CHAPTER 6
Detection and Decoding Concepts

6.1 Noise Effects in Baseband Transmission:

The term demodulation or detection in baseband signaling means recovering the pulse waveform which has been mutilated by noise and distortion, to obtain the original pulse representing proper 0 or 1. The filtering at the transmitter and the channel causes also intersymbol interference. The goal of the demodulator receiving filter is to recover the baseband pulse with the best S/N free of any ISI. Equalization is a technique to compensate for channel induced interference. Another cause of degradation is noise and interference. Thus, the task of the detector is to retrieve the bit stream from the received waveform as intact as possible, despite various impairments in the signal throughout transmission. For any binary channel, the transmitted signal over a symbol interval $(0, T_S)$ is represented by

$$s_i(t) = \begin{cases} s_1(t) & 0 \le t \le T_S \\ & \text{for binary } 1 \\ s_2(t) & 0 \le t \le T_S \\ & \text{for binary } 0 \end{cases} \qquad (6-1)$$

The received signal at the detector input $r_d(t)$ is degraded by noise $n(t)$ and possibly degraded by the impulse response of the channel $h_c(t)$

$$r_d(t) = s_i(t) * h_c(t) + n(t) \qquad i = 1,2 \ldots M \qquad (6-2)$$

where $n(t)$ is zero mean AWGN and $*$ is the convolution operation, and M is the number of states (in binary, M=2) for binary transmission over an ideal band-unlimited distortionless channel, where convolution with $h_c(t)$ produces no degradation (since for the ideal case $h_c(t)$ is an impulse function for a flat frequency response of an ideal band unlimited distortionless channel). Thus

$$r_d(t) = s_i(t) + n(t) \qquad i = 1,2 \ldots \qquad 0 \le t \le T_S \qquad (6-3)$$

Fig. (6.1) shows a typical baseband detection system. The figure consists of two blocks. Block1 is demodulate and sample block. Block 2 is decision making block (Detect block). In the first block, we recover the waveform to an undistorted baseband pulse. The goal of the receiving filter is to recover a baseband pulse with best S/N free of ISI, hence called matched or correlated filter.

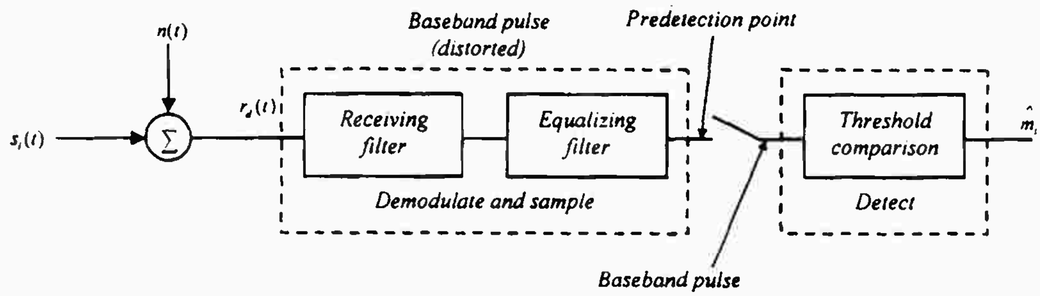


Fig. (6.1) Baseband detection block diagram

The equalizing filter compensates for distortion due to ISI. The detect block performs the decision making process of selecting the digital meaning of the waveform. The output is the estimate of the message symbol $\hat{m}_i$ (called hard decision), where $\hat{m}_i$ is the output bit or symbol. The waveform to sample transformer is made up of a demodulator (receiving filter + equalizing filter) followed by a sampler. At the end of each symbol duration T_s, the output of the sample (at the predetection point) yields a sample $z(T_s)$ which consists for a binary system of

$$z(T_s) = a_i(T_s) + n(T_s) \qquad i = 1.,2... \tag{6-4}$$

where $a_i(T_s)$ is the desired signal component and $n(T_s)$ is the noise component at the point of sampling. Since the input noise is Gaussian, the output noise is Gaussian, also $z(T_s)$ is Gaussian with mean of either A_1 or A_2 depending on whether a binary 1 or binary 0 was sent. Since n_0 is AWGN, we have

$$f(n) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n}{\sigma_0}\right)^2} \tag{6-5}$$

The conditional *pdfs* $f(z/s_1)$ and $f(z/s_2)$ are given by

$$f(z|s_1) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-A_1}{\sigma_0}\right)^2} \tag{6-6}$$

$$f(z|s_2) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-A_2}{\sigma_0}\right)^2} \tag{6-7}$$

These conditional *pdfs* are shown (Fig. 6.2)

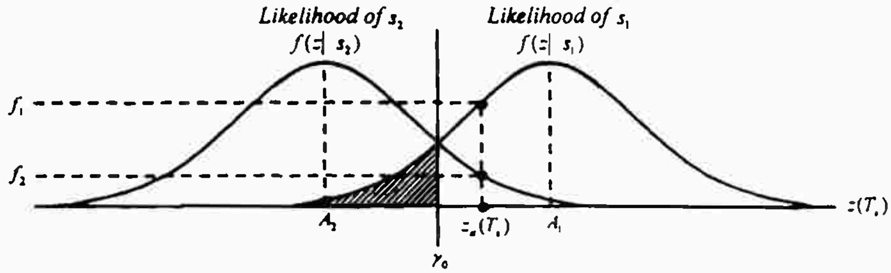


Fig. (6.2) Conditional *pdfs*

The conditional *pdf* $f(z|s_1)$ is called the likelihood of s_1, i.e., the probability of $z(T_s)$ belonging to A_1 given that s_1 was transmitted. Similarly, the conditional *pdf* $f(z|s_2)$ is the likelihood of s_2, i.e., the *pdf* of $z(T_s)$ belonging to A_2 given that the symbol s_2 was transmitted, while $z(T_s)$ represents the range of possible sample values in the output of waveform to sample transformer. After the received waveform has been transformed to a sample, the actual shape of the waveform is not important. The optimum receiving filter (Fig. 6.1) – called matched filter – maps all signals of equal energy into the same point $z(T_s)$ regardless of the shape of the waveform. Therefore, the received signal energy not its shape is the important parameter in the detection process.

Since $z(T_s)$ is a voltage signal - which is proportional to the energy of the received symbol, - the larger the magnitude of $z(T_s)$, the more error free will be the decision making process. Detection is then performed by using the threshold measurement where γ is a threshold value,

$$z(T_s) > \gamma \text{ set } z(T_s) = A_1, 1 \text{ was sent} \qquad (6-8)$$

$$z(T_s) < \gamma \text{ set } z(T_s) = A_2, 0 \text{ was sent}$$

If $z(T_s) = \gamma$ the decision can be arbitrary.

6.2 Signal Representation in Orthogonal Space:

We define an n-dimensional orthogonal space of N linearly independent orthogonal functions $[\phi_i(t)]$, called basis functions. Any arbitrary function in this space can be represented by a linear combination of these basis functions. The basis functions must satisfy the orthogonality condition

$$\int_0^{T_s} \phi_i(t)\,\phi_j(t)\,dt = K_i\,\delta_{i,j} \quad , \quad 0 \le t \le T_s \quad , \; i\,,\,j = 1 \ldots N \qquad (6-9)$$

where $$\delta_{j,k} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$

T_s Is the symbol interval and $\delta_{i,j}$ is called Kronecker delta, if $K_i = 1$. In the orthogonal space, each $\phi_i(t)$ function of the set of the basis functions must be independent of the other members of the set, so it will not interfere with any other member in the detection process. Therefore, we say $\phi_i(t)$ is mutually perpendicular to each of the other $\phi_j(t)$ for $i \ne j$. If $\phi_i(t)$ represents voltage or current, then the energy in 1Ω resistor is

$$E_i = \int_0^{T_s} \phi_i^2(t)\,dt = K_i \qquad (6-10)$$

For an orthonormal system, $K_i = 1$

$$E_i = \int_0^{T_s} \phi_i^2(t)\,dt = 1 \qquad (6-11)$$

We may represent any arbitrary finite set of waveforms $[s_i(t)]$ $(i = 1 \ldots M)$ where each member of the set is physically realizable and of duration T_s can be expressed as a linear combination of N orthogonal waveforms $\phi_1(t)$, $\phi_2(t) \ldots \phi_N(t)$, $N \le M$ such that

$$s_1(t) = a_{11}\,\phi_1(t) + a_{12}\,\phi_2(t) + \ldots\; a_{1N}\,\phi_N(t)$$
$$s_2(t) = a_{21}\,\phi_1(t) + a_{22}\,\phi_2(t) + \ldots\; a_{2N}\,\phi_N(t)$$
$$s_M(t) = a_{M1}\,\phi_1(t) + a_{M2}\,\phi_2(t) + \ldots\; a_{MN}\,\phi_N(t)$$

or

$$s_i(t) = \sum_{j=1}^{N} a_{i,j}\;\phi_j(t) \qquad \begin{matrix} i = 1, \ldots. M \\ N \le M \end{matrix} \qquad (6-12)$$

when

$$a_{i,j} = \frac{1}{K_j} \int_0^{T_s} s_i(t)\,\phi_j(t)\,dt \qquad \begin{matrix} i = 1 \ldots M \\ j = 1, \ldots N \\ 0 \le t \le T_s \end{matrix} \qquad (6-13)$$

The coefficient $a_{i,j}$ is the value of $\phi_i(t)$ component of the signal $s_i(t)$. We note that there is no unique set of $[\phi_i(t)]$ but any orthogonal set may be chosen as convenient and will depend on the signal waveforms. The set of signal waveforms

$\{s_i(t)\}$ can be viewed as vectors $\left\{\vec{s_i}\right\} = \{a_{i1}, a_{i2}, ... a_{iN}\}$ in N dimensional space, for $i = 1,...M$.

