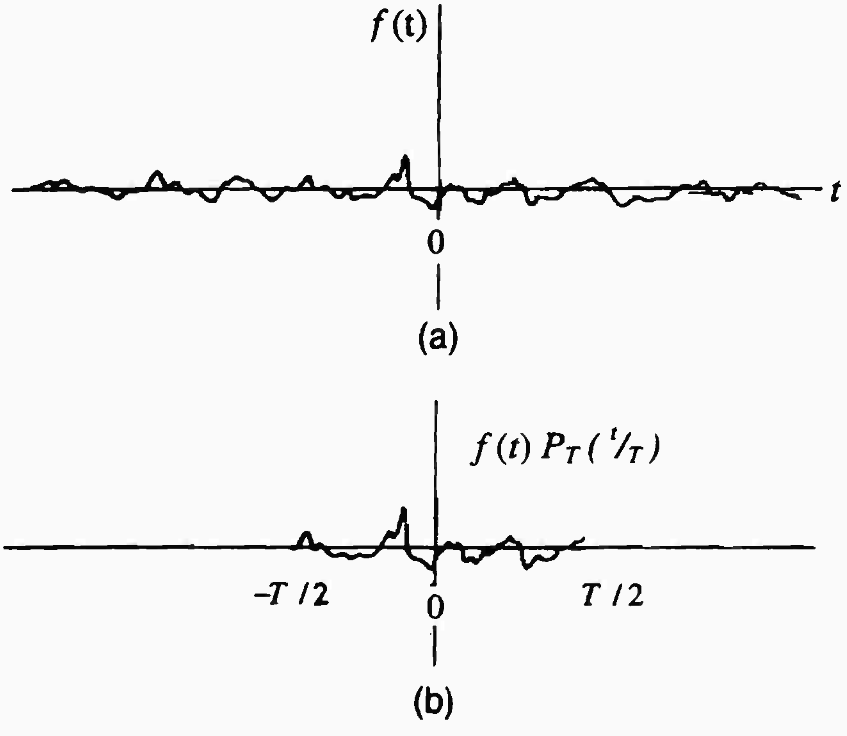


Fig 7.1 Power signal
a) extended in time b) finite interval truncation.

We define $G(\omega)$ as the cumulative amount of power for all frequency components below a given frequency ω. It is called the cumulative power spectrum or the integrated power spectrum of $f(t)$, and is given by

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\omega} S(\nu)d\nu=\lim_{T\to\infty}\frac{1}{T}\frac{1}{2\pi}\int_{-\infty}^{\omega}\left|F_T(\nu)\right|^2 d\nu \qquad (7-10)$$

Thus,

$$2\pi G(\omega)=\int_{-\infty}^{\omega} S(\nu)d\nu=\int_{-\infty}^{\omega}\lim_{T\to\infty}\frac{\left|F_T(\nu)\right|^2}{T}d\nu \qquad (7-11)$$

The average or mean power contained in any frequency interval (ω_1, ω_2) is $G(\omega_2)$ $G(\omega_1)$. From eqn. (7 11),

$$2\pi\frac{dG(\omega)}{d\omega}=S(\omega)=\lim_{T\to\infty}\frac{\left|F_T(\omega)\right|^2}{T} \qquad (7-12)$$

The above discussion holds in general. If $f(t)$ is periodic with fundamental frequency ω_o, such that

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_o t}, \qquad (7-13)$$

Parseval's theorem - eqn. (7-5) - may be rewritten as

$$P=\frac{1}{T}\int_{-T/2}^{T/2}f(t)f^*(t)dt \qquad (7-14)$$

$$=\frac{1}{T}\int_{-T/2}^{T/2}\sum_{m=-\infty}^{\infty}F_m e^{jm\omega_o t}\sum_{n=-\infty}^{\infty}F_n^* e^{-jn\omega_o t}dt \qquad (7-15)$$

$$=\sum_{m=-\infty}^{\infty}F_m\sum_{n=-\infty}^{\infty}F_n^*\frac{1}{T}\int_{-T/2}^{T/2}e^{j(m-n)\omega_o t}dt \qquad (7-16)$$

Because the exponential functions are orthogonal (see Appendix A), the integral vanishes for n different from m.

$$P=\sum_{n=-\infty}^{\infty}F_n F_n^*=\sum_{n=-\infty}^{\infty}\left|F_n\right|^2 \qquad (7-17)$$

For a periodic signal, we can use eqn. (7-17) to plot a line power spectrum (Fig. 7.2a). The corresponding cumulative power spectrum $G(\omega)$ is found by summing the terms in eqn. (7-17) over all components up to the given frequency ω. Each harmonic component adds a discrete amount of power. Thus, $G(\omega)$ is a series of step functions giving a staircase graph (Fig. 7.2b). Thus,

$$G(\omega)=\sum_{n=-\infty}^{\omega/\omega_o}\left|F_n\right|^2 u(\omega-n\omega_o) \qquad (7-18)$$

Noting that the slope of the step function $u(\omega)$ is the delta function $\delta(\omega)$, (see Appendix B) we have from eqn. (7-12),

