CHAPTER 3
Random Signals

3.1 The Real World:

In communication systems, we deal with electrical signals that carry the information from a transmitter to a receiver via a channel. These signals are called deterministic, meaning that they can be modeled as completely specified functions of time. Such a signal may be a simple function of time, or Fourier series of sinusoids, if it is periodic, or Fourier integral if it is aperiodic. But in all cases, the function has intelligence. If the time dependence is known and the value of the function at one instant of time is also known, then its value at any other instant of time can be determined, and hence the name deterministic. Other signals appear also is communication. One is called interference, which means signals from other sources seeping through the channel, or the receiver, affecting the received information. Interference coming from extraneous telephone links is called cross talk, neighbor channel interference can also occur in radio stations or from pickup of near by power lines. Another phenomenon - called distortion - causes degradation in the information, It occurs when a system or a channel treats the frequency components in the signal differently. There are amplitude, frequency and phase distortions.

Another spurious signal is noise. Noise is unwanted nonintelligible signals that cannot be totally eliminated, but may at best be reduced. Noise is associated with randomness. Thus, a noise signal cannot be formulated into a function of time or Fourier series of sinusoids. It does not have a pattern or a continuous behavior. It is erratic and unpredictable. It is not even a real function per se, but an addition of a huge amount of modes of random signals. Noise is usually associated with the nature of the electron itself. An electron in an atom for example has unpredictable motion. It does not follow a rigorous path. Quantum mechanics is a whole science developed to adapt to the notion laid down by Heisenberg known as Heisenberg Uncertainty Principe. It stipulates that the velocity and the position of an electron cannot be precisely determined at the same time. This is in contradiction to what we are used to in normal life, known as classical mechanics. Therefore, we need a different type of mechanical laws to describe the motion of the electron in an atom or in a molecule or even in a solid, i.e. in all situations in which the electron is bound. Schrödinger developed such a set of rules which is the basis of quantum mechanics. In a nutshell, we may not be able to determine an exact trajectory or orbit of an electron but we can propose a function which may tell us something about the likelihood or the probability of an electron being at a certain position or another if it has one of the discrete energy values of Bohr. We may say for example that the probability function describing the electron - denoted by Ψ^2 - is zero at the nucleus and at infinity. , or else the electron would fall onto the nucleus and the entire universe would collapse. At the other extreme the electron could not leave the atom on its own. Thus, the electron is bound within the limits of the atom. Thus, the probability that an electron exists at some point outside (henceforth called infinity) is zero. Between the two zeros the probability has a maximum. That is where we designate an orbit. But there is no way we can ascertain to the actual position of the electron at any instant of time. This logic is also explained by saying that, if we want to locate an electron, we may shine upon it a light ray to "see" it. Once we do that, the photon in the light ray collides with the electron. Since a photon has momentum and mass the electron is scattered away (Compton effect) and the reflected photon which should have located the electron would give obsolete or wrong information about the position of the electron, because the latter is no longer there. Also, a free electron moves in random motion due to its thermal energy. The random velocity is in the order of $10^5\ m/s$. The electron is a charged particle, Hence, its motion gives current. With such high velocities we would expect large values of current. But this motion is random, so its average is zero. The only current that counts is when an external supply is applied, then the electrons acquire in addition to their random velocity another velocity which is uniform for all electrons and is directed in the same direction. (AC or DC) This velocity is called drift velocity, and gives rise to a net current.

We may then ask ourselves, if the average thermal velocity is zero why bother about it? The average of a sine wave current is zero. But it is the power that counts. A sinusoidal voltage or current has zero average, but has an rms value. Also, the noise has zero average but has an rms value. The noise signal (voltage or current) and the noise power maim the useful signal. We usually define an important figure of merit called signal to noise ratio (S/N or SNR) as the ratio of the useful power to the noise power. The higher S/N, the better the system. The influence of the noise may be so grave that it overwhelms the signal altogether. We then say the signal is buried in noise. What do we do in such case? One might think that the best thing to do is to use an amplifier. An amplifier, however, amplifies the input signal, but it also amplifies the input noise. Not only that, but it augments the noise component by adding noise of its own. The amplifier has active and passive devices which have electrons in them. Thus, there is additional or excess noise which appears in the output. Therefore, the S/N at the output is less than the S/N at the input. In other words an amplifier does not help the situation when the input signal is weak. In fact, it makes it worse by adding more noise that was not present at the input. Why then use an amplifier in the first place?