For orthogonal system, $k_j = 1$, eqn (6 -13) reduces to

$$a_{ij} = \int_0^{T_s} si(t)\phi_j(t/d). \qquad i = 1....M \qquad (6-14)$$

$$j = 1....N$$

$$0 \le t < T_s$$

The normalized energy E_i associated with the waveform $s_i(t)$ over a symbol interval T_s can be expressed using eqns. (6 – 12), (6 – 13), (6 – 9), (6 – 10), (6 – 11), as

$$\mathrm{E}_i = \int_0^{T_s} s_i^2(t)\,dt \qquad (6-15)$$

$$= \int_0^{T_s} \left[\sum_j a_{ij}\,\varphi_j(t)\right]^2 dt$$

$$= \int_0^{T_s} \left[\sum_j a_{ij}\,\varphi_j(t) \sum_k a_{ik}\,\varphi_k(t)\right] dt$$

$$= \sum_j \sum_k a_{ij}\, a_{ik} \int_0^{T_s} \varphi_j(t)\,\varphi_k(t)\,dt$$

$$= \sum_j \sum_k a_{ij}\, a_{ik}\, K_j\, \delta_{jk}$$

$$= \sum_{j=1}^{N} a_{ij}^2 K_j \qquad i = 1,...M \qquad (6-16)$$

which is a special case of Parseval's theorem relating the integral of the waveform $s_i(t)$ to the sum of the squares of the orthogonal series coefficients. For orthonormal functions, $K_j = 1$ and eqn. (6 – 16) becomes

$$E_i = \sum_{j=1}^{N} a_{ij}^2 \qquad (6-17)$$

We should note that eqns. (6 – 9), (6 – 12), (6 – 13) are generalization of Fourier series. In ordinary Fourier series the $\{\phi_j(t)\}$ set uses sine and cosine harmonic functions, while in our case here $\{\varphi_j(t)\}$ set is not confined to any specific form. The only condition is orthogonality. The transmitter and receiver need only converse in terms of the set $\{\varphi_j(t)\}$ instead of the original set $\{s_i(t)\}$.

Ex 6.1

Fig. (6.3) shows three waveforms $s_1(t)$, $s_2(t)$, $s_3(t)$ where it is required to express them is terms of two functions $\varphi_1(t)$ and $\varphi_2(t)$. Verify that $\varphi_1(t)$ and $\varphi_2(t)$ are orthogonal, whereas $\{s_i\}$ are not, and then show how $\{s_i\}$ may be represented in terms of $\varphi_1(t)$ and $\varphi_2(t)$

Solution

We note

$$\int_0^{T_s} s_1(t)\, s_2(t)\, dt = \int_0^{T_s/2} s_1(t)\, s_2(t)\, dt + \int_{T_s/2}^{T_s} s_1(t)\, s_2(t)\, dt$$

$$= \int_0^{T_s/2} (-1)\,(3)\, dt + \int_{T_s/2}^{T_s} (-3)(0)\, dt = -\tfrac{3}{2}$$

Thus $\{s_i\}$ are not orthogonal. While

$$\int_0^{T_s} \phi_1(t)\, \varphi_2(t)\, dt = \int_0^{T_s/2} (1)\,(1)\, dt + \int_{T_s/2}^{T_s} (-1)\,(1)\, dt = 0$$

Using eqn. (6 – 13)

$$s_1(t) = \varphi_1(t) - \varphi_2(t)$$
$$s_2(t) = 2\varphi_1(t) + \varphi_2(t)$$
$$s_3(t) = \varphi_1(t) + \varphi_2(t)$$

6.3 Significance of the Signal Space:

In a baseband binary PAM system, (Fig. 6.4), the pulse amplitude modulator produces binary pulses with one of two possible amplitude levels. On the other hand, in a baseband M-ary PAM system, the pulse amplitude modulator produces one of M possible amplitude levels with $M > 2$. (Fig. 6.5) shows the case of baseband M-ary PAM, where every two bits (Dibit) are assigned a level and hence a specific amplitude. This is a quaternary system. If we call $00(A)$, $01(B)$, $10(C)$, $11(D)$, the bit stream 0010110111 would read ACDBD. Thus, we have in this case 4 symbols or alphabets (in general M) with each alphabet denoting an amplitude level. Consider an M-ary PAM system with M equally likely and statistically independent symbols. The symbol duration T_s is not the bit duration. We refer to $1/T_s$ as the signaling rate (symbol rate), i.e., symbols/second (Baud), whereas the bit duration is T_b and hence the bit rate is $1/T_b$ bits/second. In the quaternary system above, we find $T_s = 2T_b$. In general

$$T_s = T_b \log_2 M \qquad (6-18)$$

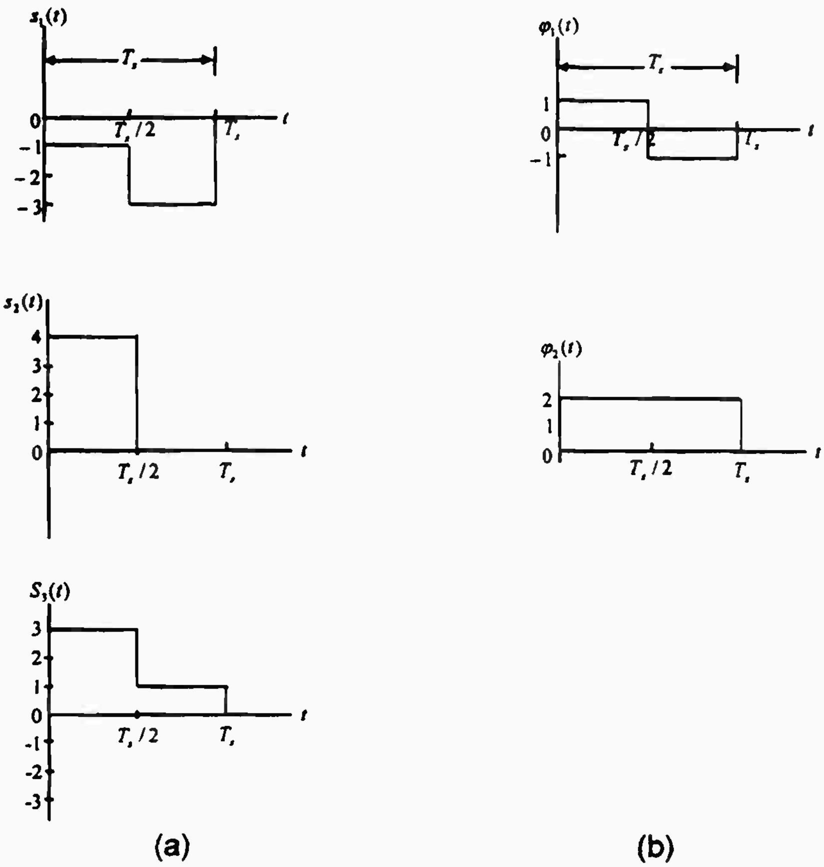


Fig. (6.3) Ex. 6.1
a) waveform set *b) orthogonal basis set*

Therefore, in a given channel bandwidth, we find that by using an M-ary PAM system we are able to transmit information at a rate that is $\log_2 M$ faster than the corresponding binary PAM system.

Referring to Fig. (6.4), in a baseband M-ary system, the sequence of symbols emitted by the information source is converted into an M level PAM pulse train by PAM modulator. The pulse train is shaped by a transmit filter, and then transmitted over the channel in which the signal waveform may be corrupted by both noise and distortion. The received signal is passed through a receive filter, and then sampled at an appropriate rate in synchronism with the transmitter.

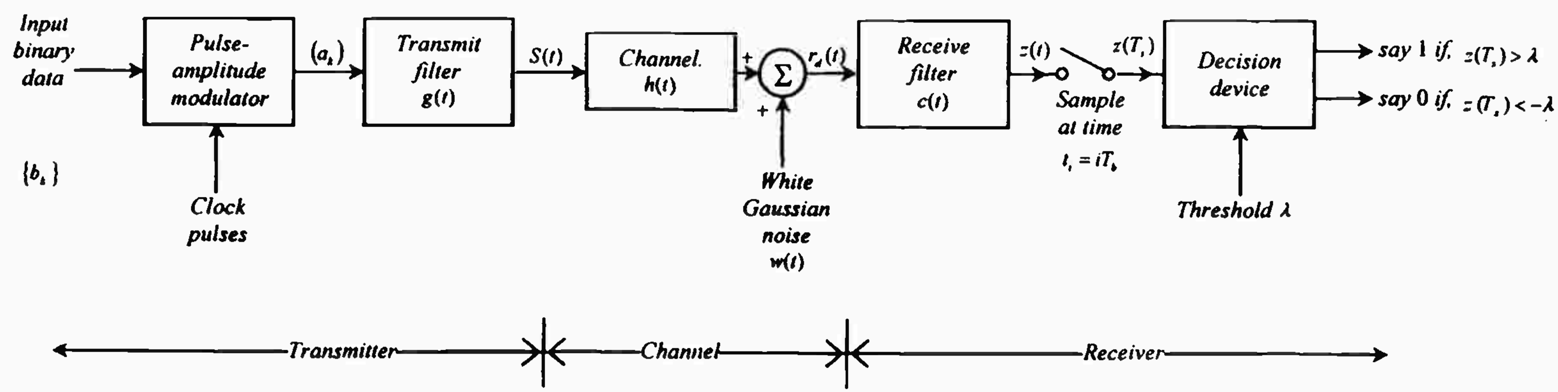
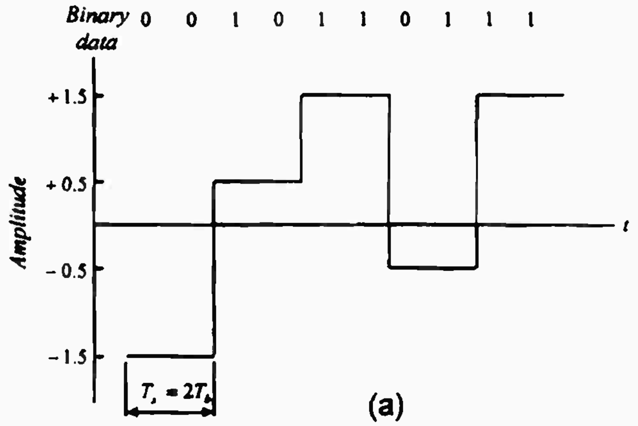


Fig. (6.4) Baseband PAM system



(a)

Dibit	Amplitude
00	-1.5
01	-0.5
10	+0.5
11	+1.5

(b)

Fig. (6.5) Quaternary system
a) waveform *b) Dibit level representation*

Each sample is compared with preset threshold values (slicing levels) and a decision is made as to which symbol was transmitted. Intersymbol interference, noise and imperfect synchronism cause errors to appear at the receiver output. The transmit and receive filters are designed to minimize these errors.

