CHAPTER 5
Additive White Gaussian Noise (AWGN)

5.1 The Central Limit Theorem:

If two random variables X and Y are added and their joint pdf is $f_{XY}(x,y)$ is known, what is the pdf, $f_Z(z)$ of their sum? We have

$$Z = X + Y \qquad (5-1)$$

The probability that Z lies in the range z to $z+dz$ is given by the volume contained under $f_{XY}(x,y)$ with the strip between the two lines $Y = z - x$ and $Y = z + dz - x$ (Fig. 5.1)

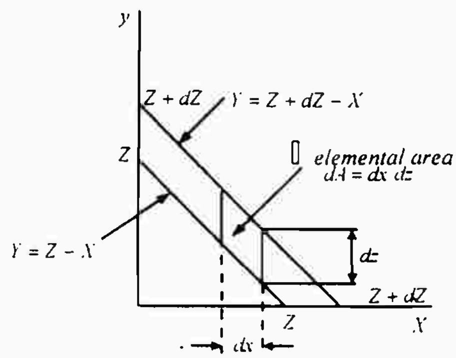


Fig. (5.1) Finding $P(z < Z \leq z + dz)$

$$P(z < Z \leq z + dz) = \int_{strip} f_{XY}(x,y)\,dA \qquad (5-2)$$

$$f_Z(z)\,dz = \int_{strip} f_{XY}(x, z - x)\,dx\,dz \qquad (5-3)$$

where $y = z - x$ and $dA = dx\,dz$

Therefore,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x)\,dx \qquad (5-4)$$

If X and Y are statistically independent, then

$$f_{XY}(x, z - x) = f_X(x)\, f_Y(z - x) \qquad (5-5)$$

Thus, eqn. (5 – 4) becomes

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z - x)\,dx \qquad (5-6)$$

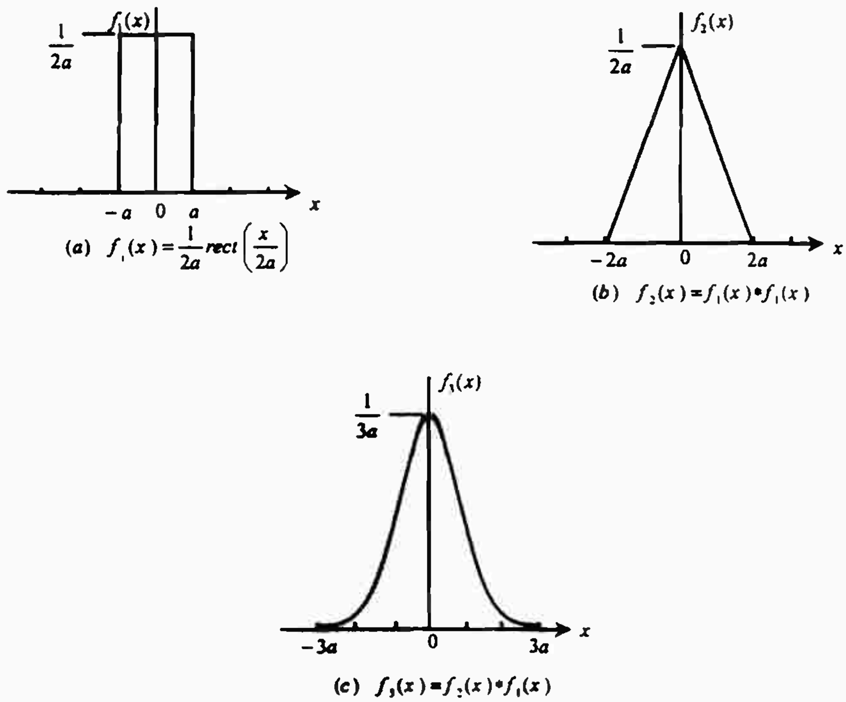


Fig. (5.2) Multiple convolutions of a rectangular pulse

Thus, the *pdf* of the sum of independent random variables is the convolution of their individual *pdfs*.

The multiple convolutions of *pdfs* arises when many independent random variables are to be added. Since convolution is an integration process it has a smoothing effect. After few convolutions, this repeated smoothing results in a distribution which approximates a Gaussian function. This is the essence of the central limit theorem. We see in Fig. (5.2) a *pdf* in the form of a rectangular pulse $f_1(x) = \frac{1}{2a} rect\left(\frac{x}{2a}\right)$. The function $f_2(x)$ is the convolution of $f_1(x)$ with itself as a result of adding a random variable x to a random variable x. The function $f_3(x)$ is the convolution of $f_2(x)$ with $f_1(x)$. We can see how close the result approximates a Gaussian.

The central limit theorem states that: if N statistically independent random variables are added, the sum will have a *pdf* which tends to a Gaussian function as N tends to infinity irrespective of the original random variable *pdfs*.

If two Gaussian random variables are added, their sum will also be a Gaussian random variable. The mean and the sum of the variance of the sum are given (as in Ex 4.2) by $\overline{Z} = \overline{X} + \overline{Y}$,

$$\sigma^2_{X \pm Y} = \sigma^2_X \pm 2\rho\sigma_X\sigma_Y + \sigma^2_Y \qquad (5-7)$$

If $\rho = 0$,

$$\sigma^2_{X \pm Y} = \sigma^2_X + \sigma^2_Y \qquad (5-8)$$

For uncorrelated (hence independent) Gaussian random variables, the variances – like the mean - are simply added. This is in harmony with the result of Ex 4.13, where the PSD of the sum is the sum of PSD's for statistically independent random variables, i.e., the powers due to individual random variables simply add up .Note that σ^2 is the mean of power. We need to state here some of the properties of Gaussian random variables.

5.2 Gaussian Processes:

The Gaussian function is

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}}\, e^{-(x-\mu_X)^2/2\sigma^2_X} \qquad (5-9)$$

and the normalized Gaussian $N(0,1)$ has $\mu_X = 0$, $\sigma_X = 1$ (Fig. 5.3)

$$N(0\,,\,1) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \qquad (5-10)$$

The following properties hold for Gaussian processes

1. If $X(t)$ is Gaussian, then

$$Y\,(t) = \int_0^t G(\tau)\, X(\tau) d\tau \qquad (5-11)$$

   $Y(t)$ is also Gaussian regardless of $G(t)$, provided the mean square value of $Y(t)$ is finite.

2. If a Gaussian process $X(t)$ is applied to a stable linear filter, then the output random process is also Gaussian

$$Y(t) = \int_0^t h(t-\tau)\; X(\tau) d\tau \qquad 0 \le t < \infty$$

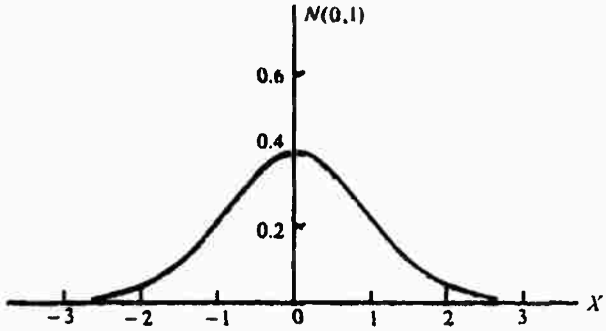


Fig. (5.3) Normalized Gaussian distribution

3. If the random variables $X(t_1), X(t_2) .... X(t_n)$ are obtained by sampling a Gaussian process $X(t)$ by many observers at times $t_1, t_2 .... t_n$ are uncorrelated, i.e.

$$E\left[\{X(t_k) - \mu_X(t_k)\}\{X(t_i) - \mu_X(t_i)\}\right] = 0 \qquad i \neq k$$

then, these random variables are statistically independent. Therefore the joint *pdf* of the set of random variables $X(t_1), X(t_2) .... X(t_n)$ can be expressed as the product of the *pdf* functions of the individual random variables.

4. For independent Gaussian variables, the *pdf* of the sum is the convolution of two Gaussian functions. This is equivalent to multiplying the Fourier transforms of the original *pdfs* and then inverse Fourier transforming the result. The Fourier transform of a *pdf* is called the characteristic function of the random variable. When a Gaussian *pdf* is Fourier transformed, the result is a Gaussian characteristic function. When Gaussian characteristic functions are multiplied the result remains Gaussian. When the Gaussian product is inverse Fourier transformed, the result is also a Gaussian *pdf*. The characteristic function $\varphi_X(v)$ of *pdf* is

$$\phi(v) = E\left[e^{-jvx}\right]$$

$$= \int_{-\infty}^{\infty} f_X(x)\, e^{-jvx}\, dx \qquad (5-12)$$

v, x play the role of ω, t of Fourier transform

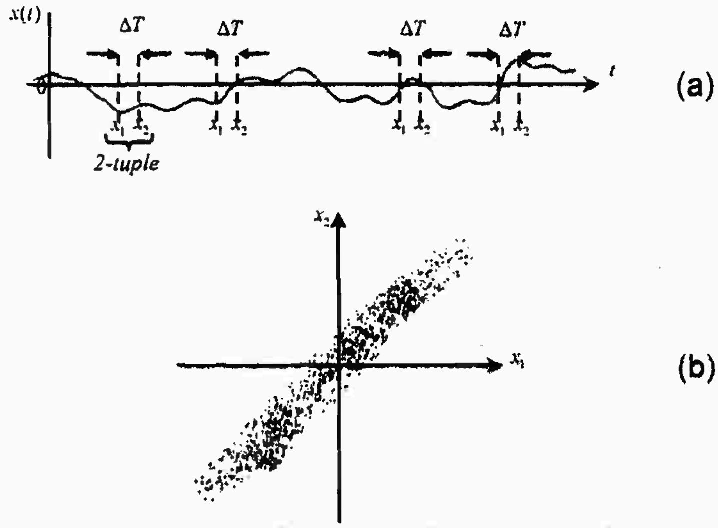


Fig. (5.4) Single sample function of ergodic strict sense Gaussian random process

a) N tuple $(N=2)$ samples with constant sample separation ΔT taken at random times from a sample function $x(t)$ of the random process $X(t)$

b) Joint N variate $(N=2)$ Gaussian scatter diagram for $f(x_1 \ , x_2)$.

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_X(v)\, e^{jvx}\, dv \qquad (5-13)$$

For Gaussian *pdf*

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_X} e^{\frac{(x-\mu_X)^2}{2\sigma_X^2}} \qquad -\infty < x < \infty \qquad (5-14)$$

The characteristic function is

$$\varphi_X(v) = e^{-jv\mu_X - \frac{1}{2}v^2\sigma_X^2} \qquad (5-15)$$

which is also Gaussian.

5. A sample function $X_i(t)$ is said to belong to a Gaussian random process $X(t)$ in the strict sense if the random variables $X_1 = X(t_1)$, $X_2 = X(t_2)$ $\dots X_N = X(t_N)$ have an N dimensional joint Gaussian *pdf*. For an ergodic process the strict sense Gaussian condition can be defined in terms of a single sample function. In this case, if the joint *pdf* of multiple sets of N

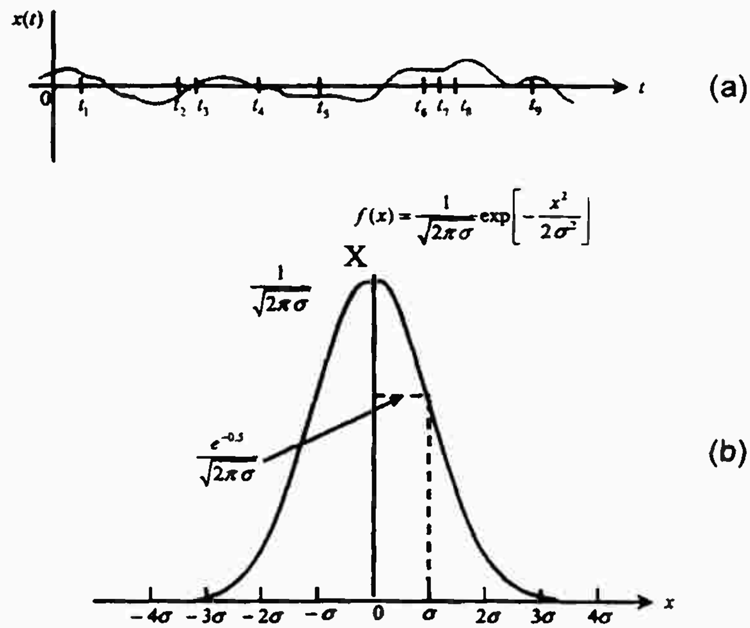


Fig. (5.5) Definition of wide sense Gaussian process

i. isolated samples $x(t_n)$ taken at random from $x(t)$

ii. Gaussian distribution of samples from above

tuple samples taken with fixed time intervals between the samples of each N tuple is N variate Gaussian, then the process is Gaussian in the strict sense. Fig.(5.4) illustrates this situation for multiple sets of sample pairs $(N = 2)$

6. A sample function $X_i(t)$ is said to belong to a Gaussian random process in the wide sense if isolated samples taken from $x_i(t)$ come from a Gaussian *pdf*.

5.3 Thermal Noise:

The random motion of electrons in a resistor gives rise to random current or random voltage whose average is zero. Since this random signal is generated by a large number of independent random variables, the overall *pdf* is Gaussian. This *pdf* gives the probability of the total noise having a certain value. From the power point of view, since these random variables are independent, the overall PSD is the sum of individual PSDs. It has been found from thermodynamics that the overall PSD of thermal noise is given by

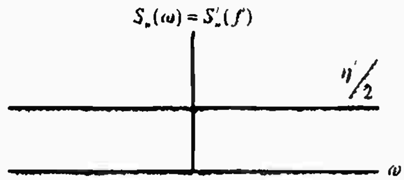


Fig. (5.6) Flat noise PSD

$$S_n'(\omega) = \frac{h|\omega|}{\pi\left[e^{h|\omega|/2\pi kT} - 1\right]} \qquad (5-16)$$

where h is Planck's constant and k is Maxwell Boltzmann's constant.

For $|\omega| << \frac{2\pi kT}{h}$ ($= 6000\ GHz$ at $T = 298°K$), eqn. (5 – 16) reduces to

$$S_n'(f) = S_n'(\omega) = 2kT \qquad (5-17)$$

If we call

$$S_n'(f) = S_n'(\omega) = \eta'/2 \qquad (5-18)$$

Then

$$\eta' = 4kT \qquad (5-19)$$

Thus, $S_n'(f) = S_n'(\omega)$ is flat for all frequencies below the quantum mechanical limit which is way too far to reach (Fig. 5.6.).

