CHAPTER 2

BASIC CONCEPTS

2.1 INTRODUCTION

This chapter outlines the basic concepts which are utilized in structural analysis. Some of the basic concepts have been taught in courses like statics and strength of materials. Other concepts are presented in order to complete the basics of structural analysis and shall extensively be used in other chapters. Presentation of all basic concepts in this chapter shall aid in easy understanding of the structural analysis methods.

2.2 SMALL DEFORMATION THEORY

It is assumed throughout this book that the geometry changes of the structure are very small and can be neglected as compared with structure's dimension. In some structures like arches, and cable structures, the geometry changes affect significantly the final stress state in the structure. Although the geometry changes in structures can be considered by methods like second-order theory and large-deformation theory, these kinds of structures are not considered in this text. However, these special structures can normally be analyzed, ignoring the effect of the deformation on the final internal actions.

2.3 ELASTIC STRUCTURES

If a structure is loaded and then is unloaded and retains its original shape without permanent deformation, the structure is said to be an elastic structure. The load-deflection curve of an arbitrary joint for this kind of structures is shown in Figure 2.1. For this behavior, one concludes that the structure is made of elastic materials whose stress-strain relation is as shown in Figure 2.2. The inelastic materials represent those which do keep strain after unloading, as shown in Figure 2.3.

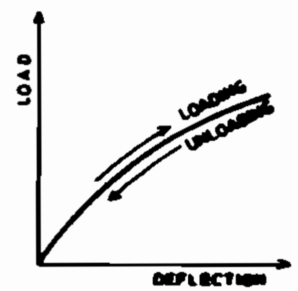


Figure 2.1

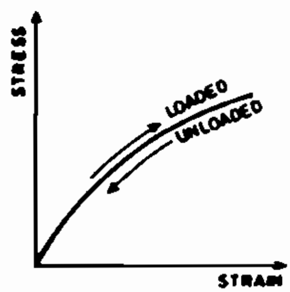


Figure 2.2

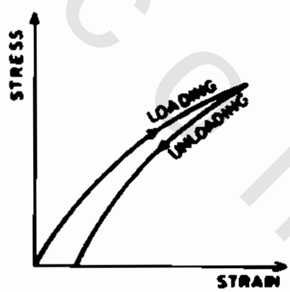


Figure 2.3

2.4 LINEAR-ELASTIC STRUCTURES

Those structures which retain their original shape after unloading such that the load-deflection relationship of any arbitrary joint is linear, are called linear-elastic structures. These structures are made of linear-elastic materials whose stress-strain relation is linear as shown in Figure 2.4.

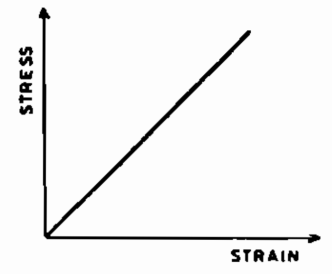


Figure 2.4

2.5 PRINCIPLE OF SUPERPOSITION

Considering the small deformation theory and linear-elastic structures, the sequence of loading a structure does not affect the final stress state. Figure 2.5 illustrates an example of this principle. It indicates that one may use this principle to make symmetry in loading the structure, as part of the final solution.

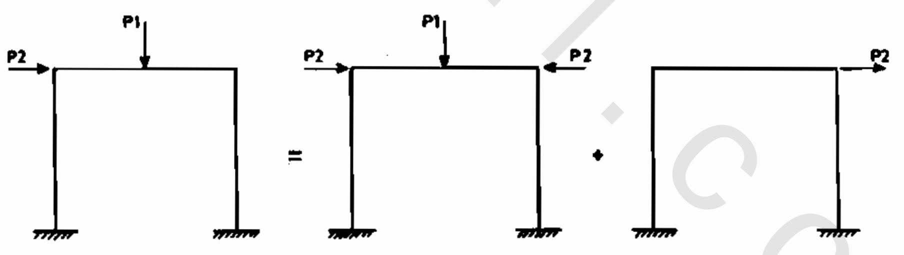


Figure 2.5

2.6 STATIC EQUILIBRIUM

A linear-elastic structure loaded gradually up to specific magnitudes of loads is said to be in a static equilibrium state if the equilibrium conditions are satisfied. For planar structures in static equilibrium one has three conditions of equilibrium stated as follows:

Summation of horizontal force components along x-axis $= \Sigma F_x = 0$ (2.1)

Summation of vertical force components along y-axis $= \Sigma F_y = 0$ (2.2)

Summation of moments about z-axis at any arbitrary point $= \Sigma M_z = 0$ (2.3)

where x and y are two Cartesian axes in the plane of the structure and z is an out of plane axis.

For space structures in static equilibrium one has six equations of equilibrium: three for forces and three for moments, respectively, along the direction of the three orthogonal Cartesian coordinates x, y, and z.

These conditions assure that the structure remains at rest and every joint or member is in static equilibrium. If these equilibrium conditions are able to determine the reactions and internal forces in the structure, the structure is called statically determinate. If the number of reactions and internal forces exceeds the number of equilibrium equations, the structure becomes statically indeterminate.

2.7 COMPATIBILITY

The structure consists of elements connected by joints. If the structure is loaded, then the deformed shape is a unique one and every point in the structure must lie on the deformed shape. If a number of members are rigidly connected at a certain joint, as shown in Figure 2.6, then after deformation, every end-member connected with that joint must have displaced the same magnitude and direction in order to keep the integrity at the joint. This principle is used as a condition to solve statically indeterminate structures by the compatibility method. This principle is also called connectivity condition to indicate the connection at each joint.

2.8 BOUNDARY CONDITIONS

Boundary conditions include conditions related to the deformation and actions at the boundary joints. For example, a hinged support which has zero deflection and

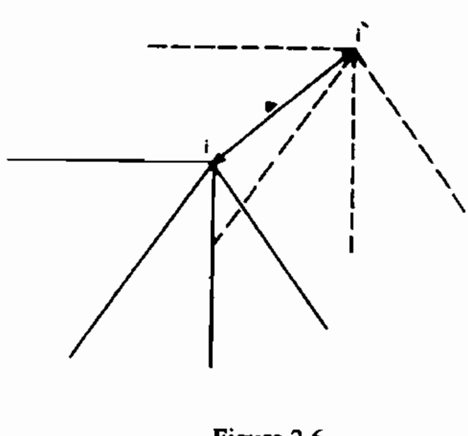


Figure 2.6

zero moment includes both deflections and actions types boundary conditions. Fixed support boundary conditions have the deflections and slopes are zero. The boundary conditions must be specified for every structure in order to obtain a unique solution for the analysis problem. Types of boundary conditions, for planar structures are summarized in Table 2.1 below.

Table 2.1 : Types of Boundary Conditions for Planar Structures

Joint Type	Actions Conditions			Deformation Conditions		
	Horizontal Force	Vertical Force	Bending Moment	Horizontal Displacement	Vertical Displacement	Rotation
Roller	0		0		0	
Hinged			0	0	0	
Fixed				0	0	0

2.9 UNIQUE SOLUTION

By satisfying the equilibrium, compatibility, and boundary conditions for a loaded structure, the deformed shape and the state of stress in the structures are unique, and can not take other forms.

2.10 REACTIONS AND INTERNAL FORCES OF DETERMINATE STRUCTURES

In this section it is shown how the equilibrium conditions are used to determine the reactions and internal forces in statically determinate structures.

2.10.1 Plane Trusses

The truss consists of a system of members connected together by hinged joints. Each member is only subjected to an axial force. The unknowns in the truss analysis represents the number of reactions and the members forces. Because each truss joint is a hinge, one can apply two equilibrium equations at each truss joint that is $\Sigma F_x = 0$ and $\Sigma F_y = 0$. From these equations the members forces in statically determinate truss can be determined. The reactions at the supports of the truss can be determined using the three equilibrium equations (2.1), (2.2), and (2.3) for the whole structure.

Example 2.1

Determine the reactions and the internal forces in the truss shown in Figure 2.7.

The support at A is hinged which has the condition that the moment about z-axis is zero, while horizontal and vertical force reactions are unknown. Support B is a roller which has the condition that the force in the horizontal direction and the moment about z-axis are zero. The vertical reaction at B is an unknown. The reactions for this truss are shown in Figure 2.8.

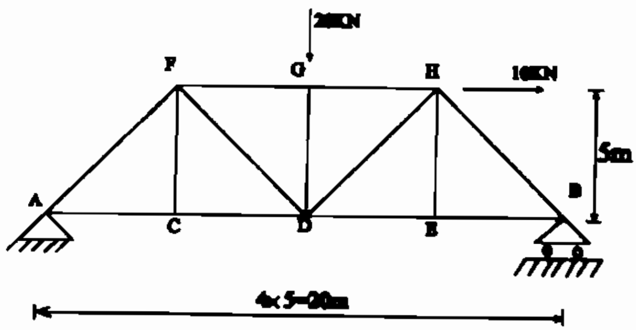


Figure 2.7

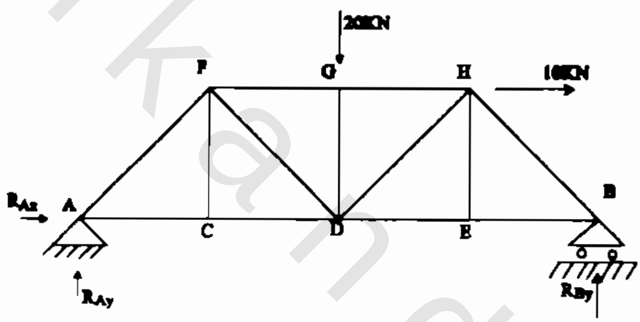


Figure 2.8

To determine the reactions, the three equilibrium equations are applied as follows:

Sum of horizontal force, ΣF_x, = 0,
$R_{AX} + 10 = 0$; $R_{Ax} = -10$ kN

The negative sign of R_{Ax} indicates the reverse of the assumed direction.

Sum of vertical forces, ΣF_y, = 0,
$R_{Ay} + A_{By} - 20 = 0$

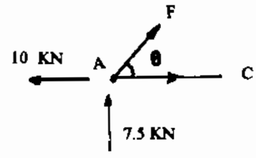
Moment of all forces about any joint = 0
By taking moment about Joint A = 0, one obtains
$20 \times R_{By} - 20 \times 10 - 10 \times 5 = 0$; $R_{By} = 12.5$ kN (↑upward)

Substituting into the sum of vertical forces equation one gets
$R_{Ay} = 20 - 12.5 = 7.5$ kN (↑ upward)

After determining the reactions, one is able to determine the members forces by studying the equilibrium of each joint in the truss. This method is called method of joints.

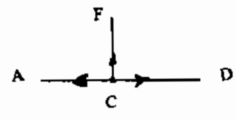
Equilibrium of Joint A

$\Sigma F_x = 0$	$F_{AC} - 10 + F_{AF} \cos\theta = 0$
$\Sigma F_y = 0$	$7.5 + F_{AF} \sin\theta = 0$
These result in	$F_{AF} = -10.6$ kN (compression)
	$F_{AC} = +17.5$ kN (Tension)



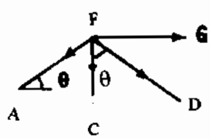
Equilibrium of Joint C

$\Sigma F_x = 0$	$F_{CD} - F_{CA} = 0$
$\Sigma F_y = 0$	$F_{CF} = 0$
	$F_{CD} = F_{CA} = 17.5$ kN (Tension)



Equilibrium of Joint F

$\Sigma F_x = 0$	$F_{FG} + F_{FD} \sin\theta - F_{FA} \cos\theta = 0$
$\Sigma F_y = 0$	$-F_{FC} - F_{FD} \cos\theta - F_{FA} \sin\theta = 0$
	$F_{FG} + 0.707\, F_{FD} - 0.707\,(-10.6) = 0$
	$-0.707\, F_{FD} - 0.707\,(-10.6) = 0$



Solving for F_{AC} and F_{FD} one obtains

$F_{FD} = 10.6$ kN (Tension)

$F_{FG} = -15$ kN (Compression)

The values of members forces are shown in Figure 2.9.

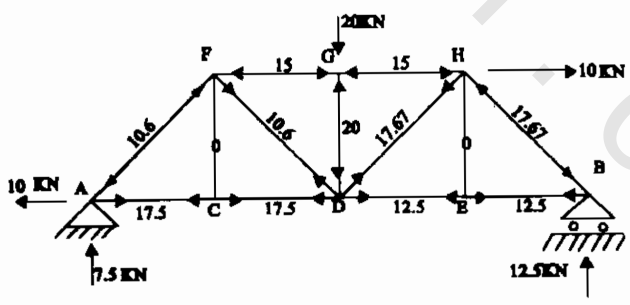


Figure 2.9

One can check the solution by testing the values of internal forces of specific members. For example, by cutting the truss into two parts through members GH, DH, and DE as shown in Figure 2.10, one can determine the internal forces in these members using the equilibrium conditions for each part of the truss.