On the other hand, not all communication systems are baseband. To allow for multiuser operation on the same communication gear, several carriers are used, one assigned to each user, hence using the hardware efficiently. Also, when wireless communication is used, carrier modulation becomes a necessity. Each carrier is assigned a bandwidth for the information, hence the name bandpass transmission. In this case, the incoming stream is modulated onto a carrier with fixed frequency limits dictated by the bandwidth of the bandpass channel. The communication channel used for bandpass data transmission may be a microwave link, optical fiber or satellite channel. Modulation in digital bandpass transmission involves keying (switching) the amplitude, frequency or phase of a sinusoidal carrier in accordance with the incoming data.

Hence, we have amplitude shift keying (ASK), frequency shift keying (FSK) and phase shift keying (PSK). In both FSK and PSK, the information is not imbedded in the carrier amplitude or envelope making both impervious to amplitude nonlinearities and noise added to the envelope as in the case of ASK. Fig. (6.6) shows a digital bandpass transmission system. We have a message source that emits one symbol every T_s second with the symbols belonging to an alphabet of

M symbols $\{m_i\}, i = 1...M$. An example is binary transmission where the two symbols are just $0,1$. Another example is the quaternary PCM encoder with an alphabet $A(00)$, $B(01)$, $C(10)$, $D(11)$, ($M = 4$) In general, the a priori probabilities $P(m_1)$, $P(m_2)... P(m_M)$ describe the probabilities of emission by the source of the various symbols. Unless otherwise specified, we assume that the M symbols of the alphabet are equally likely, i.e.,

$$P_i = P(m_i) = \frac{1}{M} \text{ For all,} \qquad (6-20)$$

The M-ary output of the message source is presented to a signal transmission encoder which produces vector $\vec{s}_i$ made up of N real elements described as projections on the basis function axes $\{\phi_j\}$ in the signal space where $N \le M$. With vector $\vec{s}_i$ as input, the modulator then constructs a distinct waveform or real valued time signal $s_i(t)$ of duration T_s as the representation of the symbol m_i generated by the message source. The signal $s_i(t)$ has finite energy

$$E_i = \int_0^{T_s} s_i^2(t)\,dt \qquad i = 1,...M \qquad (6-21)$$

The particular time signal depends on the incoming message and possibly on preceding data. With a sinusoidal carrier, the feature that is used by the modulator to distinguish one signal from another is a step in amplitude, frequency or phase of the carrier, or a combination (hybrid of two). Fig. (6.7) shows ASK, PSK, FSK in the special case of binary data where $T_s = T_b$.

We assume that the channel is linear with bandwidth wide enough to accommodate the transmission of the modulated signal $s_i(t)$ with minimum distortion. During transmission, the signal $s_i(t)$ is marred by additive zero mean Gaussian white noise $n(t)$. We call such a channel AWGN channel. We must identify here six devices (Fig. 6.8)

The encoder at the transmitter end converts the $\{m_i\}$ state to a vector $\vec{s}_i = \{a_{ij}\}$. The modulator at the transmitter changes $\vec{s}_i$ to $s_i(t)$. The sampler at the receiving end takes samples every T_s of the input $r_d(t)$. The detector formulates these samples into a vector $\vec{r}$. The decoder translates $\vec{r}$ into values of energy (or voltage levels), $z(T_s)$ is a random variable depending on the energy or voltage level of the signal to be decided at the instant of sampling $z(T_s)$ when compared with known values of the reference states. Thus,

$$z(T_s) = A_i + n_0(T_s) \qquad (6-22)$$

where $A_i(T_s)$ is the value of voltage or (energy) for the state for which $z(T_s)$ is closest, and $n_0(T_s)$ is the output noise from the decoder interfering with the decision device (estimator). Usually, the level decoder and the estimator are collectively called transmission decoder. The output of the estimator (decision device) is $\hat{m}_i$ (Fig. 6.9). The received signal energy not its shape is the important parameter in the detection process.

The function of the modulator may be described by introducing each of the j^{th} components of the vector $\bar{s}_i$ onto a basis function $\phi_j(t)$ using a multiplier (Fig. 6.10). If there is no noise in the channel then $r_d(T_s) = \hat{s}_i(T_s) = \bar{s}_i(T_s)$. The function of the detector is to convert the received time signal into vector $\hat{\bar{s}}_i$ (Fig. 6.11) using a bank of correlators. If there is noise, the input $r_d(t) = \hat{s}_i(t) + n(t)$. The output of the correlators is $\bar{r}$ with projections along $\{\varphi_i\}$ basis functions (Fig. 6.12). It is left to the decoder to find $\bar{s}_i$ closest to $\bar{r}$. By using the observation vector $\bar{r}$ together with prior knowledge of the modulation format used in the transmission and the a priori probabilities $P(m_i)$, the decoder produces an estimate $\hat{m}$. However, due to the presence of AWGN at the receiver input, the decision making process is statistical in nature causing occasional error. Our preoccupation is to design a receiver with minimum average probability of symbol error defined as

$$P_e = \sum_{i=1}^{M} P(\hat{m} \neq m_i)\, P(m_i) \qquad (6-23)$$

where m_i is the transmitted symbol, $\hat{m}$ is the estimate produced by the decision device, $P(\hat{m} \neq m_i)$ is the conditional error probability given that the i^{th} symbol was sent. This receiver is hence called optimum in the minimum probability of error sense. Usually, we assume that the receiver is time synchronized with the transmitter, i.e., the receiver knows the time when the modulation changes state. We say the receiver is phase - locked to the transmitter. This is called coherent detection, and the receiver is called coherent receiver. When there is no phase synchronism between the transmitter T_X and the receiver R_X ,we call this noncoherent detection and the receiver is called noncoherent receiver (Fig. 6.13).