The noise power is usually calculated for a certain band of frequencies $\pm B$

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n'(\omega)\,d\omega = \int_{-\infty}^{\infty} S_n'(f)\,df \qquad (5-20)$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \frac{\eta'}{2}\,d\omega = \int_{-B}^{B} \frac{\eta'}{2}\,df = \eta' B \qquad (5-21)$$

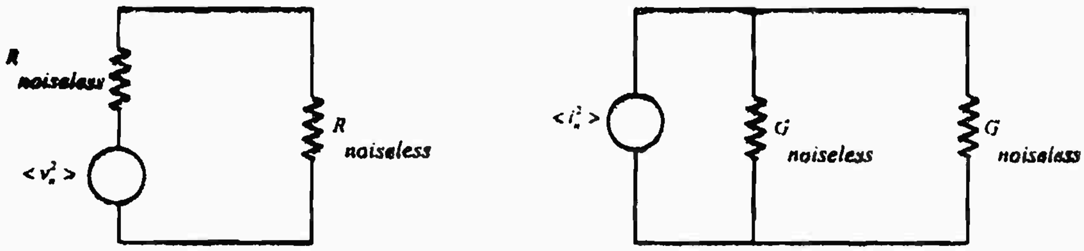
If we consider this power P_a as being available from a resistor R across which a random open circuit voltage is developed whose mean square is $< v_n^2 >$

$$P_a = \frac{< v_n^2 >}{R} = \eta' B \qquad (5\text{-}22)$$

$$< v_n^2 > = \eta' RB$$

$$= 4kTRB \qquad (5-23)$$

Similarly, a short circuit noise current develops whose mean square $\langle i_n^2 \rangle$ is given by



(a) (b)

Fig. (5.7) Noise equivalent circuits
a) Thevenin's b) Norton's

$$P_a = < i_n^2 > R = \eta' B \qquad (5-24)$$

$$< i_n^2 > = \frac{\eta' B}{R} = \eta' BG$$
$$= 4kTGB \qquad (5-25)$$

This noise power is the available power, but it is not the power deliverable to other circuit components, because the resistance is open circuit. We can now think of the actual noise power that can be delivered under matched condition by considering Thevenin's (or Norton's) equivalent circuit of noise. We have the open circuit (oc) mean square noise voltage (eqn. 5 – 23) in series with noiseless R as source resistance matched to noiseless R (Fig. 5.7a). We also have a noise current source $< i_n^2 >$ of eqn. (5 – 25) in parallel with noiseless G and matched to noiseless G (Fig. 5.7b).
Under matched condition,
the matched power P_n is given by

$$P_n = \frac{1}{4} < \frac{v_n^2}{R} >$$
$$= \frac{1}{4} \frac{< i_n^2 >}{G}$$
$$= \frac{1}{4} P_a \qquad (5-26)$$

Thus,

$$P_n = \frac{1}{4} \eta' B = \eta B = kTB \qquad (5-27)$$

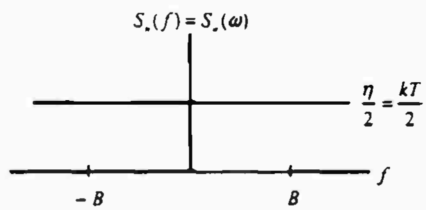


Fig. (5.8) PSD of noise under matched condition

as $$\eta = \frac{\eta'}{4} = kT \qquad (5-28)$$

Also, eqn. (5 – 17) reduces to

$$S_n(f) = S_n(\omega) = \frac{S_n'(f)}{4} = \frac{2kT}{4} = \frac{kT}{2} = \frac{\eta}{2} \qquad (5-29)$$

The PSD of noise is usually $S_n(f)$, i.e., under matched condition (Fig. 5.8)

The factor 2 takes care of positive and negative frequencies so that for finite positive sided B

$$P_n = \frac{kT}{2} \times 2B \qquad (5-30)$$

$$= kTB \qquad (5-31)$$

This type of noise shown (Fig. 5.8) is called white noise, because it is composed of all frequencies with flat response, similar to white light which is composed of all colors. This noise is additive, since the total PSD is the sum of individual PSD's, hence it is Gaussian. Therefore, it is called Additive White Gaussian Noise (AWGN).

There are different sources of noise besides thermal. But we will focus here on all those types of noise which in the end are AWGN.

Using eqn. (4 – 66) for the autocorrelation function of white noise

$$R_X(\tau) = \int_{-\infty}^{\infty} \frac{\eta}{2} e^{j2\pi f\tau} df$$

$$= \frac{\eta}{2} \delta(\tau)$$

$$= \frac{kT}{2} \delta(\tau) \qquad (5-32)$$

The same result can be obtained from eqn. (4 – 68)

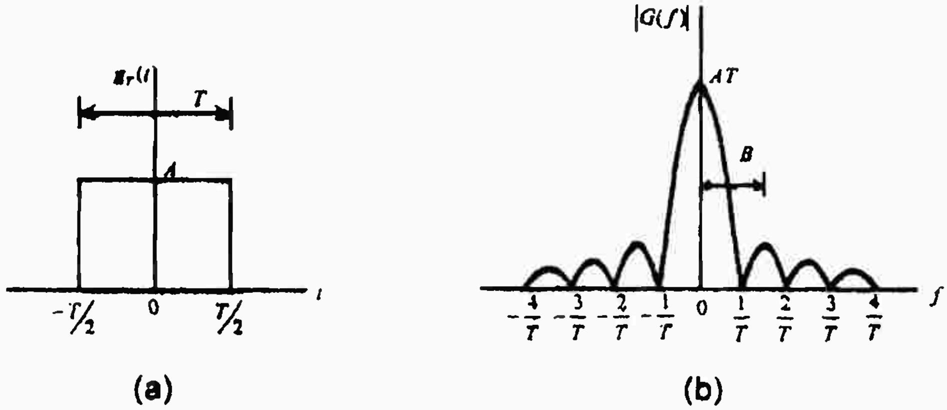


Fig. (5.9) Rectangular pulse and its Fourier transform

$$R_X(\tau) = \frac{1}{2\pi}\int \frac{\eta}{2}\, e^{j\omega\tau}\, d\omega$$
$$= \frac{\eta}{2}\delta(\tau)$$
$$= \frac{kT}{2}\delta(\tau)$$

Thus, the autocorrelation function of noise is a delta function. This means that the decorrelation time is extremely small, meaning that noise signals are completely independent statistically. In fact, the decorrelation time serves as memory for the random signal. For noise there is no memory at all. The random signal correlates only at $\tau = 0$ but decorrelates completely for $\tau \neq 0$, i.e., has zero memory. Adjacent signals are uncorrelated no matter how close. We also note that any Fourier transform pair holds conjugation relationship. For example a pulse in time has $\sin c$ function as Fourier transform (Fig. 5.9).

We note that $T \times B = 1$ (5 - 33)

where T is the pulse width in time and B is the positive sided bandwidth. This is Heisenberg uncertainty principle. As T becomes zero, i.e., the event is well localized in time the bandwidth is extended, and vice versa.

We see the same relation in the PSD, $R(\tau)$ pair (Fig. 5.10), i.e., the decorrelation time $\tau_0 = 1/2B$ and the two sided bandwidth of the PSD is $2B$, hence

$$\tau_0 \times B = 1 \qquad (5-34)$$

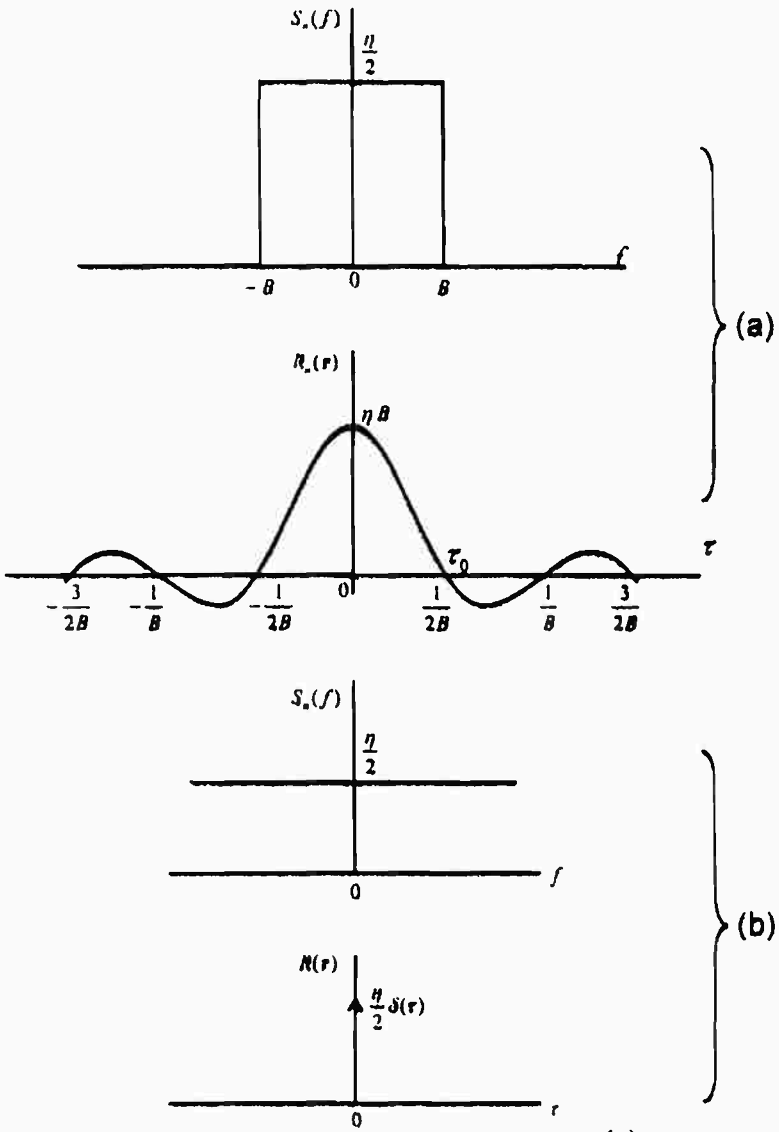


Fig. (5.10) Characteristics of PSD and $R(\tau)$ Pair for AWGN

a) finite bandwidth and decorrelation time

b) localized delta function and extended bandwidth

Thus the Bandwidth is the inverse of the decorrelation time, i.e., as $\tau_0 = 0$ for delta function, its PSD frequency spectrum is extended for all frequencies from $-\infty$ to $+\infty$.

The significance of the autocorrelation and cross correlation can best be emphasized in the following case study. The autocorrelation function is widely used in the detection or recognition of signals masked by additive noise. Consider a periodic square wave (Fig. 5.11). Its autocorrelation function is triangular. A random noise waveform $n(t)$ is added to it such that $y(t) = f(t) + n(t)$. Note that even though the square wave is immersed in noise, the autocorrelation function is clearly recognizable in the final result. We note that $f(t)$ and $n(t)$ are statistically independent that is why the autocorrelation function of $y(t)$ is the sum of the auto correlation function of $f(t)$ and $n(t)$, the latter being a delta function. That is why for $\tau > 0$ the autocorrelation is essentially that of $f(t)$.

As an example of the use of cross correlation we choose a random signal $f(t)$ (Fig. 5-12). For $y(t)$, we choose a delayed replica of $f(t)$ plus noise, so that $y(t) = f(t - t_0) + n(t)$. The receiver has a replica of $f(t)$ available in memory. On the basis of this knowledge, the receiver can make a measurement of the time delay t_0. We take the cross correlation function

$$R_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, g(t+\tau)\, dt \qquad (5-35)$$

The value of the time delay t_0 is evident by measuring the time delay from the origin to the large peak in the result.

Correlation functions thus measure the similarity of a signal $f(t)$ either with itself (in case of autocorrelation) or with another signal (in case of crosscorrelation) versus a relative shift by an amount τ. For dissimilar signals, the peak of the correlation function is an indication of how good this match is between signals. Thus, autocorrelation and cross correlation are powerful tools in recovering signals buried in noise.

5.4 Bandlimited Noise:

If white noise is passed through a LPF with bandwidth $\pm B\ Hz$, the output noise can be obtained by means of the transfer function $H(\omega)$

$$\frac{S_0(\omega)}{S_i(\omega)} = |H(\omega)|^2 \qquad (5-36)$$

Taking $|H(\omega)| = 1$, using eqn. (5 – 20)