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_o) \qquad (7-19)$$

Therefore, the power spectral density of a periodic function is a series of impulse functions with weights (areas) equal to the square of the respective Fourier series coefficients. Thus, we can convert any line power spectrum to a power spectral density simply by changing the lines to impulses (Fig 7.2c). Integrating the power spectral density,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \ , \qquad (7-20)$$

whereas from eqn. (7-19),

$$\overline{f^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_o) d\omega = \sum_{n=-\infty}^{\infty} |F_n|^2 , \qquad (7-21)$$

which is expected from Parseval's theorem, eqn. (7-1)

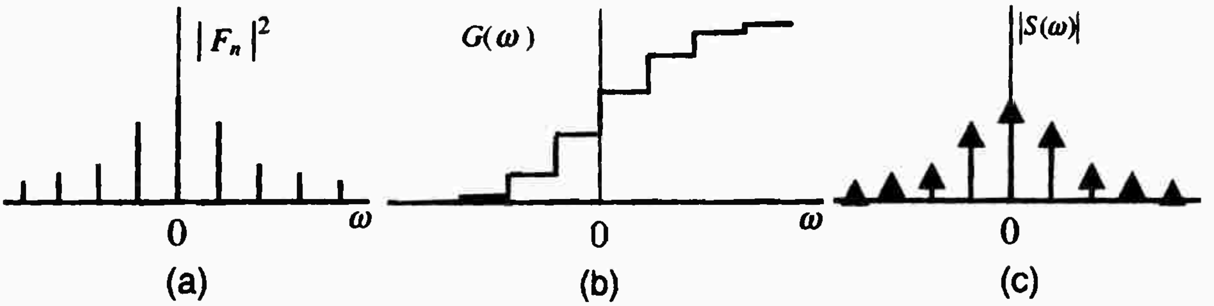


Fig. 7.2 Power spectra of periodic functions
a) line spectrum b) cumulative power spectrum c) power spectral density.

For aperiodic power signals, $G(\omega)$ is a smoothly ranging function of frequency instead of the stepped function (Fig. 7.3a). $S(\omega)$ follows from eqn. (7-12) (Fig. 7.3b).

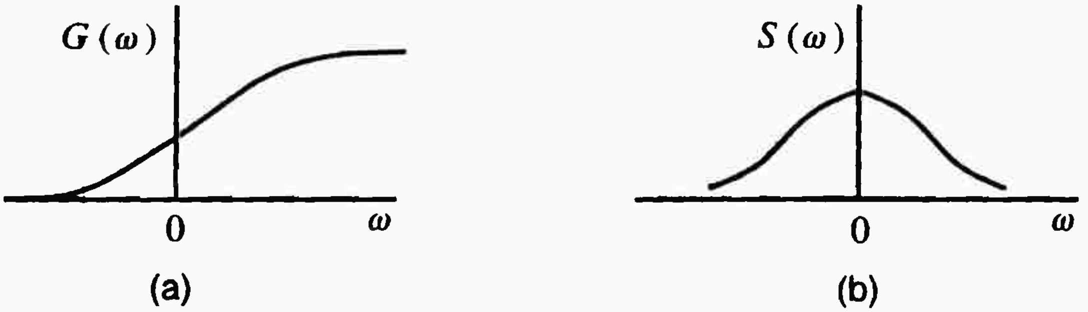


Fig. 7.3 Power spectra of an aperiodic functions
a) integrated power spectrum b) power spectral density.

7.2 Time Averaged Noise Representation:

Suppose $n(t)$ is a noise signal, then the average noise signal $\overline{n(t)}$ is given by

$$\overline{n(t)} = \lim_{T\to\infty} \frac{1}{2} \int_{-T/2}^{T/2} n(t)dt \qquad (7-22)$$

$$S_n(\omega) = \overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2 dt\,, \qquad (7-23)$$

where $S_n(\omega)$ is the noise power spectral density, and is the mean square of the noise signal $n(t)$. If we define the fluctuation $\sigma(t)$ as

$$\sigma(t) = n(t) - \overline{n(t)} \qquad (7-24)$$

From eqn. (7-22),

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\overline{n(t)} + \sigma(t)\right]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt + 2\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \overline{n(t)}\sigma(t)dt$$

$$+ \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\sigma(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\sigma(t)|^2 dt \qquad (7-25)$$

Note that $\overline{n(t)}$ is constant and the average of $\sigma(t)$ $(\overline{\sigma}(t))$ is zero.

From eqn. (7-25), the LHS is the time averaged power in $n(t)$ in one ohm resistance. The first term of RHS is the dc power, and the second term is the ac power. If $\overline{n(t)}$ is zero, then the rms value of $n(t)$ is the rms value of $\sigma(t)$. This analysis applies for random or non random signals. For periodic signals, T is taken as the period.