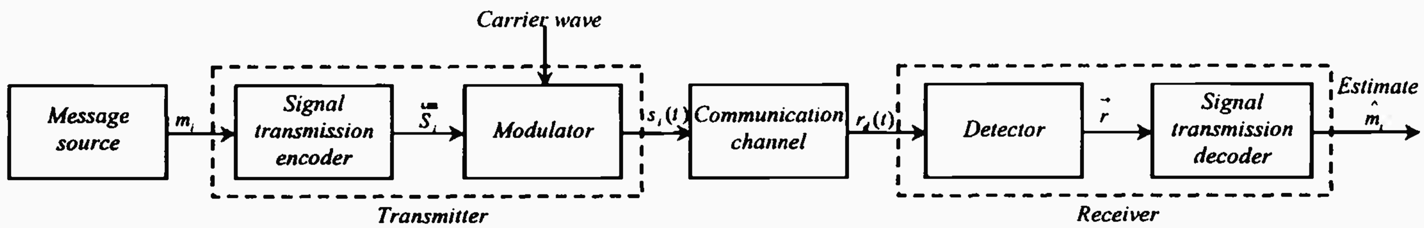


Fig. (6.6) Digital bandpass transmission system

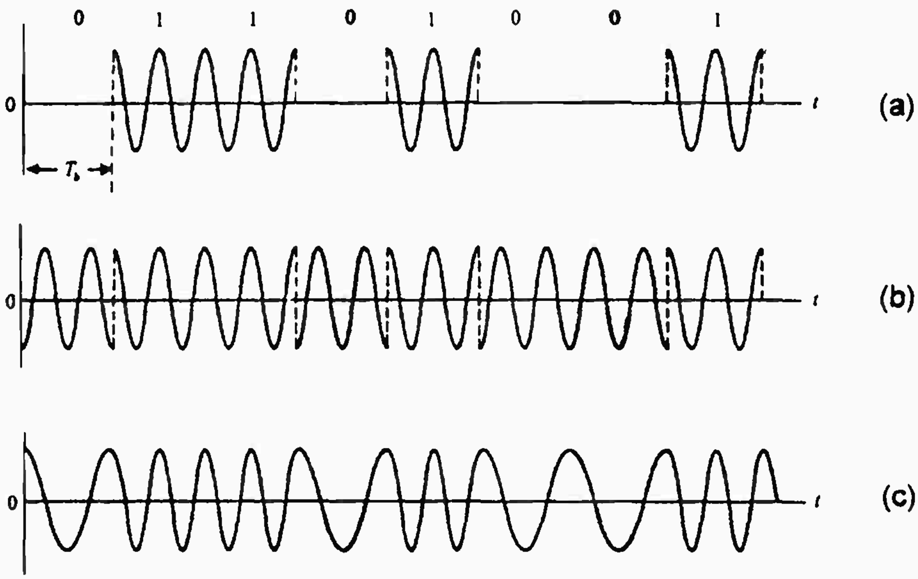


Fig.(6.7). Digital modulation for binary data

a) ASK b) PSK c) FSK

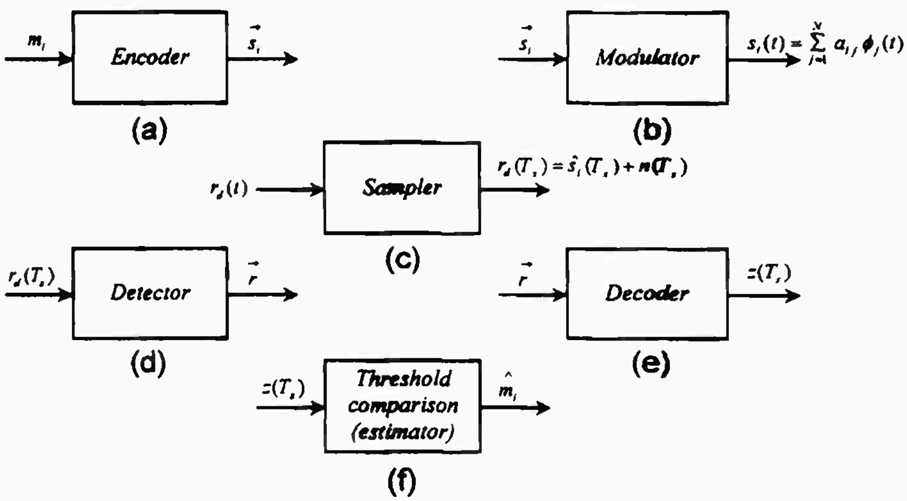


Fig.(6.8.) Components of modulation and demodulation In a general digital transmission link

a) encoder *b) modulator* *c) sampler* *d) detector*

e) decoder *f) threshold comparator (decision device or estimator)*

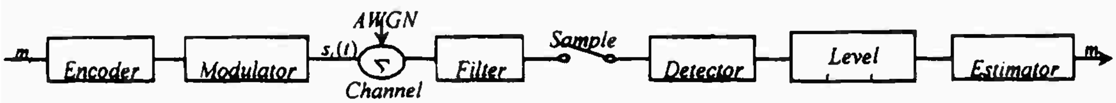


Fig. (6.9) Conceptual digital link

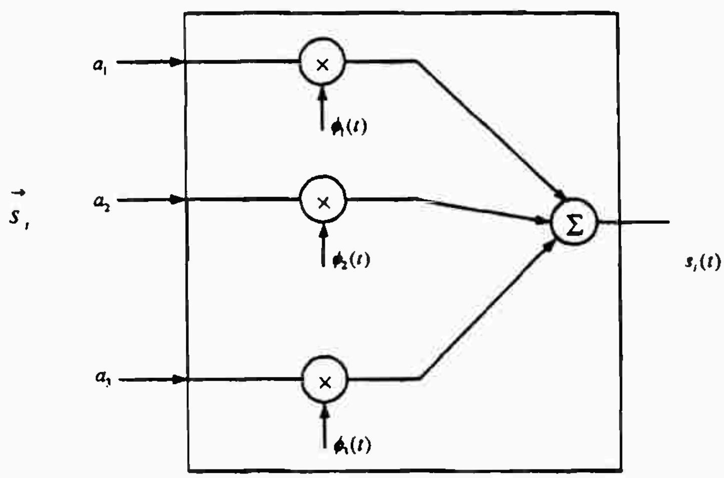


Fig. (6.10) Modulator in digital bandpass transmission without noise

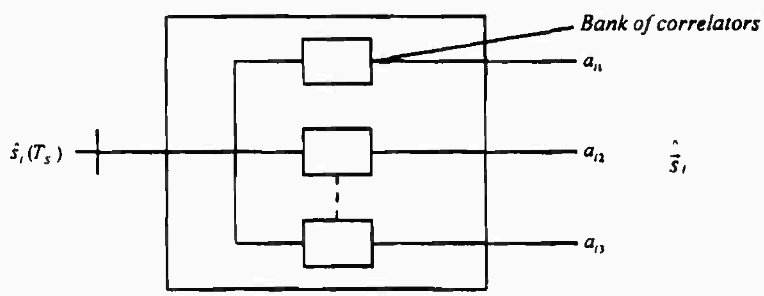


Fig. (6.11) Detector without noise

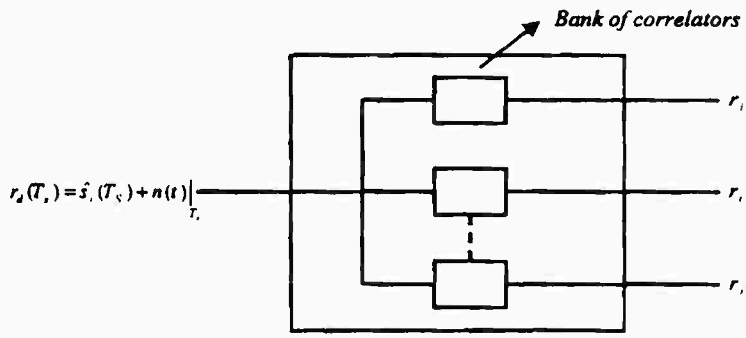


Fig. (6.12) Detector with noise

(a) (b)

Fig. (6.13) Coherent and noncoherent detection
a) coherent *b) noncoherent*

6.4 Gram Schmidt Orthogonalization Procedure:

The coefficient a_{ij} in eqn. (6 – 13) may be viewed as the j^{th} element of the N^{th} dimensional vector $\hat{s}_i$ Given the N elements of the vector $\hat{s}_i$, i.e., a_{i1}, a_{i2}, $a_{i3}...a_{iN}$ operating as input, we may use the scheme shown (Fig. 6.14a) to generate the signal $s_i(t)$. It consists of a bank of N multipliers with each multiplier supplied with its own basis function followed by a summer. Conversely, given the signals $s_i(t)$, $i = 1,2...M$ operating as input, we may use the scheme shown (Fig. 6.14b) to calculate the coefficients a_{i1}, a_{i2} ... a_{iN}, which follows directly from eqns. (6 – 9), (6 – 10). It consists of a bank of N product integrators or correlators with a common input, each one supplied with its own basis function. The Gram Schmidt orthogonalization procedure provides a convenient way in which an appropriate choice of a basis function set $\{\varphi_j(t)\}$ can be obtained for any given signal set $\{s_i(t)\}$.

To prove the Gram Schmidt orthogonalization procedure, we proceed by defining the first basis function as

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \qquad (6-24)$$

where E_1 is the energy of the signal $s_1(t)$ as given by eqns. (6 – 17).,(6 – 21)

Thus,

$$s_1(t) = \sqrt{E_1}\ \phi_1(t) \qquad (6-22)$$

$$s_1(t) = a_{11}\ \phi_1(t) \qquad (6-23)$$

where

$$a_{11} = \sqrt{E_1} \qquad (6-24)$$

Next, using the signal $s_2(t)$, we define the coefficient a_{21}, as

$$a_{21} = \int_0^{T_s} s_2(t)\ \phi_1(t)\,dt \qquad (6-25)$$

where T_s is the symbol duration. We may introduce a new function

$$\psi_2(t) = s_2(t) - a_{21}\ \phi_1(t) \qquad (6-26)$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T_s$ so that if we mulriply both sides by ϕ_1 and integrant the LHS is zero and the RHS is zero from eqn (6 – 28) . Now, we define the second basis function as

$$\phi_2(t) = \frac{\psi_2(t)}{\sqrt{\int_0^{T_s} \psi_2^2(t)\,dt}} \qquad (6-30)$$

Substituting in eqn. (6 – 29), using eqns. (6 – 28) and (6 – 25),

$$\phi_2(t) = \frac{s_2(t) - a_{21}\ \phi_1(t)}{\sqrt{E_2 - a_{21}^2}} \qquad (6-31)$$

where E_2 is the energy of the signal $s_2(t)$.

From eqn. (6 – 30), it is clear that

$$\int_0^{T_s} \phi_2^2(t)\,dt = 1 \qquad (6-32)$$

From eqns. (6 – 31), (6 -28),

$$\int_0^{T_s} \phi_1(t)\ \phi_2(t)\,dt = 0$$

Thus, $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal set. We continue

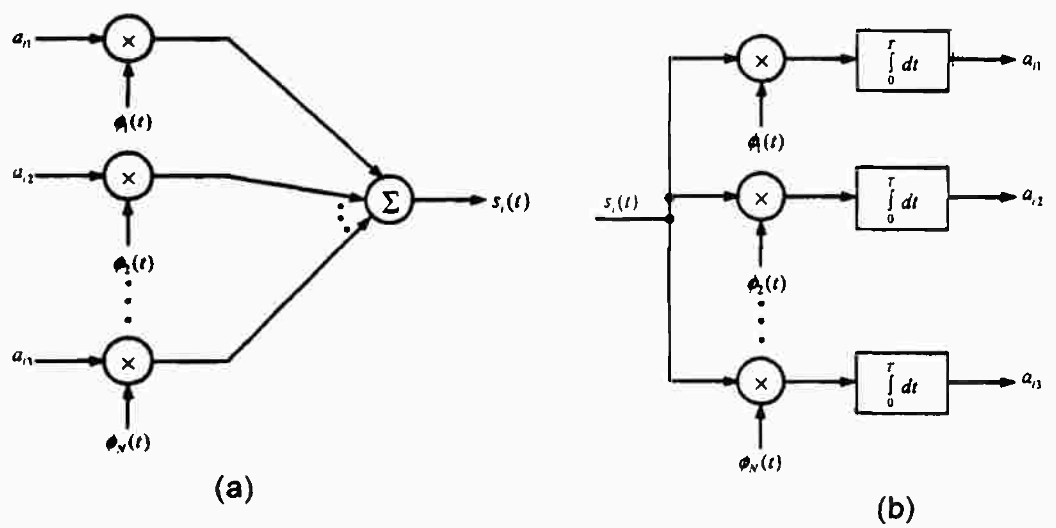


Fig. (6.14) Schemes for waveform - basis function set transformation
a) waveform generation *b) coefficient generation*

$$\psi_i(t) = s_i(t) - \sum_{j=1}^{i-1} a_{ij}\ \phi_j(t) \tag{6 - 33}$$

where

$$a_{ij} = \int_0^{T_i} s_i(t)\ \phi_j(t)\ dt \qquad j = 1,2 \ldots i-1 \tag{6 - 34}$$

Given $\psi_i(t)$ we may now define the set of basis functions

$$\phi_i(t) = \frac{\psi_i(t)}{\int_0^{T_i} \psi_i^2(t)\, dt} \qquad i = 1,2 \ldots N \tag{6 - 35}$$

$\{\varphi_i\}$ Forms an orthonormal set for $N \le M$. For a linearly independent set $N = M$. If the signals $\{s_i(t)\}$ are not linearly independent then $N < M$.