$$P_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)\, d\omega$$

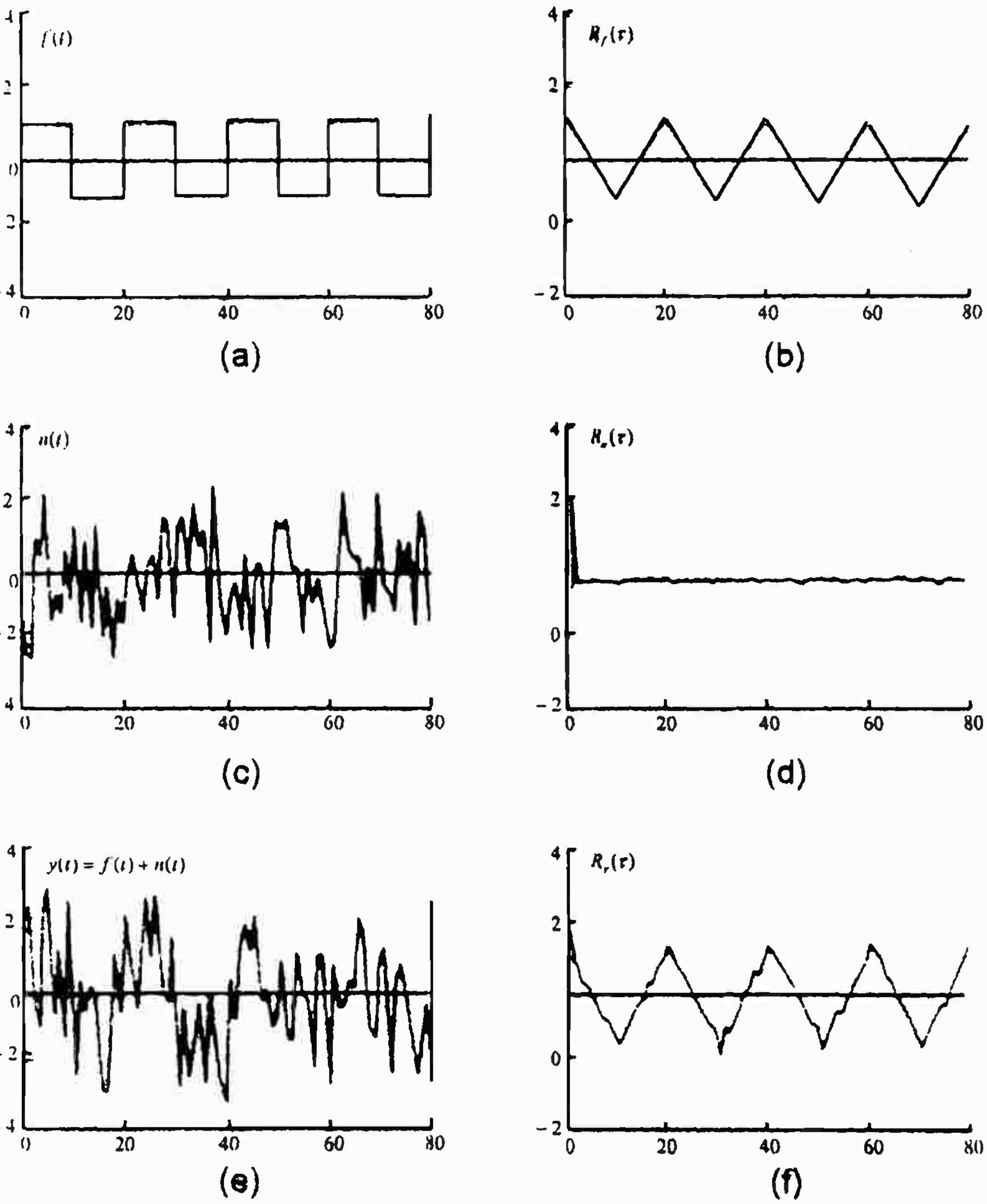


Fig. (5.11) Autocorrelation of a periodic function plus noise

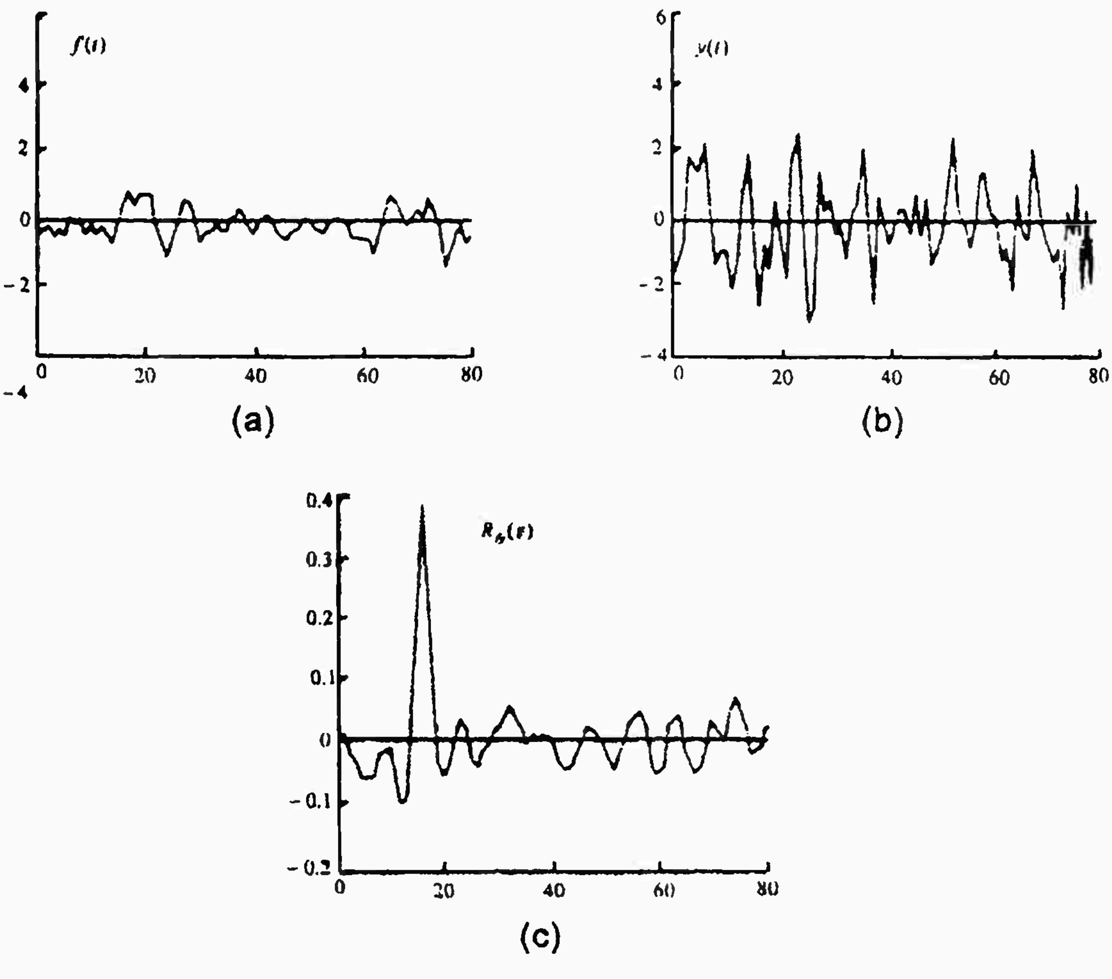


Fig.(5.12.) Cross correlation of random signal plus noise

$$= \frac{kT}{2}\int_{-\infty}^{\infty} df = kTB \qquad (5-37)$$

The autocorrelation function of the filtered white noise can be obtained using eqn. (4 – 66)

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{kT}{2}\int_{-B}^{B} e^{j2\pi f\tau}\, df$$

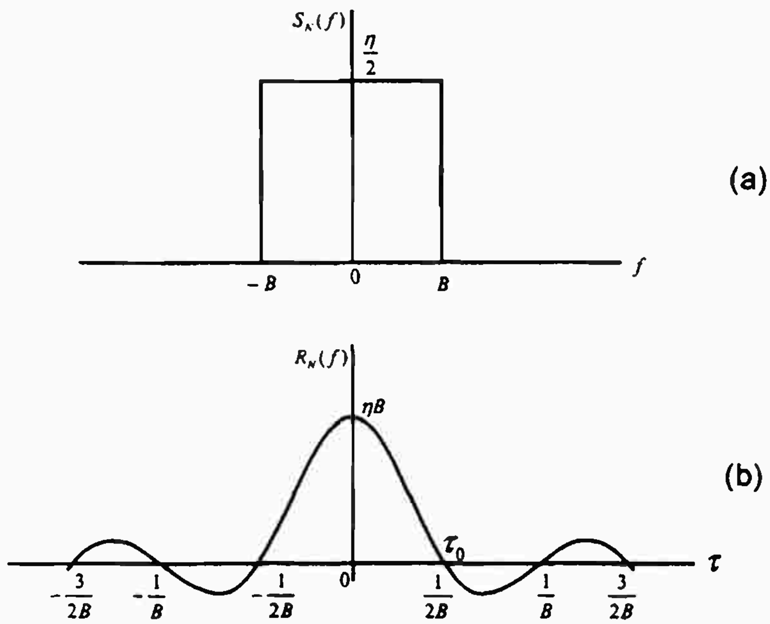


Fig. (5.13) Characteristics of low pass filtered white noise
a) PSD *b)* $R_n(\tau)$

$$= \frac{kT}{2}\left[\frac{e^{j2\pi f\tau}}{j2\pi\tau}\right]_{-B}^{B}$$

$$= kTB\ \mathrm{sinc}(2\pi B\tau) \qquad (5-38)$$

The decorrelation time is given by $\tau_0 = \frac{1}{2B}$ (Fig. 5.13). Hence, filtering an uncorrelated white noise produces correlated bandlimited white noise or so called colored noise.

Ex. 5.1
Consider a white Gaussian noise applied to an RC LPF, find the PSD and $R_n(\tau)$ of the output noise
Solution
The transfer function of an RC LPF is given by

$$H(f) = \frac{1}{1 + j2\pi fRC} \qquad (5-39)$$

The output noise PSD is given using eqn. (5 – 36) by

$$S_{n_0}(f) = \frac{kT/2}{1+(2\pi fRC)^2} \qquad (5-40)$$

We know from Fourier transform that

$$e^{-a|\tau|} \leftrightarrow \frac{2a}{a^2+(2\pi f)^2} = \frac{2/a}{1+\left(\frac{2\pi f}{a}\right)^2} \qquad (5-41)$$

Thus, $R_{n_0}(\tau)$ which is the inverse Fourier transform of S_{n_0} is given, taking $a = 1/RC$ by

$$R_{n_0}(\tau) = \frac{kT}{4RC} e^{-|\tau|/RC} \qquad (5-42)$$

This is shown (Fig. 5.14)

We see that the decorrelation time τ_0 for which $R_{n_0}(\tau)$ drops to 1% of its maximum value $kT/4RC$ is equal to 4.61 RC. Thus, if the noise appearing at the filter output is sampled at a rate $\leq 0.217/RC$ samples/s, the resulting samples are uncorrelated and being Gaussian are statistically independent.

In the case when white noise is inputted to a LPF, the PSD of the output noise is given by eqn. (5 – 36). The total output noise power is given by

$$P_{n_0} = \frac{kT}{2}\int_{-\infty}^{\infty} |H(f)|^2 \, df \qquad (5-43)$$

$$= kT \int_{-\infty}^{\infty} |H(f)|^2 \, df \qquad (5-44)$$

Assume that the LPF is idealized with a constant value $H(0)$ for an equivalent noise bandwidth of one side $|B_N|$ (Fig. 5.15) Thus, eqn. (5 – 44) reduces to

$$P_{n_0} = kT \int_0^{B_N} |H(0)|^2 \, df$$

$$= kT B_N |H(0)|^2 \qquad (5-45)$$

Thus, the onesided bandwidth B_N is given by

$$B_N = \frac{\int_0^{\infty} |H(f)|^2 \, df}{|H(0)|^2} \qquad (5-46)$$

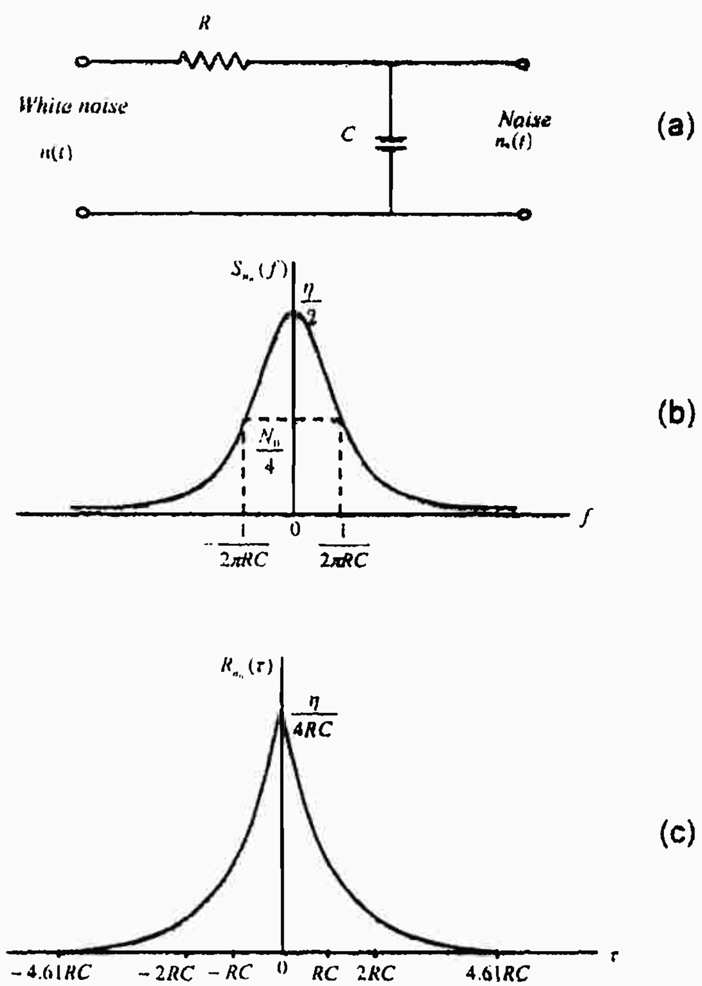


Fig.(5.14.) Characteristics of RC filtered white noise

a) RC LPF *b) PSD of filter output* *c)* $R_{n_o}(\tau)$

Ex 5.2

The input to an RC LPF is white noise. Determine the output power and equivalent noise bandwidth.

Solution

$$S_{n_o}(\omega) = \frac{kT/2}{1+(\omega RC)^2} \qquad (5-47)$$

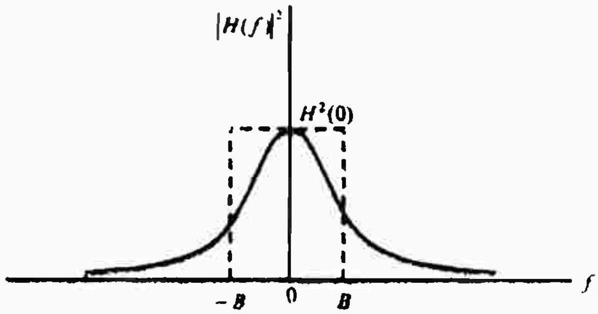


Fig. (5.15) Equivalent noise bandwidth

$$P_{n_o} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)\,d\omega = \frac{kT}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{1+(\omega RC)^2}$$

$$= \frac{kT}{4\pi RC}\int_{-\infty}^{\infty}\frac{du}{u^2+1}$$

when $u = \omega RC$

$$P_{n_o} = \frac{kT}{4\pi RC}\tan^{-1}u \Big|_{-\infty}^{\infty}$$

$$= \frac{kT}{4\pi RC}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{kT}{4RC} \qquad (5-48)$$

Applying eqn. (5- 45), noting that $H(0)=1$, the onesided bandwidth B_N is given by

$$B_N = \frac{1}{4RC} \qquad (5-49)$$

5.5 Narrowband Noise:

If Gaussian white noise is passed through a narrow BPF with bandwidth $B < f_c$, the output from the filter is known as narrowband noise. If has the spectrum shown (Fig. 5.16), which can be approximated by a finite number of noise components spaced Δf apart where $\Delta f \to 0$.

Each pair of delta functions such as $\delta(f+f_c)$ and $\delta(f-f_c)$ can be represented by a sine wave function of arbitrary phase, and the sum of all such spectral components yields the time function $n(t)$ of narrowband noise.

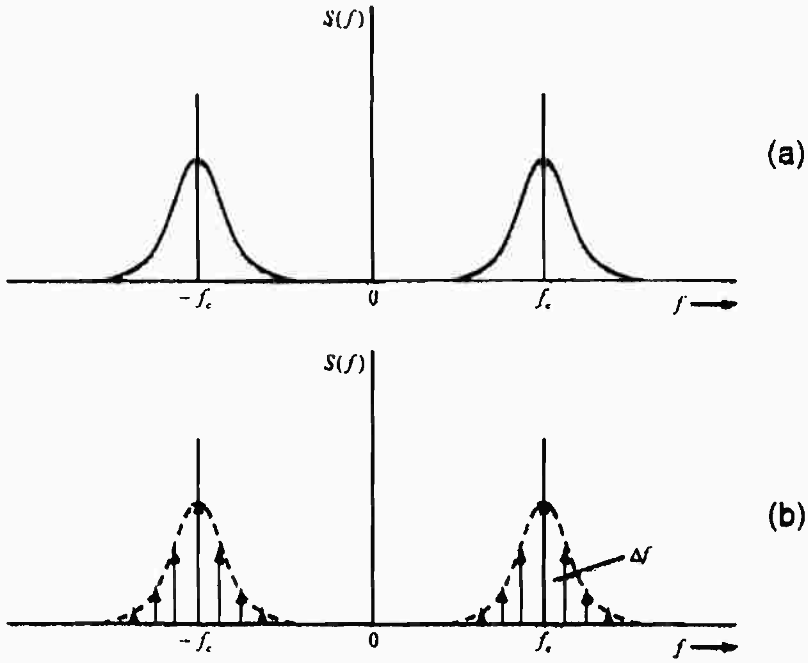


Fig. (5.16) Narrowband noise PSD

a) band limited noise *b) division to delta functions*

$$n(t) = \lim_{\Delta f \to 0} \sum_{0}^{n} a_n \{\sin[(\omega_c + 2\pi n\Delta f)t + \theta_n]\} \qquad (5-50)$$

where a_n is the amplitude of the n^{th} frequency component, θ_n is an arbitrary phase, and $n\Delta f$ is the separation from the center frequency,

$$n(t) = \lim_{\Delta f \to 0} \sum_{0}^{n} a_n \{\sin\omega_c t \cos(2\pi n\Delta f + \theta_n)$$

$$+ \cos\omega_c t \sin(2\pi n\Delta f + \theta_n)\} \qquad (5-51)$$

$$= x(t)\sin\omega_c t + y(t)\cos\omega_c t \qquad (5-52)$$

where

$$x(t) = \lim_{\Delta t \to 0} \sum_{0}^{n} a_n \cos(2\pi n\Delta f t + \theta_n) \qquad (5-53)$$

$$y(t) = \lim_{\Delta t \to 0} \sum_{0}^{n} a_n \sin(2\pi n\Delta f t + \theta_n) \qquad (5-54)$$

