CHAPTER 7
Optimum Baseband Receiver

7.1 Digital Figure of Merit:

In digital communications, we use a normalized version of S/N namely, E_b/η where E_b is the bit energy, which is the signal power S times the bit time T_b and η is the noise PSD which is the noise power N divided by the bandwidth B. Since the bit rate R_b is $1/T_b$, we have

$$\frac{E_b}{\eta}=\frac{ST_b}{N/B}=\frac{S/R_b}{N/B}=\frac{S}{N}\left(\frac{B}{R_b}\right)=\frac{S}{\eta R_b} \qquad (7-1)$$

where $\eta = kT$, $N = kT/2\times 2B = kTB = \eta B$. Eqn. (7 – 1) is one of the most important metrics of performance. The smaller the required E_b/η , the more efficient is the detection process for a given probability of error. We may discuss here why we needed a new figure of merit different from the familiar S/N . In analog communication, we deal with waveforms of infinite duration. This means that such waveforms have an infinite amount of energy. We call such signal a power signal. However, in digital communication, we transmit and receive a symbol, by using a transmission waveform within a window of time (symbol time T_s). Focusing on one symbol the power averaged over all time goes to zero. Hence, we need energy rather than power to describe a digital waveform. Also, in digital communication, the information is quantized on the bit level. The figure of merit must measure the performance per bit independently of the message length.

We note that like S/N , E_b/η is dimensionless, because E_b is in Joule (watt second) and η is watt per Hz (watt second). We shall find that the average probability of symbol error in a binary encoded PCM receiver due to additive white Gaussian noise depends solely on E_b/η . We shall derive shortly the relevant formulas for the probability of error. It can be shown for NRZ signaling that the values of probability of error as a function of E_b/N_0 can be tabulated as in table (7.1) for a bit rate of $10^5\, b/s$. From the table, it is clear that there is an error threshold $(11dB)$. For E_b/η below the error threshold the receiver suffers significant errors, and above it the noise effect is negligible. When, however, E_b/η drops below the error threshold there is a sharp increase in the error rate.

Table 7.1 Influence of E_b/η on the probability of error for bit rate $10^5\ b/s$

$E_b/\eta\ dB$	P_e	1 error every
4.3	10^{-2}	$10^{-3}\ s$
8.4	10^{-4}	$10^{-1}\ s$
10.6	10^{-6}	$10 s$
13	10^{-8}	10 minutes
13	10^{-10}	1 day
14	10^{-12}	3 months

Decision errors result in the construction of incorrect codewords, and hence, the reconstructed message at the receiver output may be quite different from the original message. The error threshold in PCM using NRZ signaling is $11 dB$, this is to be compared with the $60-70 dB$ required for high quality transmission of speech using AM. Hence, we see that PCM requires much less power even though the noise power in PCM is increased n fold due n fold increase in bandwidth, where n is the number of bits in the code word. This is really the strength of digital communication.

We also note that - unlike analog transmission where repeaters needed to counteract distance attenuation may add noise and distortion to the signal, where repeaters in a digital link regenerate the signal, rather than amplify it. Because the signal (in bit form) may be regenerated anew, the effects of amplitude, phase and nonlinear distortions in one link have practically no effect on the regenerated signal in the next link. We have also seen that the effect of channel noise can be made practically negligible by using the ratio E_b/η above threshold. Thus, for all practical purposes the performance of PCM link is almost independent of the physical length of the communication channel. It is interesting for those who have background in biology to note that, this situation is similar to pulse transmission down the axon of a neuron. The pulses are regenerated as they propagate down the axon, making distance attenuation virtually absent. At the synapses, the pulses are transformed from electrical form to chemical form and back to electrical. Thus, the sensation of pain in the brain is independent of the distance the pulse has to travel, since attenuation is practically eliminated.

Another important characteristic of a PCM system is its ruggedness to interference caused by a strong impulse or crosstalk. By providing an adequate margin over the error threshold, the system can withstand the presence of relatively large amounts of interference. Additionally, PCM provides secure communication by using special modulation schemes or encryption. The increased bandwidth does not pose a real disadvantage, since use is made of data compression, and also due to the availability of wide band links, such as satellite and optical fibers. Therefore, exchanging bandwidth for improved S/N in PCM does not bear a heavy cost. The issue of coding and compression will be treated in chapter 15 on information theory.

7.2 The Likelihood Ratio Test:

The signal source at the transmitter consists of a set $\{s_i(t)\}$ $i = 1 \ldots M$ of waveforms. A signal waveform $r_d(t) = s_i(t) + n(t)$ is received. At the receiver, the waveform is reduced to a single number $z(T_s)$, where T_s is the symbol duration. This value is compared with a threshold γ according to eqns. (6 – 8) and (6 – 9) which may be rewritten for binary PCM as

$$z(T_s) > \gamma \text{ for } \hat{s}_1 \qquad (7-2)$$

$$z(T_s) < \gamma \text{ for } \hat{s}_2$$

where s_1 and s_2 are the two possible binary states. If $z(T_s) = \gamma$ the decision is arbitrary. Choosing s_1 is equivalent to deciding that signal $s_1(t)$ was sent and hence a binary 1 is detected. Choosing s_2 is equivalent to deciding that signal $s_2(t)$ was sent, and hence, a binary 0 is detected. Fig (. 7.1) shows the components of the decision process for a case where we have a set of states $s_1, \ldots s_M$.

Because the noise is Gaussian and the receiver is linear the output is also Gaussian and $z(T_s)$ is a continuous random variable.

$$z(T_s) = A_k + n_o(T_s) \qquad (7-5)$$

Thus, the sample $z(T_s)$ is made up of a single component A_k and a noise component $n_o(T_s)$. At each $q\,T_s$ where q is an in integer the receiver uses a decision rule for deciding which signal class has been received. For simplicity, we may rewrite eqn. (7 – 3) as

$$z = A_k + n_0 \qquad (7-4)$$

where T_s dependence is implicit

We may start by establishing the receiver decision rule for the case of binary signals. The decision making step of eqn. (7 – 2) can be reformulated as

$$P(\hat{s}_1|z) \underset{\hat{s}_2}{\overset{\hat{s}_1}{\gtrless}} P(\hat{s}_2|z) \qquad (7-5)$$

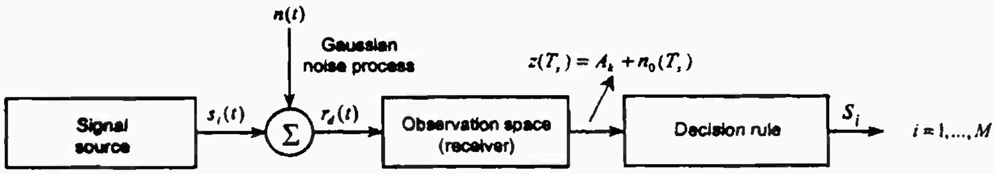


Fig. (7.1) Components of the decision process of M-ary PAM

Eqn. (7 – 5) means that we should choose state s_1 if the a posteriori probability $P(\hat{s}_1|z)$ is greater than the a posteriori probability $P(\hat{s}_2|z)$, otherwise we should choose state H_2. Using Bayes theorem

$$P(\hat{s}_i|z) = \frac{f(z|s_i)\,P(s_i)\,dz}{f(z) \quad dz} = \frac{f(z(s_i)P(s_i)}{f(z)} \qquad (7-6)$$

where $f(s_i|z)$ is the conditional *pdf* of the received continuous valued sample z, conditional on the signal class

$$f(z) = \sum_{j=1}^{2} f(z|s_i)\,P(s_i) \qquad (7-7)$$

Thus, from eqn. (7 – 5) and (7 – 6),

$$f(z|s_1)\,P(s_1) \underset{\hat{s}_2}{\overset{\hat{s}_1}{\gtrless}} f(z|s_2)\,P(s_2) \qquad (7-8)$$

Rearranging

$$\frac{f(z|s_1)}{f(z|s_2)} \underset{\hat{s}_2}{\overset{\hat{s}_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \qquad (7-9)$$

The left hand ratio is called the likelihood ratio and eqn. (7 – 9) is called the likelihood ratio test. This equation corresponds to making a decision based on a comparison of a measurement of a received signal to a threshold. Since the test is based as choosing the signal class with maximum a posteriori probability the decision criterion is called maximum a posteriori (MAP) criterion, or the minimum error criterion, since this criterion yields the minimum number of incorrect decisions. When the apriori probabilities are equal, i.e. the classes are equally likely, eqn. (7 – 9) becomes

$$\frac{f(z|s_1)}{f(z|s_2)} \underset{s_2}{\overset{s_1}{\gtrless}} 1 \qquad (7-10)$$

Fig. (7.2) illustrates the conditional *pdfs* of the binary noise perturbed output signals $z(T_s) = A_1 + n_0$ and $z(T_s) = A_2 + n_0$ from a typical receiver. The reference values A_1 and A_2 are mutually independent and equally likely signals. The noise n_0 is an independent Gaussian random variable with zero mean and variance σ_0^2. The *pdf* is given by

$$f(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n_0^2}{\sigma_0^2}\right)} \qquad (7-11)$$

We can therefore rewrite the likelihood ratio as

$$\frac{f(z|s_1)}{f(z|s_2)} = \frac{\frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-A_1}{\sigma_0}\right)^2}}{\frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-A_2}{\sigma_0}\right)^2}} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

$$= \frac{e^{-\frac{z^2}{2\sigma_0^2}} e^{-\frac{A_1^2}{2\sigma_0^2}} e^{\frac{2zA_1}{2\sigma_0^2}}}{e^{-\frac{z^2}{2\sigma_0^2}} e^{-\frac{A_2^2}{2\sigma_0^2}} e^{\frac{2zA_2}{2\sigma_0^2}}} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

$$e^{\left[z(A_1-A_2)/\sigma_0^2 - (A_1^2-A_2^2)/2\sigma_0^2\right]} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \qquad (7-12)$$

where A_1 is the receiver output signal component when $s_1(t)$ is sent and A_2 is the receiver output signal component when $s_2(t)$ is sent. Taking the natural logarithms of both sides of eqn. (7 – 12),

$$\ell n\left[\frac{f(z|s_1)}{f(z|s_2)}\right] = \frac{z(A_1 - A_2)}{\sigma_0^2} - \frac{(A_1^2 - A_2^2)}{2\sigma_0^2} \underset{s_2}{\overset{s_1}{\gtrless}} \ell n \frac{P(s_2)}{P(s_1)} \qquad (7-13)$$

when the classes are equally likely then

$$\frac{P(s_2)}{P(s_1)} = 0 \qquad (7-14)$$

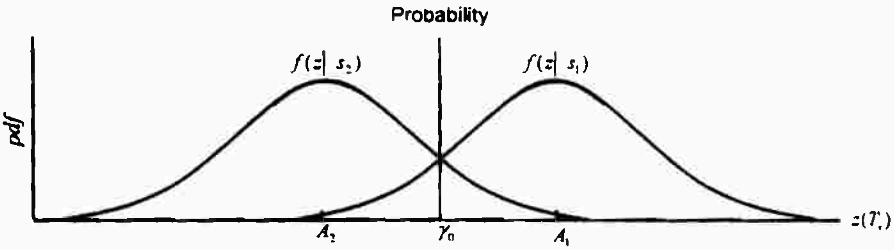


Fig. (7.2) Conditional *pdfs* **for binary receiver**

a) $\gamma_0 \neq 0$ **b)** $\gamma_0 = o$

In this case, we have

$$z \underset{s_2}{\overset{s_1}{\gtrless}} \frac{(A_1^2 - A_2^2)}{(A_1 - A_2)} = \frac{(A_1 + A_2)}{2} = \gamma_0 \qquad (7-15)$$

For NRZ (antipodal) signals $s_1(t) = -s_2(t)$ and $A_1 = -A_2$, we thus have

$$z \underset{s_2}{\overset{s_1}{\gtrless}} 0 \qquad (7-16)$$

Thus, the maximum likelihood rule for the case of equally likely antipodal signals compares the received sample to a zero threshold. This is the same as deciding $s_1(t)$ if the sample is positive and $s_2(t)$ if the sample is negative.

Ex. 7.1

Consider two classes s_1 and s_2 characterized by the triangular shaped *pdf's* $f(z|s_1)$ and $f(z|s_2)$ shown (Fig. 7.3). A signal is received which might have any value on the z axis. Develop a rule to classify received signals particularly for signals falling in the region of overlap of the two *pdfs*. Assume s_1 and s_2 are equally likely. Calculate the aposteriori probabilities.