By studying the equilibrium of the right part, one has

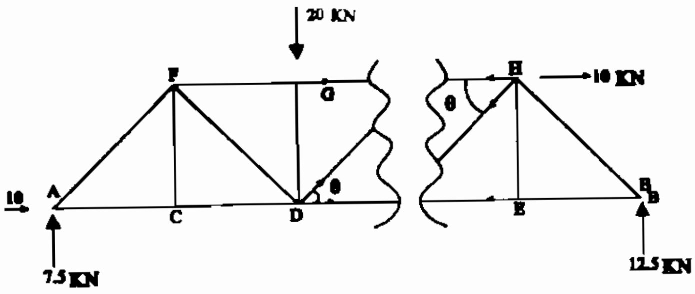


Figure 2.10

$\Sigma F_y = 0$;	$12.5 - F_{HD} \sin\theta = 0$;	$F_{HD} = 17.67$ (tension)
$\Sigma M_H = 0$;	$12.5 \times 5 - F_{ED} \times 5 = 0$;	$F_{ED} = 12.5$ kN (Tension)
$\Sigma F_x = 0$;	$F_{ED} + F_{HG} + F_{HD} \cos\theta = 10$;	$F_{HG} = -15$ kN (Compression)

which are the same results obtained from the method of joints.

2.10.2 Beams

The internal forces at any section along a beam consist of axial force along the axial axis passing through the centroid of the cross section, shear force along the vertical axis, and bending moment about the horizontal axis passing through section's centroid. The three orthogonal cartesian coordinates x, y, and z passing through sections' centroid are, respectively, the direction of axial force, shear force, and bending moment. In order to determine the internal forces at any beam's section, the reactions at the beam supports must first be determined using the static equilibrium equations. The static equilibrium equations can then be applied for any portion of the beam.

Example 2.2

Determine the reactions and the internal forces at sections C and D for the beam shown in Figure 2.11.

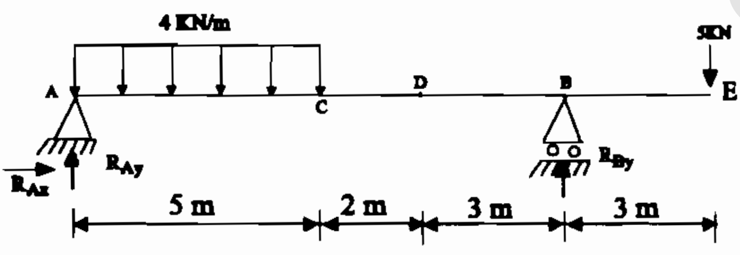


Figure 2.11

The support A is hinged and support B is a roller. By applying the static equilibrium equations, one has

$\Sigma F_x = 0$; $R_{Ax} = 0$
$\Sigma F_y = 0$; $R_{Ay} + R_{By} - 4 \times 5 - 5 = 0$
$\Sigma M_A = 0$; $4 \times 5 \times 2.5 + 5 \times 13 - R_{By} \times 10 = 10$; $R_{By} = 11.5$ kN (↑ upward)
$R_{Ay} = 25 - 11.5 = 13.5$ kN (↑ upward)

To determine the internal forces at C, the beam is cut at C and the static equilibrium conditions are applied to the left or the right part of the section as shown in Figure 2.12.

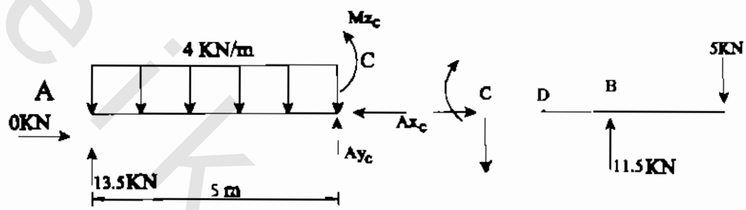


Figure 2.12

$\Sigma F_x = 0$; $R_{xc} = 0$
$\Sigma F_y = 0$; $13.5 - 4 \times 5 + A_{yc} = 0$; $A_{yc} = 6.5$ kN(↑)
$\Sigma M_z = 0$; $M_{zc} + A_{yc} \times 5 - 4 \times 5 \times 2.5 = 10$; $M_{zc} = 50 - 6.5 \times 5 = 17.5$ kN.m (↑)

Same results could be obtained from studying the equilibrium of the right part.

Similarly, to determine the internal forces at D one cuts the beam at this section and studies the equilibrium of the left or right part as shown in Figure 2.13.

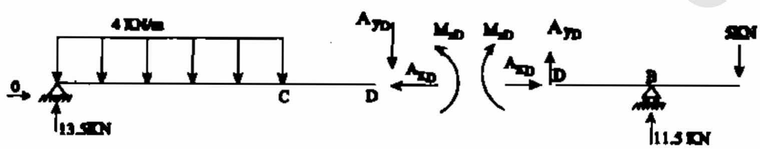


Figure 2.13

By studying the equilibrium of the right part, one has

From $\Sigma F_x = 0$; $A_{xD} = 0$
From $\Sigma F_y = 0$; $A_{yD} + 11.5 - 5 = 0$; $A_{yD} = -6.5$ kN(↓)

From $\Sigma M_B = 0$; $M_{zD} + A_{yD} \times 3 + 5 \times 3 = 0$

$M_{zD} - 6.5 \times 3 + 5 \times 3 = 0$; $M_{zD} = +4.5$ kN.m.

2.10.3 Plane Frames

The determination of internal forces in plane frames follows the same steps as in beams. Any section in a plane frame is subjected to axial force, shear force and bending moments. The internal forces at any section can be determined after calculating the reactions using the static equilibrium equations.

Example 2.3

Determine the reactions and internal forces at sections C and D for the frame shown in Figure 2.14.

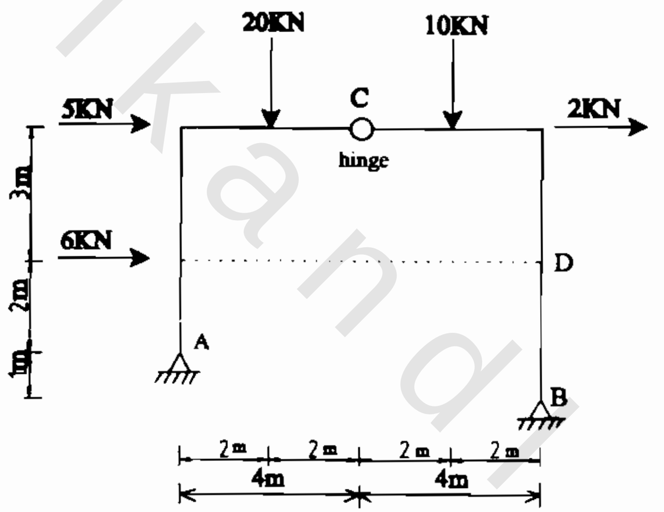


Figure 2.14

The supports at A and B are both hinged, thus one has four reactions, one horizontal and one vertical at each support. Since we have three static equilibrium equations, we must have an extra equilibrium equation, otherwise, the frame become a statically indeterminate. Since there is a hinge at C, this imposes an additional equilibrium condition that is $M_C = 0$ for any part of the frame passing through joint C, as shown in Figure 2.15.

Taking moment about C for the right part, one obtains:

$4\,R_{By} - 6\,R_{Bx} - 10 \times 2 = 0$

Taking moment about A one has:

$8\,R_{By} - R_{Bx} - 2 \times 5 - 10 \times 6 - 20 \times 2 - 5 \times 5 - 6 \times 2 = 0$

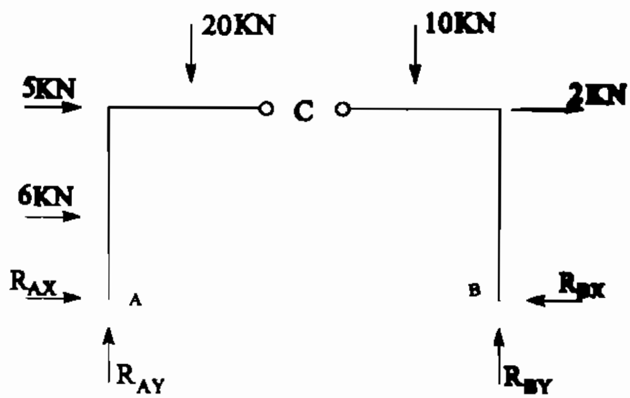


Figure 2.15

Solving the previous equation for R_{Bx} and R_{by} one obtains

$R_{Bx} = 9.727$ kN (←) , $R_{By} = 19.59$ kN (↑)

Applying the remaining equilibrium equations, one gets

$\Sigma F_x = 0$; $R_{Ax} - R_{Bx} + 5 + 2 + 6 = 0$; $R_{Ax} = -3.273$ kN (←)
$\Sigma F_y = 0$; $R_{Ay} + R_{By} - 20 - 10 = 0$; $R_{Ay} = 10.41$ kN (↑)

The internal forces at C can be obtained by studying the equilibrium of part AC as shown in Figure 2.16.

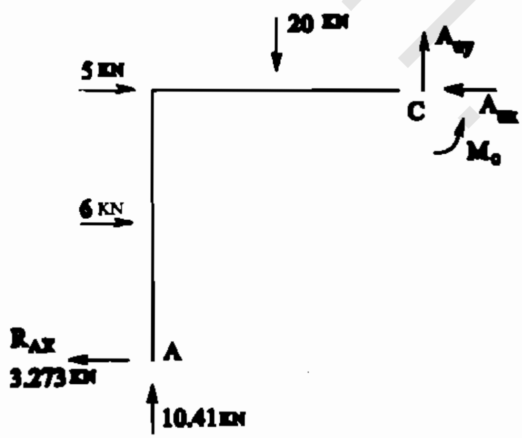


Figure 2.16

$\Sigma F_x = 0$; $5 + 6 - 3.273 - A_{Cx} = 0$; $A_{Cx} = 7.723$ kN (←)
$\Sigma F_y = 0$; $10.41 - 20 + A_{Cy} = 0$; $A_{Cy} = 9.59$ kN (↑)

The internal forces at D are determined by cutting the frame at D and studying the equilibrium of part AD or part DB. The internal forces are shown in Figure 2.17.

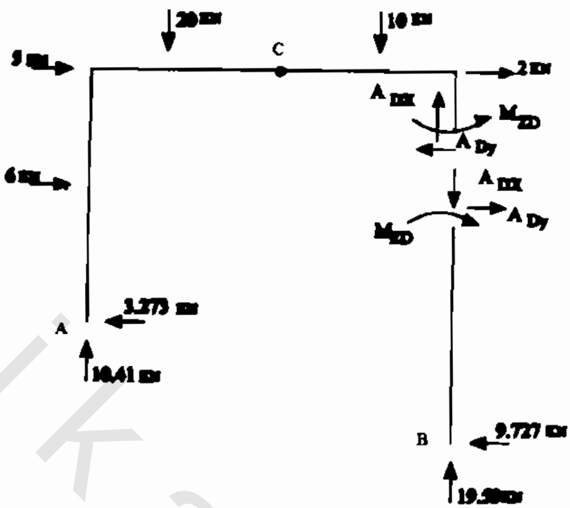


Figure 2 17

$\Sigma F_x = 0$;	$A_{Dy} - 9.727 = 0$;	$A_{Dy} = 9.727$ kN
$\Sigma F_y = 0$;	$-A_{Dx} + 19.59 = 0$;	$A_{Dx} = 19.59$ kN
$\Sigma M_B = 0$;	$M_{zD} + A_{Dy} \times 3 = 0$;	$M_{zD} = -29.181$ kN.m.

2.11 WORK AND COMPLEMENTARY WORK

If an action force or moment A_i is gradually applied to a structure resulting in deformation D_i in the same direction of A_i, and the action-deformation relation is given as shown in Figure 2.18, then the work done by A_i during the deformation D_i is given by

$$W = \int_0^{D_i} A \, dD \qquad (2.4)$$

The complementary work is defined as

$$W^* = \int_0^{A_i} D \, dA = A_i D_i - W \qquad (2.5)$$

For linear-elastic structures, the action-deformation relationship is linear as shown in Figure 2.19. In this case, the work and complementary work are both equal and each is given by

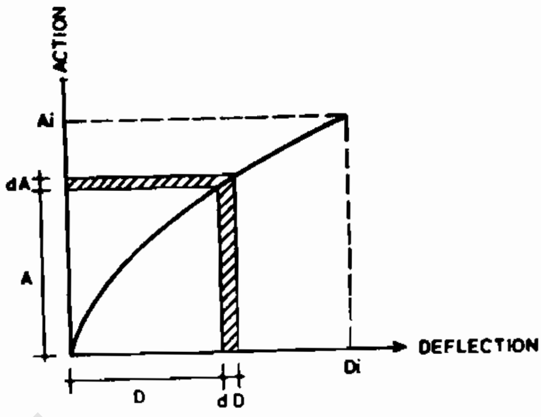


Figure 2.18

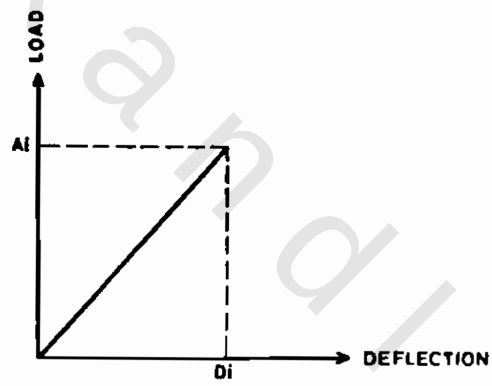


Figure 2.19

$$W = W^* = \tfrac{1}{2} A_i \ D_i \qquad (2.6)$$

If the linear-elastic structure is subjected to a set of actions (A_1, A_2, ... A_n) resulting in their directions the deformation (D_1, D_2, ..., D_n) respectively, then the work done by the actions is expressed as

$$W = \frac{1}{2}\left(A_1 \ D_1 + A_2 \ D_2 \ + \ldots\ldots + A_n \ D_n\right) = \frac{1}{2} \underline{A}^T \ \underline{D}$$
$$= \tfrac{1}{2} \underline{D}^T \ \underline{A} \qquad (2.7)$$

where $\underline{A}^T = [A_1 \ A_2 \ A_3 \ldots A_n]$; and $\underline{D}^T = [D_1 \ D_2 \ldots D_n]$.

