الباب الثانى عشر

امثلة متنوعة

Various Problems

مثال: أحسب النهاية $\lim_{x\to 0} \frac{1-\sin x}{\cos x}$

الحـــل : بوضع

$$L = \lim_{x\to 0} \frac{1-\sin x}{\cos x} = \frac{1-0}{1} = 1 \begin{cases} \because x\to 0 \\ \therefore x\neq 2 \end{cases}$$

مثال: أحسب النهاية $\lim_{x\to \pi/4} (1-\cot x)$

الحـــل : بوضع

$$L = \lim_{x\to \pi/4} (1-\cot x)$$

$$= 1+\cot(\pi/4) = 1+1 = 2 \begin{cases} x\to \pi/4 \\ \therefore x\neq \pi/4 \end{cases}$$

مثال: أحسب النهاية $\lim_{x\to 0} \frac{1+\sin x}{1+\cos x}$

الحـــل : بوضع

$$L = \lim_{x\to 0} \frac{1+\sin x}{1+\cos x} = \frac{1+\sin 0}{1+1} = 1 \begin{cases} \because x\to 0 \\ \therefore x\neq 0 \end{cases}$$

مثال: أحسب النهاية $\lim_{x\to 0^+} \cot x$

الحـــل : بوضع

$$L = \lim_{x\to 0^+} \cot x = \lim_{x\to 0^+} \frac{\cos x}{\sin x} = 1$$

$$\sin x \to 1 \text{ and } \quad \cos x \to 0 \text{ as } \quad x \to 0 \text{ and } \quad x > 0$$

وبالتالى لها خط تقريبى عند $x = 0$

مثال: أحسب النهاية $\lim_{x\to 0} \frac{1-\cos^3 x}{\sin^2 x}$

الحـــل : بوضع

$$L = \lim_{x\to 0} \frac{1-\cos^3 x}{\sin^2 x}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{1-\cos^2 x}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{(1-\cos x)(1+\cos x)} \begin{cases} \text{as} \quad x\to 0, x\neq 0 \\ \therefore \cos x\neq 1, \cos x-1\neq 0 \end{cases}$$

$$= \frac{1+\cos 0+\cos^2 0}{1+\cos 0} = \frac{1+1+1}{1+1} = \frac{3}{2}$$

مثال: أحسب النهاية

$$\lim_{x\to\pi/4} \frac{\csc^2 x - 2}{\cot x - 1}$$

الحـــل : بوضع

$$L = \lim_{x\to\pi/4} \frac{\csc^2 x - 2}{\cot x - 1} \ldots \left[\csc^2 x = 1 + \cot^2 c\right]$$

$$= \lim_{x\to\pi/4} \frac{\csc^2 x - 1}{\cot x - 1}$$

$$= \lim_{x\to\pi/4} \frac{(\cot x + 1)(\cot x - 1)}{\cot x - 1}$$

$$L = \lim_{x\to\pi/4} (\cot x + 1) \ldots \left[\text{as} \quad x\to\pi/2; x\neq\pi/4 \to \cot x \neq 1, \cot x - 1 \neq 0\right]$$

$$= 2$$

, **مثال:** أحسب النهاية

$$\lim_{x\to 0} \frac{\sin 7x}{x}$$

الحـــل : بوضع

$$. L = \lim_{x\to 0} \frac{\sin 7x}{x}$$

بضرب كل من البسط والمقام فى 7

$$L = \lim_{x\to 0} \frac{\sin 7x}{x}$$

$$= \left(\lim_{x\to 0} 7\right) . \lim_{x\to 0} \frac{\sin 7x}{7x} = 7.1 = 7$$

مثال: أحسب النهاية

$$\lim_{x\to 0}\frac{\sin 8x}{\sin 5x}$$

الحـل : بوضع

$$L=\lim_{x\to 0}\left(\frac{\sin 8x}{\sin 5x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin 8x}{\sin 5x}.8x\right).\left(\frac{5x}{\sin 5x}.\frac{1}{3x}\right)=\frac{3}{2}$$