Solution

Consider a received signals z_a and z_b

$$P(s_1) = P(s_2) = \frac{1}{2}$$

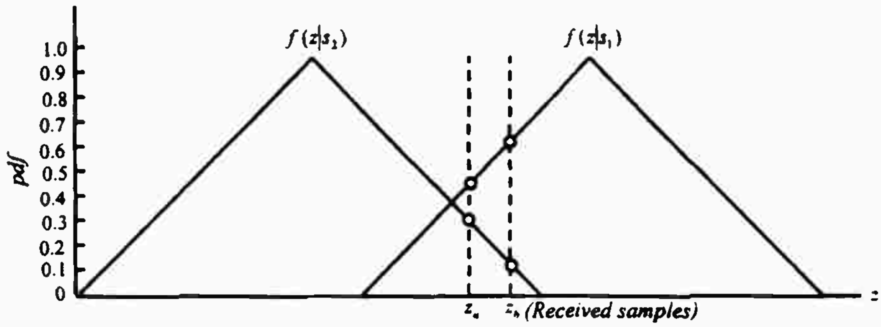


Fig. (7.3) Triangular *pdfs*

We see from Fig. (7.3) that $f(z_a|s_1) = 0.5$ and $f(z_a|s_1) = 0.3$. Thus, using eqns. (7 – 6), (7 – 7),

$$P(\hat{s}_1|z_a) = \frac{f(z_a|s_1)\,P(s_1)}{f(z_a|s_1)\,P(s_1) + f(z_a|s_2)\,P(s_2)} \qquad (7-17)$$

Thus,

$$P(\hat{s}_1|z_a) = \frac{(0.5)(0.5)}{(0.5)(0.5)+(0.3)(0.5)} = \frac{5}{8}$$

and

$$P(\hat{s}_2|z_a) = \frac{(0.3)(0.5)}{(0.5)(0.5)+(0.3)(0.5)} = \frac{3}{8}$$

Thus the received signal belongs to the class with the maximum a posteriori probability (class s_1). *For* Point z_b

$$P(\hat{s}_1|z_b) = \frac{(0.7)(0.5)}{(0.7)(0.5)+(0.1)(0.5)} = \frac{7}{8}$$

$$P(\hat{s}_2|z_b) = \frac{(0.1)(0.5)}{(0.7)(0.5)+(0.1)(0.5)} = \frac{1}{8}$$

As before the maximum likelihood rule dictates that we choose class s_1.

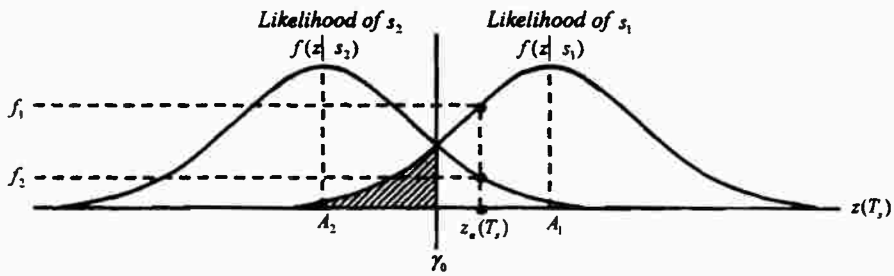


Fig. (7.4) *pdfs* **in a binary system**

7.3 Probability of Error:

For the binary decision making depicted (Fig. 7.2), there are two ways of error. An error $\in$ will occur when $s_1(t)$ is sent and the channel noise results in the receiver output signal $z(T_s)$ being less than γ_0 hence read as s_2. The probability of this event is given by

$$P(\in|s_1) = P(\hat{s}_2|s_1) = \int_{-\infty}^{\gamma_0} f(z|s_1)dz \qquad (7-18)$$

This is illustrated by the shaded area to the left of γ_0 (Fig. 7.4.). Similarly an error occurs when $s_2(t)$ is sent and the channel noise results in $z(T)$ being greater than γ_0. The probability of this event is given by

$$P(\in|s_2) = P(\hat{s}_1|s_2) = \int_{\gamma_0}^{\infty} f(z|s_2)dz \qquad (7-19)$$

The probability of error P_B is the sum of these two ways of error

$$P_e = \sum_{i=1}^{2} P(\in, s_i) = \sum_{i=1}^{2} P(\in|s_i)\, P(s_i) \qquad (7-20)$$

$$= P(\in|s_1)\, P(s_1) + P(\in|s_2)\, P(s_2) \qquad (7-21)$$

$$= P(\hat{s}_2|s_1)\, P(s_1) + P(\hat{s}_1|s_2)\, P(s_2) \qquad (7-22)$$

This means that if signal $s_1(t)$ was transmitted an error results if state $\hat{s}_2$ is chosen Also if signal $s_2(t)$ was transmitted an error results if state $\hat{s}_1$ is chosen:

For the case when the a priori probabilities are equal, i.e., $P(s_1) = P(s_2) = \frac{1}{2}$, the bit error P_B is given by

$$P_\epsilon = P_B = P(\hat{s}_2|s_1) + \frac{1}{2}P(\hat{s}_1|s_2) \qquad (7-23)$$

Because of the symmetry of *pdf*

$$P(\hat{s}_2|s_1) = P(\hat{s}_1|s_2) \qquad (7-24)$$

$$P_B = P(\hat{s}_2|s_1) = P(\hat{s}_1|s_2) \qquad (7-25)$$

The probability of a bit error P_B is numerically equal to the area under the tail of either likelihood function $f(z|s_1)$ or $f(z|s_2)$ falling on the incorrect side of the threshold. We can therefore compute P_B by integrating $P(z|s_1)$ between the limits $-\infty$ and γ_0, or by integrating $f(z|s_2)$ between the limits γ_0 and ∞ (Fig. 7.5a).

$$P_B = \int_{\gamma_0 = \frac{A_1+A_2}{2}}^{\infty} f(z|s_2)\,dz \qquad (7-26)$$

where $\gamma_0 = \frac{A_1 + A_2}{2}$ is the optimum threshold. Replacing the likelihood $f(z|s_2)$ with its Gaussian equivalent

$$f(z|s_2) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-A_2}{\sigma_0}\right)^2} dz \qquad (7-27)$$

$$P_B = \int_{\gamma_0 = \frac{A_1+A_2}{2}}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-A_2}{\sigma_0}\right)^2} dz \qquad (7-28)$$

where σ_0 here is the standard deviation of the values of z around the value A_2. This spread is due to noise, hence σ_0^2 is the variance of the noise out of the correlator. Let $u = \frac{z - A_2}{\sigma_0}$, then $\sigma_0\, du = dz$ and

$$P_B = \int_{u = \frac{A_1+A_2}{2\sigma_0}}^{u=\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \qquad (7-29)$$

$$= Q\left(\frac{A_1 - A_2}{2\sigma_0}\right) \qquad (7-30)$$

where $Q(x)$, called the complementary error function or coerror function is the probability under the tail of the Gaussian *pdf*, is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, du \qquad (7-31)$$

We note $\frac{A_1 - A_2}{2}$ is $V_P = \frac{\Delta V}{2}$ where V_P is the value of the peak of the pulse in NRZ binary system, and ΔV is peak to peak voltage, σ_0 is the standard derivation for the noise voltage, hence is the rms noise voltage, σ_0^2 is noise power for one ohm resistor.

$$V_P / \sigma_0 = \sqrt{\frac{V_P^2}{\sigma_0^2} \frac{T_b}{T_b}} \qquad (7-32)$$

But $\sigma_0^2 = kTB = kT \times \frac{1}{T_b}$ since the bandwidth is $\frac{1}{T_b}$, $V_P^2\, T_b$ is the energy in one bit, in one ohm, E_b, and kT is the thermal energy. Hence,

$$\frac{V_P}{\sigma_0} = \sqrt{\frac{E_b}{kT}} = \sqrt{\frac{E_b}{\eta}} = \sqrt{\frac{E_b}{\eta} \frac{T_b}{T_b}} = \sqrt{\frac{S_P}{N_b}} \qquad (7-33)$$

where S_p is the peak power which is energy per bit divided the bit time i.e., $\frac{E_b}{T_b}$. The noise power contained in the bandwidth corresponding to one bit pulse

$$N_b = kTB = kT \times \frac{1}{T_b} = \eta / T_b \qquad (7-34)$$

It should be noted that another form of the coerror function is frequently used

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-v^2}\, dv \qquad (7-35)$$

Thus $Q(x)$ in eqn. (7 – 31) and $erfc(x)$ in eqn. (7 – 35) are related by

$$Q(x) = \frac{1}{2}\, erfc\left(\frac{x}{\sqrt{2}}\right) \qquad (7-36)$$

Noting

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \qquad (7-37)$$

As x approaches infinity, $erf(\infty) = 1$

$$erfc(x) = 1 - erf\, x \qquad (7-38)$$

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \qquad (7-39)$$

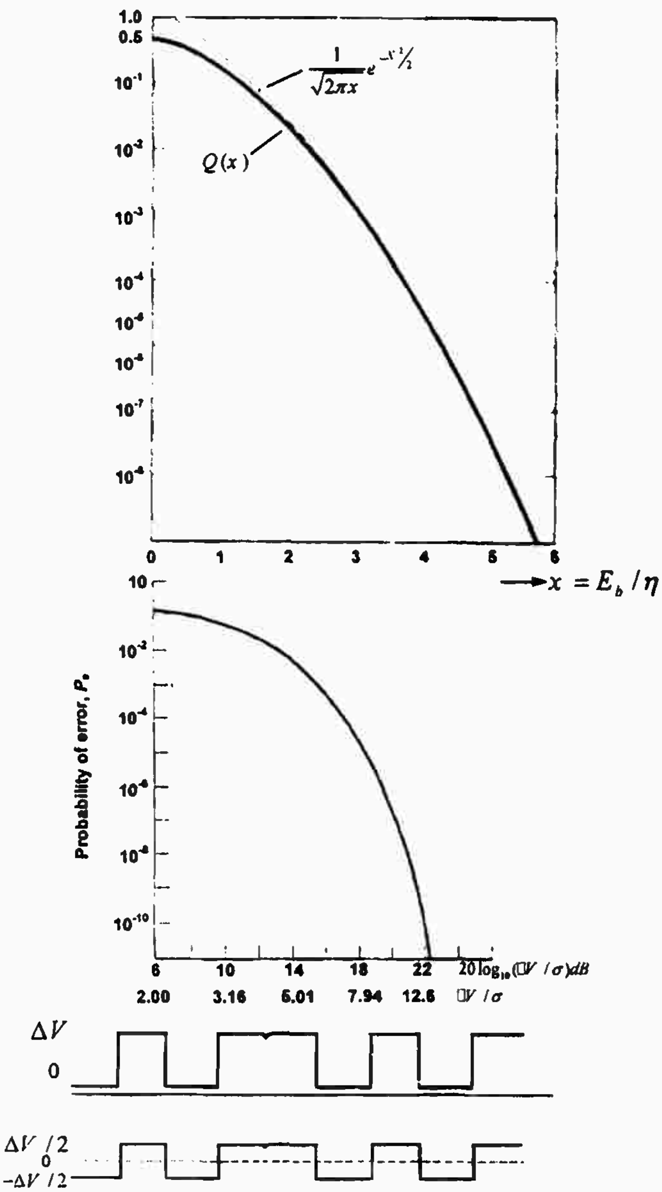


Fig. (7.5) Probability of error

a) $P_B v_s\ E_b|\eta$

b) $P_B\ v_s\ \Delta v/\sigma$

c) unipolur rectangular pulse

d) polar rectangular pulse

Thus, calling the peak to peak voltage $\Delta V = A_1 - A_2$, $u = \dfrac{A_1 - A_2}{2\sigma_0} = \dfrac{\Delta V}{2\sigma_0}$

$$P_B = \frac{1}{2} erfc\left(\frac{\Delta V}{2\sigma_0\sqrt{2}}\right) \qquad (7-40)$$

$$= \frac{1}{2}\left[1 - erf\left(\frac{\Delta V}{2\sigma_0\sqrt{2}}\right)\right] \qquad (7-41)$$

We should note that eqn. (7 – 41) is valid for both unipolar signaling (symbols represented by voltages $0, \Delta V$ and polar signaling (symbols represented by voltages $\pm\Delta V/2$). In fact, it is valid for all pulse levels and shapes provided that ΔV represents the voltage difference at the sampling instant. Fig. (7.6) shows P_B as a function of $\Delta V/\sigma_0$. Although $\Delta V/\sigma_0$ is not exactly $\sqrt{S/N}$, it involves the signal only at the sampling instant, whereas the conventional S/N uses a time averaged signal power noting that ΔV neglects any DC component. For unipolar signaling, $S_p = \Delta V^2$, $N = \sigma_o^2$

$$\frac{\Delta V}{\sigma_\circ} = \left(\frac{S_p}{N}\right)^{1/2} = \left(\frac{2s_{av}}{N}\right)^{1/2} \qquad (7-42)$$

since the average signal power $s_{av} = \frac{1}{2}S_p$ for equiprobable unipolar binary symbol. Eqn. (7 – 41) becomes

$$P_B = \frac{1}{2}\left[1 - erf\left\{\frac{1}{2}\left(\frac{s_{av}}{N}\right)^{1/2}\right.\right] \qquad (7-43)$$