2.12 STRAIN ENERGY AND COMPLEMENTARY STRAIN ENERGY

Consider an infinitesimal segment dv of a structural element subjected to final stress σ_i and the corresponding strain is ε_i as shown in Figure 2.20. The work done by the stress during the element deformation is called strain energy. The strain energy due to stress σ_i and strain ε_i is expressed as

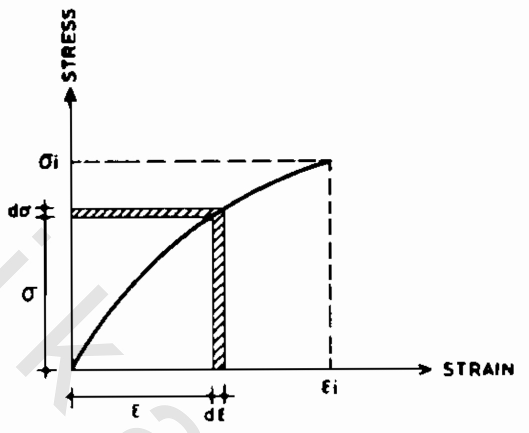


Figure 2.20

$$dU_i = \left(\int_0^{\varepsilon_i} \sigma \, d\varepsilon \right) dv \qquad (2.8)$$

The total strain energy due to stress σ_i and strain ε_i in a structure is then obtained by integrating all structural elements to give

$$U_i = \int_v \left(\int_0^{\varepsilon_i} \sigma \, d\varepsilon \right) dv \qquad (2.9)$$

The total complementary strain energy due to strain ε_i and stress σ_i in a structure is defined as

$$U_i^* = \int_v \left(\int_0^{\sigma_i} \varepsilon \, d\sigma \right) dv$$

$$= \int_v \sigma_i \, \varepsilon_i \, dv - U_i \qquad (2.10)$$

For structures made of linear-elastic material as shown in Figure 2.21, the strain energy and complementary strain energy due to stress σ_i and strain ε_i are both equal and each is given by

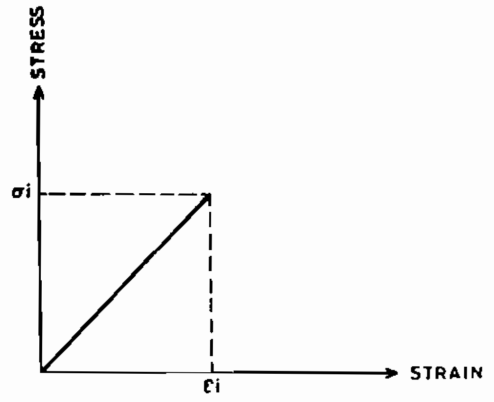


Figure 2.21

$$U_i = U_i^* = \frac{1}{2}\int_v \sigma_i\, \varepsilon_i\, dv \qquad (2.11)$$

For a linear-elastic structure subjected to several types of stresses σ_1, σ_2, ... σ_n and the corresponding strains are ε_1, ε_2, ... ε_n, the total strain energy in the structure is obtained from

$$\begin{aligned} U &= \frac{1}{2}\int_v \left(\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \ldots \sigma_n \varepsilon_n\right) dv \\ &= \frac{1}{2}\int_v \underline{\sigma}^T \underline{\varepsilon}\, dv \\ &= \frac{1}{2}\int_v \underline{\varepsilon}^T \underline{\sigma}\, dv \qquad (2.12) \end{aligned}$$

where $\underline{\sigma}^T = [\sigma_1\, \sigma_2 \ldots\ldots \sigma_n]$ and $\underline{\varepsilon}^T = [\varepsilon_1\, \varepsilon_2 \ldots\ldots \varepsilon_n]$

2.13 TYPES OF STRAIN ENERGY

In general, any member in a skeletal structure is subjected to six different types of internal actions, namely one axial force, two shear forces, one twisting moment, and two bending moments, as shown in Figure 2.22. The strain energy due to the stresses and strains resulting from these internal actions are derived in this section.

For a member of length L along the x-axis and with a uniform cross section area A, the strain energy due to the axial force A_x which causes normal stress σ_x is obtained as

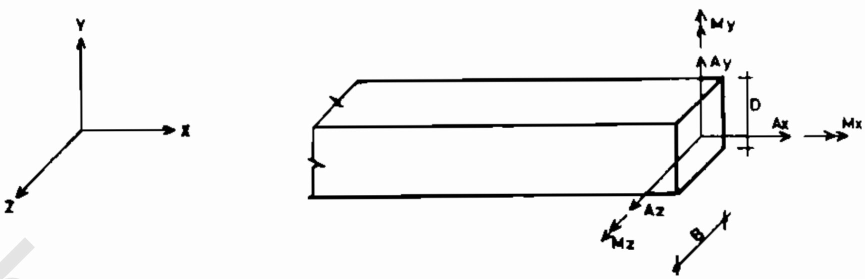


Figure 2.22

$$U_1 = U_1^* = \frac{1}{2}\int_v \sigma_x \, \varepsilon_x \, dv$$

$$= \frac{1}{2}\int_0^L \frac{A_x}{A} \times \frac{A_x}{EA} \times (A \times dx) = \frac{1}{2}\int_0^L \frac{A_x^2}{EA} \times dx \qquad (2.13)$$

in which E is the modulus of elasticity of member's material.

If the member is subjected to a shearing force A_y resulting in shear stress τ_y and shear strain γ_y, the strain energy is obtained as follows:

$$U_2 = \frac{1}{2}\int_v \tau_y \, \gamma_y \, dv$$

$$= \frac{1}{2}\int_v \frac{A_y Q_z}{I_z B} \times \frac{A_y Q_z}{G I_z B}(dy\, B\; dx)$$

$$= \frac{1}{2}\int_0^L \frac{A_y^2 \, dx}{G A_{ry}} \qquad (2.14)$$

in which Q_z represents the moment of area above or below a certain level in the section about the z axis as shown in Figure 2.23, I_z is moment of inertia about z axis, G is the shear modulus, and A_{ry} is defined as

$$\frac{1}{A_{ry}} = \int_0^{D/2} \frac{Q_z^2 \, dy}{I_z^2 \, B} \qquad (2.15)$$

Similarly, for the shear force A_z, one has

$$U_3 = \int_0^L \frac{A_z^2 \, dx}{2G A_{rz}} \qquad (2.16)$$

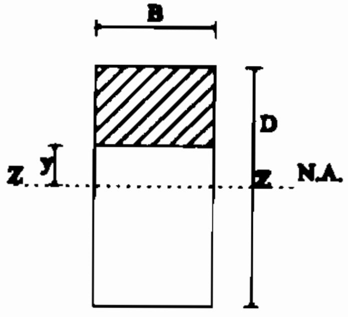


Figure 2.23

where

$$\frac{1}{A_{rz}} = \int_0^{B/2} \frac{Q_y^2\, dz}{I_y^2\, D} \tag{2.17}$$

and Q_y is the moment of area of an area about y axis, and I_y is the moment of inertia about the y-axis.

For the twisting moment M_x, the strain energy is

$$U_4 = \frac{1}{2}\int_v \left(\frac{M_x\, r}{J_x}\right)\left(\frac{M_x\, r}{G\, J_x}\right) dA\, dx$$
$$= \int_0^L \frac{M_x^2\, dx}{2\, G\, J_x} \tag{2.18}$$

where J_x is the polar moment of inertia, and r is the distance from the section centroid.

For the bending moment M_y, one has

$$U_5 = \frac{1}{2}\int_v \left(\frac{M_y\, z}{I_y}\right)\left(\frac{M_y\, z}{E\, I_y}\right) dA\, dx$$
$$= \frac{1}{2}\int_0^L \frac{M_y^2\, dx}{2\, E\, I_y} \tag{2.19}$$

Similarly, for the bending moment M_z, the strain energy is

$$U_6 = \frac{1}{2}\int_0^L \frac{M_z^2\, dx}{E\, I_z} \tag{2.20}$$

The total strain energy due to the six internal actions in the member is the sum of U_1, U_2, ..., and U_6. This is expressed as

$$U = \sum_{i=1}^{6} U_i \tag{2.21}$$

For planar structures of dimensions in the x-y plane and subjected to loads in the same plane, the total strain energy in all members becomes

$$U = \int \frac{A_x^2\,dx}{2EA} + \int \frac{M_z^2\,dx}{2EI_z} + \int \frac{A_y^2\,dx}{2GA_{ry}} \tag{2.22}$$

Values of A_{ry} or A_{rz} depend on the cross-section shape. For example, A_r = 0.833A for the rectangular section, and A_r = 0.9A for the circular section.

It is not necessary that the deformation in the structure is due to applied loading only. The temperature changes and the settlements in the footings introduce also strain energy and work for the loaded structure.

Consider for example member ij which is subjected to a rise in temperature, as shown in Figure 2.24. The axial elongation in an infinitesimal element dx is obtained as

$$\Delta_x = \alpha\left(\frac{T_1 + T_2}{2}\right)dx \tag{2.23}$$

where α is the coefficient of thermal expansion

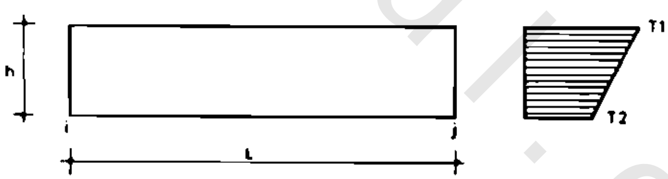


Figure 2.24

Therefore, the axial strain due to the axial deformation is obtained from

$$\varepsilon_x = \frac{\Delta x}{dx} = \alpha\left(\frac{T_1 + T_2}{2}\right) \tag{2.24}$$

Similarly, the rotation due to temperature deformation is calculated from

$$\theta_z = -\alpha\left(\frac{T_1 - T_2}{h}\right)dx \tag{2.25}$$

Where h is the member depth and the negative sign indicates that the tension is on the top fibers. Therefore, the curvature due to this rotation is obtained as

$$y'' = \frac{\theta_z}{dx} = -\alpha\left(\frac{T_1 - T_2}{h}\right) \tag{2.26}$$

The axial strain at any point of height y from the neutral axis is obtained as shown in Figure 2.25 from

$$\varepsilon_x = y''\,y = -\alpha\left(\frac{T_1 - T_2}{h}\right)y \tag{2.27}$$

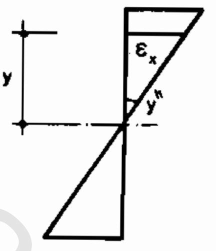


Figure 2.25

Therefore, the strain energy due to the temperature change on member ij under the internal actions of A_x, A_y, and M_z is obtained from the substitution into Eq. 2.12 as follows:

$$\begin{aligned} U &= \int_v \sigma_x\,\varepsilon_x\,dv \\ &= \int_0^L \left(\frac{A_x}{A}\right)\alpha\left(\frac{T_1 + T_2}{2}\right)A\,dx - \int_v \left(\frac{M_z\,y}{I}\right)\alpha\left(\frac{T_1 - T_2}{h}\right)y\,dv \\ &= \alpha\left(\frac{T_1 + T_2}{2}\right)\int_0^L A_x\,dx - \alpha\left(\frac{T_1 + T_2}{h}\right)\int_0^L M_z\,dx \end{aligned} \tag{2.28}$$

Similarly, if the support in any structure displaces a distance Δ, and the support is placed on an elastic material of stiffness K, then the reaction, P, at the support does an external work given by

$$W = \frac{1}{2}P\,\Delta \tag{2.29}$$

From the definition of stiffness which is a force per unit displacement as shown in Figure 2.26, one may write Equation 2.29 as

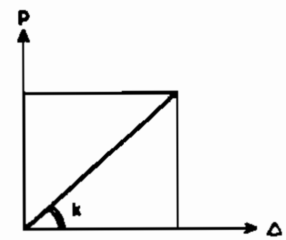


Figure 2.26

$$W = \frac{1}{2} K \Delta^2 \qquad (2.30)$$

or,

$$W = \frac{1}{2K} P^2 \qquad (2.31)$$

where $\frac{1}{K}$ is called the spring flexibility.

2.14 INTERNAL FORCES DIAGRAMS

The signs used in drawing the internal forces diagrams in this book are shown in Figure 2.27. It is noticed that the bending moment diagram is drawn on the tension side of the member which is useful to the designer of concrete structures in visualizing the locations of placing the tension steel reinforcement.