مثال: أحسب النهاية

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3$$

الحـل : بوضع

$$L=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3=1^3=1$$

مثال: أحسب النهاية

$$\lim_{x\to 0}\frac{\sin^2(x/2)}{4x^2}$$

الحـل : بوضع

$$L = \lim_{x\to 0} \frac{\sin^2(x/2)}{4x^2}$$

$$= \frac{1}{4} \lim_{x\to 0} \frac{\sin^2(x/2)}{4x^2} = \frac{1}{4} \lim_{x\to 0} \left(\frac{\sin(x/2)}{x/2}\right)^2$$

$$= \frac{1}{4}\left(\lim_{x\to 0} \frac{\sin(x/2)}{x/2} \times \frac{1}{2}\right)^2 = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

مثال: أحسب النهاية

$$\lim_{x\to 0} \frac{\sin^2(-11x)}{\tan 9x}$$

الحل : بوضع

$$L = \lim_{x\to 0} \frac{\sin^2(-11x)}{\tan 9x}$$

$$= \lim_{x\to 0} \left[\frac{\sin(-11x)}{-11x}\right] \times \left[\frac{9x}{\tan 9x} \times \frac{1}{9}\right]$$

$$= -11 \lim_{x\to 0} \left[\frac{\sin(-11x)}{-11x}\right] \times \frac{1}{9} \lim_{x\to 0} \left[\frac{9x}{\tan 9x}\right]$$

$$= -\frac{11}{9}\left[\lim_{x\to 0} \frac{\sin(-11x)}{-11x}\right] \times \frac{1}{9}\left[\lim_{x\to 0} \frac{9x}{\tan 9x}\right]$$

$$= -\frac{11}{9}[1][1] = -\frac{11}{9}$$

مثال: أحسب النهاية

$$\lim_{x\to0}\frac{\sin 3x \sin 5x}{7x^2}$$

الحـل : بوضع

$$L=\lim_{x\to0}\frac{\sin 3x \sin 5x}{7x^2}$$

$$=\frac{1}{7}\lim_{x\to0}\left[\frac{\sin(3x)}{x}\right]\times\left[\frac{\sin(5x)}{x}\right]$$

$$=\frac{1}{7}\left[\lim_{x\to0}\frac{\sin(3x)}{x}\right]\times\left[\lim_{x\to0}\frac{\sin(5x)}{x}\right]$$

$$=\frac{1}{7}\left[\lim_{x\to0}\frac{\sin(3x)}{x}\right]\times\left[\lim_{x\to0}\frac{\sin(5x)}{x}\right]$$

$$=\frac{1}{7}\left[\lim_{x\to0}\frac{\sin(3x)}{3x}\times3\right]\times\left[\lim_{x\to0}\frac{\sin(5x)}{5x}\times5\right]$$

$$=\frac{1}{7}(1\times3)(1\times5)=\frac{15}{7}$$

مثال: أحسب النهاية

$$\lim_{x\to3}\frac{\sin\left(4x^2-3x\right)}{x^2-9}$$

الحـل : بوضع

$$L = \lim_{x\to 3} \frac{\sin\left(x^2-3x\right)}{x^2-9}$$

$$= \lim_{x\to 3} \frac{\sin[x(x-3)]}{(x-3)(x+3)}$$

$$= \lim_{x\to 3}\left[\frac{\sin[x(x-3)]}{[x(x-3)]}\right]\times\left[\frac{x}{x+3}\right]$$

$$= \left[\lim_{x\to 3} \frac{\sin[x(x-3)]}{[x(x-3)]}\right]\times\left[\lim_{x\to 3}\frac{x}{x+3}\right]$$

$$= \left[\lim_{x\to 3} \frac{\sin[x(x-3)]}{[x(x-3)]}\right]\times\left[\frac{3}{3+3}\right]$$

نفترض ان

$x(x-3)=\theta$ as $x\to 3 \Rightarrow$

$x(x-3)\to 0$ i.e $\theta\to 0.$

$$= \left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right]\left[\frac{3}{6}\right]$$

$$= 1\times\frac{1}{2} = \frac{1}{2}$$

مثال: أحسب النهاية

$$\lim_{x\to 1} \frac{\left(x^2-x\right)\sin(x-1)}{x^2-2x+1}$$

الحـل : بوضع

$$L = \lim_{x\to 1} \frac{\left(x^2-x\right)\sin(x-1)}{x^2-2x+1}$$

$$= \lim_{x\to 1} \frac{x(x-1)\sin(x-1)}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{x\sin(x-1)}{(x+1)}$$

$$= \lim_{x\to 1}(x).\lim_{x\to 1} \frac{\sin(x-1)}{(x+1)}$$

$$= (1).\lim_{x\to 1} \frac{\sin(x-1)}{(x+1)}$$

Let $x-1=\theta$, when $x\to 1 \Rightarrow \theta\to 0.$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