For NRZ polar

$$S_p = \left(\frac{\Delta V}{2}\right)^2$$

$$\frac{\Delta V}{\sigma_\circ} = 2\left(\frac{S_p}{N}\right)^{1/2} = 2\left(\frac{s_{av}}{N}\right)^{1/2} \qquad (7-44)$$

since S_p the saverage power s_{av} for equally probable polar binary symbols, therefore, eqn. (7 – 41) becomes

$$P_B = \frac{1}{2}\left[1 - erf\left\{\frac{1}{\sqrt{z}}\left(\frac{s_{av}}{N}\right)^{1/2}\right.\right] \qquad (7-45)$$

In our case of binary symbols, the symbol error (SER) is the same as the bit error (BER). In general, if the symbol rate is R_s symbols/s and the bit rate is R_b bits/s, then

$$SER = P_B \ R_s \qquad (7-46)$$

$$BER = P_B \ R_b \qquad (7-47)$$

Ex. 7.1
Find the BER of a $10000k$ baud equiprobable binary polar rectangular pulse signaling if the measured S/N at the detector input is $12\,dB$

Solution

$S/N = 15.85$

From (Fig. 7 – 5b), $P_B = 3.45\times10^{-5}$

Using eqn. (7 – 47) $BER = 345$ bit errors/s

Ex. 7.2
Find probabilities of error in binary system for signal plus noise

Solution

Fig. (7.6a) shows the pdf of a binary information signal which can take on voltage levels V_0 and V_1 only. The pdf is of Gaussian noise (Fig. 7.66) with $rms\,\sigma$ (for zero mean the rms and σ are identical). Fig. (7.6c) shows the pdf of the sum of the signal plus noise, since adding random variable is tantamount to convolving their $pdfs$ (prob. 7-11). Fig. (7.6c) is the convolution of Fig. (7.6a,b). For equiprobable symbols, the optimum decision level is set at $(V_0+V_1)/2$. Given that the symbol 0 is transmitted (voltage V_0), the probability that the signal plus noise will be above the threshold at the decision instant is given by twice the shaded area under the curse $f_0(v_n)$ since $f_0(v_n)$ and $f_1(v_n)$ as defined here each represents a total probability of $\frac{1}{2}$. The pdf of signal plus noise conditional on a zero being transmitted is $2 f_0(v_n)$. Thus, from eqn. (7 – 27),

$$P_{e_1} = \int_{\frac{V_0+V_1}{2}}^{\infty} \frac{1}{\sigma_n \sqrt{2\pi}} e^{-(v_n - V_0)^2/2\sigma_n^2} \qquad (7-48)$$

Changing the variable $x = (v_n - V_0)/\sqrt{2}\,\sigma$.

$$P_{e_1} = \frac{1}{\sqrt{\pi}} \int_{(V_1-V_0)/2\sqrt{2}\sigma_\circ}^{\infty} e^{-x^2}\, dx \qquad (7-49)$$

Defining

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-x^2}\, dx \qquad (7-50)$$

$$= 1 - erf(z) \qquad (7-51)$$

$$P_{e_1} = \frac{1}{2} erfc\left[\frac{V_1 - V_0}{2\sqrt{2}\sigma_\circ}\right] \qquad (7-52)$$

$$= \frac{1}{2}\left[1 - erf\left(\frac{V_1 - V_0}{2\sqrt{2}\sigma_\circ}\right)\right] \qquad (7-53)$$

If digital 1 is transmitted (voltage V_1), then the probability P_{e_0} that the signal pulse noise will be below the threshold at the instant of decision is

$$P_{e_0} = \int_{-\infty}^{(V_0+V_1)/2} \frac{1}{\sigma_\circ\sqrt{2\pi}} e^{-(v_n-V_1)^2/2\sigma_\circ^2}\, dv_n \qquad (7-54)$$

P_{e_0} is identical to P_{e_1} and the total probability of error P_B irrespective of whether 1 or 0 is transmitted is equal to either P_{e_0} or P_{e_1}. The probably of error depends only on the symbol voltage difference not the absolute voltage level.

Thus,

$$P_B = \frac{1}{2}\left[1 - erf\left(\frac{\Delta V}{2\sqrt{2}\sigma_0}\right)\right] \qquad (7-55)$$

Which is similar to eqn.(7 – 41). It is valid for both unipolar and polar signaling as long as symbols are represented by voltages $\pm \Delta V/2$

Ex. 7.3

Find the probabilities of errors in $M = 4$ level system for signal plus noise.

Solution

For multi level (multi symbol) signaling we have a number of equally spaced allowed levels M, then the symbol plus noise *pdfs* are shown (Fig. 7.7) for $(M = 4)$. The probability of symbol error for $M-2$ inner symbols, i.e., any but the symbols represented by the lowest or highest voltage levels is now twice that in the binary case. This is because the symbol can be in error if the signal plus noise voltage is too high or too low

$$P_{\epsilon_M}\Big|_{inner\ symbols} = 2P_B \qquad (7-56)$$

The symbol error for the two outer levels, i.e., the symbols represented by the lowest and highest voltage levels

$$P_{\epsilon_M}\Big|_{outer\ symbols} = P_B \qquad (7-57)$$

For equiprobable symbols, the average probable symbol error is

$$P_{\epsilon_{M_{av}}} = \frac{M-2}{M} 2P_B + \frac{2}{M} P_B \qquad (7-58)$$

$$= \frac{2(M-1)P_B}{M} \qquad (7-59)$$

Using eqn. (7 – 55) defining ΔV as the voltage step between levels

$$P_{\epsilon_{M_{av}}} = \left(\frac{M-1}{M}\right)\left[1 - erf\left(\frac{\Delta V}{2\sigma\sqrt{2}}\right)\right] \qquad (7-60)$$

7.4 The Matched Filter:

We now consider optimizing the receiving filter in terms of increasing S/N at its output for a given transmitted symbol waveform. Consider that an input signal $\hat{s}_i(t)$ is known to be $s_i(t)$ plus noise $n(t)$ is inputted to a linear time invariant receiving filter followed by a sampler. We define a matched filter as the filter immediately preceding the decision circuit and is said to be matched to a particular symbol pulse if it maximizes the output S/N at the sampling instant when the pulse is present at the filter input (Fig. 7. 8).

$$z(T_s) = \hat{A}_i(T_s) + n_0(T_s) \qquad (7-61)$$

where A_i is the estimated signal component and $z(T_s)$ is a Gaussian random variable with a mean of A_1 or A_2 depending on whether a binary 1 or 0 was sent. The ratio of the instantaneous signal power to the average signal power $(S/N)_{T_s}$ at time $t = T_s$ is

$$\left(S/N\right)_{T_s} = \frac{A_i^2}{\sigma_0^2} \qquad (7-62)$$

We wish to find the filter transfer function $H_0(f)$ that maximizes eqn. (7 – 61). We can express the signal $z_i(t)$ at the filter output in terms of the filter transfer function $H(f)$ and the Fourier transform of the input signal $s_i(t)$,which is $F_{si}(f)$ we have

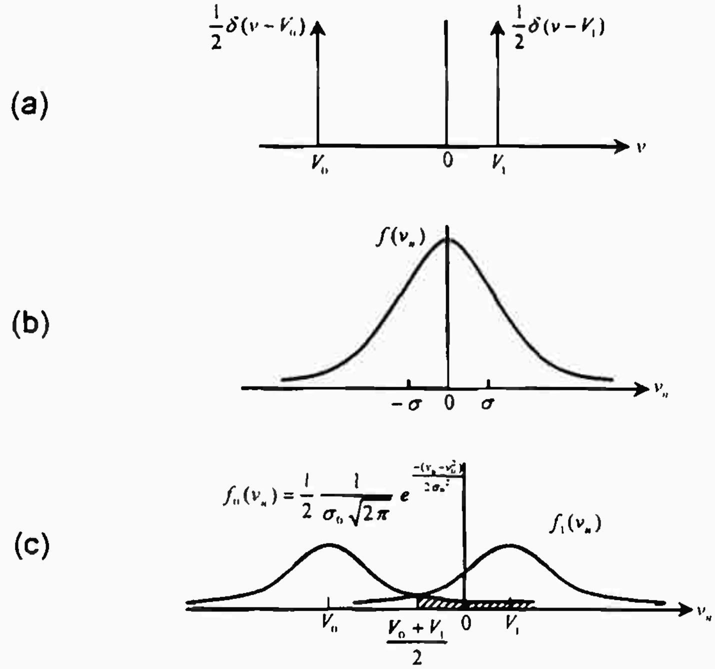


Fig. (7.6) Binary system with signal plus noise
a) Binary symbols *b) noise* *c) signal plus noise*

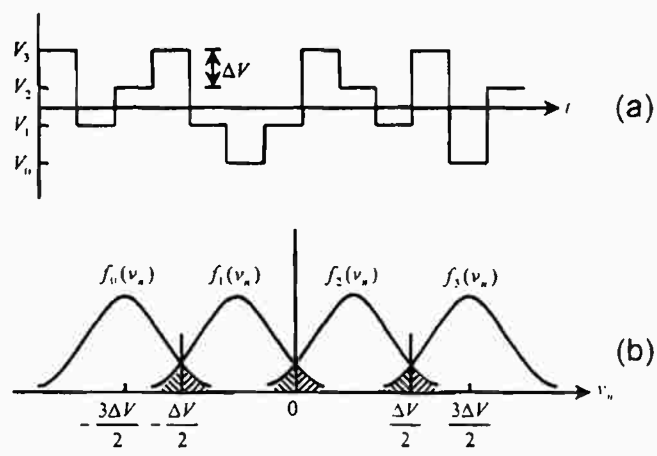


Fig. (7.7) 4-level baseband signaling
a) waveform *b) pdfs*

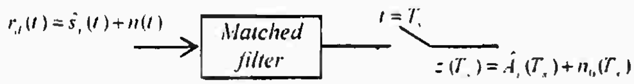


Fig. (7.8) Matched filter

$$z(t) = \int_{-\infty}^{\infty} H(f)\, F_{si}(f)\, e^{j2\pi ft}\, df \qquad (7-63)$$

The output noise power is given by

$$\sigma_0^2 = \frac{kT}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df \qquad (7-64)$$

Thus,

$$\left(\frac{S}{N}\right)_{T_s} = \frac{\left|\int_{-\infty}^{\infty} H(f)\, F_{si}(f)\, e^{j2\pi fT_s}\, df\right|^2}{\frac{kT}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df} \qquad (7-65)$$

From Schwartz inequality (Appendix 2)

$$\left|\int_{-\infty}^{\infty} f_1(x)\, f_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty} |f_1(x)|^2\, dx \int_{-\infty}^{\infty} |f_2(x)|^2\, dx \qquad (7-66)$$

The equality holds if $f_1(x) = k\, f_2^*(x)$ where k is an arbitrary constant and $f_2^*(x)$ is the conjugate of $f_2(x)$.

If we identify $H(f)$ with $f_1(x)$ and $F_{si}(f)\, e^{j2\pi fT_s}$ with $f_2(x)$,

$$\left|\int_{\infty}^{\infty} H(f)\, F_{si}(f)\, e^{j2\pi fT_s}\, df\right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\, df \int_{-\infty}^{\infty} |F_{si}(f)|^2\, df \qquad (7-67)$$

For the equality case,

$$H_o(f) = f_1(x) = k\, f_2^*(x) = k\left[F_{si}(f)\, e^{j2\pi fT_s}\right]^* \qquad (7-68)$$

$$H_o(f) = k\, F_{si}(f)\, e^{-j2\pi fT_s} \qquad (7-69)$$

Substituting eqn. (7 – 67) into eqn. (7 – 65)

$$\left(\frac{S}{N}\right)_{T_s} \le \frac{2}{kT} \int_{-\infty}^{\infty} |F_{si}(f)|^2\, df \qquad (7-70)$$

But we have

$$E = \int_{-\infty}^{\infty} |F_{si}(f)|^2\, df \qquad (7-71)$$

From eqn. (7 – 70) we have

$$\max\left(\frac{S}{N}\right)_{T_s} = \frac{2E}{kT} = \frac{2E}{\eta} = \frac{E}{\eta/2} \qquad (7 - 72)$$

E/η is called the ratio of signal energy to noise PSD. The output $(S/N)_{T_s}$ depends on the input signal energy and the PSD of the noise not on the particular shape of the waveform that is used. The equality in eqn. (7 – 71) holds only if the optimum filter transfer function $H_0(f)$ is used such that eqn. (7 – 69) holds.