Example 2.4

A linear-elastic frame is subjected to the loads shown in Figure 2.28. The cross-section of each member has a rectangular shape of 40 cm width and 100 cm depth. If $E = 2100$ kN/cm^2, determine the internal actions diagrams and the strain energy stored in the structure. Consider $G = 0.4E$.

Applying the static equilibrium equations at several sections in the frame, the axial force, shear force and bending moment diagrams are obtained as shown in Figure 2.29. Using Equations 2.13, 2.14 and 2.20, the strain energy due to each internal force is calculated as follows:

$$U_1 = \int \frac{A_x^2\, dx}{2EA} = 2 \int_0^{1000} \frac{10^2\, dx}{2(2100)(40 \times 100)} = 0.0119 \text{ kN.cm}$$

$$U_2 = \int \frac{A_y^2\, dx}{2GA_{ry}} = 2 \int_0^{500} \frac{100\, dx}{2 \times 0.4 \times 2100 \times (0.833 \times 40 \times 100)} = 0.01785 \text{ kN}$$

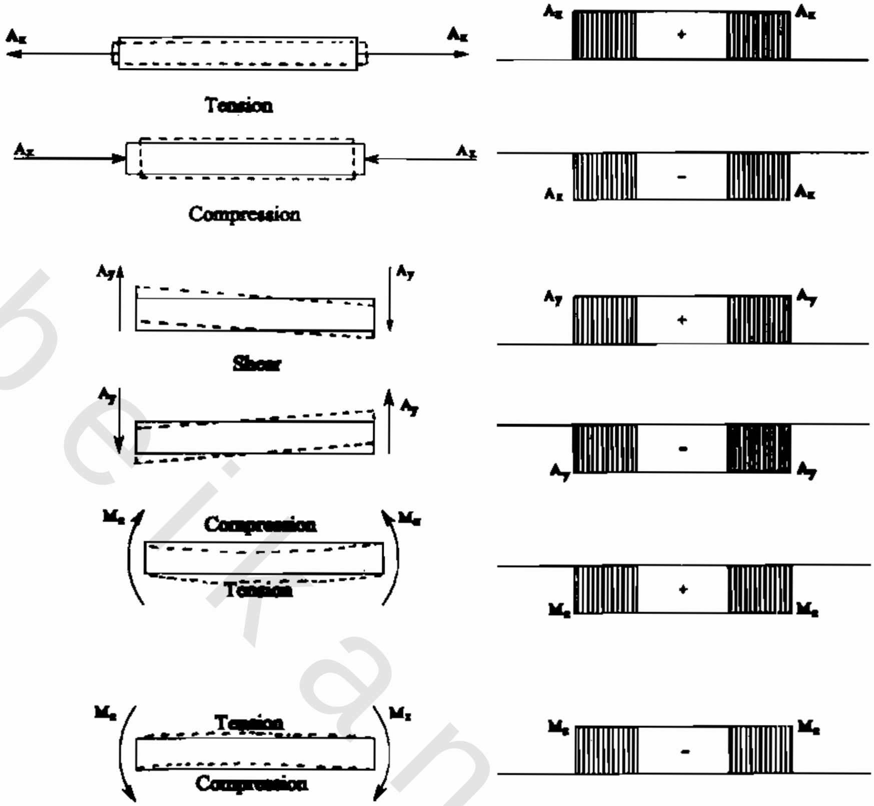


Figure 2.27

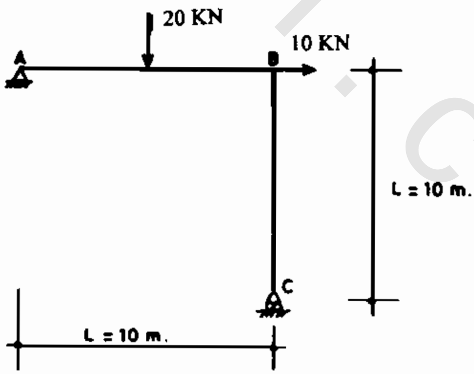


Figure 2.28

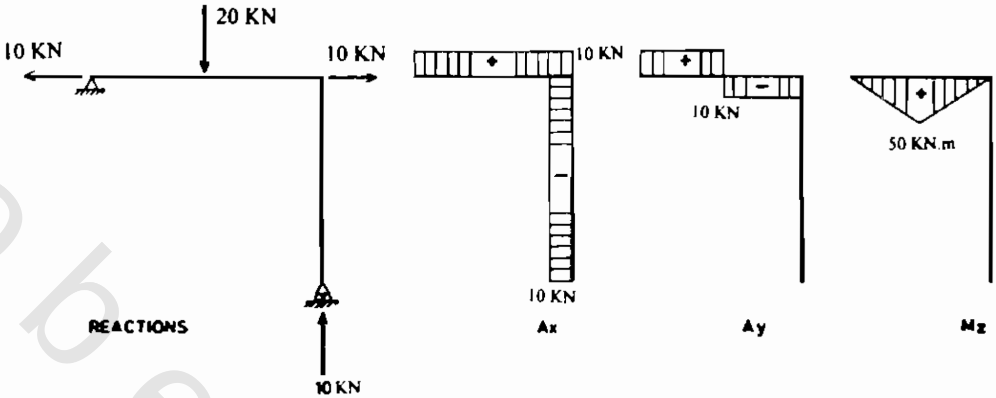


Figure 2.29

$$U_3 = \int \frac{M_z^2\,dx}{2\,E\,I_z} = 2\int_0^{500}\left(\frac{(10\,x)^2\,dx}{2\times 2100\times \frac{40\times 100^3}{12}}\right) = 0.5952\ \text{kN.cm}$$

Thus, the total strain energy stored in the structure is

$$U = U_1 + U_2 + U_3 = 0.6249\ \text{kN.cm}$$

2.15 RELATIONSHIP BETWEEN LOAD, SHEAR FORCE AND BENDING MOMENT

Considering an element dx of a member taken at a location x from the origin and subjected to distributed load $\omega(x)$, the internal forces and bending moments on the element are described in Figure 2.30.

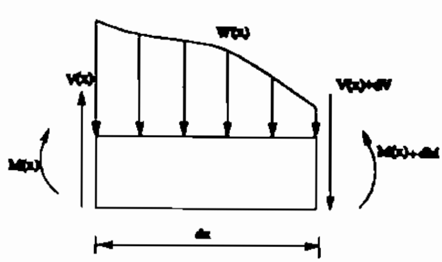


Figure 2.30

By studying the equilibrium in the vertical direction of this element, one obtains

$$V(x) - \omega(x)\,dx - \left(V(x) + dv\right) = 0$$

From which one has

$$\frac{dV}{dx} = -\,\omega(x) \qquad (2.32)$$

By studying the equilibrium of moment at any point one has

$$M(x) + dM - dx\left[V(x) + dv\right] - \omega(x)\,dx\,\frac{dx}{2} - M(x) = 0$$

Neglecting the second order terms like dx dv and dx^2 one obtains

$$\frac{dM}{dx} = V(x) \qquad (2.33)$$

Equation 2.32 indicates that the slope of the shear force diagram at any location x is the negative of the load intensity at this location. On the other hand, Equation 2.33 shows that the slope of the bending moment diagram at any location x equals the value of the shear force at this location. Equation 2.33 also indicates that the maximum value for the bending moment occurs at the location of the zero shear force.

Equations 2.32 and 2.33 can also be written in another form as follows:

$$V_{x1-x2} = \int_{x_1}^{x_2} -\,\omega(x)\,dx \qquad (2.34)$$

$$M_{x1-x2} = \int_{x_1}^{x_2} V(x)\,dx \qquad (2.35)$$

Equation 2.34 indicates that the difference in the shear force values between two locations x_1 and x_2 is the negative area of the load diagram between x_1 and x_2. Equation 2.35 shows also that the difference in bending moment between two locations x_1 and x_2 equals the area of shear force diagram between these two locations. These equations can thus be used to construct the shear force and bending moment diagrams.

Example 2.5

Determine the shear force and bending moment diagrams for the beam shown in Figure 2.31.

The reactions at the supports are first determined using the free body diagram of Figure 2.32 as follows:

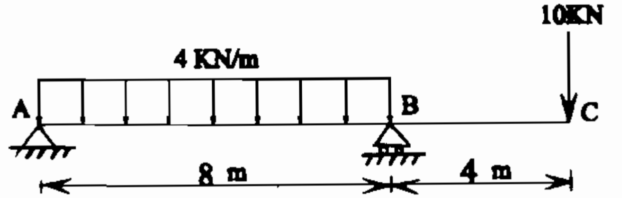


Figure 2.31

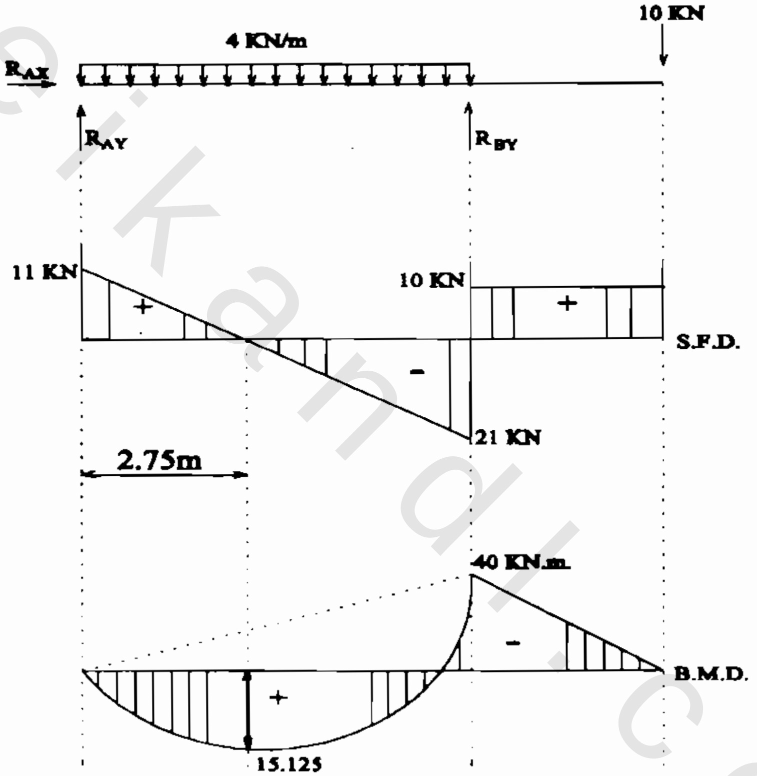


Figure 2.32

$\Sigma F_x = 0$; $R_{Ax} = 0$

$\Sigma M_A = 0$; $8\,R_{By} - 10 \times 12 - 4 \times 8 \times 4 = 0$; $R_{By} = 31$ kN

$\Sigma F_y = 0$; $R_{Ay} + R_{By} - 4 \times 8 - 10 = 0$; $R_{Ay} = 32 + 10 - 31 = 11$ kN

Using the equilibrium equations at any section along the beam, one can determine the shear force and bending moment diagrams as given in Figure 2.32.

Expressions for the shear force and bending moment at any location x from support A are obtained as follows:

$$V(x) = 11 - 4x \qquad \text{for} \qquad 0 \le x \le 8$$

$$M(x) = 11x - 2x^2 \qquad \text{for} \qquad 0 \le x \le 8$$

The magnitude of the maximum positive bending moment is obtained by looking for the location of zero shear force. From the shear force equation one has

$$V(x) = 11 - 4x = 0 \qquad ; \qquad x = 2.75\,\text{m}$$

The maximum bending moment is then obtained as

$$M_{max} = 11 \times 2.75 - 2(2.75)^2 = 15.125\,\text{kN.m.}$$

Equation 2.35 can also be used to construct the bending moment at certain locations. For example, the difference in bending moment between points B and C is the area of shear force diagram between these two points, thus

$$M_C - M_B = \int_B^C V\,dx = 10 \times 4 = 40\,\text{kN.m.}$$

Since $M_C = 0$, one gets $M_B = -40$ kN.m.

Also, the difference in bending moment between point A and the maximum positive moment is

$$M_{max} - M_A = \int_A^{x=2.75} V\,dx = \frac{1}{2} \times 11 \times 2.75 = 15.125\,\text{kN.m.}$$

which gives $M_{max} = 15.125$ kN.m. since $M_A = 0$.

Similarly, the shear force at B is obtained in reference to the point of zero shear as

$$V_B - 0 = \int_{x=2.75}^{x=8} -4\,dx = -4\,[8 - 2.75] = -21\,\text{kN}$$

which is the same value obtained in Figure 2.32.

2.16 CONSERVATIVE STRUCTURAL SYSTEMS

Conservative structural systems are those which do not lose any energy from the work done by the external actions. Most linear-elastic structures are conservative, especially those subjected to quasi-static loading. This means that the work done by external actions must equal the strain energy stored in the structure.