We use an amplifier when the signal is weak to bring it up to our level of perception, even if the cost is signal degradation. The art of communication centers on extracting information from a noisy background. Because this is real life and *nothing is ideal, then the name of the game is how to deal with information in a non* ideal ambiance and make the best of it. This requires developing techniques to characterize, represent and tackle noise. Unwanted as it is, yet it weighs heavily on all applications in electronics and communication systems. Therefore, we need a new language to deal with it before we even find ways of combating it.

3.2 Probability Axioms:

The language appropriate for noise is probabilistic, or stochastic. It is called statistical analysis. There are some important properties for probability

1. If event A has probability of occurrence $P(A)$, then the probability of the event not to occur $P(\bar{A})$ is $1-P(A)$, where $P(A)$ is ≤ 1, i.e. $P(A)+P(\bar{A})=1$ (3 – 1)
2. Probabilities of events are said to be mutually exclusive if these events can never occur simultaneously. In this case,
$$P(A_1)+P(A_2)+\ldots P(A_m)=1 \qquad (3-2)$$
where $P(A_i)\leq 1$ for $i=1,\ldots\ldots M$
3. In the case of M equally Likely events,
$$P(A_i)=\frac{1}{M} \qquad i=1,\ldots\ldots M \qquad (3-3)$$
4. For events not mutually exclusive, there is room for *A* and *B* to occur at the same time. In this case,
$$P(A+B)=P(A)+P(B)-P(AB) \qquad (3-4)$$
where $P(AB)$ is the probability of the joint event, hence called joint probability, which is the union of events A, B. Note that $P(AB)$ is subtracted for being counted twice, in $P(A)$ and in $P(B)$. We call $P(A+B)$: A or B, while we call $P(AB)$: A and B.
5. For mutually exclusive probability $P(AB)=0$ which means there is no union or intersection between A and B. In this case
$$P(A+B)=P(A)+P(B) \qquad (3-5)$$
6. Oring [Logic OR] probabilities, means addition of probabilities, hence enhancing the total probability.

7. Anding [Logic AND] probabilities means restricting probabilities which comes about by multiplication. Note that multiplying fractions decreases the total.
8. Statistically independent events have joint probability $P(AB)$ as the product of individual probabilities

$$P(AB) = P(A)\ P(B) \qquad (3-6)$$

9. We assume for binary systems,

$$P(AB) = P(BA) \qquad (3-7)$$

This is not necessarily true when we study languages. For example, the probability of the letter q followed by the letter u is greater than the probability of the letter u followed by the letter q.

10. We define $P(B/A)$-pronounced B given A- as the conditional probability of B given A, as the probability of B occurring after guaranteeing that A has occurred first. Out of the probability of A there is a smaller probability of A and B occurring together. Thus,

$$P(B/A) = P(AB)/P(A) \qquad (3-8)$$

where $P(A) \neq 0$

Similarly,

$$P(A/B) = P(AB)/P(B) \qquad (3-9)$$

11. Alternatively,

$$P(AB) = P(B/A)\ P(A) \qquad (3-10)$$

$$= P(A/B)\ P(B) \qquad (3-11)$$

Thus, the joint probability of two events may be expressed as the product of the conditional probability of one event given the other, and the elementary probability of the other. From eqns. (3 – 9) and (3 – 10), Baye's rule follows

$$P(B/A) = P(A/B)\ P(B)/P(A) \qquad (3-12)$$

$$P(A/B) = P(B/A)\ P(A)/P(B) \qquad (3-13)$$

12. For statistically independent events, the conditional probability $P(B/A)$ is simply the elementary probability of occurrence of event B,.i.e.,

$$P(B/A) = P(B)$$

and likewise,

$$P(A/B) = P(A)$$

Hence, for statistically independent events,

$$P(AB) = P(A)\ P(B) \qquad (3-14)$$

This is to be distinguished from the mutually exclusive case where $P(AB) = 0$.

Ex. 3.1

Consider a discrete memoryless source transmitting binary data into a discrete memoryless channel, find the accuracy of the received data if the channel has probability of error for the binary data equal to p and is symmetric for both 0 and 1.

Solution

By discrete memoryless source or channel, we mean that the present data is independent of the data at any previous moment. In other words, there is no history for the data that the source or the channel may keep or remember .Due to the unavoidable presence of noise, assuming symbol 1 is sent, it may be received erroneously and interpreted as 0, which is in error. The channel is said to be symmetric, if the probability of receiving symbol 1 when symbol 0 is sent is the same as the probability of receiving symbol 0 when symbol 1 is sent.