Using eqn. (7 – 62) we can get the impulse response of the filter

$$h(t) = \mathcal{F}^{-1}\left\{k\, F_{s_i}^{*}(f)\, e^{-j2\pi f T_s}\right\} \qquad (7 – 73)$$

From Fourier transform properties we have

$$X(-f) = X^{*}(f) \qquad (7 – 74)$$

And

$$\mathcal{F}[x(t - t_o)] = \int_{-\infty}^{\infty} x(t - t_o)\, e^{-j2\pi f t}\, dt$$

putting $u = t - t_o$ we can show

$$\mathcal{F}[x(t - t_0)] = X(f)\; e^{-j2\pi f t_0} \qquad (7 – 75)$$

Similarly, putting $u = t_0 - t$

$$\mathcal{F}[x(t_0 - t)] = X(-f)\; e^{-j2\pi f t_0} \qquad (7 - 76)$$

From eqns (7-73) and (7 -76),

$$h(t) = k\; s_i(T_s - t) \quad o \le t \le T_s \qquad (7 – 77)$$

Alternatively, we may derive this relation as follows

$$h(t) = \int_{-\infty}^{\infty} H(f)\; e^{j2\pi f t}\; df$$

Since,

$$H(f) = kF_{s_i}^{*}(f)e^{-j2\pi f T_s}$$

$$h(t) = k\int_{-\infty}^{\infty} kF_{s_i}^{*}(f)\, e^{j2\pi f(t - T_s)} df$$

$$= k\left[\int_{-\infty}^{\infty} F_{si}(f) e j2\pi f\; (T_s - t)\right]^{*} df$$

$$= k\, s_i(T_s - t)$$

Which is the same result as eqn.(7-77)

Thus, the impulse response of a filter that produces the maximum output S/N is the mirror image the input signal $s_i(t)$ delayed by the symbol time duration T_s. We must have $t \le T_s$,i.e., all the signal energy in $s_i(t)$ must be received before the decision is made at $t = T_s$. Note that the delay of T_s makes eqn. (7 – 77) causal ,i.e., $h(t)$ is a real function of positive time in the interval $0 \le t \le T_s$.,i.e., it dose not exist at negative time. Without the delay T_s, the response is unrealizable.

The output $z(t)$ of a causal filter can be described in the time domain as the convolution of a received input waveform $r_d(t)$ with the impulse response of the filter

$$z(t) = r_d(t) * h(t) \qquad (7-78)$$

$$= \int_0^t r_d(\tau)\, h(t-\tau)\, d\tau \qquad (7-79)$$

Using eqn. (7 – 77),

$$z(t) = \int_0^t r_d(\tau)\, s_i(T_s - t + \tau)\, d\tau \qquad (7-80)$$

when $t = T_s$

$$z(T_s) = \int_0^{T_s} r_d(\tau)\, s_i(\tau)\, d\tau \qquad (7-81)$$

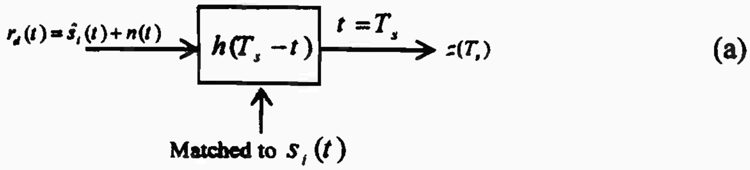
Eqn. (7 – 81) indicates that the output of the matched filter at the instant of sampling $t = T_s$ is the product integration of the received signal $r_d(t)$ with a replica of the transmitted waveform $s_i(t)$ over one symbol interval. This is the correlation of $r_d(t)$ with $s_i(t)$.

If $r_d(t)$ is correlated with each replica of $s_i(t)$ $i = 1 ... M$ using a bank of correlators, the signal whose product integration or correlation with $r_d(t)$ gives the maximum output $z_i(T_s)$ is the signal that matches $r_d(t)$ best. We note that the operation of a matched filter is a convolution of the received signal with the impulse response of the filter (eqn. 7 – 79). From eqn. (7 – 81), it is clear that we can implement the receiving filter with either a matched filter or a correlator. The output of the matched filter or correlator are the same at $t = T_s$ (Fig. 7.9)

Referring to eqn. (7 – 69),

$$|H(f)|^2 = k^2 |F_{si}(f)|^2 \left|e^{-j2\pi f T_s}\right|$$

$$= k^2 |F_{si}(f)|^2 \qquad (7-82)$$



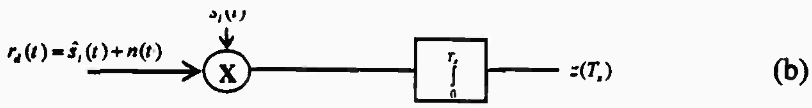


Fig. (7.9) Equivalence of matched and correlator
a) matched filter *b) filter correlator*

$$\phi_H(f)=-\phi_{s_i}(f)-2\pi f T_s \qquad (7-83)$$

Thus, the square of the amplitude response of a matched filter $(|H(f)|)^2$ has the same shape as the energy spectral density $|F_{si}(f)|^2$ to which it is matched, which is in agreement with eqn. (7 – 72). The phase response is the negative of the phase spectrum of the pulse to which the filter is matched plus an additional linear phase of $-2\pi f T_s$.

An important note should be made here. The impulse response of a matched filter is a time reversed version of the pulse to which it is matched, delayed by a time equal to the duration of the pulse. The time delay T_s ensures causality. The output of the filter is the convolution of the impulse response with the input. The convolution process involves reversing one of the functions, sliding the reversed function over the non - reversed function, and integrating the product. Since the impulse response of the matched filter is a time reversed copy of the expected input, and since convolution requires a further reversal, then the output is given by the integrated sliding product of either the input or the impulse response with an unreversed version of the function itself $r_d(t)$. This is nothing more than the autocorrelation. This explains why the matched filter and correlator are equivalent. Thus,

$$z(t)=r_d(t)*h(t) \qquad (7-84)$$

Assume that the filter is matched to $s_i(t)$, where $r_d(t)=\hat{s}_i(t)+n(t)$. Since the cross correlation of statistically independent random processes for which one has zero mean is 0 .But this does not mean that $n(t)$ is zero. In fact, the output noise power is the input noise power times $\int |H(f)|^2 df$.

$$n(t) * h(t) = 0 \quad (7-85)$$

$$z(t) = s_i(t) * h(t) \quad (7-86)$$

$$= s_i(t) * ks_i(T_s - t) \quad (7-87)$$

$$= k \int_{-\infty}^{\infty} s_i(t')s_i(T_s t' - t)dt' \quad (7-88)$$

This is in harmony with eqns. (7 – 79) and (7 – 81) .Putting $T_s - t = \tau$

$$z(t) = z(T_s - \tau) = k \int_{-\infty}^{\infty} s_i(t')\, s_i(t' + \tau)\, dt' \quad (7-89)$$

$$= k\, R(\tau) \quad (7-90)$$

Thus, the output of a filter driven by and matched to a real input pulse is the autocorrelation of the input pulse within a constant k and time shift T_s.

At $t = T_s \qquad \tau = 0 \qquad z(T_s) = k\, R(0)$ (7 – 91)

If $T_s \to 0 \qquad z(t) \to k\, R(t)$ (7 – 92)

From eqn. (7 – 86), taking Fourier transform of both sides

$$Z(f) = F_{si}(f)\ H(f) \quad (7-93)$$

Using eqn. (7 – 77)

$$Z(f) = F_{si}(f) F_{si}^*(f)\ e^{-j2\pi f T_s} \quad (7-94)$$

$$|Z(f)| = |F_{si}(f)|^2 \quad (7-95)$$

Thus, from eqn. (7 – 90), and (7 – 95), we note

$$s_i(t) \xleftrightarrow{FT} F_{si}(f) \quad (7-96)$$

$$z(t) = k\, R(\tau) \xleftrightarrow{FT} |F_{si}(f)|^2 \quad (7-97)$$

From eqn. (7 – 97) and (7 – 91),

$$z(T_s) = k\, R(o) = \int_{-\infty}^{\infty} |F_{si}(f)|^2\, df \quad (7-98)$$

Thus, $z(T_s)$is the total energy in the signal supplied to the matching filter to match the input signal $s_i(t)$(not the actual output signal from the matching filter, which contains noise as well) This is in harmony with eqn. (4 – 73). For the special case of a rectangular pulse, the matched filter may be implemented using a circuit called integrate and dump circuit (Fig. 7.10d). The integrator computes the area under the rectangular pulse and the resulting output is then sampled at time $t = T_s$, where T_s is the duration of the pulse. Immediately after $t = T_s$, the integrator is restored to its initial condition. We see that for $0 \le t \le T_s$ the output of the circuit has the same waveform as that within a constant.

We conclude by restating, a matched filter which immediately precedes the decision circuit in a digital communication receiver is matched to a particular symbol pulse (waveform) if it maximizes the output S/N at the sampling instant when that pulse is present at the filter input. The impulse response of such a filter must be time reversed version of the pulse to which it is matched delayed by a time equal to the duration of the pulse. The output of the filter driven by and matched to a real input pulse is the auto correlation of the input pulse (within a constant) within T_s.

Ex. 7.4

Find the frequency response of the filter which is matched to triangular pulse $T_r(t-1)$ noting that Fourier transform of $T_r(t/\tau)$ of base width 2τ is $\tau \sin c^2(\tau f)$.

Solution

$$F_{si}^*(f) = \mathcal{F}\left[T_r(t-1)\right]$$

Referring to Appendix A-3

$$F_{si}(f) = \sin c^2(f)\, e^{-j2\pi f(1)}$$

$$H(f) = F_{si}^*(f)\; e^{-j2\pi f T_s}$$

$$= \sin c^2(f)\, e^{-j2\pi f} \quad e^{-j2\pi f(2)}$$

$$= \sin c^2(f)\, e^{-j2\pi f}$$

Ex. 7.5

Consider a rectangular pulse of amplitude A and duration T_s (Fig7.10a) Find the output of the matched filter

Solution

The impulse response $h(t)$ of the matched filter has exactly the same waveform as the signal but shifted T_s and inverted, The output signal $z(t)$ of the matched filter has a triangular waveform since it is the convolution of the input and $h(t)$ (eqn 7-86). The maximum value of the output signal $z(t)$ is equal to $k A^2 T_s$,which is the energy of the input signal scaled by the factor k. This maximum occurs at $t = T_s$ (Fig. 7.10b).

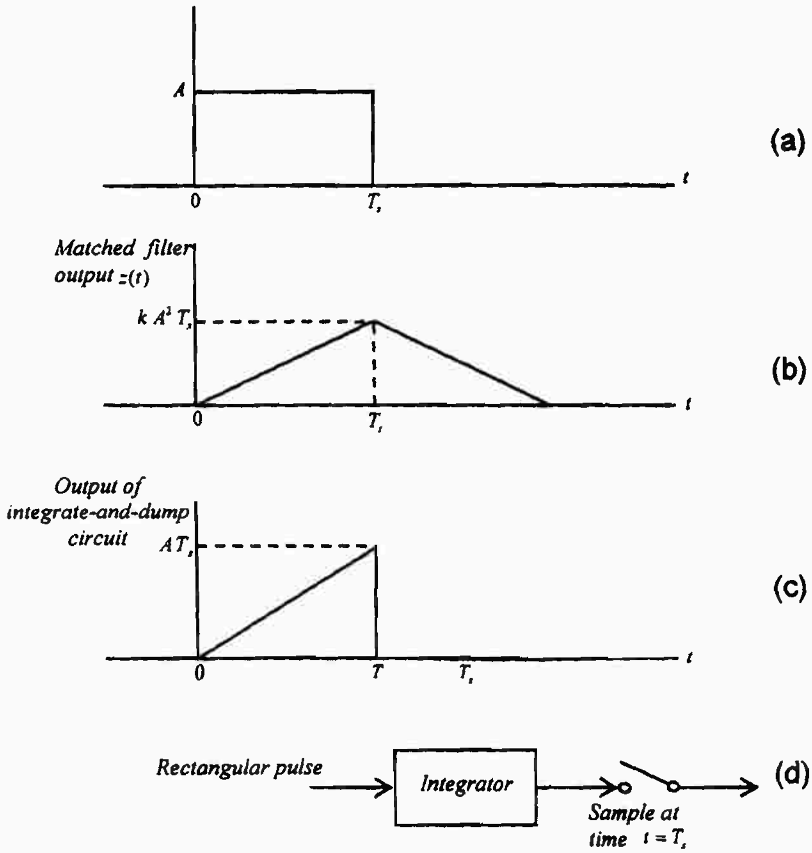


Fig. (7.10) Matched filter output

a) input rectangular pulse *b) matched filter output*

c) integrator output *c) integrate and dump circuit*

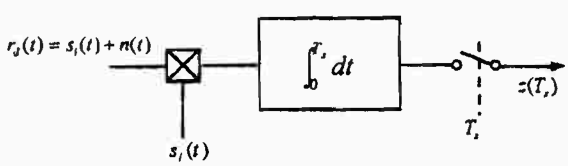


Fig. (7.11) Correlator

7.5 Correlator Design:

A block diagram of a classical correlator is shown (Fig. 7.11), where the input is $r_d(t) = s_i(t) + n(t)$. According to eqn. (7 – 91), only the peak values of the correlation function $R(\tau)$ is important, i.e., $R(0)$. Thus the correlator reaches this maximum value at the end of the input pulse, i.e., after T_s seconds. This represents the correct sampling instant which leads to an optimum (maximum) decision S/N.