Ex. 6.2

Consider the signal $s_1(t), s_2(t), s_3(t), s_4(t)$ shown (Fig. 6.15). Use the Gram Schmidt orthogonalization procedure to find an orthonormal basis for this set of signals.

Solution

From eqn. (6 – 15),

$$E_1 = \int_0^{T_s} s_1^2(t)\,dt$$

$$= \int_0^{T_s} (1)\,dt = \frac{T_s}{3}$$

The first basis function $\phi_1(t)$ is given from eqn. (6 – 24) by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{cases} \sqrt{3/T_s} & 0 \le t \le T_s/3 \\ 0 & elsewhere \end{cases}$$

From eqn. (6 – 28)

$$a_{21} = \int_0^{T_s} s_2(t)\,\phi_1(t)\,dt$$

$$= \int_0^{T_s} (1)\left(\sqrt{\frac{3}{T_s}}\right)dt$$

$$= \sqrt{T_s/3}$$

$$E_2 = \int_0^{T_s} s_2^2(t)$$

$$= \int_0^{T_s} (1)^2\,dt = \frac{2T_s}{3}$$

From eqn. (6 – 26)

$$\phi_2(t) = \frac{s_2(t) - a_{21}\,\phi_1(t)}{\sqrt{E_2 - a_{21}^2}}$$

$$= \begin{cases} \sqrt{3/T_s} & T_s/3 \le t \le 2T_s/3 \\ 0 & elsewhere \end{cases}$$

The intermediate function $\psi_i(t)$ with $i = 3$ is given according to eqn. (6 – 33) by

$$\psi_3(t) = s_3(t) - a_{31}\,\phi_1(t) - a_{32}\,\phi_2(t)$$

$$= \begin{cases} 1 & 2T_s/3 \le t \le T_s \\ 0 & elsewhere \end{cases}$$

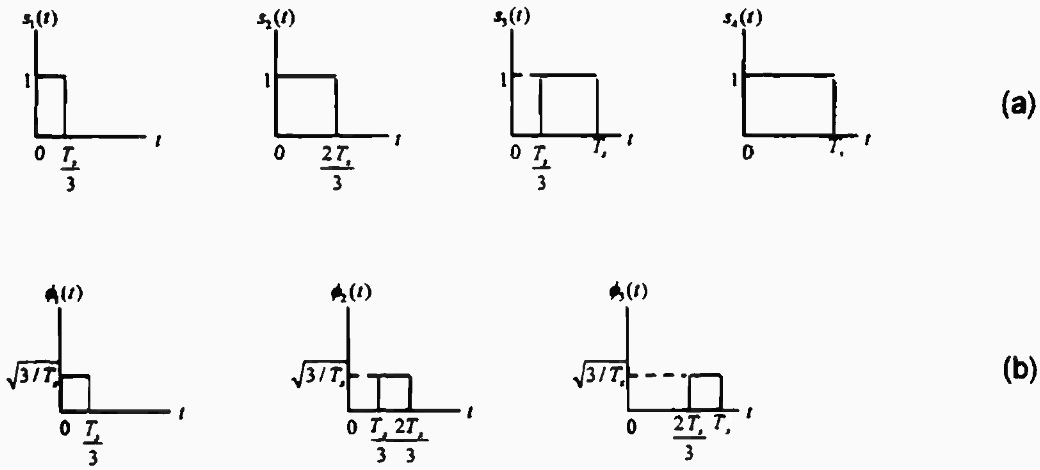


Fig. (6.15) Ex 6.2

Using eqn. (6 – 30),

$$\phi_3(t) = \frac{\psi_3(t)}{\sqrt{\int_0^{T_s} \psi_3^2(t)\, dt}}$$

$$= \begin{cases} \sqrt{3/T_s} & 2T_s/3 \le t \le T_s \\ 0 & elsewhere \end{cases}$$

For $i = 4$, we find $\psi_4(t) = 0$ and the orthogonalization process is complete.

The three basis functions $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$ form an orthogonal set. In this example, we have $M = 4$, $N = 3$, which means that the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$ do not form a linearly independent set. We should note by inspection that indeed $s_4(t) = s_1(t) + s_3(t)$. In conclusion, any of the four signals in this example can be expressed as a linear combination of three basis functions. This will make the detector function much easier, dealing only with basis functions rather than with arbitrarily varying waveforms.

6.5 Noise and the Signal Space:

We can use the notation of signal vectors $\{\bar{s}_i\}$ or signal waveforms $\{s_i(t)\}$ as convenient. A typical detection system may be viewed in terms of the reference signal vectors as shown (Fig. 6.16). Vector $\hat{\vec{s}}_i$ denotes the guessed vector of $\vec{s}_i$ in the receiver representing signals belonging to the set of M waveforms $\{s_i(t)\}$.

The receiver knows a priori the location in the signal space of each reference vector belonging to the set $\{\hat{s}_i(t)\}$. During the transmission of any signal, the signal is perturbed by noise, so that the resultant received vector is $(\hat{\vec{s}}_k + \vec{n})$, where $\vec{n}$ represents a noise vector in the $\{\phi_i\}$ space. The noise is additive and has Gaussian distribution, therefore, the resulting distribution of possible received signals is a cluster or a cloud of points around $\hat{\vec{s}}_i$. The cluster is dense in the center, and becomes sparse with increasing distance from the reference vector. The vector $\vec{r}$ represents a signal vector that might arrive at the receiver during a symbol interval. The task of the receiver is to decide whether $\vec{r}$ belongs to (is closer to) the reference $\hat{\vec{s}}_i$, i.e., one of the reference vectors in the M vector set (usually called M-ary set). The decision can be made based on a distance measurement. All demodulation or detection schemes involves the concept of distance measurement between a received signal vector $\vec{r}$ and a known set of possible transmitted vectors (or waveforms). The aim of this measurement is to decide which vector in the set $\{\hat{\vec{s}}_j\}$ is the nearest neighbor to $\vec{r}$. We can see then from Fig. (6.16) that the signal vector axes of the M dimensional space will intercept only projections of the noise along these axes, i.e., the vector signals will pick up only a small part of the noise associated with the reference signals. We must remember to start with that AWGN can be expressed as a linear combination of orthogonal waveforms in the same way as signals. For the signal detection problem, the noise can be partitioned as

$$n(t) = \hat{n}(t) + \tilde{n}(t) \qquad (6-36)$$

where $\hat{n}(t)$ is the projection of the noise components on the signal vector axes $\{\phi_j(t)\}$ or along the reference vectors $\{\hat{\vec{s}}_j\}$ in the signal space in the receiver and is given by

$$\hat{n}(t) = \sum_{j=1}^{N} n_j \ \phi_j(t) \qquad (6-37)$$

when n_j is given by

$$n_j = \frac{1}{K} \int_0^{T_s} \hat{n}(t) \ \phi_j(t) dt \qquad (6-38)$$

In eqn. (6 – 36) $\tilde{n}(t)$ is the noise outside the reference vectors

$$\tilde{n}(t) = n(t) - \hat{n}(t) \qquad (6-39)$$

We should note that

$$n(t) = \sum_{j=1}^{N} n_j \; \phi_j(t) + \tilde{n}(t) \qquad (6-40)$$

Hence $\tilde{n}(t)$ is not picked up by the detector. Thus $\hat{n}(t)$ represents only the noise that will interfere with the detection process, which constitutes the noise vector set

$$\left\{\bar{\hat{n}}\right\} = (\hat{n}_1, \hat{n}_2 \ldots \hat{n}_n) \qquad (6-41)$$

where $\bar{\hat{n}}$ is a random vector with zero mean and Gaussian distribution and the noise components $(\hat{n}_1, \hat{n}_2 \ldots \hat{n}_n)$ are independent. Therefore,

$$\int_0^{T_s} \tilde{n}(t) \; \phi_j(t) dt = 0 \text{ for all } j \qquad (6-42)$$

Hence eqn. (6 – 38) reduces to

$$n_j = \frac{1}{K_j} \int_0^{T_s} n(t) \; \phi_j(t) dt \text{ for all } j \qquad (6-43)$$

For orthonormal system $K_j = 1$

$$n_j = \int_0^{T_s} n(t) \; \phi_j(t) dt \text{ for all } j \qquad (6-44)$$

6.6 Distance Measurement:

In the signal space, we can define each vector in the N dimensional Euclidean space as

$$\bar{s}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{iN} \end{bmatrix}, \; i = 1, \ldots M \qquad (6-45)$$

The tips of such vectors form a set of M points in the, N dimensional Euclidean space, with N mutually perpendicular functional (virtual) axes $\phi_1, \phi_2 \ldots \phi_N$ which constitute the signal space.