We define the a priori probability of binary symbol 0 , 1 as

$$p(A_o) = p_o \qquad (3 - 15)$$

$$p(A_1) = p_1 \qquad (3 - 16)$$

Where A_0 , A_1 denote the events of transmitting symbols 0 , 1 respectively, we have

$$p_o + p_1 = 1 \qquad (3 - 17)$$

We define the conditional probability of error p as

$$P(B_1 / A_o) = P(B_o / A_1) = p \qquad (3 - 18)$$

where B_0 , B_1 denote the events of receiving symbols 0 , 1 , respectively. The conditional probability $P(B_1 / A_0)$ is the probability of receiving symbol 1 given that symbol 0 is sent. $P(B_0 / A_1)$ is the probability of receiving symbol 0 given that symbol 1 is sent. It is required to determine the a posteriori probabilities $P(A_0 / B_0)$ and $P(A_1 / B_1)$. The conditional probability $P(A_0 / B_0)$ is the probability that symbol 0 was sent given that symbol 0 is received. Similarly, $P(A_1 / B_1)$ is the probability that symbol 1 was sent given that symbol 1 is received. Both these conditional probabilities refer to events that are observed after the occurrence hence the name a posteriori.

Since the events B_0 and B_1 are mutually exclusive, we have

$$P(B_o / A_o) + P(B_1 / A_o) = 1 \qquad (3 - 19)$$

From eqn (3 – 18), eqn (3 – 19) becomes

$$P(B_o / A_o) = 1 - p \qquad (3 - 20)$$

$$P(B_o / A_1) + P(B_1 / A_1) = 1 \qquad (3-21)$$

From eqn (3 – 18), eqn (3 – 21) becomes

$$P(B_1 / A_1) = 1 - p \qquad (3-22)$$

We may use the transition probability diagram to represent the binary communication channel (Fig. 3.1). The term transition probability refers to the conditional probability of error which is symmetric in this case. From the diagram, we observe that the probability of receiving 0 , $P(B_0)$ is given by

$$P(B_o) = P(A_o \; B_o) + P(A_1 \; B_o)$$

From eqns (3 – 8) and (3 – 9),

$$P(B_o) = P(B_o / A_o) \; P(A_o) + P(B_o / A_1) \; P(A_1) \qquad (3-23)$$

From eqns (3 – 14), (3 – 15), (3 – 18), (3 – 20) and (3 – 22), eqn (3 – 23) becomes

$$P(B_o) = (1-p) \; p_o + pp_1 \qquad (3-24)$$

Similarly,

$$P(B_1) = P(B_1 / A_o) \; P(A_o) + P(B_1 / A_1) \; P(A_1)$$

$$= pp_o + (1-p) \; p_1 \qquad (3-25)$$

Applying Baye's rule, eqn(3 – 17), and using eqns (3 – 20), (3 – 14) and (3 – 24)

$$P(A_o / B_o) = \frac{P(B_o / A_o) \; P(A_o)}{P(B_o)}$$

$$= \frac{(1-p) \; p_o}{(1-p) \; p_o + pp_1} \qquad (3-26)$$

Similarly, using Baye's rules, eqns.(3 – 12), (3 – 22), (3 – 15),(3 -16) and (3 – 25),

$$P(A_1 / B_1) = \frac{P(B_1 / A_1) \; P(A_1)}{P(B_1)}$$

$$= \frac{(1-p) \; p_1}{pp_o + (1-p) \; p_1} \qquad (3-27)$$

These are the a posteriori probabilities of error with which we judge the accuracy of the received data. As we receive $0's$ and $1's$ how certain can we be about the correctness of what we receive? Eqns. (3 – 26) and (3 – 27) measure that accuracy. It is to be noted that if $0's$ and $1's$ are emitted from the source with equal probability, i.e., $P(A_0) = p_o = p_1 = P(B_0)$, then eqns. (3 – 27) and (3 – 28) will be identical, and the accuracy will be the same for both 0 and 1.

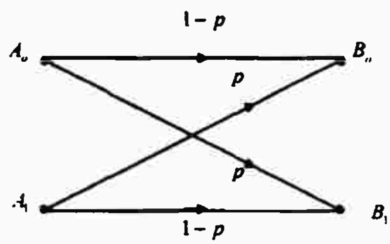

Fig. (3.1) Transition probability diagram for a binary symmetric channel

3.3 Distribution Functions:

For random signals, the outcome of an experiment is a random variable which does not depend on the input, and whose value wanders over a range of sample points called sample space. In such an experiment, the random variable may take on only a finite number of values in any finite observation interval. If, however, the random variable X takes on any value in a whole observation interval, X is called continuous random variable. An example, is the noise voltage or current at any instant of time, which is a continuous random variable whose value can be anywhere from $-\infty$ to $+\infty$. We need a probabilistic description of random variables that works equally well for discrete as well as continuous variables.