مثال: أحسب النهاية

$$\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$$

الحـل :

بوضع

$$L = \lim_{x\to 0} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{x\to 0} \frac{2\sin^2(x)}{x^2} \ldots\ldots\ldots \left[1-\cos 2x = 2\cos^2 x\right]$$

$$= 2\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)^2$$

$$= 2[1]^2 = 2$$

مثال:

أحسب النهاية

$$\lim_{x\to a} \frac{\sin x - \sin a}{x-a}$$

الحـل :

بوضع

$$L = \lim_{x \to a} \frac{\sin x - \sin a}{x-a}$$

$(\text{using} \quad \sin\alpha - \sin\beta)$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\therefore \quad L = \lim_{x \to a} \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{(x-a)}$$

$$= \lim_{x \to a} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{(x-a)}$$

$$= \lim_{x \to a} 2\cos\left(\frac{x+a}{2}\right)\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}.\frac{1}{2}$$

put $\frac{x-a}{2} = \theta$, when $x \to a, \Rightarrow \theta \to 0$

$$= 2\cos(a)\left[\lim_{\theta \to 0} \frac{\sin(\theta)}{(\theta)}.\frac{1}{2}\right]$$

$$= 2\cos(a)\left[1 \times \frac{1}{2}\right] = \cos(a)$$

مثال: أحسب النهاية

$$\lim_{x \to 0} \frac{\sin(a+x)+\sin(a-x)}{x\sin x}$$

الحـل : بوضع

$$L = \lim_{x\to 0} \frac{\sin(a+x)+\sin(a-x)-2\sin a}{x\sin x}$$

$$= \lim_{x\to 0} \frac{\sin a\cos x+\cos a\sin x+\sin a\cos x-\cos a\sin x-2\sin a-2\sin x}{x\sin x}$$

$$= \lim_{x\to 0} \frac{2\sin a\cos x-\cos a\sin x-2\sin a}{x\sin x}$$

$$= -2\sin a \lim_{x\to 0} \frac{1-\cos x}{x\sin x}$$

$$= -2\sin a \lim_{x\to 0} \frac{2\sin^2(x/2)}{x\sin x}$$

$$= -2\sin a.2 \lim_{x\to 0} \left[\frac{2\sin^2(x/2)}{x^2}\right]\left[\frac{x^2}{\sin x}\right]$$

$$= -4\sin a.\left[\lim_{x\to 0} \frac{2\sin^2(x/2)}{x^2}.\frac{1}{2}\right] \lim_{x\to 0}\left[\frac{x}{\sin x}\right]$$

$$= -4\sin a.\left[\frac{1}{4}\right][1]$$

$$= -\sin a$$

مثال: أحسب النهاية

$$\lim_{x\to 0} \frac{\sec x-1}{x^2}$$

الحـل : بوضع

$$\because \sec x = 1/\cos x \Rightarrow$$

$$L = \lim_{x\to 0} \frac{\frac{1}{\cos x}-1}{x^2}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x\cos x}$$

$$= \lim_{x\to 0} \left[\frac{1}{\cos x}\right]\left[\frac{1-\cos x}{x}\right]$$

$$= \left[\lim_{x\to 0} \frac{1}{\cos x}\right]\cdot\left[\lim_{x\to 0} \frac{1-\cos x}{x}\right]$$

$$= [1]\cdot\left[\lim_{x\to 0} \frac{1-\cos x}{x}\right]$$

$$\because \frac{1-\cos x}{x} = \frac{0}{0} \Rightarrow$$

$$= [1]\cdot\left[\lim_{x\to 0} \frac{1-\cos x}{x}\right]\cdot\left[\lim_{x\to 0} x\right]$$

$$= [1]\cdot\left[\lim_{x\to 0} \frac{1-\sin^2 x/2}{(x/2)^2}\cdot\frac{1}{4}\right]\cdot[0]$$

$$=[1]\cdot[2]\left[\lim_{x\to 0} \frac{\sin^2 x/2}{(x/2)^2}\right]^2\cdot\frac{1}{4}\cdot[0]$$