In the N dimensional Euclidean space of the receiver, we may define the lengths of vectors and angles between vectors. It is customary to denote the length (or absolute value or norm) of a signal vector $\hat{s}_i$ by the symbol $\|\hat{s}\|$. The square length of any signal vector $\hat{s}$ is defined as the dot product of $\hat{s} \bullet \hat{s}$ given by

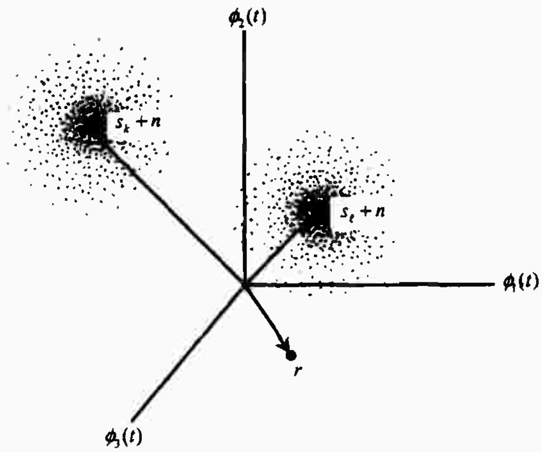


Fig. (6.16) Noise in signal space around two states s_k and s_ℓ

$$\left\|\hat{\bar{s}}_i\right\|^2 = \sum_{j=1}^{N} a_{ij}^2 \qquad (6-46)$$

The cosine of the angle between two vectors is defined as the dot product of the two vectors divided by their individual norms. If such angle is θ_{ij}

$$\cos\theta_{k\ell} = \frac{\hat{\bar{s}}_k \bullet \hat{\bar{s}}_\ell}{\left\|\hat{\bar{s}}_k\right\| \left\|\hat{\bar{s}}_\ell\right\|} \qquad (6-47)$$

The two vectors $\hat{\bar{s}}_k$, $\hat{\bar{s}}_\ell$ are orthogonal if their dot product is zero. Comparing eqn. (6 – 15) and eqn. (6 – 47), we see that the energy of a signal is equal to the squared length of the signal vector representing it. In the case of a pair of signals $s_k(t)$ and $s_\ell(t)$, the Euclidean distance $d_{k\ell}$ is given by

$$d_{k\ell}^2 = \left\|\hat{\bar{s}}_k - \hat{\bar{s}}_\ell\right\|^2 = \sum_{j=1}^{N}\left(a_{kj} - a_{\ell j}\right)^2 \qquad (6-48)$$

which can be shown (prob. 6.4) to be

$$\int_0^{T_s} \left[s_k(t) - s_\ell(t)\right]^2 dt \qquad (6-49)$$

6.7 Correlators' Response to Input Noise:

In Fig. (6.14), the input signal $s_i(t)$ was applied to a bank of multipliers-integrators. Each may be called correlator. The output of each correlator is

$a_{ij}, j = 1...N$ giving the components of $s_i(t)$ along the basis functions $\{\phi_j(t)\}$. Now, suppose that the input to the bank of N correlators is not the transmitted signal $s_i(t)$ but rather the received signal $r_d(t)$. Thus,

$$r_d(t) = s_i(t) + n(t) \qquad \begin{cases} 0 \le t \le T_s \\ i = 1,... M \end{cases} \qquad (6-50)$$

where $n(t)$ has zero mean and PSD of $\frac{kT}{2}$

The output of correlator j is the sample value of a random variable r_j which is the j^{th} projection of the vector $\overrightarrow{r}$

$$r_j = \int_0^{T_s} r(t)\ \phi_j(t)\,dt \qquad (6-51)$$

$$= a_{ij} + n_j \qquad (6-52)$$

where a_{ij} is a deterministic quantity contributed by the transmitted signal $s_i(t)$ and is given according to eqn. (6 – 14) by

$$a_j = \int_0^{T_s} s_i(t)\ \phi_j(t)\,dt \qquad (6-53)$$

and n_j is the sample of the random variable along the j^{th} axis that arises because of the presence of noise at the receiver input and is given according to eqn. (6 – 38) by

$$n_j = \int_0^{T_s} n(t)\ \phi_j(t)\,dt \qquad (6-54)$$

Consider next a new random variable $\tilde{r}_d$ which is related to the received signal $r_d(t)$ by $\tilde{r}_d = r_d(t) - \hat{n}(t)$ (6 - 55)

Thus

$$\tilde{r}_d(t) = r_d(t) - \sum_{j=1}^{N} r_j\ \phi_j(t) \qquad (6-56)$$

Substituting eqns. (6 – 50), (6 – 51) and (6 - 52) into eqn. (6 – 56) and using eqn. (6 – 12), then

$$\tilde{r}_d(t) = s_i(t) + n(t) - \sum_{j=1}^{N}\left(a_{ij} + n_j\right)\phi_j(t)$$

$$= n(t) - \sum_{j=1}^{N} n_j\ \phi_j(t) = \tilde{n}(t) \qquad (6-57)$$

where use is made of eqns. (6 – 40) and (6 – 37)

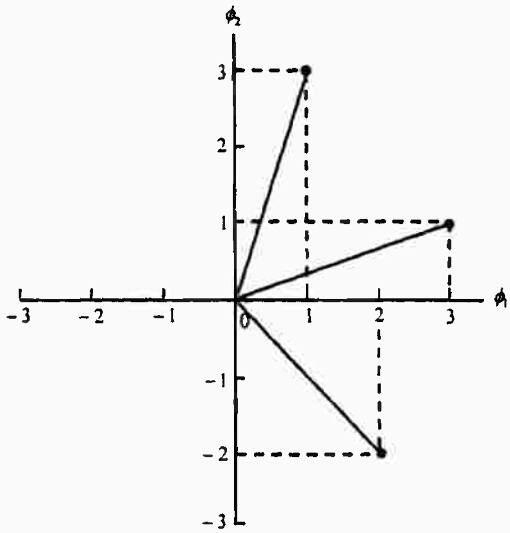


Fig. (6.17) Signal space $N = 2$, $M = 3$

The sample function $\tilde{r}_d(t)$ therefore depends only on the noise $n(t)$ at the front end of the receiver, but not at all on the transmitted signal $s_i(t)$ i.e. there is no in formation in it. According to eqns. (6 – 55) and (6 – 56), we may express the received signal as

$$r_d(t) = \sum_{j=1}^{N} r_j \ \phi_j(t) + \tilde{r}_d(t) \qquad (6-58)$$

$$= \sum_{j=1}^{N} r_j \ \phi_j(t) + \tilde{n}(t) \qquad (6-59)$$

Using eqns. (6 – 52) and (6 – 12),

$$r_d(t) = \sum_{j=1}^{N} \left(a_{ij} + n_j\right)\phi_j(t) + \tilde{n}(t) \qquad \begin{matrix} i = 1, \dots M \\ N \leq M \end{matrix} \qquad (6-60)$$

This is the received signal analyzed into projections in the signal space involving both the information $a_{ij} \ \phi_j(t)$ and the projected noise along the axes of the signal space $n_j \ \phi_j(t)$, and a remainder noise term $\tilde{n}(t)$, which is outside the axes altogether and does not affect the correlator outputs. Eqn. (6 – 59) is to be compared with eqn. (6 – 12) repeated here for convenience

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\ \phi_j(t) \qquad \begin{matrix} i = 1, \dots M \\ N \le M \end{matrix} \qquad (6-61)$$

The expansion of eqn. (6 – 61) is deterministic, whereas that in eqn. (6 – 60) is random due to noise addition n_j and $\tilde{n}$. Since $r_d(t)$ is Gaussian, it follows from eqn (6 – 52) that $\hat{r}(t)$ is also Gaussian. Hence, r_j is characterized by its mean and variance. Both $n(t)$ and $\hat{n}(t)$ have zero mean. Thus,

$$\begin{aligned} \mu_{r_j} &= E[r_j] \\ &= E[a_{ij} + n_j] \\ &= a_{ij} \end{aligned} \qquad (6-62)$$

To find the variance of r_j, we note

$$\begin{aligned} \sigma_{r_j}^2 &= \operatorname{var}[r_j] \\ &= E[(r_j - a_{ij})^2] \end{aligned} \qquad (6-63)$$

Using eqn. (6 – 52)

$$\sigma_{r_j}^2 = E[n_j^2] = \lim_{t_1 \to t_2} E[n_j(t_1) n_j(t_2)] \qquad (6-64)$$

This can be rewritten as

$$\sigma_{r_j}^2 = \lim_{t_1 \to t_2} E\left[\int_0^{T_s} n_j(t_1)\phi_j(t_1)dt_1 \int_0^{T_s} n_j(t_2)\phi_j(t_2)\,dt_2\right] \qquad (6-65)$$

Thus

$$\sigma_{r_j}^2 = \lim_{t_1 \to t_2} E\left[\int_0^{T_s}\int_0^{T_s} \phi_j(t_1)\ \phi_j(t_2)\ n_j(t_2)\ n_j(t_1)\ dt_1\ dt_2\right] \qquad (6-66)$$

Interchanging the order of integration and expectation

$$\sigma_{r_j}^2 = \lim_{t_1 \to t_2} \int_0^{T_s}\int_0^{T_s} \phi_j(t_1)\ \phi_j(t_2)\ E[n_j(t_2)\ n_j(t_1)]dt_1\ dt_2 \qquad (6-67)$$

But we know that

$$E[n_j(t_1)\ n_j(t_2)] = R_n(t_1, t_2) \qquad (6-68)$$

where $R_n(t_1, t_2)$ is the autocorrelation function of the noise $n_j(t_2)$. Since noise is stationary $R_n(t_1, t_2)$ depends only on the time difference $(t_1 - t_2 = \tau)$. Furthermore, since the noise is white, i.e., it has a constant $PDS = \frac{kT}{2}$, we have from eqn. (5 – 32).