7.3 Correlation Functions and the Wiener Khintchine Theorem:

Consider a sample function $x(t)$, which is a single pulse of a pulse train of period T. From eqn. (7-12), see Appendix C and Appendix G

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T} |F_T(\omega)|^2\,, \qquad (7-26)$$

where $F(\omega)$ is Fourier transform of $x(t)$. Now, if $x(t)$ is real, we have the transform properties:

$$F(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \qquad (7-27)$$

$$F(\omega) = -F(-\omega) \qquad (7-28)$$

$$|F(\omega)|^2 = F(\omega)F(-\omega) \qquad (7-29)$$

$$|F(\omega)|^2 = \int_{-\infty}^{\infty} x(t_1)e^{-j\omega t_1}dt_1 \int_{-\infty}^{\infty} x(t_2)e^{-j\omega t_2}dt_2 , \qquad (7-30)$$

where

$$t_2 = t_1 - \tau \qquad (7-31)$$

Hence,

$$S(\omega) = \lim_{T\to\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} x(t_1)e^{-j\omega t_1}dt_1 \int_{-T/2}^{T/2} x(t_2)e^{-j\omega t_2}dt_2\right], \qquad (7-32)$$

Interchanging the order of integration, and substituting $t_2 = t_1 - \tau$ yields

$$S(\omega) = \lim_{T\to\infty}\frac{1}{T}\left[\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} x(t_1)x(t_1-\tau)e^{-j\omega\tau}dt_1\, d\tau\right] \qquad (7-33)$$

Now, we define the autocorrelation function $R(\tau)$

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t_1)x(t_1-\tau)dt_1 \qquad (7-34)$$

Hence,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \qquad (7-35)$$

Taking the inverse transform

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega \qquad (7-36)$$

Therefore, in general, we have the same result for periodic or aperiodic functions:

$$S(\omega) = \mathcal{F}[R(\tau)] \qquad (7-37)$$

$$R(\tau) = \lim_{T\to\infty}\int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt = \mathcal{F}^{-1}[S(\omega)] \quad (7-38)$$

This is called Wiener-Khintchine theorem.

Thus, we have another method to find the power spectral density function $S(\omega)$, by determining the auto correlation function $R(\tau)$, and then taking Fourier transform. This method can be used for random and nonrandom (deterministic) functions.

Ex. 7.1:

Determine the autocorrelation function of a periodic square wave with peak to peak amplitude A, period T and mean value $A/2$.

Solution:
From eqn. (7-38),

For $-T/2 < T < 0$, $$R(\tau) = \frac{1}{T}\int_{-T/4}^{\frac{T}{4}+\tau} A^2 dt = A^2\left(\frac{1}{2} + \frac{\tau}{T}\right)$$

For $0 < \tau < T/2$, $$R(\tau) = \frac{1}{T}\int_{\tau-\frac{T}{4}}^{\frac{T}{4}+\tau} A^2 dt = A^2\left(\frac{1}{2} - \frac{\tau}{T}\right)$$

The autocorrelation function of a periodic function is periodic, while the autocorrelation function of an aperiodic function is aperiodic.

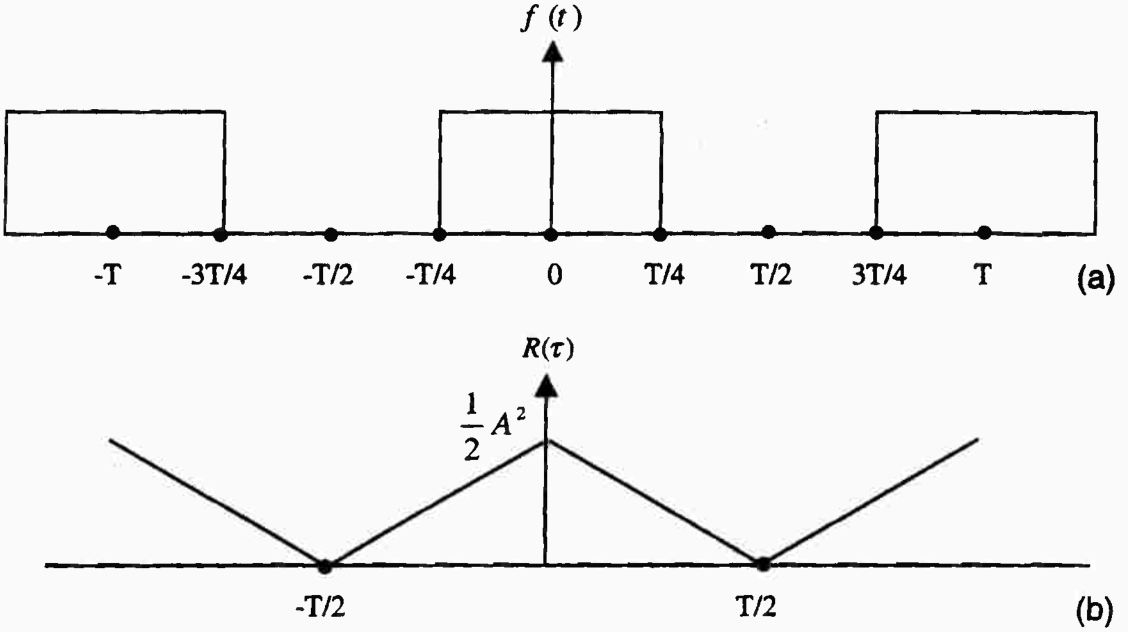


Fig. 7.4 Autocorrelation function for a square wave
a) periodic function $f(t)$ b) autocorrelation function $R(\tau)$

Ex. 7.2:

A square wave $f(t)$ (Fig. 7.5a) has an autocorrelation function $R_f(\tau)$ Fig. 7.5b. A random signal such as that shown in Fig. 7.5c has an autocorrelation function $R_n(\tau)$ as shown Fig. 7.5d. If the two signals are added the resultant is shown in Fig. 7.5e, and has an autocorrelation function as shown in Fig. 7.5f. Show how $f(t)$ can be retrieved.