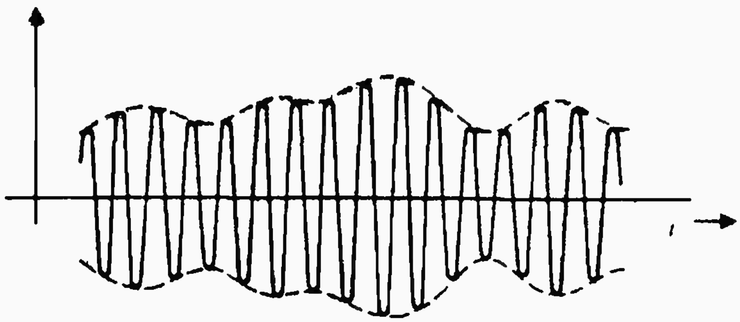


Fig. (5.17) Amplitude modulation as a result of noise passing through a BPF

Both $x(t)$ and $y(t)$ are Gaussian distributions with the same mean and variance as $n(t)$. The expression for $n(t)$ can be alternatively written in polar form by substituting

$$x(t) = R(t)\cos\varphi(t) \quad (5-55)$$

$$y(t) = R(t)\sin\varphi(t) \quad (5-56)$$

$$R(t) = \sqrt{x^2(t) + y^2(t)} \quad (5-57)$$

$$\phi(t) = \tan^{-1}\left[\frac{y(t)}{x(t)}\right] \quad (5-58)$$

$$n(t) = R(t)\sin\left[\omega_c t + \varphi(t)\right] \quad (5-59)$$

where $\varphi(t)$ varies uniformly from 0 to 2π. Hence, $n(t)$ resembles a sine wave which is amplitude modulated and whose phase is randomly modulated (Fig. 5.17). Thus, a BPF produces colored and amplitude modulated noise with Rayleigh distribution. By considering the AM output of the BPF, we may express it in phasor notation for the case when noise bandwidth is small compared to the center frequency

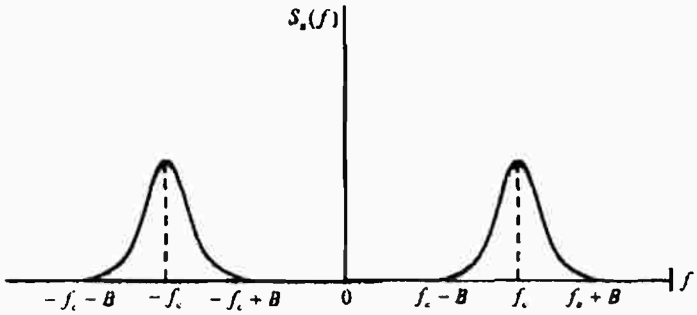
$$\left[n_I(t) + j\, n_Q(t)\right] e^{j\omega_c t} \quad (5-60)$$

where n_I is called in - phase component and $n_Q(t)$ is the quadrature component, taking $\cos\omega_c t$ as a reference

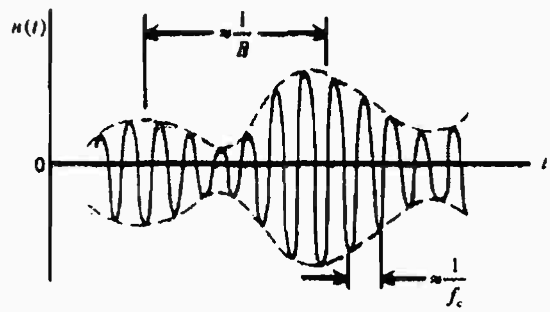
$$n(t) = \Re e\left[n_I(t) + j\, n_Q(t)\right] e^{j\omega_c t} \quad (5-61)$$

$$= \Re e\left[n_I(t) + j\, n_Q(t)\right]\left[\cos\omega_c t + j\sin\omega_c t\right] \quad (5-62)$$

$$= n_I(t)\cos\omega_c t - n_Q(t)\sin\omega_c t \quad (5-63)$$



(a)



(b)

Fig. (5.18) Envelope of narrowband filtered noise
a) noise signal *b) modulated output noise from bandpass filter*

This is called the bandpass representation of noise or the canonical equation of noise. Comparing eqn. (5 – 63) with eqn. (5 – 52)

$$n_I(t) = y(t) \qquad (5-64)$$

$$n_Q(t) = -x(t) \qquad (5-65)$$

Eqn. (5 – 63) is called the canonical equation of narrowband noise

It is to be noted that the envelope has a maximum modulating frequency of B, hence period of $1/B$ (Fig. 5.18).

There are several important properties for $n_I(t)$ and $n_Q(t)$

1. Both $n_I(t)$ and $n_Q(t)$ have zero mean.
2. If $n(t)$ is Gaussian then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian.
3. If $n(t)$ is wide sense stationary then $n_I(t)$ and $n_Q(t)$ are jointly wide sense stationary.

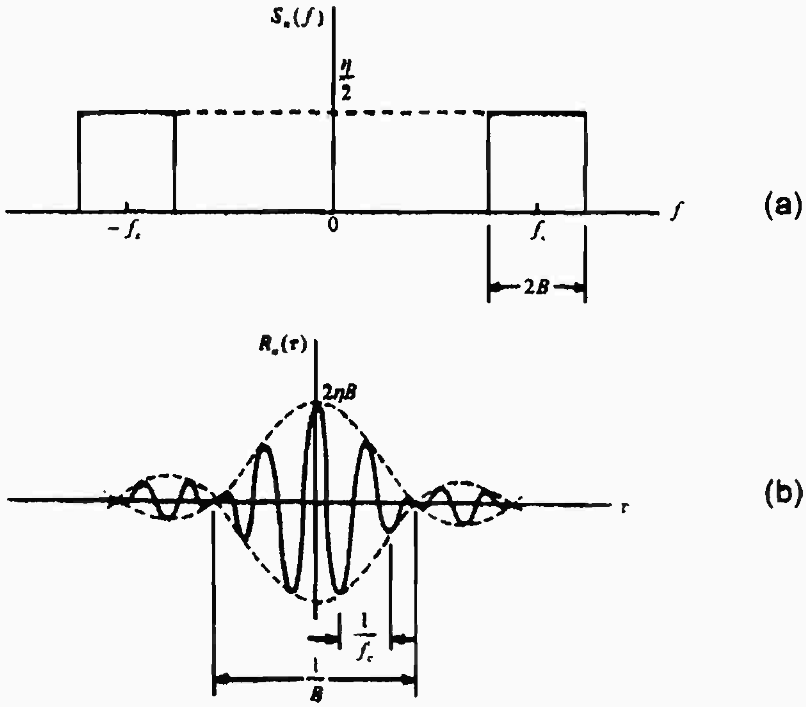


Fig.(5.19). Characteristic of Ideal BPF
a) PSD b) $R_n(\tau)$

4. Both $n_I(t)$ and $n_Q(t)$ have the same power spectral density.
5. $n_I(t)$ and $n_Q(t)$ have the same variance as $n(t)$.
6. The cross spectral densities of $n_I(t)$ and $n_Q(t)$ are imaginary.
7. If a narrowband noise $n(t)$ is Gaussian with zero mean and PSD that is locally symmetric about $\pm f_c$ then $n_I(t)$ and $n_Q(t)$ are statistically independent.

Ex. 5.3

Consider a white Gaussian noise which is passed through an ideal BPF of center frequency f_c and bandwidth $2B$. Determine the autocorrelation function of $n(t)$ and it's in - phase and quadrature components.

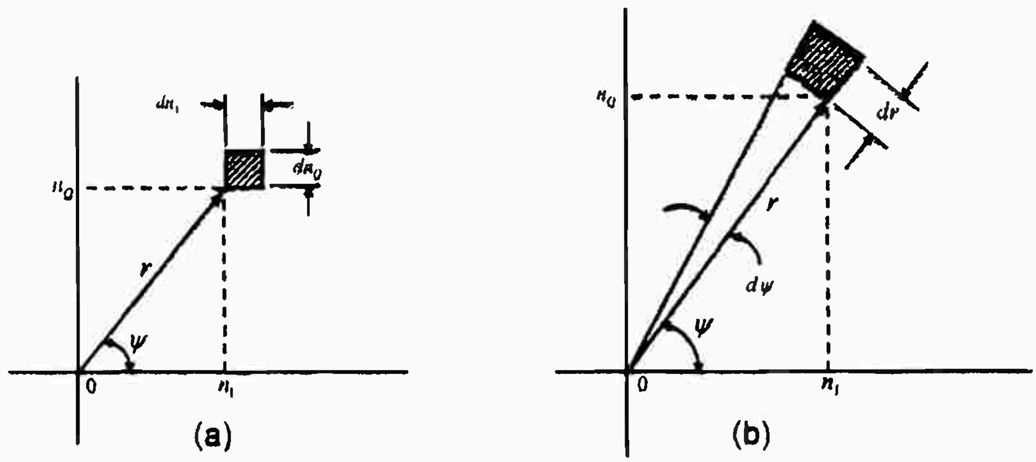


Fig. (5.20) Change to polar coordinates for n_I , n_Q

Solution

The autocorrelation function of $n(t)$ is the inverse Fourier transform of PSD shown (Fig. 5.19)

$$R_n(\tau) = \int_{-f_c-B}^{-f_c+B} \frac{kT}{2} e^{j2\pi f\tau} df + \int_{f_c-B}^{f_c+B} \frac{kT}{2} e^{j2\pi f\tau} df \qquad (5-66)$$

$$= kTB \operatorname{sinc}(2B\tau)\left[e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}\right]$$

$$= 2kTB \operatorname{sinc}(2B\tau)\cos 2\pi f_c\tau \qquad (5-67)$$

Ex 5.4

Express the noise at the output of a BPF in terms of its envelope and phase

Solution

Let N_I , N_Q represent the random variables of the in - phase and quadrature components. N_I and N_Q are independent Gaussian random variables of zero mean and variance σ^2 . The joint pdf is given by

$$f_{N_I N_Q}(n_I , n_Q) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{n_I^2+n_Q^2}{\sigma^2}\right)} \qquad (5-68)$$

The probability of the joint event that N_I lies between n_I and $n_I + d\,n_I$ and that N_Q lies between n_Q and $n_Q + d\,n_Q$ (i.e. the pair of random variables N_I, N_Q lies jointly inside the shaded area (Fig. 5.20).

$$f_{N_I N_Q}(n_I, n_Q)dn_I dn_Q = \frac{1}{2\pi\sigma^2} e^{\frac{-(n_I^2+n_Q^2)}{\sigma^2}} dn_I dn_Q \qquad (5-69)$$

We define

$$n_I = r\cos\psi \qquad (5-70)$$

$$n_Q = r\sin\psi \qquad (5-71)$$

$$dn_I dn_Q = rdrd\psi \qquad (5-72)$$

Let R and ψ denote the random variables obtained by observing the envelope $r(t)$ and phase $\psi(t)$. The probability of random variables R and Ψ lying inside the shaded area (Fig. 5.20) is given by

$$\frac{r}{2\pi\sigma^2} e^{\frac{r^2}{2\omega^2}} dr\, d\psi$$

Thus,

$$f_{R,\Psi}(r,\psi) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \qquad (5-73)$$

This pdf is independent of the angle ψ, which means that random variable R, Ψ are statistically independent. We may thus express $f_{R,\Psi}(r,\psi)$ as the product of $f_R(r)$ and $f_\Psi(\psi)$. The random variable ψ is uniformly distributed inside the range 0 to 2π

$$f_\Psi(\psi) = \frac{1}{2\pi} \quad 0 \le \psi \le 2\pi; \quad 0 \text{ elsewhere} \qquad (5-74)$$

Thus,

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} & r \ge 0 \\ 0 & \text{elsewhere} \end{cases} \qquad (5-75)$$

This pdf is called Rayleigh distribution. Let $\upsilon = r/\sigma$

$$f_v(\upsilon) = \sigma f_R(r) \qquad (5-76)$$

Then, we may rewrite Rayleigh distribution as

$$f_v(\upsilon) = \begin{cases} \upsilon\, e^{-\upsilon^2/2} & \upsilon \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

The peak occurs at $\upsilon = 1$ and is equal to 0.607. Unlike Gaussian distribution, the Rayleigh distribution is zero for negative values of υ, i.e., $r(t)$ can assume only positive values.

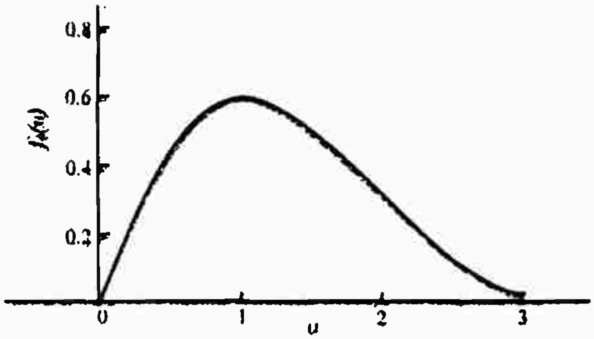


Fig. (5.21) Rayleigh distribution

Ex 5.5

A sample function $x(t) = A\cos 2\pi f_c t + n(t)$, where $n(t)$ is narrowband noise Find the *pdf* distribution

Solution

Using eqn. (5 – 63),

$$x(t) = n_I(t)\cos 2\pi f_c t - n_Q(t)\sin 2\pi f_c t \qquad (5-77)$$

$$n_I'(t) = A + n_I(t) \qquad (5-78)$$

Since $n(t)$ is Gaussian with zero mean and variance σ^2, then both $n_I'(t)$ and $n_Q(t)$ are Gaussian and statistically independent. Thus,

$$f_{N_I', N_Q}(n_I', n_Q) = \frac{1}{2\pi\sigma^2} e^{-\left[\frac{(n_I' - A)^2 + n_Q^2}{2\sigma^2}\right]} \qquad (5-78)$$

Let $r(t)$ denote the envelope of $x(t)$ and $\psi(t)$ denote its phase,

$$r(t) = \left\{\left[n_I'(t)\right]^2 + n_Q^2(t)\right\}^{1/2} \qquad (5-79)$$

$$\psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I'(t)}\right] \qquad (5-80)$$

The joint *pdf* is given by

$$f_{R,\Psi}(r, \psi) = \frac{r}{2\pi\sigma^2} e^{-\frac{\left(r^2 + A^2 - 2Ar\cos\psi\right)}{2\sigma^2}} \qquad (5-81)$$

We cannot express this joint *pdf* $f_{R,\Psi}(r, \psi)$ as a product $f_R(r)\, f_\Psi(\psi)$, because of the term $r\cos\psi$. Hence, R and ψ are dependent random variables for nonzero values of the amplitude A of the carrier.

$$f_R(r) = \int_0^{2\pi} f_{R,\psi}(r,\psi)\,d\psi$$

$$= \left\{ \frac{r}{2\pi\sigma^2} e^{-\frac{(r^2+A^2)}{2\sigma^2}} \right\} \int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos\psi}\,d\psi \qquad (5-82)$$

The integral in eqn. (5 – 81) is modified Bessel function of the first kind of zero order

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\psi}\,d\psi \qquad (5-83)$$

Letting $x = \frac{Ar}{\sigma^2}$

$$f_R(r) = \frac{r}{\sigma^2} e^{-\left(\frac{r^2+A^2}{2\sigma^2}\right)} I_0\left(\frac{Ar}{\sigma^2}\right) \qquad (5-84)$$

This is called Rician distribution

Let $\qquad \upsilon = \frac{r}{\sigma} \qquad (5-85)$

$$a = \frac{A}{\sigma}$$

$$f_v(\upsilon) = \sigma f_R(r)$$

$$= \upsilon\, e^{-\left(\frac{\upsilon^2+a^2}{2}\right)} I_0(a\upsilon) \qquad (5-86)$$

This is called normalized Rician distribution (Fig. 5.22). When a is zero, the Rician distribution reduces to Rayleigh distribution.