Let us consider the random variable X. The probability of the event that this random variable has a value $\le x$ (which is some definite level) is $P(X \le x)$ covers the probabilities of X being $\le x$ for all values of X. An example is the number of students whose marks fall below a certain threshold. We define the cumulative distribution function (cdf) or $F_X(x)$, as

$$F_X(x) = P(X \le x) \qquad (3-28)$$

We see that $F_X(x)$ has the following properties

1. $1 \ge F_X(x) \ge 0$ (3 − 29)
2. $F_X(x)$ is a monotone-nondecreasing (increasing or constant) function of x

   $F_X(x_1) \le F_X(x_2) \qquad x_1 < x_2$ (3 − 30)
3. $F_X(\infty) = 1$ (3 − 31)
4. $F_x(-\infty) = 0$ (3 − 32)

An alternative description of probability is the derivative of cdf, which is called probability density function (pdf) or $f_X(x)$ given by

$$f_X(x) = \frac{d}{dx} F_X(x) \qquad (3-33)$$

We note that the probability of the event $x_1 \le X \le x$, is given by

$$P(x_1 \le X \le x_2) = P(X \le x_2) - P(X \le x_1)$$

$$= F(x_2) - F(x_1) \qquad (3-34)$$

$$= \int_{x_1}^{x_2} f_X(x)dx \qquad (3-35)$$

The probability in an interval is, therefore, the area under *pdf* in that interval. The definition of $f_X(x)$ can thus be the probability for the random variable X to have a value between x and $x+dx$ per unit dx. Thus, $f_X(x)\,dx$ is the probability for the random variable X to have a value between x and $x+dx$. In general,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx \qquad (3-36)$$

From eqn (3 – 32), we have

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \qquad (3-37)$$

Ex. 3.2

A random variable X is uniformly distributed over the interval a, b. Find *pdf* and *cdf*.

Solution

From eqn (3 – 37), $f_X(x)$ must be defined as

$$f_X(x) = 0 \qquad x \le a$$

$$f_X(x) = \frac{1}{b-a} \qquad a < x \le b$$

$$f_X(x) = 0 \qquad x > b$$

From eqn (3 – 36), $F_X(x)$ must then be

$$F_X(x) = 0 \qquad x \le a$$

$$F_X(x) = \frac{x-a}{b-a} \qquad a < x \le b$$

$$F_X(x) = 1 \qquad x > b$$

(Fig. 3.2) shows both *pdf* and *cdf*

3.4 Statistical Averages and Random Variable Transformation:

We define the expected value or mean of a random variable X as

$$\bar{x} = \mu_X = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)dx \qquad (3-38)$$

Where the operator $E[\,]$ is the expected value, and μ_X is the center of gravity (CG) of the area under the *pdf* curve.

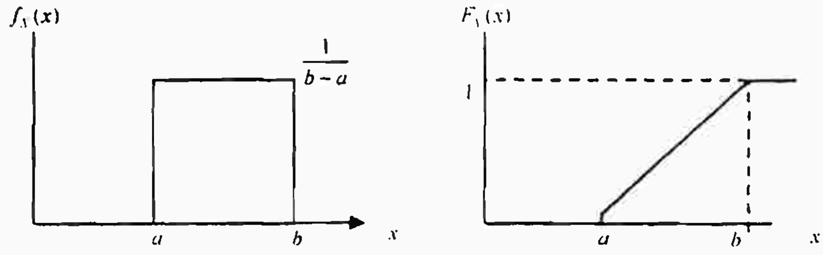

Fig. (3. 2) *pdf* **and** *cdf* **of uniform distribution**

We may define a function of random variable as Y such that $y = g(x)$. For the new random variable Y, we may define

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)dy \qquad (3-39)$$

where $f_Y(y)$ is the *pdf* of the variable Y. Alternatively, we write

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx \qquad (3-40)$$

Ex. 3.3

Let $y = g(x) = \cos x$, where X is a random variable uniformly distributed in the interval $(-\pi , \pi)$. Find $E[Y]$