$$R_n(t_1, t_2) = \frac{kT}{2}\delta(t_1, t_2) = \frac{kt}{2}\delta(\tau) \qquad (6-69)$$

Substituting eqn. (6 – 69) into eqn. (6 – 67),

$$\sigma_{r_j}^2 = \frac{kT}{2}\int_0^{T_s}\int_0^{T_s} \phi_j(t_1)\ \phi_j(t_2)\ \delta(t_1,t_2)dt_1\ dt_2 \qquad (6-70)$$

Using the sifting property of the delta function, eqn. (6 – 70) becomes

$$\sigma_{r_j}^2 = \frac{kT}{2}\int_0^{T_s} \phi_j^2(t)\,dt \qquad (6-71)$$

Since $\phi_j(t)$ has unit energy according to eqn. (6 – 12), then

$$\sigma_{r_j}^2 = \frac{kT}{2} \text{ for all } j \qquad (6-72)$$

This result is quite important. It tells us that all the correlation outputs r_j $j = 1, \ldots N$ have variance $\frac{kT}{2}$, which is the PSD of the noise $n(t)$. Moreover, since $\{\phi_j(t)\}$ form an orthogonal set, we find that r_j components are mutually uncorrelated. Invoking eqn. (4 – 6), we calculate

$$\operatorname{cov}[r_k, r_\ell] = E\left[(r_k - \mu_{r_k})(r_\ell - \mu_{r_\ell})\right] \qquad (6-73)$$

From eqn. (6 – 62)

$$\operatorname{cov}[r_k, r_\ell] = E\left[(r_k - a_{ik})(r_\ell - a_{i\ell})\right] \qquad (6-74)$$

From eqn. (6 – 52),

$$\operatorname{cov}[r_k, r_\ell] = E[n_k\ n_\ell] \qquad (6-75)$$

$$= E\left[\int_0^{T_s} n_k(t_1)\ \phi_k t_1\, dt_1 \int_0^{T_s} n_\ell(t_2)\ \phi_\ell(t_2) dt_2\right] \qquad (6-76)$$

$$= \int_0^{T_s}\int_0^{T_s} \phi_k(t_1)\ \phi_\ell(t_2)\ R_n t_1, t_2\, dt_1\, dt_2 \qquad (6-77)$$

$$= \frac{kT}{2}\int_0^{T_s}\int_0^{T_s} \phi_k(t_1)\ \phi_k(t_2)\delta(t_1 - t_2)\, dt_1\, dt_2 \qquad (6-78)$$

Using eqn. (6 – 10)

$$\operatorname{cov}[r_k, r_\ell] = \frac{kT}{2}\int_0^{T_s} \phi_k(t_1)\phi_\ell(t_1) dt_1 \qquad (6-79)$$

$$= 0 \quad j \neq k,\ k \neq \ell \qquad (6-80)$$

Thus, r_j's are uncorrelated Gaussian variables, and hence, are statistically independent. Now we define the vector of N random variables

$$\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \qquad (6-81)$$

whose elements are independent random variables with mean values equal to a_{ij} and variance equal to $\frac{kT}{2}$. Since the elements of the vector $\vec{r}$ are statistically independent, we may express the conditional *pdf* of the vector $\vec{r}$ given that the signal $s_i(t)$ (or the symbol m_i) was transmitted as the product of the conditional *pdfs* of its individual elements

$$f_{\vec{r}_d}\left(\vec{r}\,|m_i\right) = \prod_{j=1}^{N} f_{r_j}\left(r_j\,|m_i\right) \qquad i = 1,\dots M \qquad (6-82)$$

where Π denotes multiple product. The vector $\vec{r}$ is called the observation vector and r_j is called the observable element. The conditional *pdf* $f_{\vec{r}_d}\left(\vec{r}_d\,|m_i\right)$ for each transmitted massage $m_i, i = 1\dots M$ are called likelihood functions or channel transition probabilities. Since each r_j is a Gaussian random variable with mean a_{ij} and variance $\frac{kT}{2}$, we have

$$f_{r_j}\left(r_j\,|\,m_i\right) = \frac{1}{\sqrt{\pi kT}}\, e^{-\frac{1}{kT}(r_j - a_{ij})^2} \qquad \begin{matrix} j = 1,\dots N \\ i = 1,\dots M \end{matrix} \qquad (6-83)$$

Therefore, substituting eqn. (6 – 83) into eqn. (6 – 82), we find that the likelihood functions of an AWGN channel are defined by

$$f_{\vec{r}_j}\left(\vec{r}_j\,|m_i\right) = (\pi kT)^{-N/2}\, e^{-\frac{1}{kT}\sum_{j=1}^{N}(r_j - a_{ij})^2} \qquad i = 1,\dots M \qquad (6-84)$$

In eqn. (6 – 59), the first term is completely determined by the elements of the vector $\vec{r}$ given by eqn. (6 – 81). The noise term $\tilde{n}(t)$ depends only on the original noise $n(t)$. Thus, $\tilde{n}(t)$ is zero mean Gaussian. We note that the random variable $\tilde{n}(t_k)$ is in fact statistically independent of the set of random variables $\{r_j\}$, (Prob. 6.5), i.e.,

$$E\left[r_j\, \tilde{n}\left(t_{t_k}\right)\right] = 0 \qquad j = 1,\dots N \qquad (6-85)$$

$$E\left[a_{ij}\tilde{n}\left(t_k\right)\right] = 0 \qquad 0 \le t_k \le T_s \qquad (6-86)$$

Thus, the random variables $\tilde{n}(t)$ are independent of the set of random variables $\{r_j\}$ and the set of transmitted signals $\{s_i(t)\}$, then such a random variable is irrelevant to the decision on the signal m_i that was transmitted. Only the correlator outputs a_{ij} and noise projections r_j are relevant in the decision process and contain all the information in the observation vector $\vec{r}$.

6.8 Maximum Likelihood Decoder:

Now we assume that within time T_s one of M possible signals $s_1(t) \ldots s_M(t)$ is sent with equal probability, i.e., $1/M$. When this transmitted signal is applied to a bank of correlators with a common input and supplied with a set of N orthonormal basis functions, the resulting correlator outputs define the signal vector $\hat{\vec{s}}_i$. we may represent $s_i(t)$ by a point in the signal space of dimension $N \le M$, we refer to this point as the message point. The set of message points corresponding to the set of transmitted signals $\{s_i(t)\}$ $i = 1 \ldots M$ is called a signal constellation. However, the outputs of the correlators are not purely $\{s_i(t)\}$ but $\{r_d(t)\}$ where noise $\{n(t)\}$ is added to $\{s_i(t)\}$, i.e. the observation vector $\vec{r}$ is given by

$$\vec{r} = \hat{\vec{s}}_i + \hat{\vec{n}} \qquad (6-87)$$

This vector $\vec{r}$ is the representation of $\{r_d(t)\}$ in the signal space. The noise vector $\hat{\vec{n}}$ represents that portion of the noise $n(t)$ that will interfere with the detection process. The remaining portion of noise $\tilde{n}(t)$ is tuned out by the correlators. Thus, the vector $\vec{r}$ space is called the received signal point. The received signal point wanders about the message point in a completely random fashion within a Gaussian distributed cloud centered on the message point (Fig. 6.18).

The detector's task is this: for a given observation vector $\vec{r}$, we have to perform a mapping from $\vec{r}$ to an estimate $\hat{m}$ of the transmitted symbol m_i in a way that would minimize the probability of error in the decision making process, assuming that all the M transmitted symbols are equally likely. Given the observation vector $\vec{r}$, we make the decision $\hat{m} = m_i$. The probability of error $P_e(m_i, \vec{r})$ represents the event that false judgment is made namely that m_i was thought to be received whereas it was not sent, while the vector received is $\vec{r}$,i.e, we note $|\vec{r}|$ is the value which the decoder has to assign to the closest m_i

$$P_e(m_i, |\vec{r}|) = P(m_i \ not \ transmitted \,\|\vec{r}|)$$

$$= 1 - P(m_i \ transmitted \,\|\vec{r}|) \qquad (6-88)$$

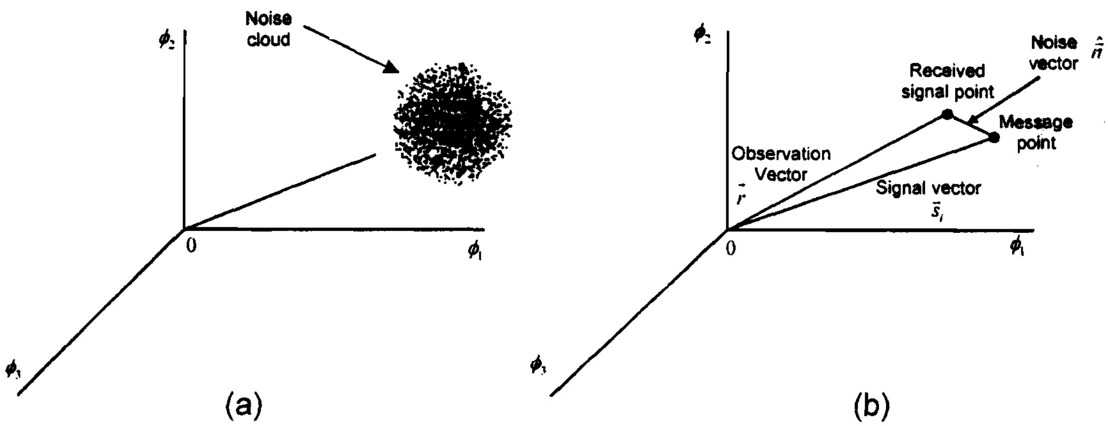


Fig. (6.18) Noise perturbation in the signal space
a) Noise cloud *b) received signal point and message point*

The decision making criterion is to minimize this probability of error for each given observation vector $\bar{r}$. The optimum decision rule states

Set $\hat{m} = m_i$ if

$$P\left(m_i\ transmitted \| \bar{r}|\right) \geq P\left(m_k\ transmitted \| \bar{r}_j|\right) \quad (8-89)$$

for all $k \neq i$, $k = 1, \ldots M$

This is called the maximum a posteriori probability rule.