Consider a linear-elastic member whose length is L and is subjected to a gradually increasing axial force up to value A_x, as shown in Figure 2.33. The work done by A_x during the elongation ΔL is obtained using Equation 2.6 to get

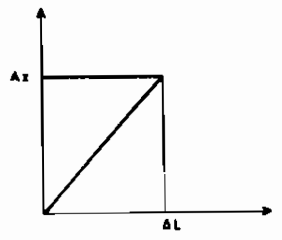


Figure 2.33

$$W = \frac{1}{2} A_x \, \Delta L$$

The strain energy is obtained from Equation 2.11 as

$$U = \frac{1}{2} \int \sigma_x \, \varepsilon_x \, dv$$
$$= \frac{1}{2} \int \left(\frac{A_x}{A}\right)\left(\frac{\Delta L}{L}\right)(A \, dL) = \frac{1}{2} A_x \, \Delta L$$

Therefore the strain energy U equals the external work done, W. This proves the conservation of energy for a linear-elastic member. The same result is applicable for a linear-elastic structure.

Non-conservative structural systems are those which lose some energy through radiation, heat, or dissipation. This class of systems is not treated in this text.

Example 2.6

It is required to determine the horizontal deflection of point B, using the law of conservation of energy for the frame shown in Figure 2.34. (Consider the cross section area $A = 40 \times 100 \text{ cm}^2$ and $E = 2100 \text{ kN/cm}^2$).

The internal actions diagrams are determined as shown in Figure 2.34. From Equations 2.6 and 2.11 one determines, respectively, the strain energy and the work done as follows:

$$U = \frac{1}{2} \int \frac{A_x^2 \, dx}{EA} = \frac{1}{2} \int_0^{1000} \frac{100 \, dx}{2100 \times 40 \times 100} = 0.00595 \text{ kN.cm}$$

$$W = \frac{1}{2}(10 \times \Delta_x)$$

Since U = W, thus $\Delta_x = \frac{U}{5} = 0.0019\,\text{cm}$

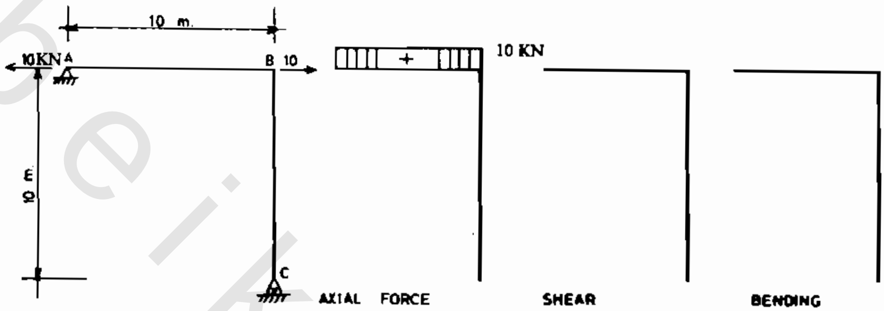


Figure 2.34

2.17 PRINCIPLE OF VIRTUAL WORK

If a rigid body (non-deformable) is subjected to a system of actions and the body is in static equilibrium, then the work done by these actions during a small virtual displacement is equal zero. This principle can be proved by using the condition of equilibrium between the applied actions and the reactions.

The principle can also be applied to a deformable body. An easy derivation for this principle makes use of the conservation of energy which was proved in the previous section, as follows:

A conservative structural system is in equilibrium under a set of actions. The relationship between the external work done by the actions and the strain energy stored in the structure is given by

$$W = U \tag{2.36}$$

A small variation in W must equal to the corresponding variation in U. If this variation is due to a compatible virtual displacement $\delta\, \underline{D}$, then, one has

$$\delta\, W = \delta\, \underline{D}^T\, \underline{A} \tag{2.37}$$

The corresponding variation in the strain energy due to a strain vector variation $\delta\, \underline{\varepsilon}$ is

$$\delta U = \int_v \delta \underline{\varepsilon}^T \underline{\sigma} \, dv \tag{2.38}$$

where $\delta \underline{\varepsilon}$ is the strain variations due to $\delta \underline{D}$.

The first form for the principle of virtual work can thus read

$$\delta \underline{D}^T A = \int_v \delta \underline{\varepsilon}^T \underline{\sigma} \, dv \tag{2.39}$$

If the variation in the work done is due to a set of virtual loads $\delta \underline{A}$, then one has the second form of the principle of the virtual work as

$$\delta \underline{A}^T D = \int_v \delta \underline{\sigma}^T \underline{\varepsilon} \, dv \tag{2.40}$$

where $\delta \underline{\sigma}$ is the variation in stresses due to the virtual loads $\delta \underline{A}$.

2.18 THE UNIT DISPLACEMENT METHOD

As a special case of the principle of virtual work, when the virtual displacements are zero at all points except to be unity at a specific point i, Equation 2.38 can be written as

$$1 \times A_i = \int_v \delta \underline{\varepsilon}_1^T \underline{\sigma} \, dv \tag{2.41}$$

where $\underline{\varepsilon}_1$ is the strain vector due to the unit displacement at i, and A_i is the load in the direction of unit displacement applied at point i.

This method is used in developing the stiffness coefficients which is used in the stiffness method.

Example 2.7

Determine the stiffness coefficients associated with the actions A_1 and A_2 in the truss shown in Figure 2.35.

Applying the unit horizontal displacement at B, as shown in Figure 2.36, one gets

$$1 \times A_1 = \begin{bmatrix} \dfrac{1}{2L} & \dfrac{-1}{2L} \end{bmatrix} \begin{bmatrix} \dfrac{EA\,D_1}{2L} \\ \dfrac{-EAD_1}{2L} \end{bmatrix} AL = \frac{EA}{2L} D_1$$

$$1 \times A_2 = 0$$

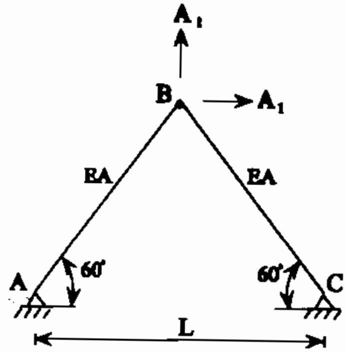


Figure 2.35

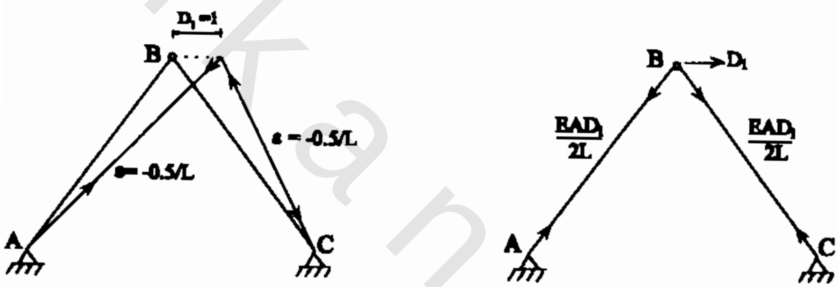


Figure 2.36

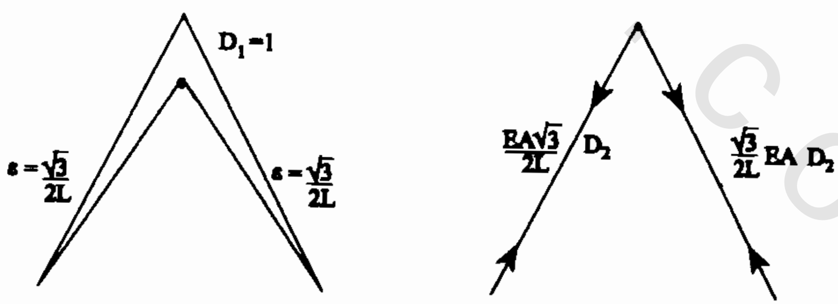


Figure 2.37

Similarly, applying a unit vertical displacement at B as shown in Figure 2.37, one obtains

$$1 \times A_2 = \begin{bmatrix} \frac{\sqrt{3}}{2L} & \frac{\sqrt{3}}{2L} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}\,EA\,D_2}{2L} \\ \frac{\sqrt{3}\,EA\,D_2}{2L} \end{bmatrix} AL = \frac{3\,EA}{2L} D_2$$

$$1 \times A_1 = 0$$

Thus, the force-displacement relationship for the truss at joint B is given by

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{2L} & 0 \\ 0 & \frac{1.5\,EA}{L} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

2.19 THE UNIT LOAD METHOD

If a unit load is applied at point i as a virtual load in the virtual work principle, then Equation 2.40 becomes:

$$1 \times D_i = \int_v \underline{\sigma}_1^T \, \underline{\varepsilon} \, dv \qquad (2.42)$$

where $\underline{\sigma}_1$ is the stress vector due to the unit load applied at i.

By using Equation 2.42 one is able to calculate the deflection at certain points, and also one can develop the flexibility coefficients needed for the flexibility method.

Example 2.8

Determine the flexibility coefficients associated with displacements D_1 and D_2 at joint B in the truss shown in Figure 2.38.

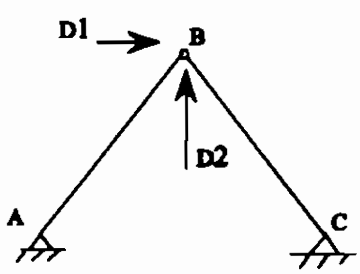


Figure 2.38

Solution

Apply a unit load A_1 = 1 kN at B in direction of D_1 as in Figure 2.39. The internal forces are obtained using static equilibrium equations.

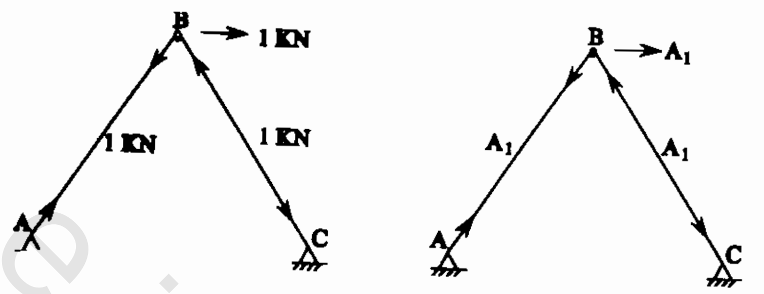


Figure 2.39

Applying the virtual work principle, one obtains

$$1 \times D_1 = \int_v \underline{\sigma}_1^T \underline{\varepsilon} \, dv$$

$$= \begin{bmatrix} \frac{1}{A} & \frac{-1}{A} \end{bmatrix} \begin{bmatrix} \frac{A_1}{EA} \\ \frac{A_1}{EA} \end{bmatrix} \times AL = \frac{2L}{EA} A_1$$

$$1 \times D_2 = 0$$

Similarly, applying a unit load at B in direction of D_2 as shown in Figure 2.40, one obtains the members forces from static equilibrium equations.

Applying the unit load method one gets

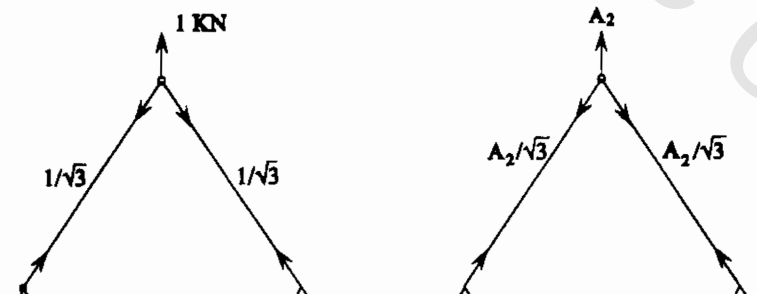


Figure 2.40

$$1 \times D_2 = \int_v \underline{\sigma}_1^T \, \underline{\varepsilon} \, dv$$

$$= \begin{bmatrix} \dfrac{1}{\sqrt{3}\,A} & \dfrac{1}{\sqrt{3}\,A} \end{bmatrix} \begin{bmatrix} \dfrac{A_2}{\sqrt{3}\,EA} \\ \dfrac{A_2}{\sqrt{3}\,EA} \end{bmatrix} AL = \frac{2L}{3\,EA} A_2$$

$$1 \times D_1 = 0$$

These relations can be put in a matrix form to represent the flexibility coefficients at joint B.

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2L}{EA} & 0 \\ 0 & \dfrac{2L}{3\,EA} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Considering that a unit load results in planar structures the internal actions a_x, a_y, m_z and the loading produces internal forces A_x, A_y and M_z, Equation 2.42 yields

$$1 \times D_i = \int \begin{bmatrix} \dfrac{a_x}{A} & \dfrac{a_y\,Q}{I_z\,b} & \dfrac{m_z}{I_z}\,y \end{bmatrix} \begin{bmatrix} \dfrac{A_x}{EA} & \dfrac{A_y\,Q}{G\,I_z\,b} & \dfrac{M_z\,y}{I_z\,E} \end{bmatrix}^T dv$$

$$= \int \frac{a_x\,A_x\,dx}{EA} + \int \frac{a_y\,A_y\,dx}{G\,A_{ry}} + \int \frac{m_z\,M_z\,dx}{EI_z} \tag{2.43}$$

In case of having linear rise in temperature as described in Figure 2.24 the deformation due to unit load is

$$D_i = \alpha \frac{(T_1 + T_2)}{2} \int a_x\,dx - \alpha\,\frac{(T_1 - T_2)}{h} \int m_z\,dx \tag{2.44}$$

in which T_1 and T_2 are the temperature changes of the upper and lower surface of the member, α is the coefficient of thermal expansion, and h is the member depth.