For binary systems employing two nonzero pulses, a possible implementation would include two filters or correlators, one matched to each pulse type (Fig 7.12a). In a binary system, we take voltage A_1 to indicate state 1 or $s_1(t)$ and voltage $-A_1$ to indicate state 0 or $s_0(t)$. Therefore, we may define the filter output $z(T_s)$ as

$$z(T_s) = \begin{cases} C\int_0^{T_s} s_0^2(t)\,dt - C\int_0^{T_s} s_0(t)\; s_1(t)\,dt & for\ 0 \\ -C\int_0^{T_s} s_1^2(t)\,dt + \int_0^{T_s} C\; s_1(t)\; s_0(t)\,dt & for\ 1 \end{cases} \qquad (7-99)$$

when C is a constant to account for the dimensions of both sides. If the signal pulses $s_0(t)$ and $s_1(t)$ are orthogonal but contain equal normalised energy E_s (in 1Ω load), then the sampling instant voltages will be $\pm E_s$. If the pulses are antipodal, i.e., $s_1(t) = -s_0(t)$. Then from eqn. (7 – 99) the sampling voltages will be $\pm 2E_s$.

We may also use one filter or correlator (Fig. 7.24b). In this case the filter is matched to the difference waveform $s_1(t) - s_0(t)$. For multisymbol signaling the number of channels may be extended to M (Fig. 7.12c). We note that if we use a single correlator then E_s is replaced by E_s which is the energy of the reference signal inputted to the multiplier.

In the case of orthogonal signal pulses the cross product terms in eqn. (7 – 99) vanishes and if the conversion constant C is numerically equal to 1,

$$|z(T_s)| = E_s \qquad (7-100)$$

The sampling instant normalized signal power at the correlator output within a conversion constant is

$$|z(T_s)|^2 = E_s^2 \qquad (7-101)$$

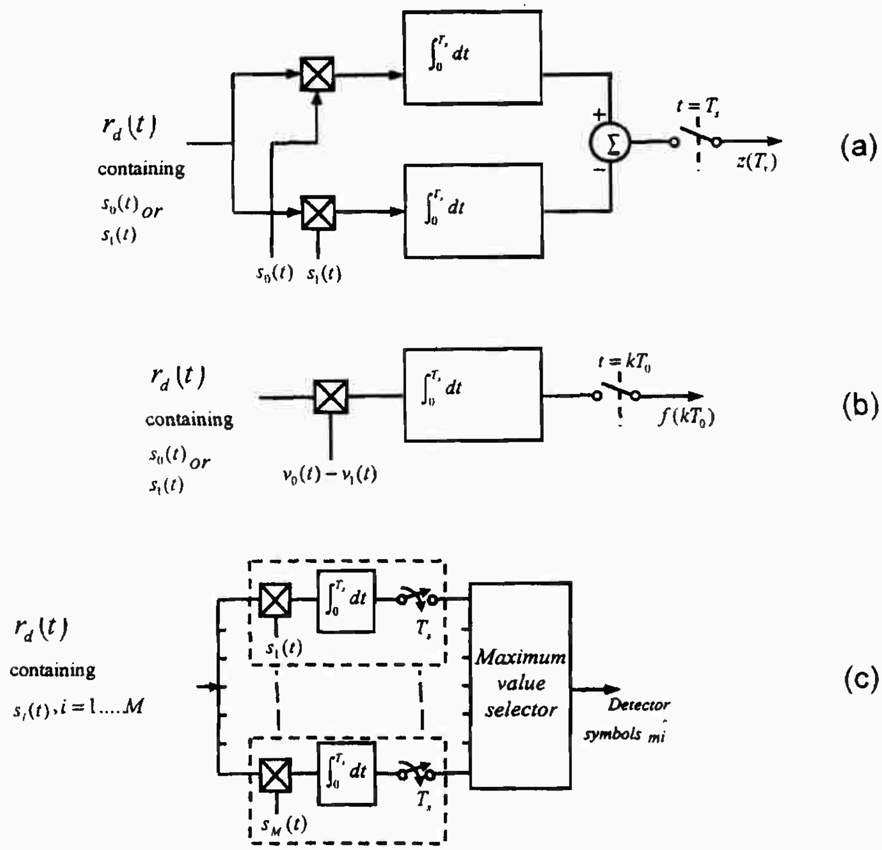


Fig. (7.12) Correlator configurations

a) two channels *b) one channel* *c) M - ary signals*

We may obtain the autocorrelation function of the input noise R_n as

$$R_n(\tau) = < n(t)\ n(t+\tau) > \qquad (7-102)$$

Assuming that $n(t)$ is white with double sided noise PSD of $\frac{kT}{2}$

$$R_n(\tau) = \frac{kT}{2}\ \delta(\tau) \qquad (7-103)$$

The input of the circuits of Fig.(7.12) is $r_d(t)$ which contains $\hat{s}_i(t)$ and noise $n(t)$. Multiplying $r_d(t)$ by $s_i(t)$ results in noise $n(t)$ being multiplied by $s_i(t)$.

The autocorrelation of noise after multiplication with $s_i(t)$ is

$$R_{x_i}(\tau) = < x_i(t)\ x_i(t+\tau) > \quad (7-104)$$

where $x_i(t) = n(t)\ s_i(t)$, with multiplier constant equal unity

$$R_{x_i}(\tau) = < n(t)\ s_i(t)\ n(t+\tau)\ s_i(t+\tau) > \quad (7-105)$$

Since $n(t)$ and $s_i(t)$ are independent processes,

$$R_{x_i}(\tau) = \frac{kT}{2}\ \delta(\tau)\ R_{s_i}(\tau)$$

$$= \frac{kT}{2}\ \delta(\tau)\ R_{s_i}(0) \quad (7-106)$$

Where $R_{s_i}(0)$ is the mean square of $s_i(t)$

$$R_{x_i}(\tau) = \frac{kT}{2}\ \delta(\tau)\ \frac{1}{T_s}\int_0^{T_s} s_i^2(t)dt$$

$$R_{x_i}(\tau) = \frac{kT}{2}\ \delta(\tau)\ \frac{E_s}{T_s} \quad (7-107)$$

Since

$$R_{x_i}(\tau) \xleftrightarrow{FT} S_{x_i}(f) \quad (7-108)$$

Thus, the PSD $S_{xi}(t)$ of $R_{x_i}(\tau)$ becomes the Fourier transform of the right hand side of eqn. (7 – 107)
Thus,

$$S_{x_i}(f) = \frac{kT}{2}\ \frac{E_s}{T_s} \quad (7-109)$$

We must note that $s_{x_i}(t)$ is the input noise to the integrator. We want now to find the output noise at the output of the integrator. Therefore, we must find $H(f)$ of the integrator from the noise point of view. To find $H(f)$ we may first find $h(t)$ of the integrator. This $h(t)$ is the response of a delta function at the input, to give an output in the form of a rectangle, i.e., constant from 0 to T_s. This is called time windowed integrator (moving average filter). This may be realized by the system shown in (Fig. 7.13). Thus, the multiplier must have dimension V^{-1} for the output to be voltage. The integrator has dimension of s^{-1}. Thus, the overall comersion constant is $1 \quad V/V^2 s$ to make $z(T_s)$ indicate energy.

This way $R(\tau)$ is assigned units of V^2 and $S_{xi}(f) \quad V^2/H_z$.

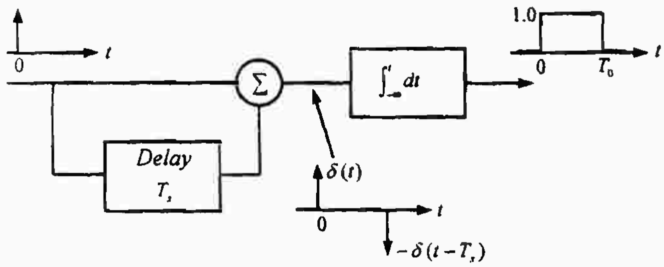


Fig. (7.13) Time windowed integrator (moving average filter)

Such a system will have $H(f)$ as

$$H(f) = T_s \sin c(T_s f)\, e^{-j2\pi f T_s/2} \tag{7-110}$$

$$|H(f)| = T_s |\sin c(T_s f)| \tag{7-111}$$

The noise PSD at the integrator input is given by eqn. (7 – 109). The noise PSD at the integrator output is given by

$$S_{y_t}(f) = S_{x_t}(f)\, |H(f)|^2 \tag{7-112}$$

$$= \frac{kT}{2} E_s\, T_s \sin c^2 (T_s f) \tag{7-113}$$

The total noise power at the correlator output is

$$N_o = \frac{kT}{2} E_s\, T_s \int_{-\infty}^{\infty} \sin c^2 (T_s f) df \tag{7-114}$$

Noting that $u(o) = \int_{-\infty}^{\infty} U(f) df$ which is called value at the origin theorem (Appendix 3), the integral of the $\sin c^2$ is $1/T_s$. Hence,

$$N_o = \frac{kT}{2} E_s \tag{7-115}$$

The standard deviation of the noise at the correlator output σ_o is

$$\sigma_o = \sqrt{N_o} = \sqrt{\frac{kT}{2} E_s} = \sqrt{\frac{\eta E_s}{2}} \tag{7-116}$$

Thus, at the decision instant, we have from eqns. (7 – 100) and (7 – 116)

$$\frac{z(T_s)}{\sigma_o} = \sqrt{\frac{2E_s}{kT}} = \sqrt{\frac{2E_s}{\eta}} \tag{7-117}$$

Thus, the decision instant S/N is given by

$$S/N = \frac{|z(T_s)|^2}{\sigma^2} = \frac{2E_s}{kT} = \frac{2E_s}{\eta} \qquad (7-118)$$

Note that the noise picked up is smaller than the noise at the input by a factor $1/T_S^2$ (eqns.7 – 109 and 7 – 115).

We see that the decision instant S/N at the output of the correlator or matched filter depends only on the pulse energy and input noise PSD and is independent of the pulse shape. This result (eqn.7-118) is in harmony with eqn. (7-72).

It should not bother us that $z(T_s)$is often referred to as voltage or as energy while $|z(t)|^2$ as power since the measure that applies to signal applies as well to noise. A conversion factor is understood to exist (often taken numerically equal to unity) to account for the dimensionality problem. We should also note that $z(T_s)$ indicate signal content not noise. To calculate noise we had to follow in a separate procedure. It should also be clear that the noise in this calculation is $\hat{n}$ not all of the noise input, because $\tilde{n}$ is discarded due to the orthagenality property of the $\{\phi_i\}$ reference system. It is this $\hat{n}$ that is relevant which was multiplied by s_i in the correlator or matched filter.

Ex. 7.6

What is the sampling instant S/N at the output of a filter matched to a triangular pulse of height $100mV$ and width $1ms$ if the noise PSD is $100nV^2/Hz$.

Solution

The energy in one pulse is

$$E_s = \int_0^{T_s} s_i^2(t)\,dt = \int_0^{T_s/2} s_i^2(t)\,dt + \int_{T_s/2}^{T_s} s_i^2(t)\,dt$$

$$= \int_0^{0.5\times10^{-3}} [200t]^2\,dt + \int_{0.5\times10^{-3}}^{1\times10^{-3}} [0.2^{-2} - 200t]^2\,dt$$

$$= 4\times10^4\left[\frac{t^3}{3}\right]_o^{0.5\times10^{-3}} + 4\times10^{-2}[t]_{0.5\times10^{-3}}^{1\times10^{-3}}$$

$$-80\left[\frac{t^2}{2}\right]_{0.5\times10^{-3}}^{1\times10^{-3}} + 4\times10^4\left[\frac{t^3}{3}\right]_{0.5\times10^{-3}}^{1\times10^{-3}}$$

$$= 3.3\times10^{-7}\ V^2s$$

Using eqn. (7 – 118)

$$S/N = \frac{2E_s}{kT} = \frac{2\times 3.3\times 10^{-7}}{100\times 10^{-9}} = 6.67 = 8.2dB$$

Ex. 7.7

Using a an integrate and dump circuit as a detector for a rectangular pulse embedded in AWGN, derive a formula for error rate due to noise in binary NRZ – PCM system. Note that this is a simplification of the matched filter case.