Near $\upsilon = a$ and when a is large i.e. $A >> \sigma$, the Rician distribution reduces to Gaussian.

5.6 Partition of Noise:

Taking eqn. (5 – 63) and noting that $n_I(t)$ and $n_Q(t)$ are low pass voltages whose fluctuations are limited by the bandwidth of the bandpass noise, we multiply both sides by $\cos\omega_c t$

$$n(t)\cos\omega_c t = n_I(t)\cos^2\omega_c t - n_Q(t)\sin\omega_c t\cos\omega_c t \qquad (5-87)$$

If we retain only the LPF terms

$$n(t)\cos\omega_c t\,\Big|_{LPF} = \frac{1}{2}n_I(t) \qquad (5-88)$$

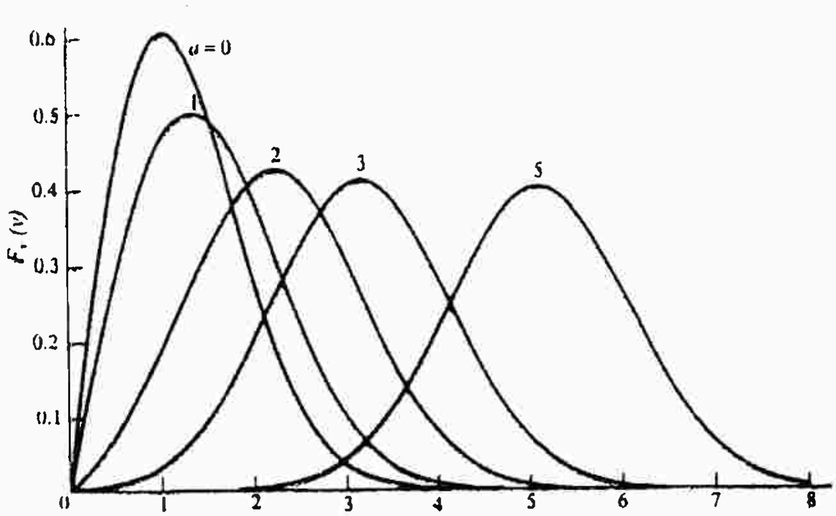


Fig. (5.22) Rician distribution

We know that

$$\mathcal{F}[f(t)\cos\omega_c t] = \frac{1}{2}[F(\omega+\omega_c)+F(\omega-\omega_c)] \qquad (5-89)$$

We also know from eqn. (4 – 86)

$$S_f(\omega) = \lim_{T\to\infty} \frac{|F_T(\omega)|^2}{T} \qquad (5-90)$$

Therefore, taking Fourier transform of both sides of eqn. (5 – 88)

$$\mathcal{F}[n(t)\cos\omega_c t]\Big|_{LPF} = \frac{1}{2}\mathcal{F}[n_I(t)]$$

Using eqn. (5 – 89),

$$\frac{1}{2}[N(\omega+\omega_c)+N(\omega-\omega_c)]_{LPF} = \frac{1}{2}N_I(\omega) \qquad (5-91)$$

where $N(\omega)$ is $\mathcal{F}[n(t)]$ and $N_I(\omega)$ is $\mathcal{F}[n_I(t)]$

Squaring both sides of (5 – 91) and using the format of (5 – 90),

$$\lim_{T\to\infty} \frac{|N_{I_T}(\omega)|^2}{T}\Big|_{LPF} = \lim_{T\to\infty} \frac{[N_T(\omega-\omega_c)+N_T(\omega+\omega_c)]^2}{T}\Big|_{LPF} \qquad (5-92)$$

Noting that the average of the cross product $N_T(\omega-\omega_c)N_T(\omega+\omega_c)$ represents the product of Fourier transform of two random signals, i.e., the Fourier transform of their convolution. Such convolution is zero for they are uncorrelated. Thus, using eqn. (5 – 90), eqn. (5 – 92) becomes

$$S_{n_I}(\omega)\Big|_{LPF} = \left[S_n(\omega-\omega_c)+S_n(\omega+\omega_c)\right]\Big|_{LPF} \qquad (5-93)$$

Similarly, multiplying (5 – 63) by $\sin\omega_c t$ for both sides and following the same procedure

$$S_{n_Q}(\omega)\Big|_{LPF} = \left[S_n(\omega-\omega_c)+S_n(\omega+\omega_c)\right]\Big|_{LPF} \qquad (5-94)$$

Thus,

$$S_{n_I}(\omega) = S_{n_Q}(\omega) = \left[S_n(\omega-\omega_c)+S_n(\omega+\omega_c)\right]\Big|_{LPF} \qquad (5-95)$$

This is illustrated in Fig. (5.23)

If we call the area of each flank A, then the area under $S_n(\omega)$ $=2A$. Also, the area under $S_n(\omega-\omega_c)$ is $2A$ and the area under $S_n(\omega+\omega_c)$ is $2A$. Thus, the area under $S_n(\omega+\omega_c)+S_n(\omega-\omega_c)$ is $4A$ which is $2A$ under the central flank and A for each flank. The LPF retains only the area under the central flank. We thus see

$$\begin{aligned} n^2(t)\Big|_{LPF} &= \overline{n_I^2(t)}\Big|_{before\ LPF} \\ &= \overline{n_Q^2(t)}\Big|_{before\ LPF} \\ &= \overline{n^2(t)}\Big|_{before\ LPF} \end{aligned} \qquad (5-96)$$

Also,

$$\overline{n^2}(t)\Big|_{LPF} = \frac{1}{2}\overline{n_I^2(t)}\Big|_{before\ LPF} + \frac{1}{2}\overline{n_Q^2}\Big|_{before\ LPF} = \overline{n^2(t)}\Big|_{before\ LPF} \qquad (5-97)$$

Eqn. (5 – 96) can be interpreted by saying that $n_I(t)$, $n_Q(t)$, $n(t)$ all have the same σ. Eqn. (5 – 97) can be interpreted by saying that the noise power after LPF is $\frac{1}{2}$ the in-phase power and $\frac{1}{2}$ the quadrature power before LPF.

It is also clear that the noise power is equally divided between the in-phase component and the quadrature component. This is due to the random nature of noise which tends to distribute the noise components over both cosine and sine terms equally, since $n_I(t)$ and $n_Q(t)$ are statistically independent and orthogonal.

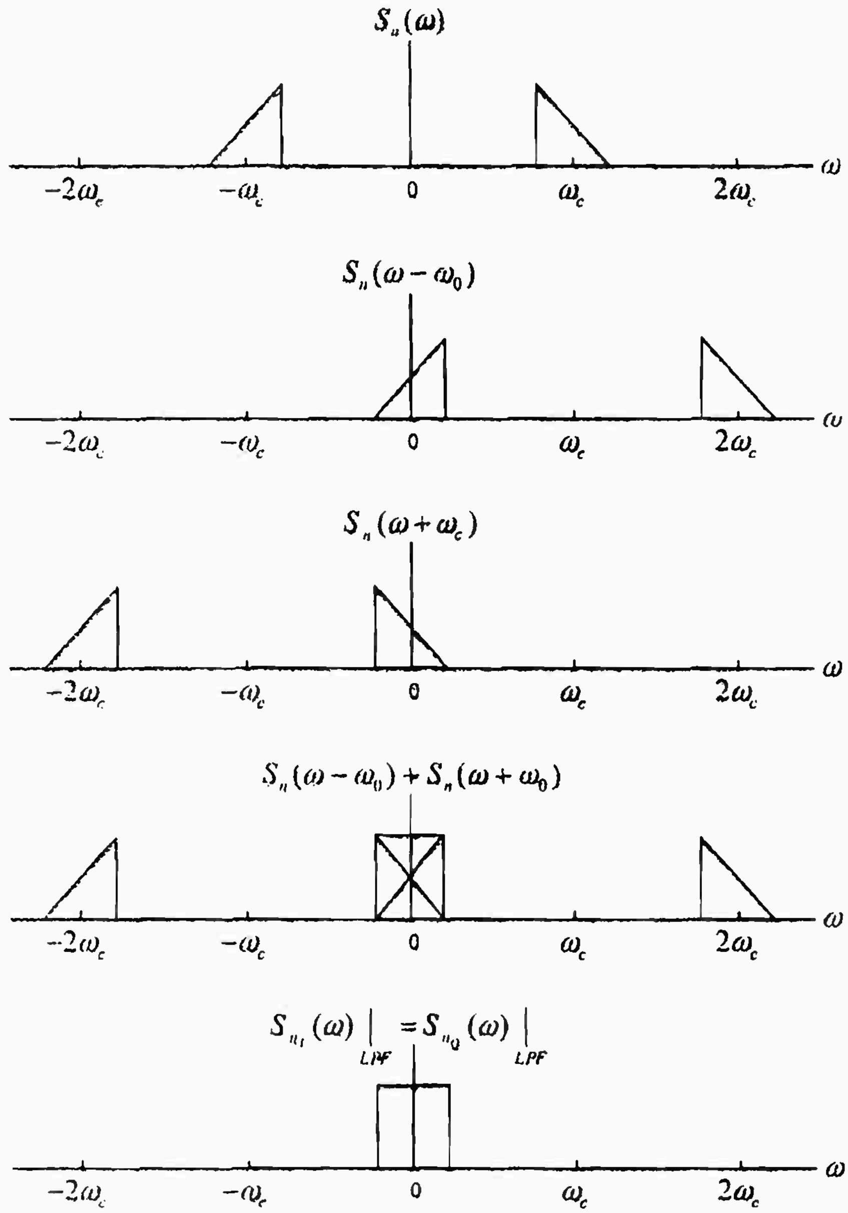


Fig. (5.23) In-phase and quadrature noise spectral densities for bandpass noise

$$n(t) = n_I(t)\cos\omega_c t - n_Q(t)\sin\omega_c t$$

Hence

$$\overline{n^2}(t) = \frac{1}{2}\overline{n_I^2}(t) + \frac{1}{2}\overline{n_Q^2}(t) \qquad (5-98)$$

While

$$\overline{n_I^2}(t) = \overline{n_Q^2}(t) \qquad (5-99)$$

Or

$$S_{n_I}(f)\Big|_{LPF} = S_{n_Q}(f)\Big|_{LPF} = \begin{cases} S_n(f+f_c) + S_n(f-f_c) \\ \qquad -B \le f < B \\ 0 \qquad elsewhere \end{cases} \qquad (5-100)$$

$$S_n(f)\Big|_{LPF} = \frac{1}{2}S_{n_I}(f)\Big|_{LPF} + \frac{1}{2}S_{n_Q}(f)\Big|_{LPF} \qquad (5-101)$$

The multiplication of $n(t)$ by $\cos 2\pi f_c t$ eqn. (5 – 88), and multiplication by $\sin 2\pi f_c t$ can be represented by an analyzer (Fig. 5.24a), while the canonical equation of forming the bandpass noise can be represented by a synthesizer (Fig. 5. 24b).

Ex 5.6

For an ideal BPF, consider white Gaussian noise. Find the in-phase and quadrature PSD and autocorrelation and the total noise power

Solution

$S_{n_I}(f)\Big|_{LPF} = S_{N_Q}(f)\Big|_{LPF}$ shown in Fig. (5.25)

From eqn. (5 – 67),

$$R_n(\tau) = 2kTB\,\mathrm{sinc}(2B\tau)\cos 2\pi f_c\tau$$

From eqn. (5 – 38), noting that the height here for S_{n_I} or S_{n_Q} is kT not $kT/2$,

$$R_{n_I}(\tau) = R_{n_Q}(\tau) = 2kTB\,\mathrm{sinc}(2B\tau) \qquad (5-102)$$

The total noise in the output is $\frac{kT}{2} \times 4B = 2kTB$. The noise power in the in-phase component or the quadrature component after LPF is also $2kTB$. This is in agreement with eqn (5 – 96). From eqn. (5 – 97), $\overline{n^2}(t)$ is kTB from the in-phase component and kTB from the quadrature component.

If one component is missing by considering only the in-phase component for example, we get only half the noise i.e. kTB. This is the case of what is known as coherent or synchronous detection, which is superior to any other from of detector because it picks up only half the noise output.

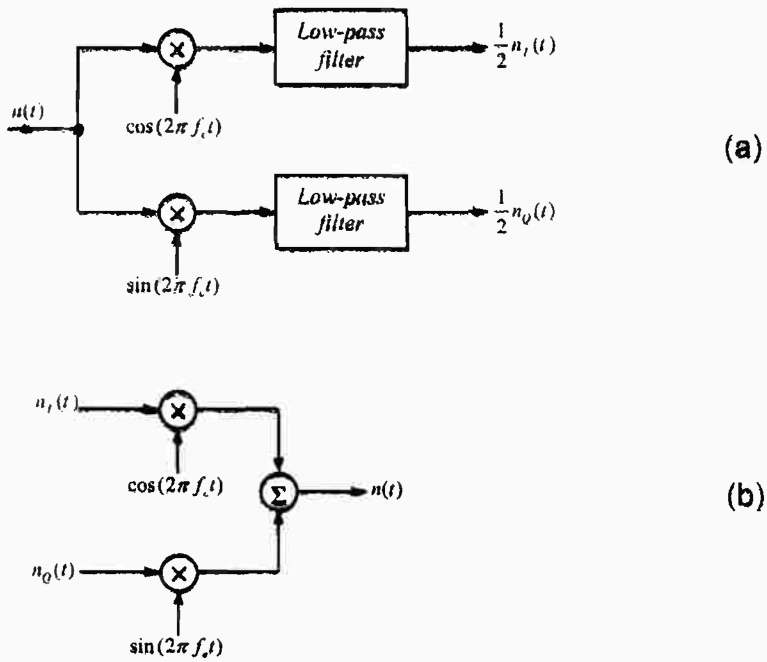


Fig. (5.24) Extraction and generation of the in-phase and quadrature noise outputs of BPF
a) analyzer *b) synthesizer*

5.7 Figures of Merit:

One of the most important figures of merit in communication is signal to noise ratio (S/N or SNR). It measures the signal power to noise power. For a weak signal S/N can become especially important, for if S/N falls below 1, then the signal in buried is noise.