The deformation of trusses are determined from Equations 2.43 and 2.44 by retaining only the terms containing axial force as

$$D_i = \Sigma \frac{a_x\,A_x\,L}{EA} \tag{2.45}$$

The deformation due to thermal changes is obtained from

$$D_i = \sum \alpha\,T\,a_x\,L \tag{2.46}$$

in which T is the change in temperature of a specific truss member.

Thus, in order to determine the deformation at a point i, in a certain direction, one has to do the following:

1) Determine the internal actions due to applied loads.

2) Place a unit virtual load or moment at point i, in the required direction and determine the internal virtual actions.

3) Use the proper equations to determine the required deformation.

Example 2.9

For the beam shown in Figure 2.41, use the unit load method to determine:

(a) the slope at A,
(b) the deflection at mid-span.

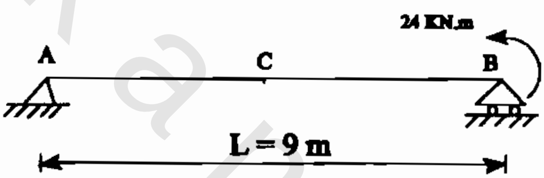


Figure 2.41

Solution

From Figure 2.42 one can determine the reactions and moment at section x from the left support.

$$R_A = \frac{8}{3}\,\text{kN} \quad ; \quad M = \frac{8}{3}x$$

(a) To determine slope at A, apply a unit moment at A as shown in Figure 2.43.

$m = 1 - x/L$

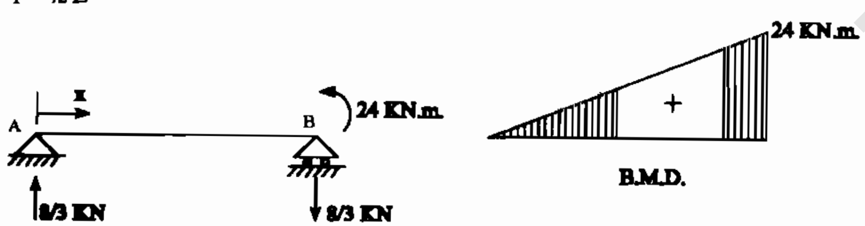


Figure 2.42

Substituting into Equation 2.39 considering bending moment only, one has

$$\theta_A = \int_0^L \frac{m\,M}{EI}\,dx$$

$$\theta_A = \frac{1}{EI}\int_0^L (1 - x/L)\left(\frac{8}{3}x\right)dx = \frac{36}{EI}\ \text{rad.}$$

The positive value for the answer indicates that the slope is in the same direction as the unit moment applied at A.

(b) To determine the vertical deflection at mid-span, apply a unit load at C as shown in Figure 2.44.

$$m = x/2 \qquad \text{for} \qquad 0 \le x \le \frac{L}{2}$$

$$m = \frac{1}{2}(L - x) \qquad \text{for} \qquad \frac{L}{2} \le x \le L$$

$$D_c = \frac{1}{EI}\left[\int_0^{L/2} (x/2)\left(\frac{8}{3}x\right)dx + \int_{L/2}^{L} \frac{1}{2}(L - x)\left(\frac{8}{3}x\right)dx\right] = \frac{121.5}{EI}\ \text{m.}$$

The answer indicates that the deflection at mid-span is downward.

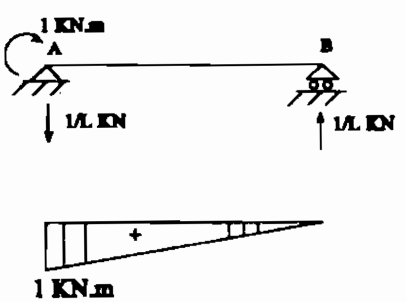


Figure 2.43

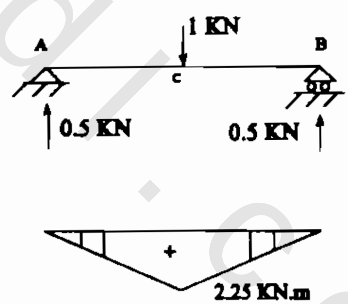


Figure 2.44

2.20 INTEGRATION BY DIAGRAMS

Using mathematical integrations to evaluate deformation using Equation 2.43 is a time consuming process. A method which is based on performing the integrations using internal forces diagrams is presented here.

Suppose it is required to evaluate the contribution of the third term in Equation 2.43. In this case, the bending moment diagrams M_z and m_z are used. Since m_z is constructed due to a unit load, this diagram is always formed of straight lines segments. Whereas, M_z could be curved or straight line segments depending on the kind of loading on the structure whether distributed or concentrated loads. The diagrams on a segment of a member of length S are shown in Figure 2.45. The deformation due to bending moment, denoted by D_m, is calculated from

$$1 \times D_m = \int_0^S M_z \frac{m_z\, dx}{EI}$$

$$= \int_0^S \left(\frac{M_z\, dx}{EI}\right) m_z$$

$$= \frac{1}{EI} \times \text{Area of } M_z \times \bar{y} \qquad (2.47)$$

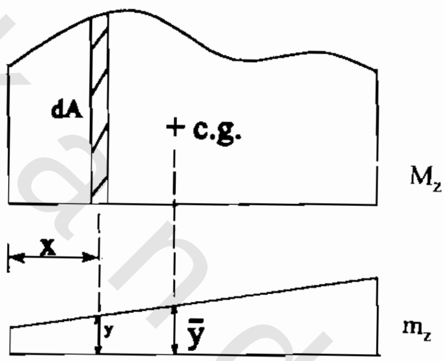


Figure 2.45

The deformation D_m can thus be obtained from multiplying the area of M_z diagram times the ordinate of m_z at the centroid of M_z diagram $(\bar{y})$ divided by the value of EI of the segment S. Using this law requires the knowledge of the areas of the simple shapes and the location of their centroids. The moment diagram m_z must be a continuous straight line from the start to the end of the segment of integration. Integrating different shapes of diagrams are given in Tables 2.2 and 2.3.

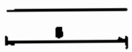
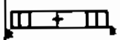
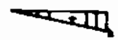
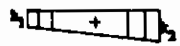
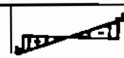
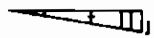
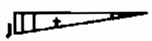
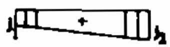
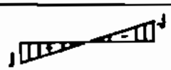
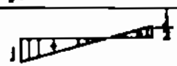
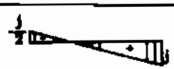
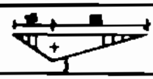
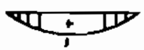
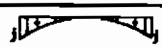
Example 2.10

Determine the deflection at point C for the beams shown in Figures 2.46 and 2.47 using the contribution of bending moment only (EI = 10^5 kN.m^2).

Solution

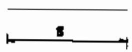
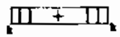
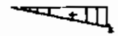
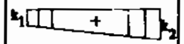
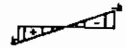
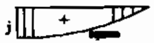
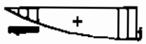
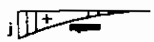
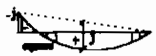
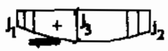
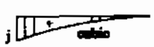
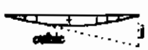
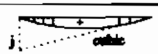
The deflection for the beam in Figure 2.46 is obtained using Equations 2.43 and 2.47 as follows:

Table* 2.2 Values of Integrals : $\int M_j\, M_k\, ds = s \times$ terms given in tables

No.								$\int j^2\, ds$
1		jk	$jk/2$	$j(k_1+k_2)/2$	0	$jk/2$	$jk/2$	j^2
2		$jk/2$	$jk/3$	$j(k_1+2k_2)/6$	$-jk/6$	0	$jk(1+\alpha)/6$	$j^2/3$
3		$jk/2$	$jk/6$	$j(2k_1+k_2)/6$	$jk/6$	$jk/4$	$jk(1+\beta)/6$	$j^2/3$
4		$k(j_1+j_2)/2$	$k(j_1+2j_2)/6$	$[j_1(2k_1+k_2)+j_2(k_1+2k_2)]/6$	$k(j_1-j_2)/6$	$j_1k/4$	$k[j_1(1+\beta)+j_2(1+\alpha)]/6$	$\left(j_1^2+j_1j_2+j_1^2\right)$
5		0	$-jk/6$	$j(k_1-k_2)/6$	$jk/3$	$jk/4$	$jk(1-2\alpha)/6$	$j^2/3$
6		$jk/4$	0	$jk_1/4$	$jk/4$	$jk/4$	$jk(1-\alpha+2\beta)/12$	$j^2/4$
7		$jk/4$	$jk/4$	$Jk_2/4$	$-jk/4$	$-jk/8$	$jk(1+2\alpha-\beta)/12$	$j^2/4$
8		$jk/2$	$jk/4$	$j(k_1+k_2)/4$	0	$jk/8$	$jk(3-4\alpha^2)/12\beta$	$j^2/3$
9		$jk/2$	$\frac{jk(1+\gamma)}{6}$	$j(k_1(1+\delta)+k_2(1+\gamma))/6$	$jk(1-2\gamma)/6$	$jk(1+2\delta-\gamma)/12$	$\frac{jk}{6\beta\gamma}\left(2\gamma-\gamma^2-\alpha^2\right)\ \gamma \ge \alpha$	$j^2/3$
10		$jk/3$	$jk/3$	$j(k_1+k_2)/3$	0	$jk/6$	$jk(1+\alpha\beta)/3$	$8j^2/1$
11		$jk/3$	$jk/6$	$j(k_1+k_2)/6$	0	$jk/12$	$jk(1-2\alpha\beta)/6$	$j^2/5$

*Reference 9.

Table* 2.3 Values of Integrals : $\int M_j M_k \, ds = s \times$ terms given in tables

No.								$\int j^2 ds$
12		2jk/3	jk/4	$j(5k_1+3k_2)/12$	jk/6	7jk/24	$jk(5-\alpha-\alpha^2)/12$	$8j^2/15$
13		2jk/3	5jk/12	$j(3k_1+5k_2)/12$	-jk/6	jk/24	$jk(5-\beta-\beta^2)/12$	$8j^2/15$
14		jk/3	jk/4	$j(k_1+3k_2)/12$	-jk/6	-jk/24	$jk(1+\alpha+\alpha^2)/12$	$j^2/5$
15		jk/3	jk/12	$j(3k_1+k_2)/12$	jk/6	5jk/24	$jk(1+\beta+\beta^2)/12$	$j^2/5$
16		jk/6	jk/6	$jk_2/6$	-jk/6	-jk/12	$jk[1-\beta(1-2\alpha)]/6$	$j^2/5$
17		jk/6	0	$jk_1/6$	jk/6	jk/6	$jk[1-\alpha(1-2\beta)]/6$	$j^2/5$
18		$k(j_1+4j_2+j_3)/6$	$k(2j_2+j_3)/6$	$[j_1k_1+2j_2(k_1+k_2)+j_2k_2]/6$	$k(j_1-j_3)/6$	$k(2j_1+2j_2-j_3)/12$	$\frac{k}{6}[j_1\beta+2j_2+j_3\alpha\ \alpha\beta(j_1-2j_2+j_3)]$	....
19		Jk/4	jk/5	$j(k_1+4k_2)/20$	-3jk/20	-jk/20	$\frac{jk}{20}(1+\alpha)(1+\alpha^2)$	$j^2/7$
20		jk/4	jk/20	$j(4k_1+k_2)/20$	3jk/20	7jk/40	$jk(1+\beta)(1+\beta^2)/20$	$j^2/7$
21		jk/4	2jk/15	$j(7k_1+8k_2)/60$	-jk/60	jk/20	$jk(1+\alpha)(7/3-\alpha^2)/20$	$8j^2/105$
22		jk/4	7jk/60	$j(8k_1+7k_2)/60$	jk/60	3jk/40	$jk(1+\beta)(7/3-\beta^2)/20$	$8j^2/105$

*Reference 9.

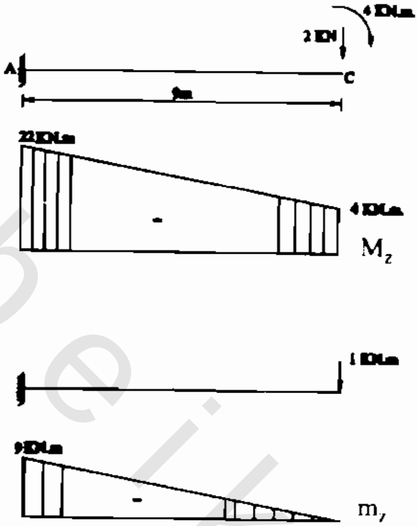


Figure 2.46

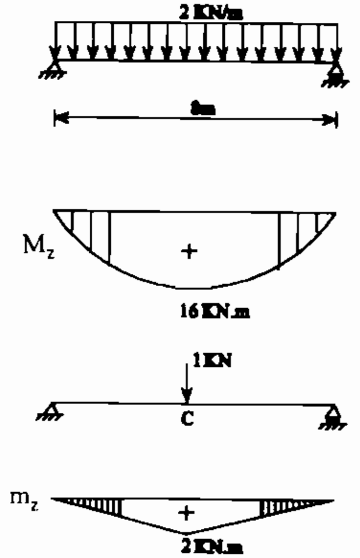


Figure 2.47

$$1 \times D_C = \frac{1}{EI}\left[(-4 \times 9) \times (-4.5) + (-18 \times 9 \times \frac{1}{2}) \times \frac{2}{3} \times (-9)\right] = 0.00648 \text{ m.}$$

The deflection for the beam in Figure 2.47 is similarly obtained as

$$1 \times D_C = \frac{1}{EI}\left[(\frac{2}{3} \times 16 \times 4) \times (\frac{5}{8} \times 2)\right] \times 2 = 0.0010667 \text{ m}$$

Example 2.11

Determine the angle of rotation at A for the beam shown in Figure 2.48.