Solution

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & -\pi < x < \pi \\ 0 & otherwise \end{cases}$$

$$E[Y] = \int_{-\pi}^{\pi} \cos x\left(\frac{1}{2\pi}\right)dx$$

$$= \frac{1}{2\pi}\sin x \int_{-\pi}^{\pi}$$

$$= 0$$

3.5 Statistical Moments:

If we take the special case of $g(x) = x^n$, we obtain the n^{th} moment of the probability distribution of the random variable X

$$E\left[X^n\right]=\int_{-\infty}^{\infty} x^n f_X(x)dx \tag{3-41}$$

If $n=1$, we get the mean of the random variable or first moment (eqn. 3 – 38). If we put $n=2$, we get the mean square value of X

$$E\left[X^2\right]=\int_{-\infty}^{\infty} x^2 f_X(x)dx \tag{3-42}$$

We may also define the central moments which are simply the moments of the difference between a random variable X and its mean μ_X

$$E\left[(X-\mu_X)^n\right]=\int_{-\infty}^{\infty}(x-\mu_X)^n f_X(x)dx \tag{3-43}$$

For $n=1$, the central moment is zero

For $n=2$, the central moment is called the variance $\mathrm{var}[X]$ or σ_X^2, where σ_X is the standard deviation .

$$\sigma_X^2=\mathrm{var}[X]=E\left[(X-\mu_X)^2\right]=\int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)\,dx \tag{3-44}$$

Thus,

$$\begin{aligned}\sigma_X^2 &= E\left[X^2-2\mu_X X+\mu_X^2\right]\\ &= E\left[X^2\right]-2\mu_X E[X]+\mu_X^2\\ &= E\left[X^2\right]-\mu_X^2\end{aligned} \tag{3-45}$$

For noise, $\mu_X=0$

$$\sigma_X^2=E\left[X^2\right] \tag{3-46}$$

Thus, σ_X^2 is a measure of the power of the random signal when the average of the random signal is zero. (for 1 Ω resistor with x representing noise current or voltage)

3.6 The Gaussian Distribution:

One of the most important distributions is the normal (Gaussian) distribution, given by

$$f_X(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2} \tag{3-47}$$

This curve is shown both for $f_X(x)$ and $F_X(x)$ (Fig. 3.3).

It has the following properties

1. $\int_{-\infty}^{\infty} f_X(x)dx=F_X(\infty)=1$ (3 – 48)

2. The peak of $f_X(x)$ lies at μ and the peak value is $1/\sigma\sqrt{2\pi}$
3. The mean value of x following the distribution $f_X(x)$ is μ
4. The standard deviation of $f_X(x)$ is σ
5. $F_X(o) = 0.5$

For the Gaussian distribution we will show that the mean is μ and the standard deviation is σ

The average $E(X) = \bar{x} = \mu_x = \int_{-\infty}^{\infty} x f_x(x) dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-(x-\mu)^2/2\sigma^2} dx$$

To integrate, let $u = \dfrac{x-\mu}{\sqrt{2\sigma^2}}$

$$x = \sigma\sqrt{2}\,u + \mu$$

$$dx = \sigma\sqrt{2}\,du$$

$$\bar{x} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sigma\sqrt{2}\,u + \mu\right) e^{-u^2} \sigma\sqrt{2}\,du$$

$$\bar{x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sigma\sqrt{2}\,u + \mu) e^{-u^2} du$$

$$= \sigma\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} u\, e^{-u^2} du + \frac{2\mu}{\sqrt{\pi}} \int_{o}^{\infty} e^{-u^2} du$$

The first integral is zero since it is an odd function. The second integral $\int_0^{\infty} e^{-u^2} du$ equals $\dfrac{\sqrt{\pi}}{2}$

$$\bar{x} = \frac{2\mu}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \mu \qquad (3-49)$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2\, f_X(x) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2\, e^{-x^2/2\sigma^2} dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2\sigma^2} dx$$

Put $u = \frac{x^2}{2\sigma^2}$ $du = \frac{2x}{2\sigma^2} dx$

$$E\left[X^2\right] = \frac{2}{\sigma\sqrt{2\pi}} \sigma^2 \sqrt{2}\sigma \int_{-\infty}^{\infty} u^{1/2} e^{-u} du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{1/2} e^{-u} du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \gamma\left(3/2\right)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \gamma\left(\frac{1}{2}\right)$$

where gamma function $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Thus, $E\left[X^2\right] = \sigma^2$ (3 – 50)

If $\mu = 0$, we note that

$$P(-x_1 < x < x_1) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-x_1}^{x_1} e^{-x^2/2\sigma^2} dx$$