Solution

The received signal is

$$r_d(t) = \begin{cases} +A + n(t) & 1 \ is \ sent \\ -A + n(t) & 0 \ is \ sent \end{cases}$$

where A is the height of the pulse and is constant during bit (symbol) duration T_b. Given the noisy signal $r_d(t)$, the receiver is required to make a decision on each signaling interval as to whether the transmitted symbol is 1 or 0.

The receiver (Fig. 7.14) consists of a matched filter followed by a sampler and finally a decision device. The filter is matched to a rectangular pulse of amplitude A and duration T_b. The resulting matched filter output is sampled at the end of each signaling interval. We note that $r_d(t)$ is a random signal due to $n(t)$. Let $y(T_s)$ be the sample value at the end of signaling interval. This sample value is compared to threshold γ_o in the decision device. Suppose that symbol 0 was sent, then

$$r_d(t) = -A + n(t) \qquad\qquad 0 \le t \le T_b$$

The integrator output sample at $t = T_b$ is

$$y(T_b) = \frac{1}{T_s}\int_0^{T_b} r_d(t)\,dt$$

$$= -A + \frac{1}{T_b}\int_0^{T_b} n(t)\,dt \qquad (7-119)$$

Since y is random variable with Gaussian distribution and mean of $-A$, its variance is

$$\sigma_y^2 = E\left[(y+A)^2\right] \qquad (7-120)$$

From eqn. (7 – 110), eqn. (7 – 111) becomes

$$\sigma_y^2 = \frac{1}{T_b^2} E\left[\int_0^{T_b}\int_0^{T_b} n(t_1)\, n(t_2) dt_1\, dt_2\right] \qquad (7-121)$$

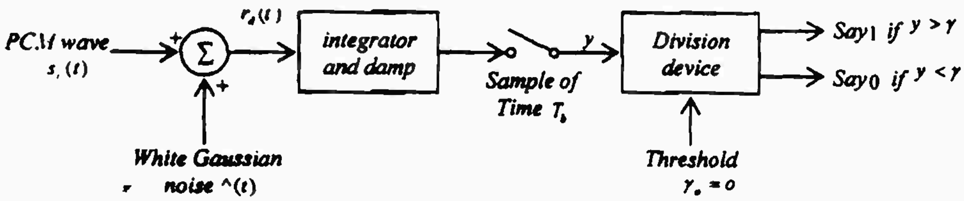


Fig. (7.14) Time windowed integrator

$$= \frac{1}{T_b^2} \int_0^{T_b} \int_0^{T_b} E\left[n(t_1)\, n(t_2)\right] dt_1\, dt_2 \tag{7-122}$$

$$= \frac{1}{T_s^2} \int_0^{T_b} \int_0^{T_s} R_n(t_1, t_2)\, dt_1\, dt_2 \tag{7-123}$$

where $R_n(t_1, t_2)$ is the autocorrelation function of noise. Since $n(t)$ is white, S/N in this case is calculated as signal power is A^2 and the noise power is given by eqn. (7 -117). Thus, $S/N = \frac{A^2 \times 2T_b}{\eta}$

Since,

$$A^2 T_b = E_b$$

$$S/N = \frac{2E_b}{\eta}$$

which is identical to eqn.(7 – 118)

$$R_n(t_1, t_2) = \frac{kT}{2}\, \delta(t_1 - t_2) \tag{7-124}$$

Hence,

$$\sigma_y^2 = \frac{1}{T_s^2} \int_0^{T_b} \int_0^{T_b} \frac{kT}{2} \delta(t_1 - t_2)\, dt_1\, dt_2 \tag{7-125}$$

$$= \frac{kT}{2T_b} = \frac{\eta}{2T_b} \tag{7-126}$$

The pdf of random variable Y given that symbol 0 was sent is therefore

$$f_y(y|o) = \frac{1}{\sqrt{\pi \frac{\eta}{T_b}}}\, e^{-\frac{(y+A)^2}{\eta / T_b}} \tag{7-127}$$

This function is shown in (Fig. 7.15a)

Let $P_{\epsilon 0}$ denote the conditional probability of error given that 0 was sent which is the shaded area under the curve of $f_y(y|o)$ from γ to ∞. Due to noise, y may exceed γ causing error. The probability of this happening is

$$P_{\epsilon 0} = P(y > \gamma |\ 0\,sent)$$

$$= \int_{\lambda}^{\infty} f_y(y|0)\,dy \qquad (7-128)$$

$$= \frac{1}{\sqrt{\pi(\eta/T_b)}} \int_{\gamma}^{\infty} e^{-\frac{(y+A)^2}{\eta/T_b}}\,dy \qquad (7-129)$$

Note that the a priori probabilities of binary symbols 0, 1 are p_0, p_1 such that $p_0 + p_1 = 1$. In case of equal probability $p_0 = p_1 = 1/2$. As $\gamma = 0$, eqn. (7 – 129) becomes

$$P_{\epsilon 0} = \frac{1}{\sqrt{\pi(\eta/T_b)}} \int_{0}^{\infty} e^{-\frac{(y+A)^2}{\eta/T_b}}\,dy \qquad (7-130)$$

Defining $$\xi = \frac{y+A}{\sqrt{\eta/T_b}} \qquad (7-131)$$

Then

$$P_{\epsilon 0} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/\eta}}^{\infty} e^{-\xi^2}\,d\xi \qquad (7-132)$$

where E_b is the transmitted energy per bit defined by

$$E_b = A^2\,T_b \qquad (7-133)$$

Since

$$erfc\ x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-x^2}\,dx \qquad (7-134)$$

Thus,

$$P_{\epsilon 0} = \frac{1}{2}\,erfc\,\sqrt{E_b/\eta} \qquad (7-135)$$

Assume now that 1 was transmitted. This time the Gaussian random variable has mean $+A$ and variance $\eta/2T_b$ which is exactly the same as before

$$f_y(y|1) = \frac{1}{\sqrt{\pi\eta/T_b}}\,e^{-(y-A)^2/(\eta/T_b)} \qquad (7-136)$$

which is shown (Fig. 7.15b). Let $P_{\epsilon 1}$ be the conditional probability of error given that symbol 1 was sent. This is defined by the shaded area under the curve of $f_y(y|1)$ from $-\infty$ to γ (i.e., from $-\infty$ to 0). Due to noise y may have a value $< \gamma$ causing an error. The probability of this error conditional on sending symbol 1 is defined by

$$P_{\epsilon 1} = P_0(y < \gamma | 1 sent) \quad (7-137)$$

$$= \int_{-\infty}^{0} f_y(y|1)\,dy \quad (7-138)$$

For $\gamma = 0$,

$$= \frac{2}{\sqrt{\pi(\eta/T_b)}} \int_{-\infty}^{0} e^{-\frac{(y-A)^2}{\eta/T_b}}\,dy \quad (7-139)$$

Putting

$$\frac{y-A}{\sqrt{\eta/T_b}} = -\xi \quad (7-140)$$

We see that $P_{\epsilon 1} = P_{o1}$ for equiprobable symbols 0 , 1. The channel for which the conditional probabilities $P_{\epsilon 1}$ and P_{o1} are equal is said to be binary symmetric.

To determine the average probability of symbol error in the receiver, we note that the two kinds of error are mutually exclusive. Thus, the average probability of symbol error P_ϵ in the receiver is given by

$$P_\epsilon = p_0\, P_{\epsilon 0} + p_1\, P_{\epsilon 1} \quad (7-141)$$

since $p_0 = p_1 = \frac{1}{2}$ and $P_{\epsilon 0} = P_{\epsilon 1}$. Thus,

$$P_\epsilon = P_{\epsilon 1} = P_{\epsilon 0} \quad (7-142)$$

i.e., using eqn. (7 - 37),

$$P_\epsilon = \frac{1}{2} erfc\left(\sqrt{E_b/\eta}\right) = Q\left(\sqrt{2E_b/\eta}\right) = \frac{1}{2}\left[1 - erf\sqrt{E_b/\eta}\right] \quad (7-143)$$

Thus, the average probability of symbol error in a binary encoded PCM receiver depends solely on (E_b/η) which is the ratio of the transmitted signal energy per bit to the noise PSD. Fig. (7.16) shows a plot of the average probability of symbol error P_e versus the dimensionless ratio (E_b/η). The probability of error decreases very rapidly as the ratio (E_b/η) is increased so that a very small increase in the transmitted signal energy will make the reception of binary pulses almost error free (check table7 – 1).

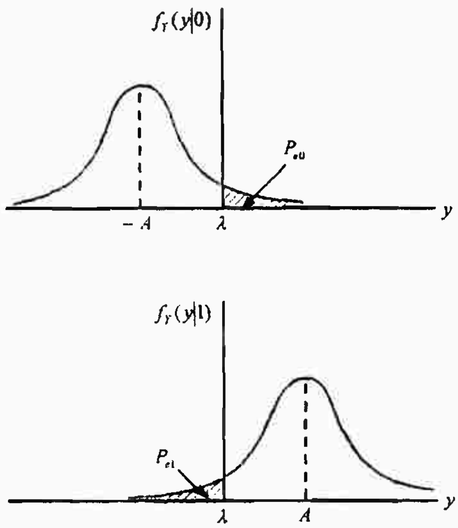


Fig. (7.15) pdf at filter output

a) When 0 is sent pdf at filter output b) when 1 is sent

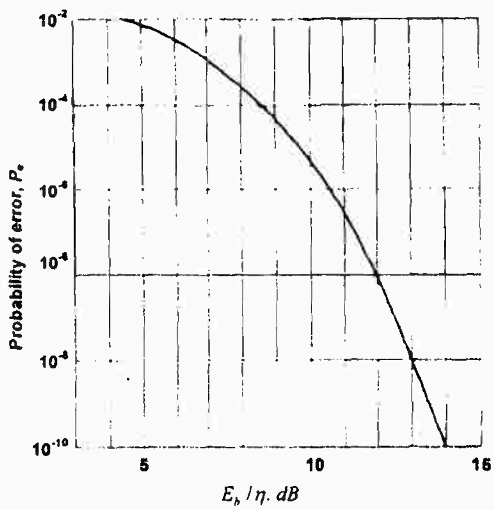


Fig. (7.16) Probability of error in a PCM receiver

7.6 Receiver Performance:

A general formula for the probability of symbol error for an optimum binary receiver whether a matched filter or correlator can be obtained by considering the single channel correlator shown (Fig. 7.12b) matched to the binary symbol difference $s_1(t) - s_0(t)$. When a binary 1 is present at the receiver input, the output at the decision instant is given within a dimensionality constant by

$$z(T_s) = \int_0^{T_s} s_1(t)\left[s_1(t) - s_0(t)\right] dt \qquad (7-144)$$

$$= E_{s1} - \int_0^{T_s} s_1(t)\, s_0(t)\, dt \qquad (7-145)$$

where E_{s1} is the binary symbol energy When a binary 0 is present at the receiver input

$$z(T_s) = -E_{so} + \int_0^{T_s} s_1(t)\, s_0(t)\, dt \qquad (7-146)$$

where E_{s0} is the binary 0 symbol energy. The second term of both equations (7 – 145) and (7 – 146) represents the cross correlation between symbols. We define the normalized correlation coefficient ρ as

$$\rho = \frac{1}{\sqrt{E_{s0}\, E_{s1}}} \int_0^{T_s} s_1(t)\, s_0(t)\, dt \qquad (7-147)$$

Thus, eqns. (7 – 145) and (7 – 146) become

$$z(T_s) = \begin{cases} E_{s1} - \rho\sqrt{E_{s0}\, E_{s1}} & \textit{for binary } 1 \\ -E_{so} + \rho\sqrt{E_{s0}\, E_{s1}} & \textit{for binary } 0 \end{cases} \qquad (7-148)$$

For OOK, where $s_0(t) = 0$, $\vartheta = 0$. In general, the difference in decision instant voltages representing binary 1 and 0 is

$$\Delta V = E_{s1} + E_{s0} - 2\rho\sqrt{E_{s1}\, E_{s0}} \qquad (7-149)$$

where ΔV is the peak to peak voltage difference. We define E_s' as the energy in the reference pulse of the single channel correlator

$$E_s' = \int_0^{T_s} \left[s_1(t) - s_0(t)\right]^2 dt = \Delta V \qquad (7-150)$$