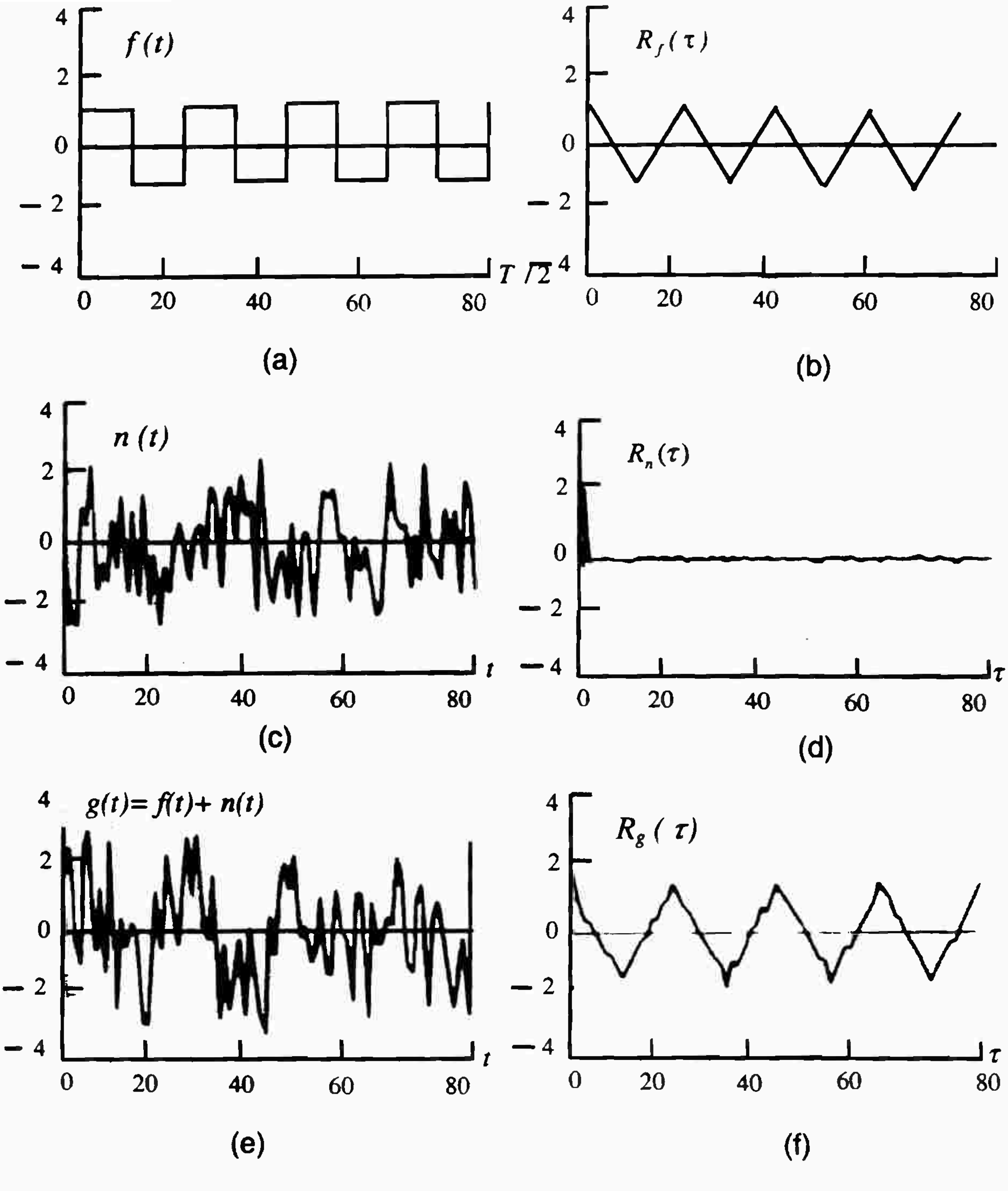


Fig. 7.5 Autocorrelation of a periodic signal plus noise

a) $f(t)$ b) $R_f(\tau)$ c) $g(t)=f(t)+n(t)$ d) $R_g(\tau)$

Solution:

We see that the useful signal $f(t)$ is submerged totally in noise background. Comparing Fig. 7f and Fig. 7b, we deduce that through the use of autocorrelation of the combined signal $g(t)$, the useful signal $f(t)$ is fairly retrieved, (prob. 7.3).

7.4 Crosscorrelation Function:

A relevant correlation function - called crosscorrelation function $R_{fg}(\tau)$ between two waveforms $f(t)$ and $g(t)$ - is given by

$$R_{fg}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t) g(t+\tau) dt \qquad (7-39)$$

Ex. 7.3:

To demonstrate the importance of crosscorrelation function, let $f(t)$ be a random waveform (Fig 7.6a). The second function is $g(t)$. Let it be a delayed replica of $f(t)$ plus random waveform $n(t)$, so that

$$g(t) = f(t - t_o) + n(t)$$

The composite waveform $g(t)$ is shown in Fig. 7.6b. We assume that the receiver has a replica of $f(t)$ stored. We wish to have the receiver make a measurement of the time delay t_o. We take the crosscorrelation function $R_{fg}(\tau)$. The result is shown in Fig. 7.6c.