It might be tempting to think that to remedy this situation the signal is could be inputted to an amplifier to bring up the signal amplitude. But this thinking is false because the amplifier will bring up the signal amplitude and the input noise as well. Not only that, but the amplifier being an electronic device will add up noise of its own (called excess noise) causing the noise in the output to exceed the amplified input noise. Thus, the S/N will be degraded.

$$S/N\big|_0 = \frac{GS_i}{GN_i + N_e}$$

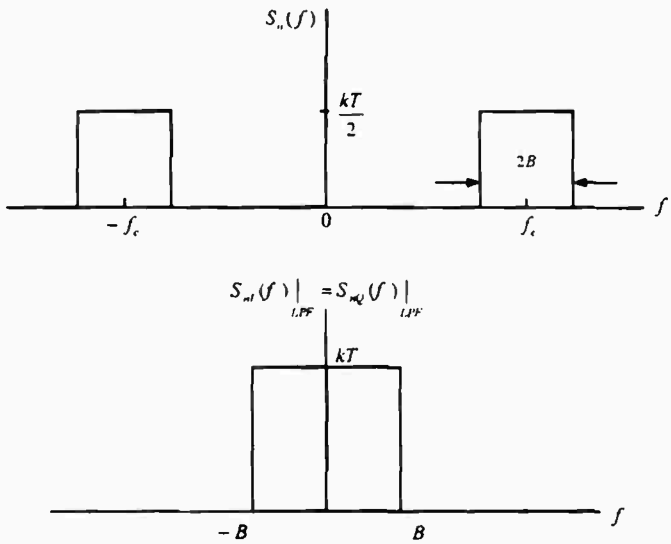


Fig. (5.25) PSD for $n(t), n_I(t), n_Q(t)$

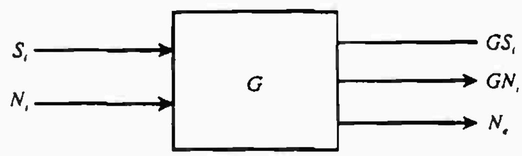


Fig. (5.26) S/N **calculation**

$$= \frac{S_i}{N_i} \frac{1}{1+\frac{N_e}{GN_i}} \tag{5-103}$$

where N_i is the input noise power and N_e is the excess noise. Defining the noise figure F

$$F = \frac{S/N|_i}{S/N|_0} = 1 + \frac{N_e}{GN_i} > 1 \tag{5-104}$$

Noise figure measures the amount of degradation of S/N. It is always > 1. We may define

$$N_i = kT_i B \qquad (5\text{–}105)$$

$$\frac{N_e}{G} = kT_e B \qquad (5\text{–}106)$$

where T_e is excess temperature, which measures the activation of excess noise in the amplifier referred to the input. Thus, eqn. (5 – 104) becomes

$$F = 1 + \frac{T_e}{T_i} \qquad (5\text{–}107)$$

Ex 5.7

If the noise output in the analyzer is considered only as the in - phase component, find the output noise power. What is the *pdf* for n_I , n_Q?

Solution

From eqn. (5 – 87), $n_0(t)\Big|_{LPF} = \frac{1}{2} n_I(t)$

$$N_0 = \overline{n_0^2}(t) = \frac{1}{4}\overline{n_I^2}(t) = \frac{1}{4} N_I \qquad (5\text{–}108)$$

We should note that the signal $m(t)$ is also multiplied by $\cos\omega_c t$, then the output signal power is $\overline{(m(t)\cos\omega_c t)^2} = \frac{1}{2}\overline{m^2}(t) = \frac{1}{2} S_i$

Since $\overline{n_I^2(t)}$ is $2kTB$, then the output noise power is $\frac{1}{4} \times 2kTB = \frac{kTB}{2}$

Thus, S/N is

$$\frac{S_i/2}{N_i/4} = 2S_i/N_i \qquad (5\text{–}109)$$

Comparing this result with that in Ex 5.6, in the present example we have totally removed the quadrature component whereas in the previous example both components are present and the total noise power is merely divided between two modes. This shows the power of coherent detector in reducing the noise output. We note

$$f_{n_I}(x) = f_{n_Q}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad (5\text{–}110)$$

where $\sigma^2 \overline{n_1^2}(t) = 2kTB$

However, the two Gaussian random variables are uncorrelated, hence, independent although they have the same σ.

Fig. (5.27) Baseband transmission

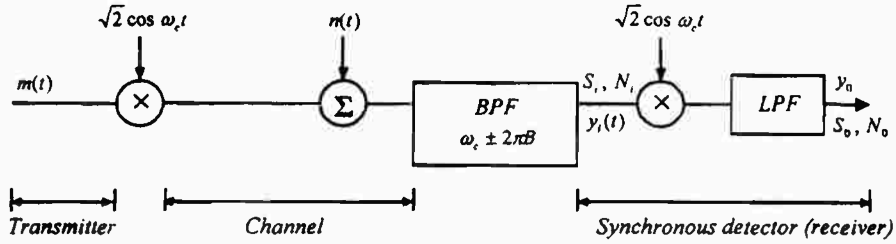


Fig. (5.28) DSBSC transmission

5.8 AM Detection:

Consider first baseband transmission (Fig. 5.27). The LPF at the beginning limits the bandwidth of the input signal. The baseband signal $m(t)$ is assumed to be zero mean wide stationary signal whose band is limited to B. For the passive nature of the link components

$$S_0 = S_i \tag{5-111}$$

Neglecting excess noise and assuming ideal resistors so that the noise at the output comes from the channel.

$$N_i = N_0 = 2\int_0^B S_n(f)\,df = 2\times\frac{kTB}{2} = kTB \tag{5-112}$$

$$\frac{S_0}{N_0} = \frac{S_i}{kTB} = \gamma \tag{5-113}$$

Thus, γ is a measure of the transmitted power and is a reference against which the performance of other detection systems is rated.

Consider now DSBSC (Fig. 5.28). The signal at the modulator input $\sqrt{2}\,m(t)\cos\omega_c t + n(t)$, where $n(t)$ is the bandpass channel noise.

$$S_i = \overline{\left[\sqrt{2}\,m(t)\cos\omega_c t\right]^2} = \overline{m^2(t)} \tag{5-114}$$

Because $n(t)$ is bandpass noise at ω_c we can express it in canonical form. Thus,

$$y_i(t) = \left[\sqrt{2}\ m(t) + n_I(t)\right]\cos\omega_c t - n_Q(t)\sin\omega_c t \qquad (5\text{–}115)$$

where the signal is multiplied by $\sqrt{2}\cos\omega_c t$, which is the case of synchronous detector, then low pass filtered

$$y_0(t) = m(t) + \frac{1}{\sqrt{2}}\, n_I(t) \qquad (5\text{–}116)$$

$$S_0 = \overline{m^2(t)} = S_i \qquad (5\text{–}117)$$

$$N_0 = \frac{1}{2}\,\overline{n_I^2(t)} = \frac{1}{2} N_i = \frac{1}{2} \times \frac{kT}{2} \times 4B \qquad (5\text{–}118)$$

$$= kTB \qquad (5\text{–}119)$$

Thus,

$$\frac{S_0}{N_0} = \frac{S_i}{kTB} = \gamma \qquad (5\text{–}120)$$

Thus, for the same transmitted power at the demodulator input the S/N at the demodulator output is the same as that for baseband and the bandwidth requirement is the same as that for baseband. Thus, DSBSC has all advantages for AM transmission without carrier power loss, yet same merits of baseband.

We should note from eqns. (5 – 117) and (5 – 118) that

$$\frac{S_0}{N_0} = \frac{S_i}{\frac{1}{2}N_i} = \frac{2S_i}{N_i} \qquad (5\text{–}121)$$

Thus, the synchronous detector improves S/N in DSBSC by a factor of 2. This improvement comes about from the fact that the coherent (synchronous) detector rejects the out of phase (quadrature) noise component in the input noise thereby halving the mean square noise power.

For synchronous demodulation of DSBLC, this is identical to DSBSC but now we have a carrier. The received signal is $\sqrt{2}\left[A + m(t)\right]\cos\omega_c t$. When this is multiplied by $\sqrt{2}\cos\omega_c t$, the demodulated output is $m(t)$

$$S_0 = \overline{m^2(t)} \qquad (5\text{–}122)$$

The output noise is exactly as in DSBSC eqn. (5 – 119),

$$N_0 = kTB \qquad (5\text{–}123)$$

The received signal has input power

$$S_i = \frac{(\sqrt{2})^2\,\overline{\left[A + m(t)\right]^2}}{2}$$

$$= \overline{\left[A + m(t)\right]^2}$$

$$= A^2 + \overline{m^2(t)} + 2A\,\overline{m(t)} \tag{5-124}$$

For signals with zero mean

$$S_i = A^2 + \overline{m^2(t)} \tag{5-125}$$

$$\frac{S_0}{N_0} = \frac{\overline{m^2(t)}}{kTB}\frac{S_i}{S_i} = \frac{\overline{m^2(t)}}{A^2 + \overline{m^2(t)}}\frac{S_i}{kTB}$$

Using eqn. (5 – 20)

$$\frac{S_0}{N_0} = \frac{\overline{m^2(t)}}{A^2 + \overline{m^2(t)}}\gamma \tag{5-126}$$

Let the maximum value of $m(t)$ be $\hat{m}$. For maximum S/N, $A = m$

$$\left.\frac{S_0}{N_0}\right|_{max} = \frac{\gamma}{\frac{\hat{m}^2}{m^2} + 1} \tag{5-127}$$

$$\frac{S_0}{N_0} \le \frac{\gamma}{2} \tag{5-127}$$

Thus, S_0/N_0 for DSB-LC is at best half that for DSBSC. For sinusoidal $m(t)$, we find

$$S_0/N_0 = \gamma/3 \tag{5-128}$$

Ex 5.8

In DSBSC system, $f_c = 500\,kHz$, $m(t)$ has uniform PSD and band limited to $4\,kHz$. The modulated signal is transmitted over a distortionless channel with $S_n = 1/(\omega^2 + a^2)$, $a = 10^6\pi$. The useful signal power at the receiver input is $1\,\mu W$. The received signal is bandpass filtered, multiplied by $2\cos\omega_c t$ and low pass filtered. Determine S_0/N_0.

Solution

If the received signal is $k\,m(t)\cos\omega_c t$, the demodulator input is $[k\,m(t) + n_I(t)]\cos\omega_c t - n_Q(t)\sin\omega_c t$. When multiplying by $2\cos\omega_c t$ and low pass filtered, the output is $s_0(t) + n_0(t) = k\,m(t) + n_I(t)$

Hence,
$$S_0 = k^2\,\overline{m^2(t)} \tag{5-129}$$

$$N_0 = \overline{n_I^2(t)} \tag{5-130}$$

The power of the received signal $k\,m(t)\cos\omega_c t$ is $1\,\mu W$

$$\frac{k^2\,\overline{m^2(t)}}{2} = 10^{-6} \qquad (5\text{–}131)$$

From $$S_0 = 2\times10^{-6}\,W \qquad (5\text{–}132)$$

$$N_0 = \overline{n^2(t)} = 2\int_{496000}^{504000}\frac{1}{f^2+a^2}df$$

$$= \frac{1}{\pi}\int_{2\pi\times496000}^{2\pi\times504000}\frac{1}{\omega^2+a^2}d\omega$$

$$= \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Big|_{2\pi\times496000}^{2\pi\times504000}$$

$$= 8.25\times10^{-10}$$

$$\frac{S_0}{N_0} = \frac{2\times10^{-6}}{8.25\times10^{-10}} = 2.42\times10^{3} = 33.83\,dB$$

Now for envelope detection, assume the received signal to be $[A+m(t)]\cos\omega_c t$. The demodulator input is $y_i(t) = [A+m(t)]\cos\omega_c t + n(t)$. Thus,

$$y_i(t) = [A+m(t)+n_I(t)]\cos\omega_c t - n_Q(t)\sin\omega_c t \qquad (5\text{–}131)$$

The desired signal at the demodulator input is $[A+m(t)]\cos\omega_c t$. Thus, for $\overline{m(t)} = 0$

$$S_i = \frac{\overline{[A+m(t)]^2}}{2} = \frac{A^2+\overline{m^2(t)}}{2} \qquad (5\text{–}132)$$

To find the envelope of $y_i(t)$, we rewrite

$$y_i(t) = E_i(t)\cos[\omega_c t + \theta_i(t)]$$

$$E_i(t) = \sqrt{[A+m(t)+n_I(t)]^2 + n_Q^2(t)} \qquad (5\text{–}133)$$

Where $E_i(t)$ is the envelope detector output signal. We may now distinguish between two cases:

a) Low noise case $[A+m(t)] >> n(t),\ n_Q(t),\ n_I(t)$

$$E_i(t) \sim A + m(t) + n_I(t) \qquad (5\text{–}134)$$

The DC component A of the envelope detector output E_i is blocked by a capacitor yielding $m(t)$ as the useful signal and $n_I(t)$ as noise

$$S_o = \overline{m^2(t)} \qquad (5\text{–}135)$$

$$N_0 = \overline{n_I^2(t)} = 2kTB \qquad (5\text{–}136)$$

$$\frac{S_0}{N_0} = \frac{\overline{m^2(t)}}{2kTB}\frac{S_i}{S_i}$$

$$= \frac{\overline{m^2(t)}}{2kTB}\frac{S_i}{\frac{1}{2}\left[A^2 + \overline{m^2(t)}\right]}$$

$$= \frac{\overline{m^2(t)}}{A^2 + m^2(t)}\frac{S_i}{kTB}$$

$$= \frac{\overline{m^2(t)}}{A^2 + \overline{m^2(t)}}\frac{S_i}{kTB} \qquad (5-137)$$

which is identical to eqn (5 – 126) for synchronous detector. Therefore for AM when noise is small, the performance of the envelope detector is identical to that of the synchronous detector. For large noise case $n(t) >> A + m(t)$ (Fig. 5.29) Hence, $n_I(t)$ and $n_Q(t) >> A + m(t)$.

$$E_i(t) = \sqrt{[A + m(t) + n_I(t)]^2 + n_Q^2(t)} \qquad (5-138)$$

$$= \sqrt{n_I^2(t) + n_Q^2 + 2n_I(t)[A + m(t)]} \qquad (5-139)$$

$$E_n(t) = \sqrt{n_I^2(t) + n_Q^2(t)} \qquad (5-140)$$

$$\theta_n(t) = -\tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right] \qquad (5-141)$$

$$n_I(t) = E_n(t)\cos[\theta_n(t)] \qquad (5-142)$$

$$n_Q(t) = E_n(t)\sin[\theta_n(t)] \qquad (5-143)$$

$$E_i(t) = \sqrt{E_n^2(t) + 2E_n(t)[A + m(t)]\cos\theta_n(t)} \qquad (5-144)$$

$$= E_n(t)\left\{1 + \frac{2[A + m(t)]\cos\theta_n(t)}{E_n(t)}\right\}^{1/2} \qquad (5-145)$$