The rms noise voltage at the output of the receiver is given by eqns. (7–115), (7–116)

$$\sigma_o = \sqrt{N_o} = \sqrt{\frac{\eta E_s'}{2}} = \sqrt{\frac{\eta \Delta V}{2}} \qquad (7-151)$$

$$= \left[\frac{\eta}{2} \left(E_{s1} + E_{s0} - 2\rho \sqrt{E_{s1}\, E_{s0}} \right) \right]^{1/2} \tag{7-152}$$

The quantity $\Delta V / \sigma_o$ is given by

$$\frac{\Delta V}{\sigma_o} = \left[\frac{2}{\eta} \left(E_{s1} + E_{s0} - 2\rho \sqrt{(E_{s1}\, E_{s0})} \right) \right]^{1/2} \tag{7-153}$$

For binary symbol of equal energy E_s

$$\frac{\Delta V}{\sigma_o} = 2 \sqrt{\frac{E_s}{\eta}(1-\rho)} \tag{7-154}$$

For center point detection eqn. (7 – 41)

$$P_e = \frac{1}{2}\left[1 - erf\left(\frac{\Delta V}{2\sigma_o \sqrt{2}} \right) \right] = \frac{1}{2}\left[1 - erf \sqrt{\left[\frac{E_s}{2\eta}(1-\rho) \right]} \right] \tag{7-155}$$

We may use the same equation for OOK signaling if E_s is the average energy per symbol (½ energy of 1). For all orthogonal signaling schemes including OOK $\rho = o$. For antipodal schemes $[s_1(t) = -s_0(t)]$ $\rho = -1$. Eqn. (7 – 153) may be used to compare the performance of a baseband matched filter receiver with simple center point detection of rectangular pulse. For unipolar NRZ, (1 is represented by a pulse of duration T_s, O no pulse)

$$\left.\frac{\Delta V}{\sigma_o}\right|_{matched\ filter} = \left(\frac{2 E_{s1}}{\eta} \right)^{1/2} \tag{7-156}$$

$$= \left(\frac{2\, \Delta V^2 T_s}{\sigma_o^2 / B} \right)^{1/2} \tag{7-157}$$

$$= \sqrt{2} (T_s B)^{1/2} \left(\frac{\Delta V}{\sigma_o} \right)_{CPD} \tag{7-158}$$

Where $\left.\frac{\Delta V}{\sigma_o}\right|_{CPD}$ is the center point detection S/N (eqn.7-42) and B is the bandwidth, Note that $\Delta V / \sigma_o$ is infinite for white noise unless the bandwidth is limited to a finite value. Thus,

$$\frac{\left. \Delta V / \sigma_0 \right|_{matched\ filter}}{\left. \Delta V / \sigma_0 \right|_{CPD}} = \sqrt{2} (T_s B)^{1/2} \tag{7-159}$$

We note that the output of the correlator and the matched filter are not the same – we have

$$z(t)\Big|_{correlator} = \int_0^t s_i^2(t)\, dt \qquad (7-160)$$

$$z(t)\Big|_{matched\ filter} = \int_0^t s_i(t')\, s_i(t' + T_s - t)\, dt' \qquad (7-161)$$

Yet, this difference has no bearing on P_ϵ, since P_ϵ depends on S/N not on the pulse shape provided there are no errors in symbol timing.

Ex. 7.8

Compare the probability of error in a matched filter for both orthogonal and antipodal signaling and show that the purpose of matching is to maximize the distance between the two vectors $\vec{s_1}$ and $\vec{s_0}$.

Solution:

To optimize the filter or minimize P_ϵ we need to maximize S/N for the filter and use the optimum decision threshold. In a binary case $P_\epsilon = P_B$ and the optimum decision threshold is given by $\frac{A_1 + A_0}{2}$ resulting in eqn. (7 – 30),. To minimize P_B it is required to choose the filter that maximizes $(A_1 - A_2)$, i.e., the argument of $Q(x)$ as in eqn. (7 – 30). Since the square of the difference signal $(A_1 - A_2)$ is the instantaneous power of the difference signal, we equivalently need to maximize $\frac{(A_1 - A_2)^2}{\sigma_0^2}$. The matched filter is the one that maximizes S/N for a given known signal. i.e., maximizes the difference between two possible signal outputs. From eqns. (7 – 62), (7 – 64), and (7 – 72), for a filter matched to the input difference $s_1(t) - s_0(t)$

$$S/N\Big|_{T_s} = \frac{(A_1 - A_2)^2}{\sigma_0^2} = \frac{2\,E_d}{\eta} \qquad (7-162)$$

where E_d is the energy of the difference signal at the filter input

$$E_d = \int_0^{T_b} [s_1(t) - s_0(t)]^2\, dt \qquad (7-163)$$

By maximizing the output S/N, the matched filter provides the maximum distance (normalized by noise) between the two candidate outputs A_1 and A_2.

Using eqns. (7 – 30) and (7 – 162)

$$P_B = Q\left(\sqrt{\frac{E_d}{2\eta}}\right) \qquad (7-164)$$

Let us define the cross correlation coefficient ρ as a measure of similarity between the two signal $s_1(t)$ and $s_0(t)$ as in eqn. (7 – 147)

$$\rho = \frac{1}{E_b}\int_0^{T_b} s_1(t)\, s_0(t)\, dt \qquad (7-165)$$

when $-1 \le \rho \le 1$. As $s_1(t)$ and $s_0(t)$ are viewed as signal vectors $\bar{s}_1$ and $\bar{s}_0$, then ρ may be expressed as

$$\rho = \cos\theta \qquad (7-166)$$

The vectors $\bar{s}_1$ and $\bar{s}_0$ are separated by the angle θ. For zero (or small) θ the vectors are lightly correlated (similar to each other), and for large angular separation they are quite dissimilar. Expanding eqn. (7 – 163),

$$E_d = \int_0^{T_b} s_1^2(t)\,dt + \int_0^{T_b} s_0^2(t)\,dt - 2\int_0^{T_b} s_1(t)\, s_0(t)\, dt \qquad (7-167)$$

Recalling that each of the first two terms in eqn. (7 – 215) represents the energy associated with a bit E_b

$$E_b = \int_0^{T_b} s_1^2(t)\,dt = \int_0^{T_b} s_0^2(t)\,dt \qquad (7-168)$$

Substituting eqns. (7 – 165) and (7 – 168) into eqn. (7 – 167),

$$E_d = E_b + E_b - 2\rho E_b$$

$$= 2E_b\,(1-\rho) \qquad (7-169)$$

Substituting eqn. (7 – 169) into eqn. (7 – 164),

$$P_B = Q\left(\sqrt{\frac{E_b\,(1-\rho)}{\eta}}\right) \qquad (7-170)$$

The case of $\rho = -1$ corresponds to $s_1(t)$ and $s_0(t)$ being anticorrelated over a symbol time. The angle between the signal vectors them is 180°. The signals are called in this case antipodal (Fig. 7.17). For $\rho = 0$, the angle between $s_1(t)$ and $s_0(t)$ is zero. This is called orthogonal case.

$$\int_0^{T_b} s_1(t)\, s_0(t)\, dt = 0 \qquad (7-171)$$

For antipodal signals ($\rho = -1$),

$$P_B = Q\left(\sqrt{\frac{2E_b}{\eta}}\right) \qquad (7\text{–}172)$$

For orthogonal signals ($\rho = 0$)

$$P_B = Q\left(\sqrt{\frac{E_b}{\eta}}\right) \qquad (7\text{–}173)$$

Fig. (7.17) shows the signal magnitudes as $\sqrt{E_b}$. The error performance is thus a function of the distance between $\vec{s}_1$ and $\vec{s}_0$. The larger the distance the smaller will be P_B. When the signals are antipodal, the distance between them is $2\sqrt{E_b}$, and the energy E_b associated with this distance is $4E_b$. When we substitute $E_d = 4E_b$ in eqn. (7 –164) the result is eqn. (7 – 172). When the signals are orthogonal, the distance between them is $\sqrt{2E_b}$, $E_d = 2E_b$. When we substitute $E_d = 2E_b$ into eqn. (7 – 164) the result is eqn. (7 – 173)

Ex. 7.9 compare the unipolar and bipolar binary signaling and show how we can use basis functions in the correlator instead of reference signal.

Solution

Fig. (7.18) shows an example of baseband orthogonal (unipolar) signaling where

$$s_1(t) = A \qquad 0 \le t \le T_s \qquad \textit{for binary } 1$$

$$s_o(t) = 0 \qquad 0 \le t \le T_s \qquad \textit{for binary } 0$$

In this case $s_1(t)$ and $s_0(t)$ have zero correlation over T_s as in eqn. (7 – 171). The correlator multiplies and integrates the incoming signal $r_d(t)$ with the difference of the prototype signals $[s_1(t) - s_2(t)] = A$. After T_s a sample yields the test statistic $z(T_s)$ which is then compared to the threshold γ.

$$z(T_s) = \int_0^{T_b} r_d(t)\, s(t)\, dt \qquad (7\text{–}174)$$

For the case of $s_1(t)$, we have

$$r_d(t) = s_1(t) + n(t)$$

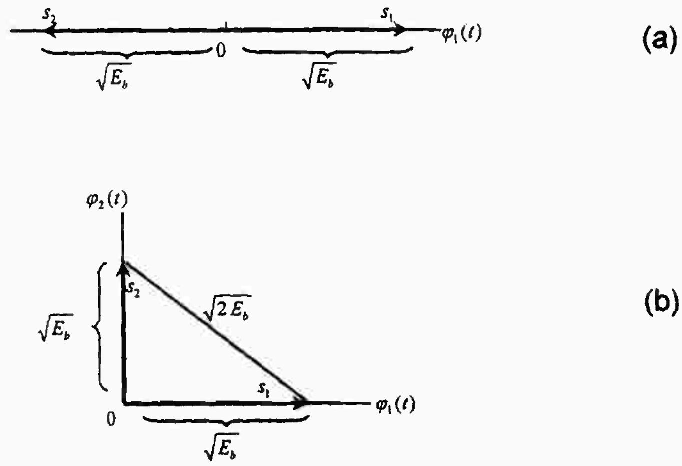


Fig. (7.17) Binary signal vectors

a) antipodal *b) orthogonal*

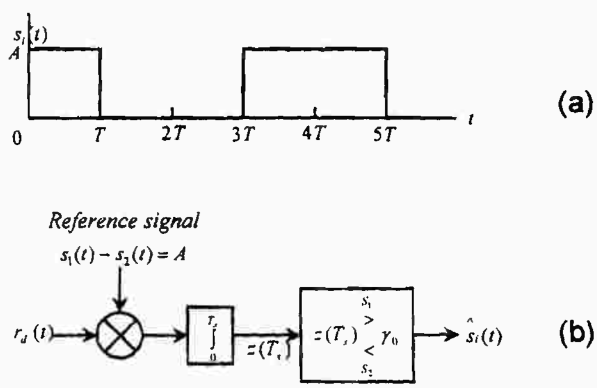


Fig. (7.18) Detection of unipolar baseband signaling

a) unipolar waveform example b) correlator detector

Thus noting that $E\,[n(t)] = 0$, we have

$$A_1 = E\,[z(T_s)]\,[s_1(t)] \tag{7-175}$$

$$= E\left[\left\{\int_0^{T_s} A^2 + A\,n(t)\right\}dt\right] \tag{7-176}$$

$$= A^2\,T_s \tag{7-177}$$

For $s_0(t)$, we have $A_0 = 0$. Thus, is this case the optimum threshold is $\gamma = \frac{A_1 + A_0}{2} = \frac{1}{2} A^2 T_s$. If the test statistic $z\,(T_s) > \gamma$, the signal is declared to be $s_1(t)$. Otherwise, it is declared to be $s_0(t)$. The energy difference signal in eqn. (7 – 163) is $E_d = A^2\, T_s$. Then the bit error performance is given by eqn. (7 – 164) as

$$P_B = Q\left(\sqrt{\frac{E_d}{2\eta}}\right) = Q\left(\sqrt{\frac{A^2\, T_s}{2\eta}}\right) = Q\left(\sqrt{\frac{E_b}{2\eta}}\right) \qquad (7\text{–}178)$$