Putting $u = \frac{x}{\sigma\sqrt{2}}$ $du = \frac{dx}{\sigma\sqrt{2}}$

$$P(-x_1 < x < x_1) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-x_1}^{x_1} e^{-u^2} \sigma\sqrt{2} du$$

$$P(-x_1 < x < x_1) = \frac{1}{\sqrt{\pi}} \int_{-u_1}^{u_1} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^u e^{-u^2} du$$

This integral is called the error function $erf(x)$ as $u = \frac{1}{\sigma\sqrt{2}} x$ (Fig. 3.3),

$$= \frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-u^2} du \qquad (3-51)$$

For cdf we have

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} dx \qquad (3-52)$$

Letting $\frac{x-\mu}{\sigma} = z$ and $dx = \sigma dz$

$$f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{z^2/2} \qquad (3-53)$$

For value of $z = y$

$$F_Z(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{z^2/2} dz \qquad (3-54)$$

It is convenient to use the function $Q(y)$

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-z^2/2} dz \qquad (3-55)$$

This function is the area under $f_X(x)$ from y to ∞ (Fig. 3.3a). From the symmetry of $f_z(z)$ around the origin and the fact that the total area under $f_z(z) = 1$

$$Q(-y) = 1 - Q(y) \qquad (3-56)$$

Eqn. (3 – 54) becomes

$$F_Z(y) = 1 - Q(y) \qquad (3-57)$$

Thus,

$$P(X \le x) = P(Z \le y) = 1 - Q(y) \qquad (3-58)$$

$$P(X > x) = P(Z > y) = Q(y) \qquad (3-59)$$

We define the complementary error function $erfc(y)$

$$erfc(y) = \frac{2}{\sqrt{\pi}} \int_{y}^{\infty} e^{z^2} dz \qquad (3-60)$$

Comparing eqn. (3 – 60) with eqn. (3 – 55), putting $\frac{z}{\sqrt{2}} = u$, then for value $z = y$,

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{z^2/2} dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{y/\sqrt{2}}^{y_\infty} e^{-u^2} du \qquad (3-61)$$

We also have

$$erfc(y) = \frac{2}{\sqrt{\pi}} \int_{y}^{\infty} e^{-z^2} dz \qquad (3-62)$$

Putting $\frac{y}{\sqrt{2}} = y'$, $y = \sqrt{2}\, y'$ in eqn. (3 – 55)

$$Q\left(\sqrt{2}\; y'\right) = \frac{1}{\sqrt{\pi}} \int_{y'}^{\infty} e^{-u^2} du$$

Putting $y = y'$ in eqn. (3 – 62)

$$erfc(y') = \frac{2}{\sqrt{\pi}} \int_{y'}^{\infty} e^{-u^2} du \qquad (3-63)$$

Comparing eqns. (3 – 62) and (3 – 63) and putting y' back as y (dummy variable)

$$erfc(y) = 2Q(\sqrt{2}\; y) \qquad (3-64)$$

Alternatively we may write

$$Q(\sqrt{2}\; y) = \frac{1}{2}\, erfc(y) \qquad (3-65)$$

Putting $\sqrt{2}\, y = y''$, $y = y'' / \sqrt{2}$

$$Q(y'') = \frac{1}{2}\, erfc(y'' / \sqrt{2})$$

Putting y'' back to y (dummy variable)

$$Q(y) = \frac{1}{2} erfc\left(\frac{y}{\sqrt{2}}\right) \qquad (3-66)$$

We note that

$$erf(y) = \frac{2}{\sqrt{\pi}} \int_{0}^{y} e^{-z^2} dz$$

$$erf(\infty) = \frac{z}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} dz = 1$$

$$erfc(y) = \frac{z}{\sqrt{\pi}} \int_{y}^{\infty} e^{-z^2} dz = 1 - erf(y) \qquad (3-67)$$

Thus, eqn. (3 – 66) becomes

$$Q(y) = \frac{1}{2}\left[1 - erf\left(\frac{y}{\sqrt{2}}\right)\right] \qquad (3-68)$$