The value of the time delay t_o is measured from the origin to the large peak in the crosscorrelation function. The peak is an indicator of the match between the two functions: $f(t)$ and $g(t)$.

7.5 Correlation Functions For Finite Energy Signals:

The concept of correlation can be extended to signals of finite energy. We define the autocorrelation function $r_f(\tau)$ for a signal $f(t)$ of finite energy as

$$r_f(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau) dt \qquad (7-40)$$

Similarly, for signals $f(t)$ and $g(t)$, both of finite energy. We define the crosscorrelation function $r_{gf}(\tau)$ as

$$r_{fg}(\tau) = \int_{-\infty}^{\infty} f^*(t) g(t+\tau) dt \qquad (7-41)$$

Taking Fourier transform of eqn. (7-40)

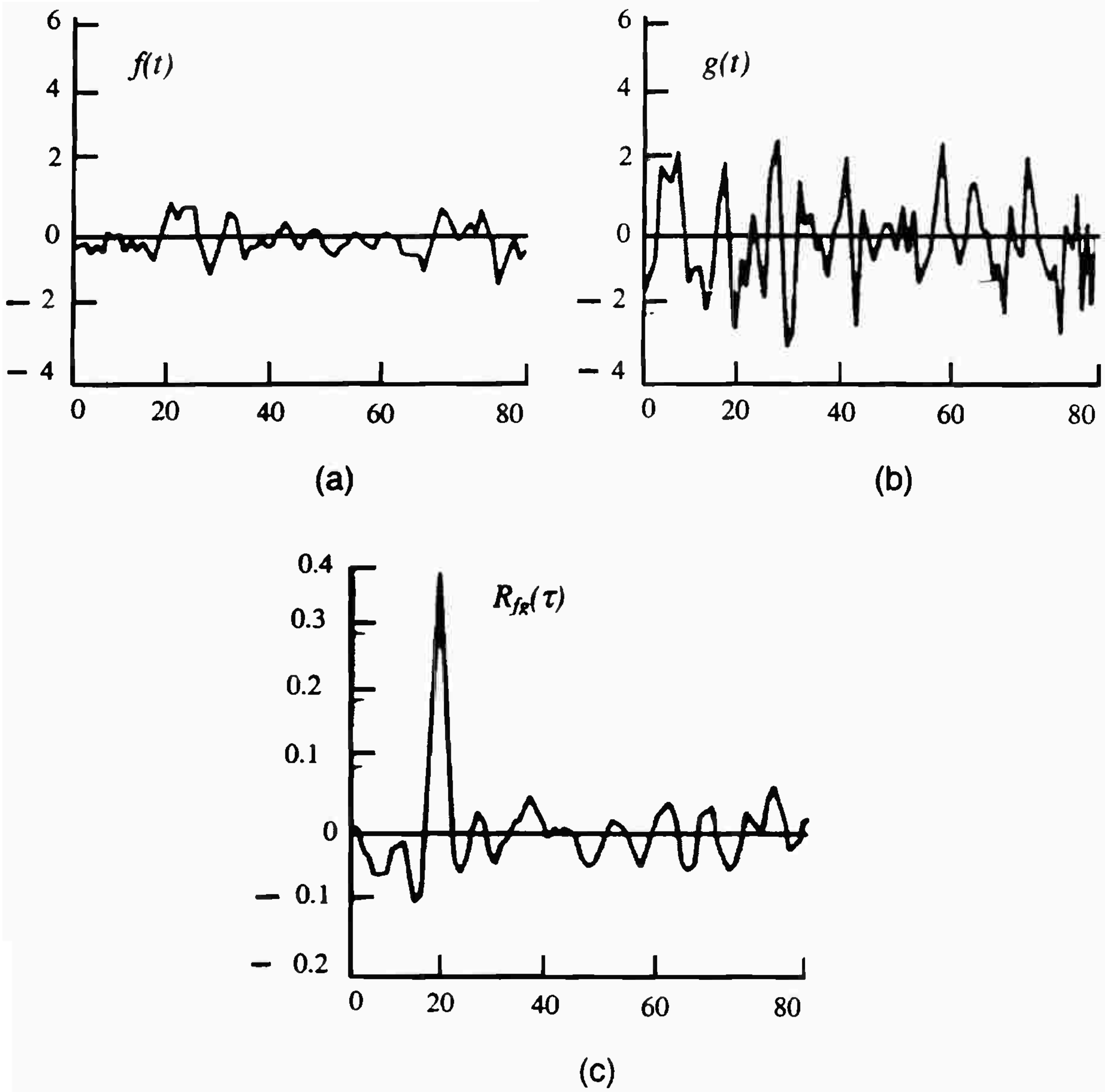


Fig. 7.5 Crosscorrelation of a random signal plus noise

a) random $f(t)$ b) $g(t) = f(t - t_0) + n(t)$

c) crosscorrelation $R_{fg}(\tau)$

$$\mathcal{F}[r_f(\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(t) f(t+\tau) e^{-j\omega\tau} dt$$

$$= \int_{-\infty}^{\infty} f^*(t) \int_{-\infty}^{\infty} f(t+\tau) e^{-j\omega\tau} d\tau \; dt \qquad (7-42)$$

$$= \int_{-\infty}^{\infty} f^*(t) F(\omega) e^{-j\omega t} dt \qquad (7-43)$$

$$= |F(\omega)|^2 \qquad (7-44)$$

We conclude that the energy spectral density is the Fourier transform of the autocorrelation function for finite energy signals.