The condition of eqn. (6 – 89) may by expressed in terms of the a priori probabilities of the transmitted signals and in terms of the likelihood functions. We may rewrite eqn. (6 – 89) as

Set $\hat{m} > m_i$ if

When $P\left(m_k\ transmitted |\bar{r}|\right)$ is maximum set $k = i$

Using Bayes rule,

$$f\left(m_k \| \bar{r}|\right) = \frac{f\left(|\bar{r}|\,|m_k\right) P\left(m_k\right)}{f\left(\bar{r}\right)} \text{ is maximum then set } k = i \quad (6-90)$$

where $P(m_k)$ is the a priori probability of occurrence of symbol m_k, $f\left(|\bar{r}_j|\,|m_k\right)$ is the likelihood function that results when symbol m_k is transmitted, and $f\left(\bar{r}\right)$ is the unconditional joint pdf of the random vector $\bar{r}$. We note that the denominator of eqn. (6 – 90) is independent of the transmitted signal and that the a priori probability $P(m_k)=P(m_i)$ when all the signals are transmitted with equal probability. The decision rule (eqn. 6 – 90) becomes

Set $\hat{m}=m_i$ if

$$f\left(|\bar{r}|\,|m_k\right) \text{ is maximum for which } k=i \qquad (6-91)$$

Taking the natural logarithm of the likelihood function and calling it the metric, eqn. (6 – 91) become

Set $\hat{m}=m_i$ if

$$\ell n\left[f\left(|\bar{r}_j|\,|m_k\right)\right] \text{ is maximum for which } k=i \qquad (6-92)$$

This is called the maximum likelihood decision. The device which implements this rule is the maximum likelihood decoder which computes the metrics for all transmitted messages, compares them and then decides in favor of the maximum. To obtain a graphical representation of the maximum likelihood decision rule, let Z denote the N dimensional space of all possible observation vectors $\bar{r}$. This is called the observation space. To assign $\hat{m}=m_i$, $i=1\ldots M$, we partition the Z space into M decision regions denoted by $Z_1\ldots Z_M$. The decision rule (eqn. 6 – 92) becomes

Observation vector $\bar{r}$ lies in region Z_1 if

$$\ell n\left[f\left(|\bar{r}|\,|m_k\right)\right] \text{ is maximum for which } k=i \qquad (6-93)$$

If the observation vector $\bar{r}$ falls on the boundary between any two decision regions Z_i and Z_k either guess is valid

From (eqn.6 – 84), we have

$$f\left(|\bar{r}|\,|m_k\right)=(\pi N)^{-N/2}\, e^{-\frac{1}{N}\sum_{j=1}^{N}(r_j-a_{kj})^2} \qquad k=1\ldots M \qquad (6-94)$$

Therefore,

$$\ell n\left[f\left(|\bar{r}|\,|m_k\right)\right]=-\frac{N}{2}\ln(\pi N)-\frac{1}{N}\sum_{j=1}^{N}\left(r_j-a_{kj}\right)^2 \qquad k=1\ldots M \quad (6-95)$$

The first term in eqn. (6 – 95) is constant. Thus, the maximum likelihood decision rule becomes

observation vector $\bar{r}$ lies in region Z_i if

$$-\frac{1}{N}\sum_{j=1}^{N}\left(r_j-a_{kj}\right)^2 \text{ is maximum then set } k=i \qquad (6-96)$$

Alternatively, we may rewrite the rule as

observation vector $\bar{r}$ lies in region Z_i if

$$\sum_{j=1}^{N}\left(r_j - a_{kj}\right)^2 \text{ is minimum then set } k = i \qquad (6-97)$$

We note that

$$\sum_{j=1}^{N}\left(r_j - a_{kj}\right)^2 = \left\|\vec{r} - \hat{\vec{s}}_k\right\|^2 \qquad (6-98)$$

where $\left\|\vec{r} - \hat{\vec{s}}_k\right\|^2$ is the square of Euclidean distance between the received signal point and the message point represented by the vector $\vec{r}$ and $\hat{\vec{s}}_k$ respectively. Thus, the decision rule may be rewritten as

observation vector $\vec{r}$ lies in region Z_i if

the Euclidean distance $\left\|\vec{r} - \hat{\vec{s}}_k\right\|$ is maximum for $k = i$ (6 – 99)

This is the rule is to choose the message point closest to the received signal point
We note that

$$\sum_{j=1}^{N}\left(r_j - a_{kj}\right)^2 = \sum_{j=1}^{N} r_j^2 - 2\sum_{j=1}^{N} r_j\, a_{kj} + \sum_{j=1}^{N} a_{kj}^2 \qquad (6-100)$$

The first summation term is independent of the index k, and therefore may be ignored. The second term is the dot product of the observation vector $\vec{r}$ and the signal vector $\hat{\vec{s}}_k$. The third summation term: is the energy of the transmitted signal $\vec{s}_k$. Thus, the decision rule may be rewritten as

observation vector $\vec{r}$ lies in region Z_i if

$$\sum_{j=1}^{N} r_j\, a_{kj} - \frac{1}{2}E_k \text{ is minimum then set } k = i \qquad (6\text{-}101)$$

where E_k is the energy of the signal $s_k(t)$,i.e.

$$E_k = \sum_{j=1}^{N} a_{kj}^2 \qquad (6-102)$$

Fig. (6.19) shows an example of decision regions for and $N = 2$ assuming that the signals are transmitted with equal energy E and equal probability. A vector $\vec{r}$ lies in region Z because $\left\|\vec{r} - \hat{\vec{s}}_1\right\|$ is minimum.

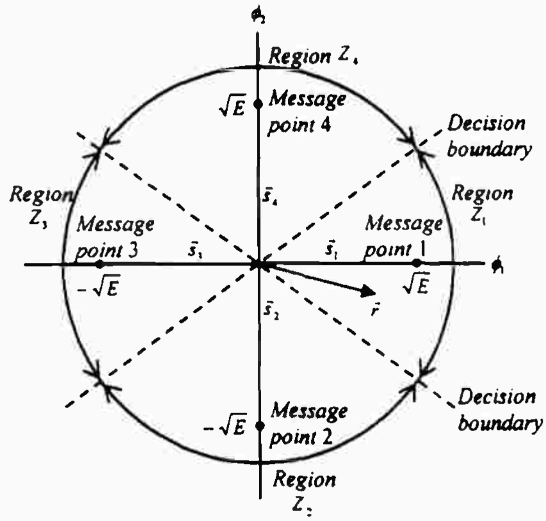


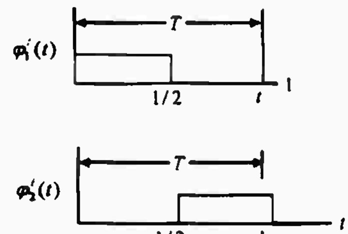
Fig. (6.19) Partioning of the observation space into decision regions when $N = 2$, $M = 4$

Problems

1. Resolve Ex 6.1 with ϕ_1' , ϕ_2' as shown

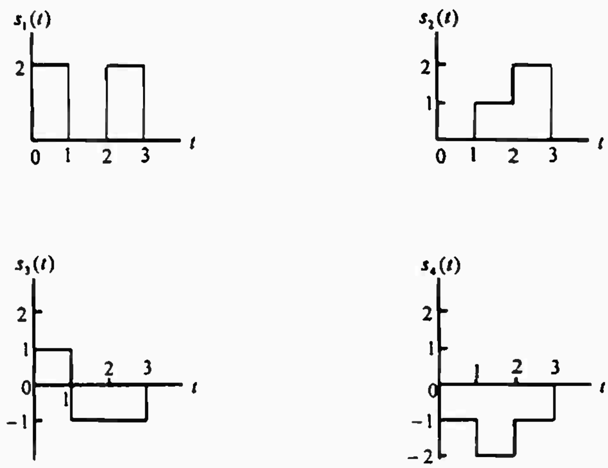
2. Determine the orthonormal basis for the two signals shown.

3. Given the four signals shown, find are orthonormal basis.



Prob. 6.1

Prob. 6.2



Prob. 6.3

4. Verify eqn. (6 – 42)

5. Verify eqns. (6 – 76) and (6 – 77)

6. For the basis functions of Ex 6.2, express the vectors $\vec{s}_1$,$\vec{s}_2$, $\vec{s}_3$, $\vec{s}_4$. Find the angles and distances between each pair of vectors.

7. In the problem above white noise is added to the received signal, with $kT = 25\,meV$. Find the noise components along the axes of the basis functions. What is the variance of the detected noise?

8. Calculator $\hat{n}$ and $\tilde{n}$ in the above problem.

9. In problem 7 sketch the constellation diagram and show the decision boundaries. Find the range of $\vec{r}$ to which Z_1 is assigned.

10. In problem 7 apply the maximum likelihood criteria to decide on a vector making an angle 130° with ϕ_1

References

1. "Communication Systems," S. HayKin, 3rd ed, J. Wiley, N.Y. 1994.
2. "Digital Communication," B. Sklar, 2nd ed, Prentice Hall, Upper Saddle River, N.J. 2001.
3. "Telecommunications Demystified," C. Nassar, LLH Technology Publishing, Eagle Rock, Virginia, 2001.
4. "Digital Communication," I. Glover, P. Grant, Prentice Hall, London, 1998.
5. "Noise," F. Connor, 2nd ed., Arnold, London, 1982.