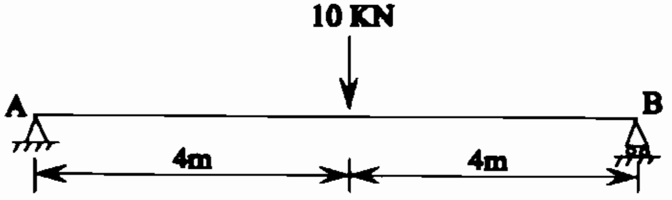


Figure 2.48

Solution

Apply a unit rotation at A and construct m_z diagram as shown in Figure 2.49. The integration with M_z diagram gives the angle of rotation at A as follows:

$$1 \times \theta_A = \frac{1}{EI}\left[\left(\frac{1}{2} \times 20 \times 8\right) \times \left(\frac{1}{2}\right)\right] = \frac{40}{EI} \text{ rad.}$$

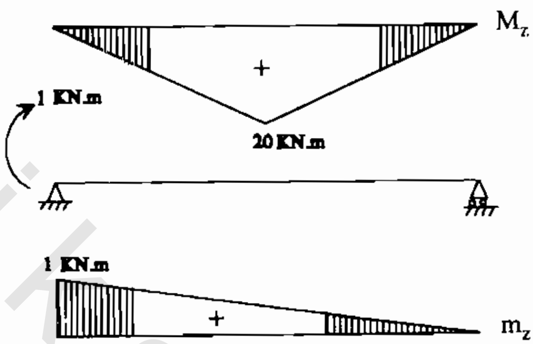


Figure 2.49

Example 2.12

Determine the relative angle of rotation at the hinge C for the frame shown in Figure 2.50 due to the rise in temperature shown ($\alpha = 10^{-5}/^{\circ}c$).

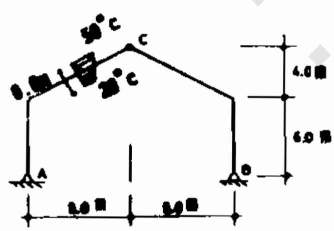


Figure 2.50

Solution

To determine the relative angle of rotation at the hinge C apply two equal and opposite unit moments at C as shown in Figure 2.51. The axial force and bending moment diagrams are given in Figure 2.52.

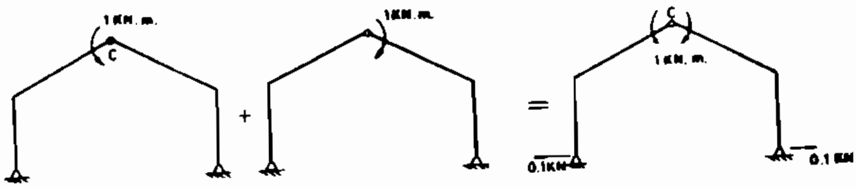


Figure 2.51

Figure 2.52

$$\theta_{cr} = \alpha\left(\frac{T_1 + T_2}{2}\right)\int a_x \, dx - \alpha\left(\frac{T_1 - T_2}{h}\right)\int m_z \, dx$$

$$= \alpha\left(\frac{50+20}{2}\right)\left[0.09 \times 4\sqrt{5} \times 2\right] - \alpha\left(\frac{50-20}{0.8}\right)\left[\frac{0.6 \times 6}{2} \times 2 + \frac{(1+0.6)}{2} \times 4\sqrt{5} \times 2\right]$$

$$= -615.3\alpha$$

$\theta_{cr} = -0.00615$ rad. (opposite to the assumed direction).

Example 2.13

Determine the deflection at C and the relative displacement between A and B for the truss shown in Figure 2.53 due to a rise in temperature of 20°C in the top chord only ($\alpha = 10^{-5}/°c$).

Solution

To determine the deflection at C apply a unit vertical load at C. The axial forces are obtained and shown in Figure 2.54.

The deflection at C is obtained using Equation 2.46 as follows:

$$D_c = \alpha\,(20)\left[-1 \times 5 - 1 \times 5 + 0 + 0\right] = -200\,\alpha = -0.2\,\text{cm} \qquad \text{(upward)}$$

To determine the relative displacement between A and B one applies two equal and opposite unit loads at A and B as shown in Figure 2.55.

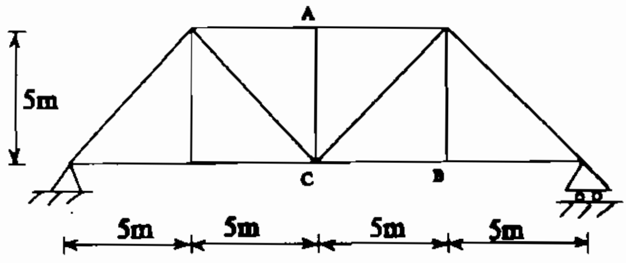


Figure 2.53

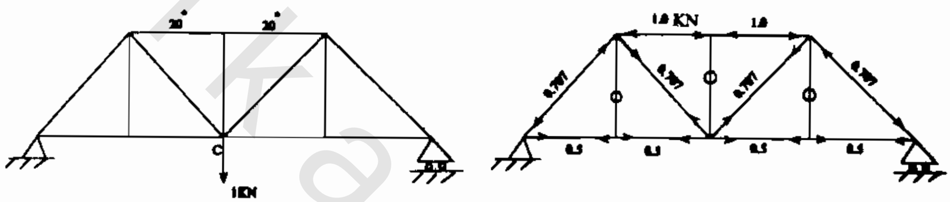


Figure 2.54

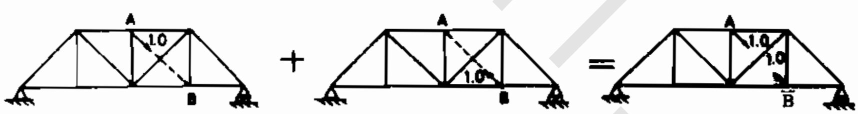


Figure 2.55

Using the axial forces a_x, the relative displacement between A and B is, obtained from

$$D_{A/B} = \alpha\left(-0.707 \times 5 \times 20\right) = -0.0707 \text{ cm} \qquad \text{(in opposite directions to the unit loads)}.$$

2.21 CASTIGLIANO'S THEOREMS

The principle of conservation of energy can also be used to derive Castigliano's theorems. Considering a linear-elastic conservative structure, the variation of strain energy must equal the variation in the work done, that is

$$\delta U = \delta W \tag{2.48}$$

The variation due to a small deformation δD_i, as shown in Figure 2.56, at a certain point i, can be written as

$$\frac{\partial U}{\partial D_i}\partial D_i = A_i\,\partial D_i \qquad (2.49)$$

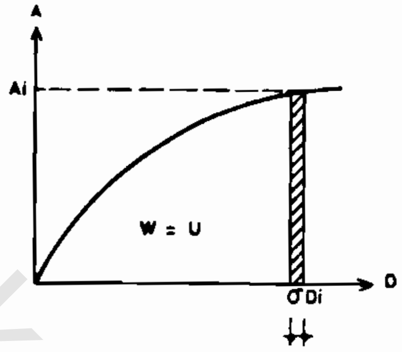


Figure 2.56

In Equation 2.49, the left hand side is the mathematical definition of the variation in U due to δD_i, and the right hand side is the actual variation in the work done due to δD at i.

From Equation 2.49, Castigliano's first theorem is stated as follows: "in an elastic system, the partial derivative of strain energy U with respect to any selected displacements D_i gives the action A_i in the direction D_i". This can mathematically be expressed as

$$\frac{\partial U}{\partial D_i} = A_i \qquad (2.50)$$

This theorem can also be derived using an engineering analysis as done in References [1, 20].

If the variation is due to additional load δA applied at point i as shown in Figure 2.57, Equation 2.48 becomes

$$\frac{\partial U^*}{\partial A_i}\delta A_i = D_i\,\delta A_i \qquad (2.51)$$

where U^* is the complementary strain energy.

For linear-elastic structures, where $U = U^*$, one has

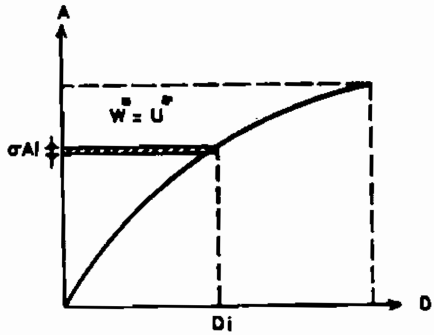


Figure 2.57

$$\frac{\partial U}{\partial A_i} = D_i \qquad (2.52)$$

This states a special form for Castigliano's second theorem, which states that "the partial derivative of the complementary strain energy with respect to any selected action A_i gives the displacement D_i in the direction of A_i".

The two Castigliano's theorems have introduced the basic tools for matrix analysis of structures. The first theorem is used in the stiffness method. The stiffness coefficient S_{ij} is determined from the term $(\partial A_i/\partial D_j)$ which is the action at i due to unit displacement at j. Therefore

$$S_{ij} = \frac{\partial A_i}{\partial D_j} = \frac{\partial}{\partial D_j}\left(\frac{\partial U}{\partial D_i}\right) = \frac{\partial^2 U}{\partial D_i \, \partial D_j} \qquad (2.53)$$

and

$$A_i = S_{ij} \; D_j \qquad (2.54)$$

Similarly, the force method has been developed using the second Castigliano's theorem. For a linear elastic structure, the flexibility coefficient $f_{ij} = (\partial D_i / \partial A_j)$ is the displacement at i due to a unit load at j. Thus,

$$f_{ij} = \frac{\partial D_i}{\partial A_j} = \frac{\partial}{\partial A_j}\left(\frac{\partial U}{\partial A_i}\right) = \frac{\partial^2 U}{\partial A_i \, \partial A_j} \qquad (2.55)$$

and

$$D_i = f_{ij} \, A_j \qquad (2.56)$$

Example 2.14

Determine the flexibility coefficient for the truss given in example 2.7 using Castigliano's second theorem.

Solution

The internal forces in the members are found in terms of the forces A_1 and A_2 using static equilibrium conditions as shown in Figure 2.58. The forces in the members are called A_{x1} and A_{x2}. These forces are

$$A_{x1} = \left(A_1 + \frac{A_2}{\sqrt{3}} \right)$$

$$A_{x2} = \left(\frac{A_2}{\sqrt{3}} - A_1 \right)$$

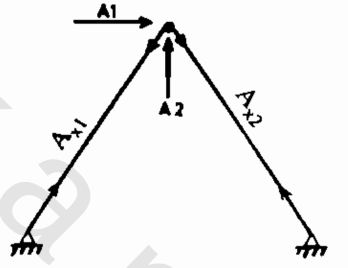


Figure 2.58

The strain energy is calculated using Equation 2.13 as

$$U = \int \frac{A_x^2 \, dx}{2EA} = \left(A_1 + \frac{A_2}{\sqrt{3}} \right)^2 \frac{L}{2EA} + \left(\frac{A_2}{\sqrt{3}} - A_1 \right)^2 \frac{L}{2EA}$$

Using Castigliano's second theorem, one has

$$\frac{\partial U}{\partial A_1} = D_1 = \frac{2L}{2EA} \left(A_1 + \frac{A_2}{\sqrt{3}} \right) - \frac{L}{2EA} \left(\frac{A_2}{\sqrt{3}} - A_1 \right)$$

$$\frac{\partial D_1}{\partial A_1} = f_{11} = \frac{\partial^2 U}{\partial A_1^2} = \frac{2L}{2EA} + \frac{2L}{2EA} = \frac{2L}{EA}$$

$$\frac{\partial U}{\partial A_2} = D_2 = 2 \left(A_1 + \frac{A_2}{\sqrt{3}} \right) \frac{L}{2\sqrt{3}\,EA} + 2 \left(\frac{A_2}{\sqrt{3}} - A_1 \right) \frac{L}{2\sqrt{3}\,EA}$$

$$\frac{\partial D_1}{\partial A_2} = 0 \quad , \quad \frac{\partial D_2}{\partial A_1} = 0$$

$$\frac{\partial D_2}{\partial A_2} = f_{22} = \frac{\partial^2 U}{\partial A_2^2} = \frac{2L}{3EA}$$

Therefore, the flexibility matrix between A_1, A_2, and D_1, D_2 is

$$\text{Flexibility Matrix} = \frac{L}{EA}\begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

Example 2.15

Determine the stiffness coefficients associated with D_1 and D_2 for the truss given in example 2.7.