(Fig. 3.3) shows the Gaussian pdf, $Q(y)$ and cdf

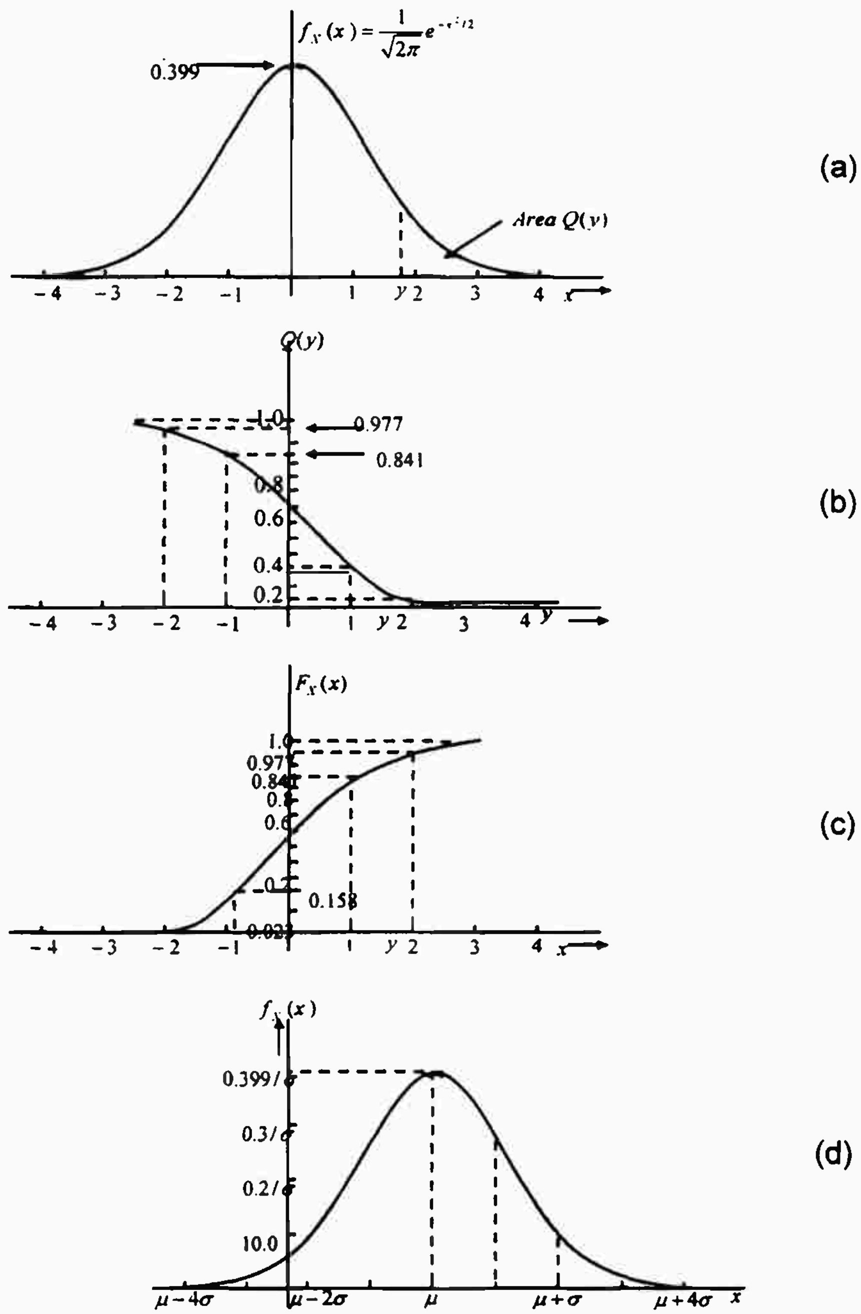

Fig. (3.3) Gaussian distribution

a) pdf *b) $Q(y)$* *c) cdf* *d) Gaussian with mean* $\neq 0$

3.7 Several Random Variables:

Thus far, we have focused our attention on situations involving a single random variable. We now consider two random variables X , Y . We define the joint distribution function $(j\,cdf)$ as $F_{X,Y}(x,y)$ as the probability that the random variable $X \le$ a specified value x, and that the random variable $Y \le$ a specified value y. The joint distribution function $F_{X,Y}(x,y)$ is the probability that the output of an experiment will result in a sample point lying in the quadrant $-\infty < X \le x$ and $-\infty < Y \le y$ of the joint sample space.