7.6 White Noise:

Random noise signals are known to have a uniform power distribution over a very wide range of frequencies (up to 10^{13} Hz). This is called white noise by analogy with white light, which has a uniform power distribution over the band of optical frequencies. This is known is Gaussian white noise.

If η is the noise power spectral density per Hz for positive frequencies, the noise power spectral density is $\eta / 2$. Hence, from eqn. (7-36),

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\eta e^{j\omega\tau}}{2} d\omega$$

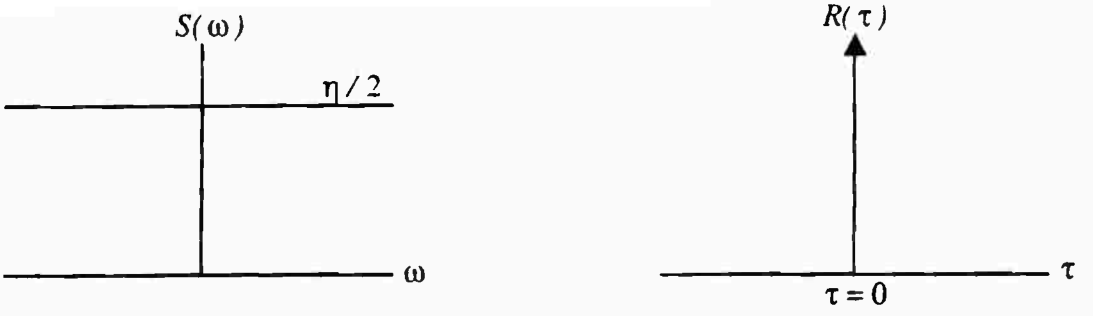
$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta e^{j\omega\tau} d\omega = \frac{\eta}{2}\delta(\tau) \qquad (7-45)$$

Since $R(\tau)$ has a value at τ=0 only, there is no correlation between any two samples of white noise separated by an interval τ>0 and they are, therefore, statistically independent. From Fig (7.7a) it will be observed that the average power - given by $P_{av} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) d\omega$ - becomes infinite and cannot be physically realized in any practical circuits.

7.7 Band Limited White Noise:

As most communication circuits are band limited, it is more practical to consider the results of passing white noise through a filter with some defined bandwidth. The output noise is called band limited white noise or colored noise.

If we have a constant power spectral density of η Watts per Hz (for positive frequencies) and $\overline{n(t)} = 0$, then the power spectral density of white noise in 1 ohm resistance is $S_n(\omega) = \frac{\eta}{2}$ for all ω.



(b) (a)

Fig. 7.7 White noise and its correlation function

a) $S(\omega)$ b) $R(\tau)$

The factor 1/2 is necessary to have a two-sided power spectral density (for positive and negative frequencies). Thus,

$$\overline{n^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\eta}{2} d\omega \rightarrow \infty \tag{7 - 46}$$

Hence, the power is infinite. In an actual case, the system has a limited bandwidth B Hz. Hence, noise in such band limited systems is given by.

$$P_n = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \frac{\eta}{2} d\omega = \eta B \tag{7 - 47}$$

If this is developed across a resistor R , the mean square noise voltage is

$$\overline{\overline{v_n^2}} = \overline{n^2(t)} = RP_n = \eta RB \tag{7 - 48}$$

If $n(t)$ is current.

$$\overline{\overline{i_n^2}} = \overline{n^2(t)} = \frac{P_n}{R} = \eta GB \tag{7 - 49}$$

For a system with transfer function $H(\omega)$

$$S_{n_o} = S_{n_i}(\omega)|H(\omega)|^2 \tag{7 - 50}$$

$$\overline{n_o^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_o}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_i}(\omega)|H(\omega)|^2\, d\omega \tag{7 - 51}$$

For white noise and one ohm resistor,

$$\overline{n_o^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\eta}{2}|H(\omega)|^2 d\omega$$

$$= \frac{\eta}{2\pi}\int_0^{\infty}|H(\omega)|^2 d\omega \qquad (7-52)$$

It is known from thermodynamics that for thermal noise.

$$S_n(\omega) = \frac{h|\omega|}{\pi\left[e^{h|\omega|/2\pi kT} - 1\right]} \qquad (7-53)$$

For

$$|\omega| << \frac{2\pi kT}{h}(6000\ GHz \quad \text{at} \quad T = 296^o K)$$

Thus,

$$S_n(\omega) = 2\ kT = \eta/2 \quad WHz^{-1} \qquad (7-54)$$

$$\eta = 4KT \qquad (7-55)$$

$$\overline{\overline{V_n^2}} = R\eta B = 4kTRB \qquad (7-56)$$

$$\overline{\overline{i_n^2}} = G\eta B = 4kTGB \qquad (7-57)$$

Note that $\eta = 4kT$ before power matching, whereas $\eta = kT$ after matching (see section 7.9)

Ex. 7.4:

If white noise is passed through a low pass ideal filter with a bandwidth $\pm B$ Hz, determine the output power.