For equally likely signaling, the average energy per bit is $E_b = A^2 T_s / 2$. Eqn. (7 – 178) is in agreement with eqn. (7 – 173) for orthogonal signals. Now consider (Fig. 7.19) which illustrates an example of baseband antipodal (bipolar) signaling where

$$s_1(t) = +A \qquad 0 \le t \le T_s \qquad \textit{for binary } 1$$

$$s_0(t) = -A \qquad 0 \le t \le T_s \qquad \textit{for binary } 0$$

Thus, the antipodal refers to binary signals that are mirror images of one another, i.e., $s_1(t) = -s_0(t)$. We use two correlators (Fig. 7.19b), one multiplies and integrates the incoming signal $r_d(t)$ with the prototype $s_1(t)$. The second multiplies and integrates $r_d(t)$ with $s_0(t)$.In general, the received signal (unknown basis function plus noise) is sent down a correlator bank. The receiver draws the largest output (the best match) to make detection. For the binary system, we need only two correlators. For a 4-ary system, we need four and so on. The correlator outputs in the binary system are designated $z_i\,(T_s)\,\Big|_{i=1,2}$. The test statistic formed from the difference of the correlator output is

$$z(T_s) = z_1\,(T_s) - z_2\,(T_s) \qquad (7\text{–}179)$$

where $z_2\,(T_s)$ is in the case of binary $z_0\,(T_s)$. For antipodal signals $A_1 - A_2$, $\gamma = 0$. Thus, if test statistic $z(T_s)$ is positive, the signal is declared to be $s_1(T_s)$, and if it is negative if is declared to be $s_2(T_s)$ or $s_0(T_s)$. The energy difference signal is $(2\,A)^2\ T_s$. Then, the bit error performance at the output can be obtained using eqn. (7 – 164)

$$P_B = Q\left(\sqrt{\frac{E_d}{2\eta}}\right) = Q\left(\sqrt{\frac{2\,A^2\ T_s}{\eta}}\right) = Q\left(\sqrt{\frac{2\,E_b}{\eta}}\right) \qquad (7\text{–}180)$$

where the average energy per bit is $E_b = A^2\ T_s$. Eqn. (7 – 180) is in agreement with eqn. (7 – 172) for antipodal signaling.

Now to generalize, instead of designating $s_i(t)$ as the reference signals in the correlator of Fig. (7.19), we may use the concept of basis functions. Binary signaling with unipolar or bipolar pulse provides simple cases where we need only one basis function if we normalize the signaling space choosing $K_j = 1$ (eqn. 6 – 11).

$$E_1 = \int_0^{T_s} \varphi_1^2(t)\, dt = K_1 = 1 \qquad (7\text{– }181)$$

Thus,

$$\varphi_1(t) = \frac{1}{\sqrt{T_s}} \qquad (7\text{– }182)$$

From eqn. (6 – 35), to have $s_1(t) = A$

$$a_{11} = \int_0^{T_s} s_1(t)\, \varphi_1(t)\, dt$$

$$= A\,\frac{1}{\sqrt{T_s}} \times T_s = A\sqrt{T_s} \qquad (7\text{– }183)$$

$s_2(t) = a_{21}\ \varphi_1(t)$, then $a_{21} = 0$. Hence $s_1(t)$ and $s_2(t)$ which is $s_0(t)$ are orthogonal. For bipolar signaling $s_1(t) = A$, $s_2(t)$ which is $s_0(t)$ is $-A$

$$s_1(t) = a_{11}\ \phi_1(t) = A\sqrt{T_s} \times \frac{1}{\sqrt{T_s}} = A \qquad (7\text{– }184)$$

$$s_2(t) = a_{21}\ \varphi_1(t) = -A\sqrt{T_s} \times \frac{1}{\sqrt{T_s}} = -A \qquad (7\text{– }185)$$

For the case of antipodal pulses (Fig. 7.19a) with the reference function $1/\sqrt{T_s}$, and for the case when $s_1(t)$ is sent and $E\left[n(t)\right] = 0$, we have according to eqn. (7 – 175),

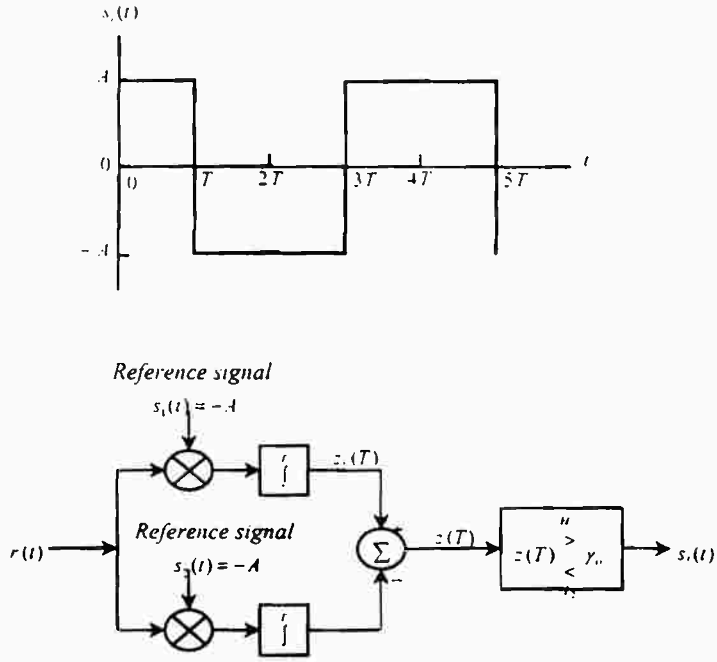
$$V_1(T_s) = E\left[r_d(t)\,|\,s_1(t)\right] = E\left\{\left[\int_0^{T_s} \frac{A}{\sqrt{T_s}} + \frac{n(t)}{\sqrt{T_s}}\right] dt\right\} \qquad (7-186)$$

$$= A\sqrt{T_s} \qquad (7\text{– }187)$$

Since for antipodal signaling $E_b = A^2\, T_s$, then

$$V_1(T_s) = \sqrt{E_b} \qquad (7\text{– }188)$$

$$V_2(T_s) = -\sqrt{E_b} \qquad (7\text{– }189)$$



(a)

(b)

Fig. (7.19) Detection of bipolar baseband signaling
a) bipolar signaling example *b) correlator detector*

When treated this way $V_i(T_s)$ has units of volts proportional within a constant to $\sqrt{E_b}$ .Thus, $z(T_s)$ is proportional to the received signal energy. Once again the dimensions are corrected for in the multiplier – integrator.

Fig. (7.20) illustrates curves of P_B versus E_b/η for bipolar (eqn.7-172) and unipolar signaling. At a certain value of E_b/η, we see that the unipolar signaling yields P_B eqn. (7-173) in the order of 10^{-3}, but the bipolar yields P_B in the order 10^{-6}. Hence, bipolar signaling has a better performance than unipolar signaling. Also, for the same P_B say 10^{-5}, the unipolar signaling of each received bit would require E_b/η about 12.5dB, while bipolar would require E_b/η of only $9.5dB$ i.e. uses less power for the same error tolerance.

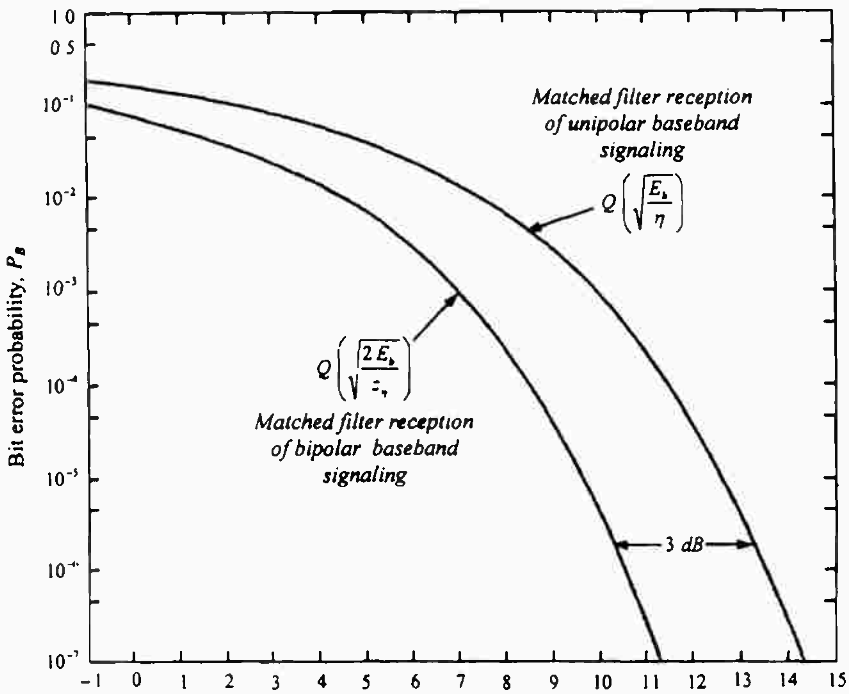


Fig. (7.20) Bit error performance of unipolar and bipolar signaling

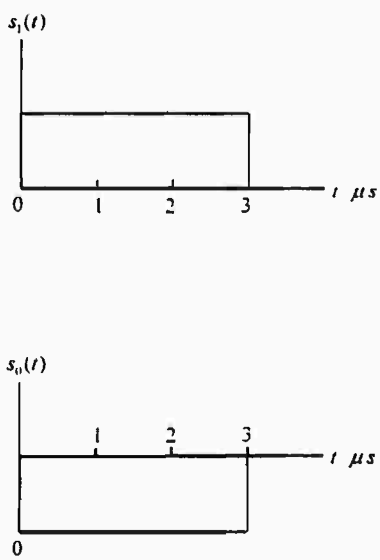
Problems

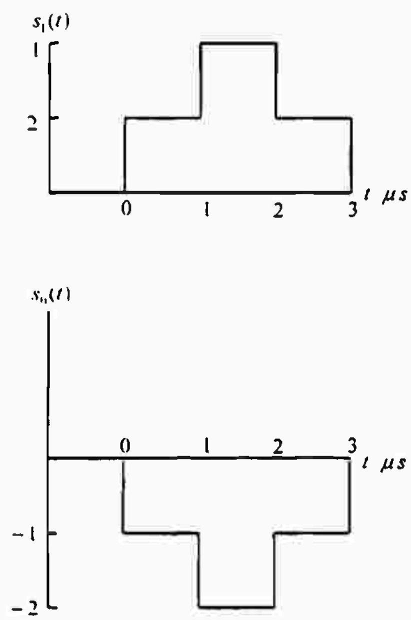
1. Verify Table (7.1).

2. A baseband binary system consists of a positive triangular pulse for digital1and a negative pulse for digital 0. If the absolute widths, peak pulse voltages and noise PSD at the input of an ideal correlator are all the same as in Ex 7.5 find the probability of bit error.

3. Consider_a binary system that receives equally likely signals $s_1(t)$ and $s_0(t)$ plus AWGN $\eta = 10^{-12}\ W/Hz$. Use the values of the received signal voltage to compute the bit error probability in a matched filter.

4. In the problem above the waveform is changed as shown, recalculate P_B. What do you conclude?





Prob. 7.3 **Prob. 7.4**

5. If the input to a filter is $e^{j\omega t}$ the output is the convolution of the input and the impulse response of the filter which gives $H(\omega)\, e^{j\omega t}$. This is called

eigenvalue problem where $H(\omega)$ is the eigenvalue and $e^{j\omega t}$ is the eigenfunction. Explain its significance and show what happens when $e^{j\omega t}$ is inputted to a matched filter.

6. A bipolar binary signal $s_i(t)$ is $+1$ or -1 during interval 0 to T_s. AWGN is added with density $10^{-3}\ W/Hz$. If the received signal is detected with a matched filter determine the maximum bit rate that can be sent with a bit error probability of $P_B \le 10^{-3}$.

7. Specify a baseband Nyquist channel having a piecewise linear amplitude response, an absolute bandwidth of $10^4\ Hz$ and is appropriate for a baud rate of $10k$ baud. What is the channels excess bandwidth.?

8. What absolute bandwidth is required to transmit information at a rate of $8kb/s$ using 64 level baseband signaling over a raised cosine channel with a roll off factor 40%?

9. A binary information source consists of statistically independent equiprobable symbols. If the bandwidth of the baseband channel over which the symbols are to be transmitted is $3kHz$, what baud rate will be necessary to achieve a spectral efficiency of 2.5 bits/Hz? Is ISI free reception at this baud rate possible? What must be the minimum size of the source symbol alphabet to achieve ISI free reception and a spectral efficiency of 10 bits/Hz?

10. A baseband binary PCM system is used to transmit a single $3.4kHz$ voice signal. If the sampling rate for the voice signal is $8kHz$ and 256 level quantization is used, calculate the bandwidth for raised cosine characteristic.

11. Show that the pdf of the sum of two signals is the convolution of the pdf of the signals.

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