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) \qquad (3-69)$$

We now define for - continuous $F_{X,Y}(x,y)$ and its derivative - the joint probability density function $(j\,pdf)$ $f_{X,Y}(x,y)$ as

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \qquad (3-70)$$

The function $F_{X,Y}(x,y)$ is a monotone non decreasing (increasing or constant) function of both x and y. Then, $F_{X,Y}(x,y)$ is a non negative (positive or zero). The total volume under the graph of $j\,pdf$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \qquad (3-71)$$

$$F_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy \qquad (3-72)$$

Differentiating both sides with respect to x

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \qquad (3-73)$$

Thus, the pdf $f_X(x)$ is obtained from the $j\,cdf$ function $F_{X,Y}(x,y)$ by integrating it over all possible values of the "other" random variable Y. Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \qquad (3-74)$$

The $pdf's$ $f_X(x)$ and $f_Y(y)$ are called marginal densities. The $j\,pdf$ $f_{X,Y}(x,y)$ contains all the possible information about the joint random variables X and Y. We define the conditional pdf of "y given x" as

$$f_Y(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \qquad (3-75)$$

Where $f_X(x)$ is >0

This is similar to eqn. (3 – 7). We have two requirements for $f_Y(y/x)$ as follows

$$f_Y(y/x) \geq 0 \qquad (3-76)$$

$$\int_{-\infty}^{\infty} f_Y(y/x)dy = 1 \qquad (3-77)$$

If the random variable X and Y are statistically independent then knowledge of the outcome of X can in no way affect the distribution of Y. In this case, the conditional *pdf* $f_Y(y/x)$ reduces to the marginal density $f_Y(y)$, i.e.,

$$f_Y(y/x) = f_Y(y) \qquad (3-78)$$

Thus eqn. (3 – 69) reduces to

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \qquad (3-79)$$

Which is similar to eqn (3 – 5). Equivalently, we say that if the *j pdf* of the random variables X and Y equals the product of their marginal densities, then X and Y are statistically independent.

It is often the case that Y is given as a function of another random variable X. For example

$$Y = g(X) \qquad (3-80)$$

Let X be a continuous random variable with *pdf* $f_X(x)$. Let $Y = g(X)$ be a monotone, differentiable function of X. Taking the differentials of eqn. (3 – 74),

$$y + dy = g(x) + \frac{dg}{dx}dx \qquad (3-81)$$

Consider the event $y < Y \leq y + dy$, the probability of this event in $P(y < Y \leq y + dy)$. The event $x < X \leq x + dx$ has probability $P(x < X \leq x + dx)$. Because of the one to one transformation of the random variable,

$$P(y < Y \leq y + dy) = P(x < X \leq x + dx) \qquad (3-82)$$

This may be rewritten as

$$f_Y(y)dy = f_x(x)dx \qquad (3-83)$$

It is assumed here that $g(x)$ is a monotone increasing function. If $g(x)$ is a monotone decreasing function, then

$$f_Y(y)dy = -f_X(x)dx \qquad (3-84)$$

Thus

$$f_Y(y)|dy| = f_X(x)|dx| \qquad (3-85)$$

This is called conservation of probability. Then,

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \qquad (3-86)$$

Then, to get $f_Y(y)$, we substitute $x = g^{-1}(y)$.

Problems

1. In a binary symmetrical channel, the probability of error is $.01$ and the frequency of transmitting 0 is 0.3. Find the accuracy of receiving 0 and 1 at the receiver. Repeat if the probability of error is 0.5. What do you conclude?

2. Repeat the above problem if the transmitter is in the place of the receiver. What do you conclude?

3. A random variable has a triangular pdf in the interval a, b, find its pdf.

4. For a parabolic cdf, find pdf.

5. Consider the transformation $Y = \cos X$, where X is uniformly distributed in the interval $-\pi, \pi$, find the pdf of Y.

6. Find $E[Y]$ for the problem above. What do you conclude?

7. A sinusoidal voltage is sampled randomly. The peak value is $\pm A$. Determine the mean and mean square of the sampled output.

8. Show that the mean of the product of two independent random variables is the product of their means.

9. Rayleigh pdf is defined as $f_r(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-r^2/2\sigma^2} & r \geq 0 \\ 0 & r < 0 \end{cases}$

$$f_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & otherwise \end{cases}$$

   Derive this function by considering two random variables X, Y with two independent and identical Gaussian $pdf's$. Sketch the output $f_r(r)$ and $f_\theta(\theta)$.

10. Find the mean of X, Y in the problem above.

References

1. "Digital Communications," B. Sklar, 2nd ed., Prentice Hall, Upper Saddle River, N.J. , 2001.
2. "Digital Communications," S Haykin, Wiley, N.Y., 1988.
3. "Communication Systems," S. Haykin, 3nd ed. Wiley, N..Y. 1994
4. "Digital and Analog Communication Systems." L. Couch, 6th ed., Pearson Education, Prentice Hall, Upper Saddle River, N.J., 2001.
5. "Modern Digital and Analog Communication Systems," B. Lathi, 3rd ed, Oxford, N.Y. 1998.