Since $E_n(t) >> A + m(t)$

$$E_i(t) \cong E_n(t)\left[1 + \frac{A + m(t)}{E_n(t)}\cos\theta_n(t)\right] \qquad (5-146)$$

$$= E_n(t) + [A + m(t)]\cos\theta_n(t) \qquad (5-147)$$

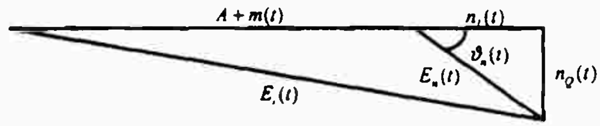


Fig. (5.29) Phasor diagram for envelope detector-Large noise case

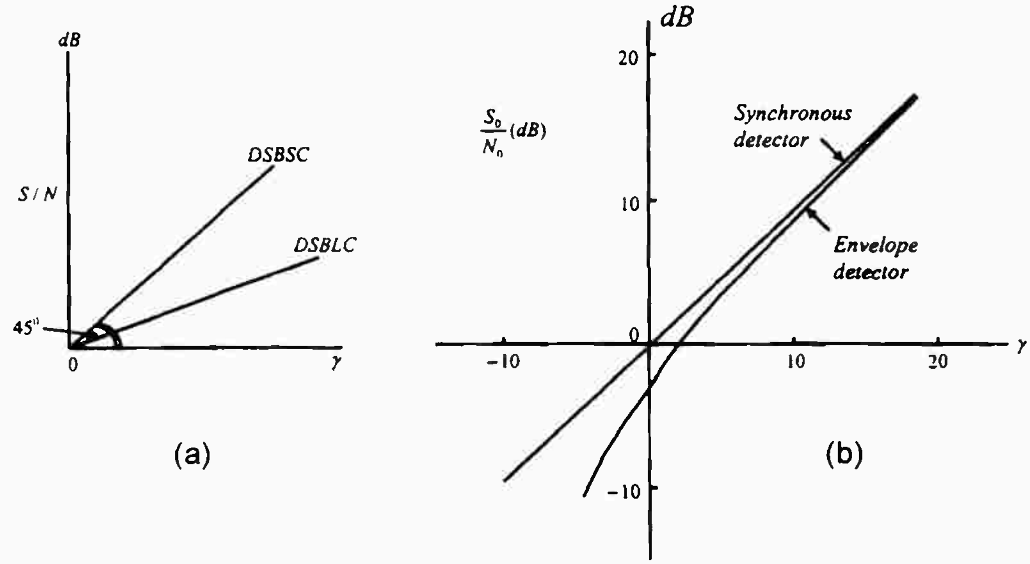


Fig. (5.30) S/N **for Am detection**
a) synchronous detector *b) envelope detector*

The output contains no term proportional to $m(t)$. The signal $m(t)\cos\theta_n(t)$ represents $m(t)$ multiplied a time varying function (noise signal $\cos\theta_n(t)$), hence is of no use is recovering $m(t)$. In the previous case, the output signal, contained a term of the form a $m(t)$, where a is a constant. Furthermore, the output noise is additive, even for envelope detection with small noise. Here, the noise is multiplicative. In this situation, the useful signal is a badly mutilated. This is called the threshold phenomenon, when the signal quality at the output undergoes rapid deterioration when the input noise increases beyond a certain level when γ drops below a certain value. From Fig. (5.30b), the threshold effect is clearly seen when $\gamma \sim 10$ or less

Ex. 5.9

Find γ at threshold in tone modulated AM with modulation index 100%, if the onset of threshold occurs when $E_n > A$, with probability 1%, where E_n is the noise envelope.

Solution

We note that $E_n = \sqrt{n_I^2(t) + n_Q^2(t)}$, and both $n_I(t)$ and $n_Q(t)$ are Gaussian with σ_n^2. Hence, E_n has a Rayleigh *pdf*

$$f_r(E_n) = \frac{E_n}{\sigma^2} e^{-E_n^2/2\sigma^2} \quad E_n > 0 \tag{5-148}$$

$$P(E_n \geq A) = \int_A^\infty \frac{E_n}{\sigma^2} e^{-E_n^2/2\sigma_n^2} dE_n \tag{5-149}$$

$$= e^{-A^2/2\sigma_n^2} = 0.01 \tag{5-150}$$

$$A^2/2\sigma_n^2 = 4.605 \tag{5-151}$$

The variance σ_n^2 of bandpass noise $= 2kTB$

$$\frac{A^2}{4kTB} = 4.605$$

For tone modulation of 100% index of modulation

$$m(t) = A\cos(\omega_m t + \theta)$$

$$S_i = \frac{A^2 + \overline{m^2(t)}}{2} = \frac{A^2 + 0.5A^2}{2} = \frac{3A^2}{4} \tag{5-152}$$

$$\gamma = \frac{S_i}{kTB} = \frac{3A^2}{4kTB} = 13.8$$

$$= 12.4\,dB$$

Ex 5.10

Show that the maximum improvement in S/N in AM for sinusoidal modulation is $2/3$ and $S_0/N_0 = m_a^2 S_C/N_C$ where S_C/N_C is the carrier to noise ratio with maximum $S_0/N_0 = S_C/N_C$

Solution

We see from eqn. (5 – 135),

$$S_0 = \overline{m^2(t)}$$

$$N_0 = \overline{n_I^2(t)} = N_i$$

From eqn. (5 – 132),

$$S_i = \frac{1}{2}A^2 + \frac{1}{2}\overline{m^2(t)}$$

Thus,

$$\frac{S_0}{N_0} = \frac{2\,\overline{m^2(t)}}{A^2 + \overline{m^2(t)}}\,\frac{S_i}{N_i} \tag{5-153}$$

For sinusoidal modulation, $m(t) = m_a\ A\cos\omega_m t$

$$\frac{S_0}{N_0} = \frac{2\,m_a^2}{2 + m_a^2}\,\frac{S_i}{N_i}$$

The maximum improvement is $2/3$ when $m_a = 1$. Sometimes, it is convenient to refer the output S/N to carrier to noise ratio (CNR) at the output of the Note $S_C = A^2/2$, $N_C = N_i$ at the output the IF amplifier

$$\frac{S_0}{N_0} = \frac{\overline{m^2(t)}}{N_i}\,\frac{S_C}{S_C}$$

$$= \frac{\overline{m^2(t)}}{A^2/2}\,\frac{S_C}{N_C}$$

For sinusoidal modulation with modulation index m_a

$$\frac{S_0}{N_0} = m_a^2\,\frac{S_C}{N_C} \tag{5-154}$$

The maximum S_0/N_0 is

$$S_0/N_0\Big|_{AM} = S_C/N_C = \frac{A^2}{4kTB} \tag{5-155}$$

Ex 5.11

Show that for AM modulation $\frac{S_0}{N_0} = 2\frac{S_i}{N_i}$ where $\frac{S_i}{N_i}$ represents the S/N of the signal in the modulated form-not the baseband S/N just at the input of the envelope detector.

Solution

$$v_i(t) = V_c(1 + m_a\sin\,\omega_c t)\sin\omega_c t \tag{5-156}$$

where m_a is the modulation index

$$v_i(t) = V_c^2\,[1 + m_a\sin\,\omega_m t]^2\sin^2\omega_c t$$

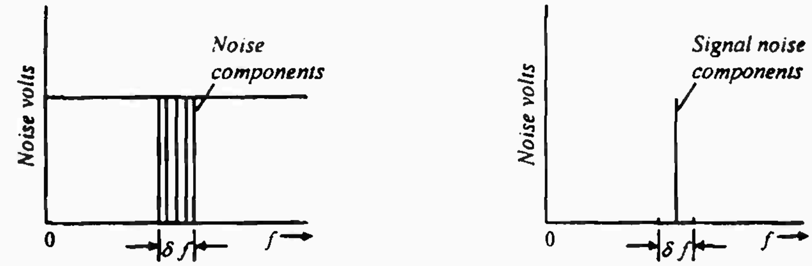


Fig. (5.31) Division of noise PSD into delta functions

$$\overline{v_i^2(t)} = \frac{1}{2}V_c^2\overline{\left[1 + 2m_a \sin\omega_m t + m_a^2 \sin^2\omega_m t\right]}$$

$$= \frac{1}{2}V_c^2\left[1 + \frac{m_a^2}{2}\right] \quad (5\text{-}157)$$

The useful signal power is

$$S_i = \frac{m_a^2 V_c^2}{4} \quad (5\text{-}158)$$

The useful output voltage from the envelope detector is the modulation signal

$$V_o(t) = m_a V_c \sin\omega_m t \quad (5\text{-}159)$$

$$S_0 = \frac{1}{2}m_a^2 V_c^2 \quad (5\text{-}160)$$

Assuming the input noise having a uniform spectral distribution, we can divide the spectrum into delta functions δf wide (Fig. 5.31).

$$S_n(f)\Delta f = \frac{kT}{2}\Delta f \quad (5\text{-}161)$$

$$v_n(t) = V_n \cos[2\pi(f_{IF} + \Delta f_n)t + \theta_n] \quad (5\text{-}162)$$

where Δf_n is the separation from the center frequency

$$\overline{v_n^2(t)} = \frac{V_n^2}{2} = 2\times\frac{kT}{2}\Delta f = kT\Delta f \quad (5\text{-}163)$$

As Δf tends to 0, all noise components cover the IF bandwidth of $2B$ continuously

$$N_i = 2B\times kT = 2kTB \quad (5\text{-}164)$$

$$\frac{S_i}{N_i} = \frac{m^2 V_c^2}{4} \bigg/ 2kTB = \frac{m^2 P_c}{4kTB} \qquad (5\text{–}165)$$

where $$P_C = \overline{v_c^2(t)} = \frac{1}{2} V_c^2 \qquad (5\text{–}166)$$

is the average carrier power

To obtain the output noise power of the detector, each noise component $v_n(t)$ within the bandwidth will appear as a modulation signal for the carrier

$$v_n(t) = V_c \left[1 + \frac{V_n}{V_c} \cos \omega_n t \right] \sin \omega_c t \qquad (5\text{–}167)$$

The noise voltage in the envelope appears as output noise

$$n_o(t) = V_n \cos \omega_n t$$

The noise power δN_0 in a bandwidth Δf is

$$\Delta N_0 = \frac{V_n^2}{2} = kT \Delta f \qquad (5\text{- }168)$$

$$N_0 = \int_{f_c - B}^{f_c + B} kT \Delta f = 2kTB \qquad (5\text{–}169)$$

$$\frac{S_0}{N_0} = \frac{m^2 V_c^2}{2} \bigg/ 2kTB = \frac{m^2 V_c^2}{4kTB} \qquad (5\text{–}170)$$

$$= \frac{m_2 P_c}{2kTB} \qquad (5\text{–}171)$$

Using eqn. (5 – 171) and eqn. (5 – 165),

$$\frac{S_0}{N_0} \bigg|_{\substack{after \\ de\,mod\,ulator}} = 2 \frac{S_i}{N_i} \bigg|_{\substack{befor \\ de\,mod\,ulator}} \qquad (5\text{–}172)$$

The improvement at the detector is clear from eqns. (5 – 158) and (5 – 160) since in AM, the power is divided as $\frac{1}{2} V_c^2 \times 1$ and $\frac{1}{2} V_c^2 \times \frac{m_a^2}{4} \times 2$ (for two side bands) whereas for the modulated signal the power is $\frac{1}{2} V_c^2 \; m_a^2$.

5.9 FM Detection:

An FM system is shown (Fig. 5.32)

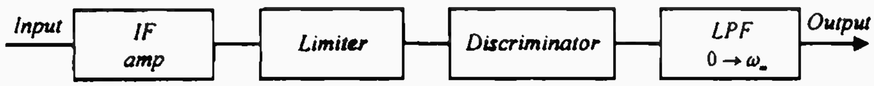


Fig. (5.32) FM system

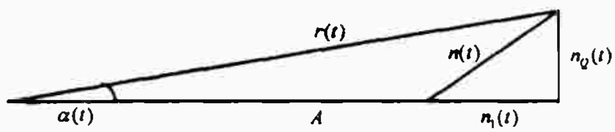


Fig. (5.33) Phasor diagram for noise in FM

At the discriminator input

$$s_i(t) = A\cos\theta(t)$$

$$= A\cos\left[\omega_c t + k_f \int_0^t m(\tau)d\tau\right] \qquad (5-173)$$

The discriminator output is proportional to the difference between the instantaneous frequency of $s_i(t)$ and the carrier frequency

$$s_o(t) = \frac{d\,\theta(t)}{dt} - \omega_c = k_f\, m(t) \qquad (5-174)$$

$$S_0 = \overline{s_o^2(t)} = k_f^2\ \overline{m^2(t)} \qquad (5-175)$$

Next, we turn to calculating the mean output noise power in the presence of an unmodulated carrier.

$$A\cos\omega_c t + n_i(t) = A\cos\omega_c t + n_I(t)\cos\omega_c t - n_Q(t)\sin\omega_c t$$

$$= r(t)\cos[\omega_c t - \alpha(t)] \qquad (5-176)$$

Thus, the addition of noise introduces both amplitude noise in $r(t)$ and phase noise in $\alpha(t)$. In the case of AM, we are interested in the effects of $r(t)$, but in the FM case we may assume amplitude limiting and look into $\alpha(t)$ only.

$$\alpha(t) = \tan^{-1}\frac{n_Q(t)}{A + n_I(t)} \qquad (5-177)$$

Assuming that noise is small, i.e. $n_I(t),\ n_Q(t) << A$

$$\alpha(t) \sim \tan^{-1}\frac{n_Q(t)}{A} \sim \frac{n_Q(t)}{A} \qquad (5-178)$$

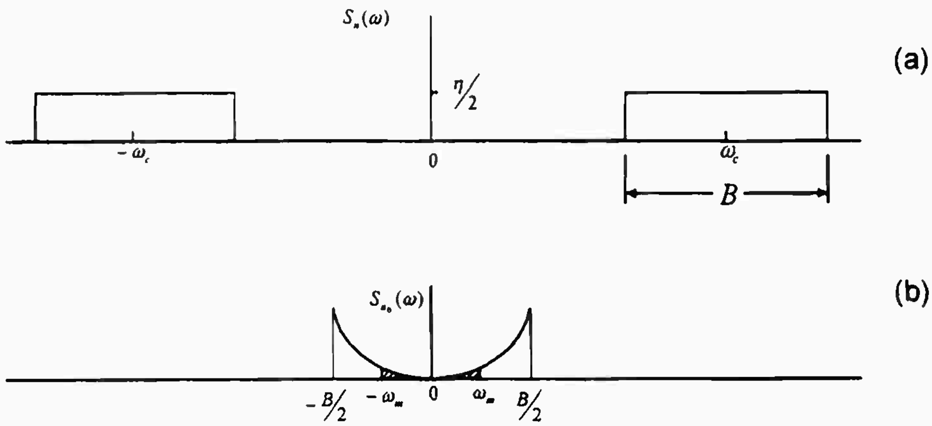


Fig. (5.34) Input – Output PSD of noise in FM
a) input *b) output*

The discriminator output is proportional to the difference between the instantaneous frequency and the carrier frequency

$$n_0(t) = \frac{d\,\alpha(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left[n_Q(t)\right] \tag{5- 179}$$

$$S_{n_0}(\omega) = \frac{1}{A^2} S_{n_Q}(\omega) \left|H(\omega)\right|^2 \tag{5- 180}$$

where $\left|H(\omega)\right|^2 = \omega^2$ corresponding to time differentiation

$$S_{n_0}(\omega) = \frac{1}{A^2}\omega^2\; S_{n_Q}(\omega) \tag{5- 181}$$

Thus, the spectral components at the higher frequencies are emphasized because of the differentiation. The bandwidth of the discriminator output is limited by a LPF whose cut off frequency is ω_m.