$$= \frac{1}{2}[1]^2\cdot[0] = 0$$

<u>مثال:</u> أحسب النهاية

$$\lim_{x\to 0} \frac{\tan x-\sin x}{x^3}$$

<u>الحـل :</u> بوضع

$$L = \lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left[\frac{\sin x}{\cos x} - \sin x\right]$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left[\frac{\sin x - \sin x \cos x}{\cos x}\right]$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left[\frac{\sin x(1-\cos x)}{\cos x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\tan x(1-\cos x)}{x^3}\right]$$

$$= \lim_{x\to 0}\left[\frac{\tan x}{x}\right].\left[\frac{1-\cos x^2}{x^2}\right].\left[\frac{1}{1+\cos x}\right]$$

$$L = \left[\lim_{x\to 0}\frac{\tan x}{x}\right].\left[\lim_{x\to 0}\frac{\sin^2 x}{x^2}\right].\left[\lim_{x\to 0}\frac{1}{1+\cos x}\right]$$

$$= \left[\lim_{x\to 0}\frac{\tan x}{x}\right].\left[\lim_{x\to 0}\frac{\sin x}{x^2}\right]^2.\left[\lim_{x\to 0}\frac{1}{1+\cos x}\right]$$

$$= 1.[1]^2.\left[\frac{1}{1+\cos 0}\right]$$

$$= 1.[1]^2.\left[\frac{1}{1+1}\right] = \frac{1}{2}$$

مثال: أحسب النهاية

$$\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x^3}$$

الحـل : بوضع

$$L = \lim_{x \to 0} \frac{3\sin x - \sin 3x}{x^3}$$

$$= \lim_{x \to 0} \frac{3\sin x\left(3\sin x - 4\sin^3 x\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{4\sin^3 x}{x^3}$$

$$= \lim_{x \to 0} 4\left(\frac{\sin x}{x}\right)^3$$

$$= 4\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^3$$

$$= 4.1 = 4$$

Exponential Limits Problems

امثلة محلولة على نهاية الدوال الاسية

If $\quad a_n = \left(1 + \frac{1}{n}\right)^n \Rightarrow \lim_{h \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k,$

if $\quad$ we $\quad$ put $\frac{1}{n} = \alpha \Rightarrow$ as $n \to \infty, \quad \alpha \to 0$

$$\therefore \lim_{\alpha \to 0} (1 + \alpha)^{1/n} = e,$$

$$\lim_{\alpha \to 0} (1 + k\alpha)^{1/\alpha} = e^k$$

وبالتالى يمكن اثبات ان:

$$\lim_{x\to 0}\frac{e^x-1}{x}=1,$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log a, \qquad (a>0)$$

$$\lim_{x\to 0}\frac{a^{4x}-1}{x}=\lim_{x\to 0}\frac{a^{4x}-1}{4x}\times 4=\log a\times 4$$

مثال: أحسب النهاية

$$\lim_{x\to 0}\frac{3^{2x}-1}{\sin x}$$

الحـل : بوضع

$$L=\lim_{x\to 0}\frac{3^{2x}-1}{\sin x}$$

$$=\lim_{x\to 0}\frac{3^{2x}-1}{x}\times\frac{x}{\sin x}$$

$$=\lim_{x\to 0}\frac{3^{2x}-1}{x}\times 2\times\frac{x}{\sin x}$$

$$=2\left(\lim_{x\to 0}\frac{3^{2x}-1}{x}\right)\left(\lim_{x\to 0}\frac{x}{\sin x}\right)$$

$$=2(\log 3)(1)\ldots\ldots\left[\because x\to 0,\Rightarrow 2x\to 0,\text{Let}\quad \theta=2x\Rightarrow \lim_{\theta\to 0}\frac{e^{\theta}-1}{\theta}\right]$$