Solution:

We have $\frac{S_o(\omega)}{S_i(\omega)} = |H(\omega)|^2$ and $|H(\omega)| = 1$

$$P_{av} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\eta}{2}d\omega = \frac{\eta}{2}\int_{-B}^{B} df = \eta B$$

The autocorrelation function $R(\tau)$ of the filtered white noise is

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_i(\omega)|H(\omega)|^2 e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\eta}{2}e^{j\omega\tau}d\omega = \frac{\eta}{2}\int_{-B}^{B} e^{j2\pi f\tau}df$$

$$= \frac{\eta}{2}\left[\frac{e^{j2\pi f t}}{j2\pi\tau}\right]_{-B}^{B} = \eta B\frac{\sin 2\pi B\tau}{2\pi B\tau}$$

$$= \eta B \frac{\sin X}{X} \quad , \text{ with } X = 2\pi B \tau$$

Since the shape of $R(\tau)$ is $\frac{\sin X}{X}$, there is correlation on either side of $\tau = 0$. Hence, filtering uncorrelated white noise produces correlated band limited white noise (colored), see Appendix H

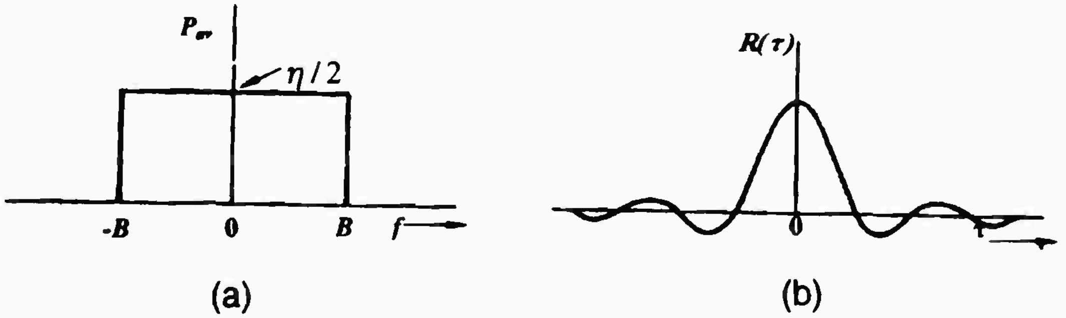


Fig. 7.8 Band limited white noise and its correlation function

a) $S(\omega)$ b) $R(\tau)$

Comparing Fig. 7.8 with Fig. 7.7, we see that band limiting the noise signal produces correlation in an otherwise uncorrelated (delta function) case.

7.8 Equivalent Noise Bandwidth:

Expressions (7-56) and (7-57) assume an ideal filter of bandwidth B. In practice, it is convenient to combine the band limiting characteristics of a system by defining an equivalent bandwidth B_N as that of an ideal filter, which gives the same noise power as the actual system. Within the white noise assumption, the input power spectral density is $\eta/2$. The mean square voltage output of a linear spectrum across one-ohm resistor is

$$\overline{\overline{v_{n_0}^2}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\eta}{2} |H(\omega)|^2 d\omega$$

$$= \frac{\eta}{2\pi} \int_0^{\infty} |H(\omega)|^2 d\omega \qquad (7-58)$$

To simplify the analysis, we define an equivalent noise bandwidth B_N, such that the power spectral density at the filter output is white within the bandwidth B_N, and zero elsewhere, forming a rectangular approximation. The area within this rectangular spectral density is equal to the area under the spectral density at the filter output (Fig. 7.9).

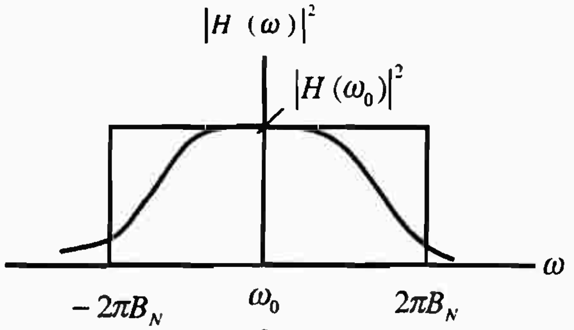


Fig. 7.9 Definition of noise equivalent bandwidth

$$\overline{v_{n_o}^2}(t) = \frac{1}{2\pi}\int_{-2\pi B_N}^{2\pi B_N} \frac{\eta}{2}|H(\omega_o)|^2 d\omega \quad = \eta |H(\omega_o)|^2 B_N \qquad (7-59)$$

From eqns. (7-57) and (7-59),

$$B_N = \frac{1}{2\pi}\frac{\int_0^\infty |H(\omega)|^2 d\omega}{|H(\omega_o)|^2} \qquad (7-60)$$

Ex. 7.5:

The input to an RC LPF is white noise. Determine the output power density, average noise power and noise equivalent bandwidth.