$$S_n(\omega) = \frac{kT}{2} \tag{5- 182}$$

$$S_{n_Q}(\omega) = kT \tag{5- 183}$$

$$S_{n_0}(\omega) = \frac{1}{A^2} kT\,\omega^2 \tag{5- 184}$$

For white noise spectral density at the discriminator input, the output noise spectral density is parabolic (Fig. 5.34).

$$N_0 = \overline{n_0^2(t)} = \frac{2}{2\pi A^2} kT \int_0^{\omega_m} \omega^2 \tag{5-185}$$

$$= \frac{kT\,\omega_m^3}{3\pi A^2} \tag{5-186}$$

$$S_i = A^2/2 \tag{5-187}$$

We see from eqn. (5 – 185) that the noise power is inversely proportional to the mean carrier power in FM. This effect of decease in the output noise power as carrier power increass is called noise quieting. From eqn. (5 – 186),

$$\left.\frac{S_0}{N_0}\right|_{FM} = 3\pi A^2 \frac{k_f^2\, \overline{m^2(t)}}{kT\,\omega_m^3} \tag{5-188}$$

The peak frequency deviation is proportional to k_f for the wide band case, the bandwidth increases in proportion to k_f. For wide band FM, we conclude that the output S/N increases as the square of bandwidth. In particular, when $m(t)$ is sinusoidal

$$m(t) = a\cos\omega_m t \tag{5-189}$$

$$\Delta\omega = a\,k_f \tag{5-190}$$

$$\left.\frac{S_0}{N_0}\right|_{FM} = 3\pi A^2 \frac{3\pi A^2 (\Delta\omega)^2}{2kT\,\omega_m^3} \tag{5-191}$$

For a comparison between FM and AM, let the modulation be sinusoidal in both case, and let us define the mean square noise at the IF for AM case

$$N_c = \frac{2}{2\pi} \int_{\omega_c-\omega_m}^{\omega_c+\omega_m} \frac{kT}{2} d\omega = \frac{kT\,\omega_m}{\pi} = 2kTf_m \tag{5-192}$$

But $$S_c = \frac{A^2}{2} \tag{5-193}$$

$$\beta = \frac{\Delta\omega}{\omega_m} \tag{5-194}$$

$$S_c / N_c = A^2 / 4kTf_m \tag{5-195}$$

Eqn. (5 – 190) becomes

$$\left.\frac{S_0}{N_0}\right|_{FM} = 3\beta^2 \frac{S_c}{N_c} \tag{5-196}$$

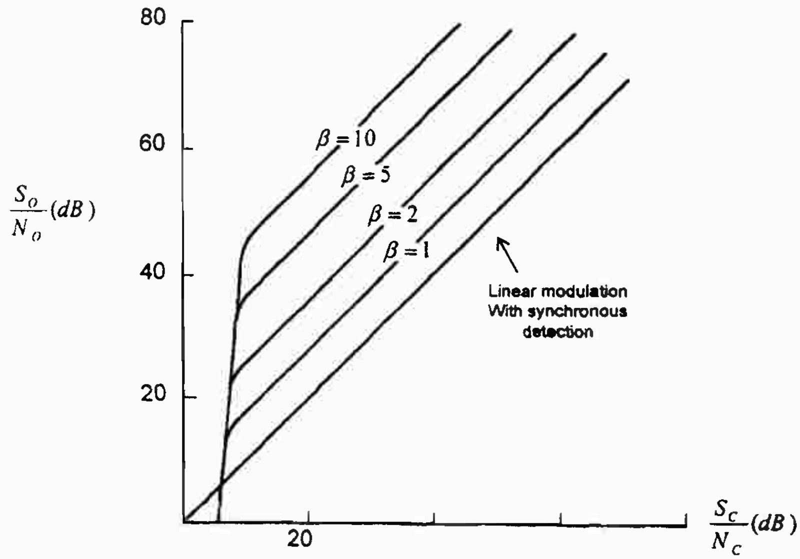


Fig. (5.35) S/N for wideband FM

Under the most favorable conditions in AM when the modulation index is 100%, from eqn. (5 – 155),

$$\left.\frac{S_0}{N_0}\right|_{FM} = 3\beta^2 \left.\frac{S_c}{N_c}\right|_{AM(\max)} \qquad (5\text{– }197)$$

From this, we conclude that the output S/N in FM can be made higher than in AM by increasing β. The factor $3\beta^2$ is the noise quieting factor. But we have derived this expression for low noise. As we increases β the bandwidth is increased. The FM system provides an improvement at the expense of bandwidth. Also we note that as the bandwidth increases more noise is invited, but the quieting effect overrides this noise increase. For example, when $\beta = 5$, the output FM S/N is 75 times that of an equivalent AM system, but the bandwidth required is 8 times larger. Thus, the use of FM allows us to exchange bandwidth for S/N. To realize S/N improvement in FM over AM, we must have $\beta > 1/\sqrt{3} = 0.577$. This condition is the transition point between NBFM and WBFM. Thus, NBFM provides no S/N improvement over AM. The exchange of bandwidth for S/N in FM cannot be continued indefinitely. Since the noise power increases with increased receiver bandwidth, the noise power in a real system eventually becomes comparable to the signal power, and the results of the above analysis no longer holds. As the noise power becomes larger in angle modulation, the phase variation of the noise takes over and the performance of the system becomes very poor. This is called threshold effect. Fig. (5.35) shows S/N for FM.

Ex. 5.12

Obtain S/N for PM and compare it with that of AM for sinusoidal case at maximum modulation index.

Solution

$$s_i(t) = A\cos\lfloor\omega_c t + k_p\, m(t)\rfloor \tag{5 -198}$$

$$s_o(t) = \theta(t) - \omega_c t = k_p\, m(t) \tag{5 -199}$$

Referring to Fig. (5.33), for small noise

$$\alpha(t) \sim \frac{n_Q(t)}{A} = n_0(t) \tag{5- 200}$$

$$S_{n_0}(\omega) = \frac{1}{A^2} S_{n_Q}(\omega) \tag{5- 201}$$

$$\overline{n_0^2} = \frac{2}{2\pi} \int_0^{\omega_m} \frac{1}{A^2} kT\, d\omega \tag{5- 202}$$

$$= \frac{\eta\, \omega_m}{\pi A^2} \tag{5- 203}$$

$$\left.\frac{S_0}{N_0}\right|_{PM} = \frac{\overline{s_o^2(t)}}{\overline{n_0^2(t)}} = \frac{\pi A^2 k_p^2\, \overline{m^2(t)}}{kT\, \omega_m} \tag{5- 204}$$

For sinusoidal case,

$$\overline{m^2(t)} = \frac{a^2}{2} \tag{5- 205}$$

$$\Delta\theta = a\, k_p \tag{5- 206}$$

$$\left.\frac{S_0}{N_0}\right|_{PM} = \frac{A^2}{4\,kTB} (\Delta\theta)^2 \tag{5- 207}$$

Compared to AM for 100% modulation using eqn. (5 – 155),

$$\left.\frac{S_0}{N_0}\right|_{PM} = (\Delta\theta)^2 \frac{S_c}{N_c} = (\Delta\theta)^2 \left.\frac{S_0}{N_0}\right|_{AM\,(\max)} \tag{5- 208}$$

5.10 Preemphasis – Deemphasis:

For FM broadcasting, it is found that signals arising from speech and music have most of the energy at the lower frequencies. The output of the FM modulator however, has a parabolic PSD (eqn. 5 – 184). Therefore, the PSD of noise output is largest in the frequency range when the signal PSD is smallest, i.e., at high frequencies. This results in an inefficient system. We need, therefore, to emphasize the high frequency in the input signal at the transmitter before noise is introduced. At the output of the FM demodulator in the receiver, the inverse operation is performed to deemphasize the high frequency components. Thus, the signal spectrum is restored to its original shape. But the noise which was added after the preemphasis process is now reduced because of the deemphasis. The preemphasis at the transmitter must have $|H(\omega)|^2 = \omega^2$ which means that we need a filter whose transfer function is constant for low frequencies and behaves like a differentiator at high frequencies as in Fig. (5.36). A reasonable choice for $f_1 = \omega_1 / 2\pi$ is usually the frequency at which the signal PSD drops by $3dB$, while $f_2 = \omega_2 / 2\pi$ is well above the highest audio frequency to be transmitted. We see that noise behave as white noise beyond ω_1 due to deemphasis. We want to see now how much improvement in S/N we gain due to preemphasis – deemphasis. Assume that the two networks are chosen properly, so that there will be no net change in the signal. The noise PSD is decreased by the transfer function of the deemphasis.

$$N_0' = \frac{2}{2\pi}\int_0^{\omega_m} S_{n_0}(\omega)|H(\omega)|^2\, d\omega \qquad (5-209)$$

where from eqn. (5 – 184)

$$S_{n_0}(\omega) = \frac{kT\,\omega^2}{A^2}$$

$$H(\omega) = \frac{1}{1 + j\omega/\omega_1} \qquad (5-210)$$

$$N_0' = \frac{kT}{\pi A^2}\int_0^{\omega_m} \frac{\omega^2}{1 + \left(\frac{\omega}{\omega_1}\right)^2}\, d\omega \qquad (5-211)$$

Without deemphasis

$$N_0 = \frac{kT}{\pi A^2}\int_0^{\omega_m} \omega^2\, d\omega \qquad (5-212)$$

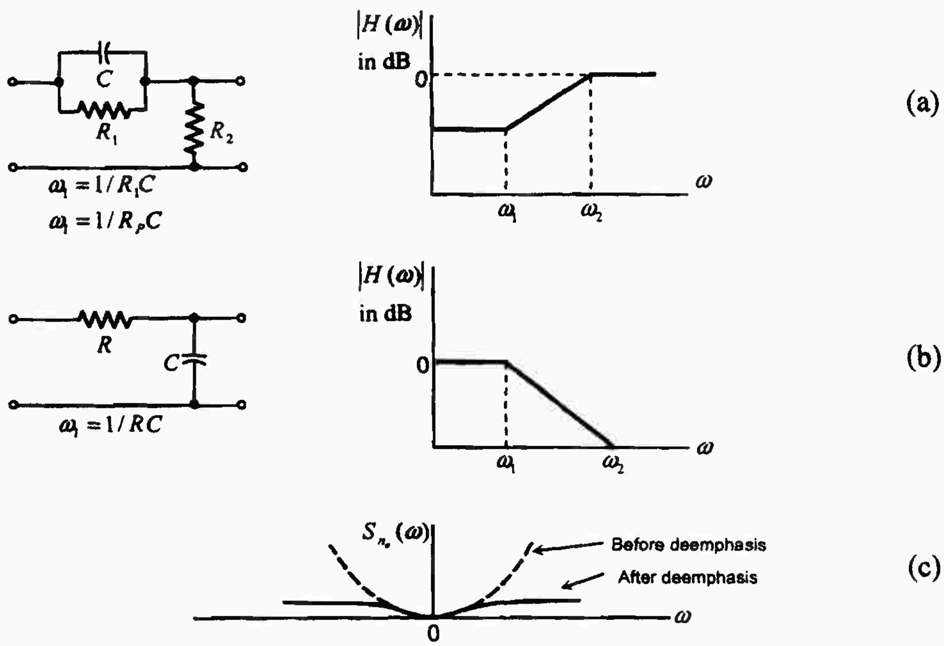


Fig. (5.36) Preemphasis – Deemphasis
a) preemphasis b) deemphasis c) noise PSD after deemphasis

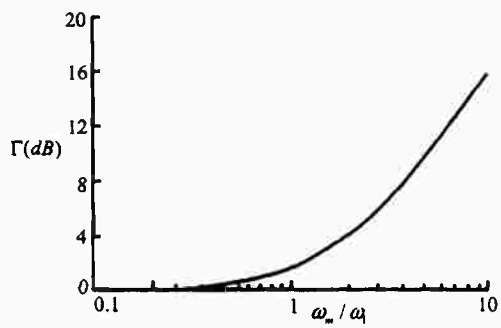


Fig. (5.37) S/N improvement using deemphasis

Defining the improvement factor Γ

$$\Gamma = \frac{N_0}{N_0'} = \frac{1}{3}\,\frac{(\omega_m/\omega_1)^3}{(\omega_m/\omega_1) - \tan^{-1}(\omega_m/\omega_1)} \qquad (5\text{–}213)$$

This is shown in Fig. (5.37). It represents increases in S/N without any required increase in the transmitted power.

Problems

1. $x(t)$ And $y(t)$ are zero mean random currents. When applied separately to 1Ω resistor they dissipate $9W$ and $1W$ of power respectively. When they are both applied to the load simultaneously, the power is $5W$. What is the correlation between X and Y?

2. Two independent random voltages have a uniform pdf given by $f(V) = \begin{cases} 0.5 & |V| \le 1 \\ 0 & |V| > 1 \end{cases}$.

   Find the pdf of their sum by finding the characteristic function and its inverse.

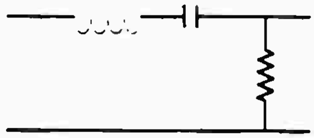
3. Verity eqn. (5 – 10).

4. A random signal in the form of a square wave is transmitted and received buried in noise. Show that, we can measure the delay time by cross correlating the received signal with a replica of the transmitted signal. Show also that we can thus detect the transmitted signal even when buried in noise. Use Matlab to illustrate the result.

5. A random process $X(t)$ consists of a sinusoidal wave $A\cos(2\pi f_c t + \theta)$ and white Gaussian noise, θ is a uniformly distributed random variable from $-\pi$ to π. Obtain the autocorrelation function of the output, and the decorrelation time. What do you conclude?

6. Deduce the equivalent noise bandwidth of a BPF whose transfer function is triangular centered at $\pm f_c$ and extending to $\pm\Delta f$ around f_c.

7. Consider a band pass LCR filter shown

   Find the PSD of the output, PSD of n_I, PSD of n_Q.

8. In the above problem find $R_n(\tau)$, $R_{n_I}(\tau)$, $R_{n_Q}(\tau)$.

9. Find $R_X(\tau)$ and $S_X(f)$ for each case, then compare and comment. Also find the decorrelation time in each case of the following.
   a) sinusoidal process of unit frequency and random phase.
   b) random binary wave of unit symbol duration.
   c) RC LPF filtered white noise.
   d) deal bandpass filtered white noise.

10. In a triangular BPF with white Gaussian noise, find the in - phase and quadrature PSD and the autocorrelation for each and for the total noise. What is the total noise power?

11. In a DSBSC system, $f_c = 1000 kHz$, $m(t)$ has uniform PSD and band limited to $4 kHz$. The channel has white noise. The useful signal power at the received input is $10 \mu W$. The signal is coherently detected. Find S/N.

12. Repeat the problem above if the signal is detected by an envelope detector. What do you conclude?

13. A sinusoidal AM modulated signal has modulation index 50%. Find S/N if received by a synchronous detector and if received by an envelope detector. What is the improvement in S/N before and after the detector?

14. In FM, $\beta = 5$ find the improvement of S/N over AM, and show how the bandwidth is traded for S/N improvement.

15. For $f_m = 15 kHz$, the preemphasis network has $f_1 = 2.1 kHz$. Find the improvement in S/N.

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