$$=\log 3^2=\log 9$$

مثال: أحسب النهاية

$$\lim_{x\to0}\frac{a^x-b^x}{x},\qquad a>0,\quad b>0$$

الحــل : بوضع

$$L=\lim_{x\to0}\frac{a^x-b^x}{x}$$

$$=\lim_{x\to0}\frac{a^x-1-b^x+1}{x}$$

$$=\lim_{x\to0}\frac{a^x-1}{x}-\left(\lim_{x\to0}\frac{b^x-1}{x}\right)$$

$$=(\log a)-(\log b)$$

$$=\log(a/b)$$

مثال: أحسب النهاية

$$\lim_{x\to0}\frac{6^x-3^x-2^x+1}{x^2}$$

الحــل : بوضع

$$L=\lim_{x\to0}\frac{6^x-3^x-2^x+1}{x^2}$$

$$=\lim_{x\to0}\frac{(2\times3)^x-3^x-2^x+1}{x^2}$$

$$=\lim_{x\to0}\frac{(2)^x(3)^x-3^x-2^x+1}{x^2}$$

$$= \lim_{x\to 0} \frac{(3)^x\left(2^x-1\right)-\left(2^x-1\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(3^x-1\right)-\left(2^x-1\right)}{x^2}$$

$$= \left(\lim_{x\to 0} \frac{3^x-1}{x}\right)\left(\lim_{x\to 0} \frac{2^x-1}{x}\right)$$

$$= (\log 3)(\log 2)$$

<u>مثال:</u> أحسب النهاية

$$\lim_{x\to 0} \frac{a^x+a^{-x}-2}{x^2}$$

<u>الحـل :</u> بوضع

$$L = \lim_{x\to 0} \frac{a^x+a^{-x}-2}{x^2} = \lim_{x\to 0} \frac{a^x+1/a^x-2}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(a^x\right)^2-2(a)^x+1}{\left(a^x\right)x^2}$$

$$= \left(\lim_{x\to 0} \frac{1}{\left(a^x\right)}\right)\left(\lim_{x\to 0} \frac{\left(a^x\right)^2-2(a)^x+1}{\left(a^x\right)x^2}\right)$$

$$= \left(\lim_{x\to 0} \frac{a^x-1}{x}\right)^2 \ldots\ldots.[x \to 0, \therefore x \neq 0]$$

$$= \frac{1}{1}(\log a)^2 = (\log a)^2$$

مثال: أحسب النهاية $\lim_{x\to 0}(1+2x)^{5/x}$

الحـل : بوضع

$$L=\lim_{x\to 0}(1+2x)^{5/x}$$
$$=\lim_{x\to 0}\left[(1+2x)^{1/x}\right]^{5}$$
$$=\left(e^{2}\right)^{5}$$
$$=e^{10}$$

مثال: أحسب النهاية $\lim_{x\to 0}\left(\frac{1+7x}{1-9x}\right)^{1/x}$

الحـل : بوضع

$$L=\lim_{x\to 0}\left(\frac{1+7x}{1-9x}\right)^{1/x}$$
$$=\lim_{x\to 0}\frac{(1+7x)^{1/x}}{1+(-9)x^{1/x}}$$
$$=\frac{\lim_{x\to 0}(1+7x)^{1/x}}{\lim_{x\to 0}\left(1+(-9)x^{1/x}\right)}$$
$$=\left(\frac{e^{7}}{e^{9}}\right)=e^{7+9}=e^{16}$$

مثال: أحسب النهاية $\lim_{x \to 0} (x)^{\frac{1}{x-1}}$

الحـل : بوضع

$$L = \lim_{x \to 0} (x)^{\frac{1}{x-1}} \ldots \left[\text{it is } 1^{\infty} \text{ form}\right]$$

$$= \text{put } x - 1 = h, \text{ as } x \to 1, h \to 0$$

$$= \lim_{h \to 0} (1+h)^{\frac{1}{h}}$$

$$= e$$

مثال: أحسب النهاية $\lim_{x \to e} \frac{\log x - 1}{x - e}$

الحـل : بوضع

$$L = \lim_{x \to e} \frac{\log x - 1}{x - e} \ldots \left[\text{it is } \frac{0}{0} \text{ form}\right]$$

$$= \text{put } x - e = h, \text{ as } x \to e, h \to 0, \text{also put } 1 = \log e.$$

$$= \lim_{h \to 0} \frac{\log(e+h) - \log e}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\log \frac{e+h}{e}\right)$$

$$= \log\left[\lim_{h \to 0} \frac{1}{h}\left(1 + \frac{1}{e}h\right)^{1/h}\right]$$

$$= \log\left[e^{1/e}\right] = \frac{1}{e}\log e = 1/e \times 1 = 1/e.$$