Solution:

$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega CR}$$

$$|H(\omega)|^2 = \frac{1}{1+(\omega CR)^2}$$

$$S_{n_o}(\omega) = S_{n_i}(\omega)|H(\omega)|^2 = \frac{\eta/2}{1+(\omega RC)^2}$$

$$P_{av} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_o}(\omega)d\omega = \frac{\eta}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{1+(\omega RC)^2}$$

Substituting $u = \omega RC$, $du = RCd\omega$

$$P_{av} = \frac{\eta}{4\pi RC}\int_{-\infty}^{\infty}\frac{du}{u^2+1} = \frac{\eta}{4\pi RC}\left[\tan^{-1} u\right]_{-\infty}^{\infty}$$

$$= \frac{\eta}{4\pi RC}\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= \frac{\eta}{4\pi RC}\left[\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right] = \frac{\eta}{4RC}$$

From eqn. (7-59),

$$B_N = \frac{P_{av}}{\eta |H(\omega_o)|^2} = \frac{1}{4RC}$$

7.9 Available Power and Noise Temperature:

From eqns. (7-55) and (7-47), with B_N as the equivalent noise bandwidth,

$$P_n = 4kTB_N \qquad (7-61)$$

Yet, the maximum available power can be extracted under matched condition when noiseless R equal to noisy R is used. In this case, the voltage is 1/2 the oc voltage, and thus, the maximum case of the available power P_a is 1/4 P_n, or

$$P_a = kTB \qquad (7-26)$$

A convenient way to describe the input noise power is to specify its noise temperature. Thus, the noise temperature specifies the thermal noise power for matched resistance (Fig. 7.10).

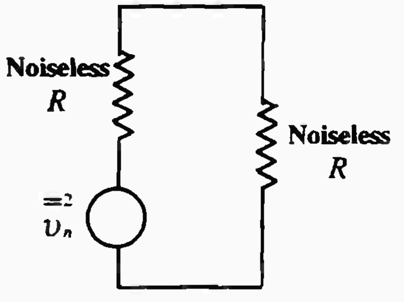


Fig. 7.10 Matched noise power

The noise temperature T_e is the effective temperature of a white thermal noise source at the system input needed to produce the same noise power at the output of an equivalent noiseless system. Such a temperature is not the ambient temperature. For low noise amplifiers, noise temperatures - as low as 10^oK - may be encountered. In broadcast receivers, temperatures may reach 1000^oK. Cooling is often used to reduce the noise temperature.
The ambient noise temperature at the input is T_i

$$T_i = \frac{P_{ai}}{kB} \qquad (7-63)$$

Hence, we may express the available excess noise in an amplifier referred to the input side by

$$P_e = kT_eB \qquad (7-64)$$

Thus, the total available noise power at the input side S_{ni}

$$S_{ni} = P_{ai} + P_e = kB(T_i + T_e) \qquad (7-65)$$

At the output, we have

$$S_{n_o} = G_P kTB(T_i + T_e), \qquad (7-66)$$

Where G_p is the power gain.

Problems:

1- A low level amplifier has a bandwidth 0.01 – 20Hz and a noise power spectral density of $S_n(\omega)$ = 10^{-13} ω^{-1} W/Hz. Calculate the signal to noise ratio of the amplifier output for 1μV rms sinusoidal input at 10Hz and at 1Hz.

2- Find the power spectral density of the signal:

a) $f(t) = A\cos(\omega_o t + \theta)$

b) $f(t) = 1 + \cos\omega_o t$

c) $f(t)A\ e^{j\omega_o t}$

d) $f(t) = A\ e^{j\omega_o t}\, p_T\,(t\,/T\,)$

3- Show that autocorrelation function of $\sqrt{2}$ cos $(\omega_o t + \theta_o)$ is $\cos\omega_o\tau$

4- Calculate the rms noise voltage arising from thermal noise in two resistors 100 Ω and 200 Ω at T=300$^o k$ within a bandwidth of 1MHz, if the resistors are connected in series. Repeat when they are connected in parallel.

5- Calculate the rms noise voltage developed across a capacitor C when it is connected in parallel with a noisy resistor R. Show that the result is independent of R. Explain.

6- Repeat for an L-R circuit.

7- Repeat for a parallel resonant circuit.

8- Calculate the equivalent noise bandwidth in an $L-R$ filter.

9- Calculate the equivalent noise bandwidth if

a) $H(\omega) = \dfrac{1}{\sqrt{1+\omega^2}}$ b) $H(\omega) = \dfrac{1}{\sqrt{1+\omega^4}}$

10- In a cascaded amplifier the first stage power gain is 20dB. Succeeding stages have gain up to 20dB. The noise power at the output is 20 mW. Determine the number of stages for T_e = 500 B_N = 12MHz

Hence, show that the noise performance of cascaded amplifiers is primarily dependent on the noise performance of the first stage.

11- Starting with eqn. (7-26), write down $\mathcal{F}^{-1}\left[S(\omega)\right]$, then show that it is $R(\tau)$.

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