Solution

The axial deformation in the truss members are determined from the geometry of Figure 2.59 as follows:

$$\Delta_1 = 0.5\,D_1 + \frac{\sqrt{3}}{2} D_2$$

$$\Delta_2 = -\,0.5\,D_1 + \frac{\sqrt{3}}{2} D_2$$

The strain energy is calculated using Equation 2.13 as

$$U = \frac{\Delta_1^2\,EA}{2L} + \frac{\Delta_2^2\,EA}{2L}$$

$$= \frac{EA}{2L}\left[\left(0.5\,D_1 + \frac{\sqrt{3}}{2} D_2\right)^2 + \left(-0.5\,D_1 + \frac{\sqrt{3}}{2} D_2\right)^2\right]$$

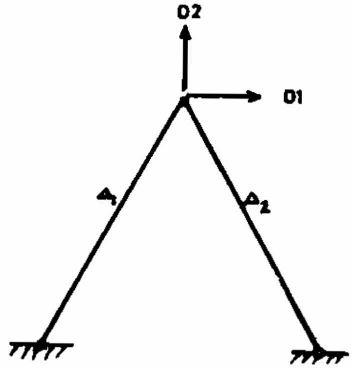


Figure 2.59

Applying Castigliano's first theorem, one has

$$\frac{\partial U}{\partial D_1} = A_1 = \frac{EA}{2L}\left[1\left(0.5\,D_1 + \frac{\sqrt{3}}{2}D_2\right) - 1\left(-\,0.5\,D_1 + \frac{\sqrt{3}}{2}D_2\right)\right]$$

$$S_{11} = \frac{\partial A_1}{\partial D_1} = \frac{\partial^2 U}{\partial D_1^2} = \frac{EA}{2L}\,[0.5 + 0.5] = \frac{EA}{2L}$$

$$S_{12} = \frac{\partial A_1}{\partial D_2} = 0$$

$$\frac{\partial U}{\partial D_2} = A_2 = \frac{EA}{2L}\left[\frac{2\sqrt{3}}{2}\left(0.5\,D_1 + \frac{\sqrt{3}}{2}D_2\right) + \frac{2\sqrt{3}}{2}\left(-0.5\,D_1 + \frac{\sqrt{3}}{2}D_2\right)\right]$$

$$S_{22} = \frac{\partial A_2}{\partial D_2} = \frac{\partial^2 U}{\partial D_2^2} = \frac{EA}{2L}\left[\frac{3}{2} + \frac{3}{2}\right] = \frac{3}{2}\,\frac{EA}{L}$$

$$S_{21} = \frac{\partial A_2}{\partial D_1} = 0$$

The stiffness matrix between A_1, A_2, and D_1, D_2 is

$$\text{Stiffness matrix} = \frac{EA}{L}\begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

One should note that the stiffness matrix is the inverse of the flexibility matrix, in any structure.

The next example shows that Castigliano's second theorem can also be used to analyze statically indeterminate structures. Although this is handled in more detail in Chapter 3, for the sake of completeness of this topic, the next example is offered.

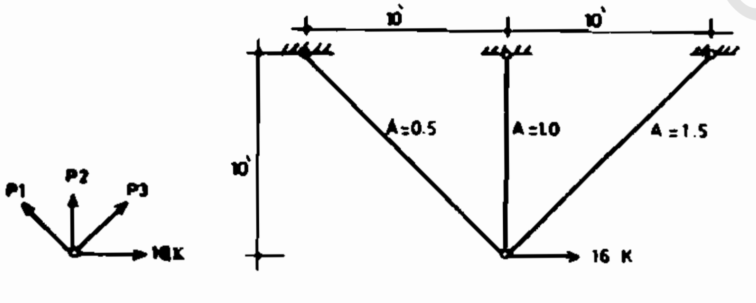


Figure 2.60

Example 2.16

Analyze the truss shown in Figure 2.60 using Castigliano's theorems.

Solution

Member 3 is cut and the forces in the members are found in terms of A_{x3} as shown in Figure 2.60. The members forces are all assumed tension. The equilibrium of the free joint gives

$$A_{x1} = 16\sqrt{2} + A_{x3}$$

$$A_{x2} = -\left(16 + A_{x3}\sqrt{2}\right)$$

The strain energy, using Equation 2.13, is calculated as follows:

$$U = \sum_{i=1}^{2} \frac{A_{xi}^2 L_i}{2EA_i}$$

$$= \frac{\left(16\sqrt{2} + A_{x3}\right)^2 10\sqrt{2}}{E} + \frac{\left(16 + \sqrt{2}\,A_{x3}\right)^2 10}{2E}$$

Applying Castigliano's Second Theorem, one obtains

$$\frac{\partial U}{\partial A_{x3}} = -D_3$$

where the negative sign indicates that the deformation D_3 is in opposite direction to the force A_{x3}, since the tension member is subjected to elongation.

$$D_3 = \frac{A_{x3} L_3}{EA_3}$$

$$\frac{\partial U}{\partial A_{x3}} = \frac{1}{E}\left[20\sqrt{2}\left(16\sqrt{2} + A_{x3}\right) + 10\left(16 + \sqrt{2}\,A_{x3}\right)\sqrt{2}\right] = -D_3$$

$$\frac{1}{E}\left(866.274 + 48.284\,A_{x3}\right) = \frac{-10\sqrt{2}\,A_{x3}}{1.5E}$$

$$A_{x3} = -\,15.01\text{ K}$$

$$A_{x1} = 16\sqrt{2} - 15.01 = 7.614\text{ kN}$$

$$A_{x2} = -\left(16 - 15.01\sqrt{2}\right) = 5.224\text{ kN}$$

Example 2.17

Determine the deflection at mid-span and the angle of rotation at the left support for the beam shown in Figure 2.61 using Castigliano's theorem. ($EI = 10^5$ kN.m^2).

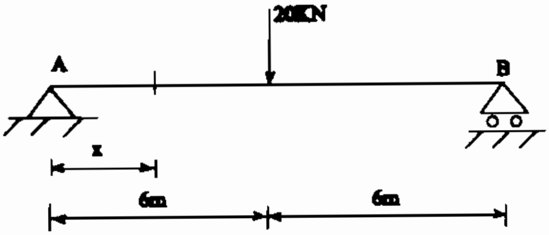


Figure 2.61

Solution

(a) Since the load 20 kN is applied at mid-span, one may use it in calculating the deflection. Calling this load A_c, the strain energy is determined as follows:

The moment at any section x from the left support is

$$M(x) = \frac{A_C}{2} x \qquad \text{for} \qquad 0 \le x \le 6$$

The strain energy in the beam is

$$U = 2\int_0^6 \frac{M^2(x)\,dx}{2EI} = 2\int_0^6 \frac{A_c^2\,x^2\,dx}{4 \times 2EI} = \frac{A_c^2}{4EI}\left[\frac{x^3}{3}\right]_0^6 = \frac{18\,A_c^2}{EI}$$

Applying Castigliano's theorem one gets

$$D_C = \left.\frac{\partial U}{\partial A_C}\right|_{A_C = 20} = 0.72 \text{ cm}$$

The positive sign indicates that the deflection is in direction of A_c.

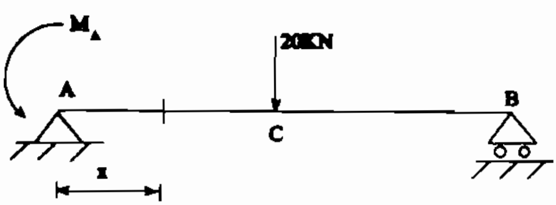


Figure 2.62

(b) To determine the angle of rotation at A, apply a virtual moment M_A at A as shown in Figure 2.62.

The moment at section x from left support is given by

$$M_1(x) = \left(\frac{M_A}{12} + 10\right)x - M_A \qquad \text{for} \qquad 0 \le x \le 6$$

The moment at distance z from right support is

$$M_2(z) = \left(10 - \frac{M_A}{12}\right)z \qquad \text{for} \qquad 0 \le z \le 6$$

The strain energy is thus

$$U = \int_0^6 \frac{M_1^2(x)\,dx}{2\,EI} + \int_0^6 \frac{M_2^2(z)\,dz}{2\,EI}$$

From Castigliano's Second Theorem, the angle of rotation is obtained by carrying out the integration and then put $M_A = 0$.

$$\theta_A = \left.\frac{\partial U}{\partial M_A}\right|_{M_A=0} = \left[\int_0^6 \frac{2M_1(x)\left(\frac{1}{12}x - 1\right)dx}{2EI} + \int_0^6 \frac{2M_2(z)\left(\frac{-1}{12}\right)z\,dz}{2EI}\right]_{M_A=0}$$

$$= \frac{1}{EI}\int_0^6 \left(\frac{x}{12} - 1\right)(10x)\,dx + \frac{1}{EI}\int_0^6 \left(\frac{-1}{12}\right) \times 10\,z^2\,dz$$

$$= \frac{1}{EI}\left[\frac{10\,x^3}{36} - \frac{10\,x^2}{2}\right]_0^6 - \frac{1}{EI}\left[\frac{10\,z^3}{36}\right]_0^6 = \frac{-360}{2\,EI} = -0.0018 \text{ rad}$$

which indicates that the slope is in opposite direction to M_A direction.

2.22 MAXWELL'S AND BETTI'S LAWS

Betti's law states that the work done by a system of action $\underline{\mathbf{A}}_m$ during the deformation state $\underline{\mathbf{D}}_n$ which is caused by another system of actions $\underline{\mathbf{A}}_n$ is equal to the work done by the $\underline{\mathbf{A}}_n$ actions during the deformation $\underline{\mathbf{D}}_m$ state which is caused by the $\underline{\mathbf{A}}_m$ actions. This can be expressed as

$$\mathbf{A}_m^T\,\underline{\mathbf{D}}_n = \mathbf{A}_n^T\,\underline{\mathbf{D}}_m \qquad (2.57)$$

Maxwell's law is a special form of Betti's law. It states that the work done by a unit action at i due to deformation caused by another unit action at j (D_{ij}) equals the work done by the unit action at j due to deformation caused by unit action at i (D_{ji}). This can be expressed as

$$(1)_i \, D_{ij} = (1)_j \, D_{ji} \tag{2.58}$$

which indicates that $D_{ij} = D_{ji}$.

It can be restated that the deflection at i due to unit action at j equals the deflection at j due to unit action at i.

This theorem serves in concluding that the stiffness and flexibility matrices are symmetric about their diagonal. This is obvious from

$$f_{ij} = f_{ji} = \frac{\partial^2 U}{\partial A_i \, \partial A_j} \tag{2.59}$$

$$S_{ij} = S_{ji} = \frac{\partial^2 U}{\partial D_i \, \partial D_j} \tag{2.60}$$

Maxwell's and Betti's laws had great influence on developing the concepts of influence lines and Müller-Breslau principles.

2.23 DEFORMATION-ACTIONS RELATIONSHIP

The previous concepts will now be used in developing a general deformation-action relationship for an element ij in skeletal structures. The member is fixed at i and is subjected to the actions A_j as shown in Figure 2.63. The strain energy due to these actions is obtained from Equations 2.13 to 2.21 as follows:

$$U = \int_0^L \frac{A_x^2 \, dx}{2EA} + \int_0^L \frac{A_y^2 \, dx}{2GA_{rz}} + \int_0^L \frac{M_x^2 \, dx}{2GJ_x} + \int_0^L \frac{(M_z + A_y x)^2 \, dx}{2EI_z} + \int_0^L \frac{(M_y - A_z x)^2 \, dx}{2EI_y} \tag{2.61}$$

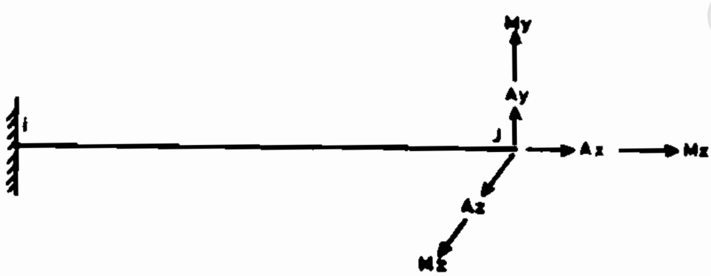


Figure 2.63

Applying Castigliano's Second Theorem one obtains

$$D_x = \frac{\partial U}{\partial A_x} \quad , \quad D_y = \frac{\partial U}{\partial A_y} \quad , \quad D_z = \frac{\partial U}{\partial A_z}$$
$$\theta_x = \frac{\partial U}{\partial M_x} \quad , \quad \theta_y = \frac{\partial U}{\partial M_y} \quad , \quad \theta_z = \frac{\partial U}{\partial M_z} \tag{2.62}$$

For example, D_x and D_y are computed as follows:

$$D_x = \frac{L}{EA} A_x$$

$$D_y = \left(\frac{L}{G A_{sy}} + \frac{L^3}{3\,EI_z} \right) A_y + \frac{L^2}{3\,EI_z} M_z$$

If the shear deformation term is neglected, one obtains the following simple matrix relationship:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3EI_z} & 0 & 0 & 0 & \frac{L^2}{2EI_z} \\ 0 & 0 & \frac{L^3}{3EI_y} & 0 & \frac{-L^2}{2EI_y} & 0 \\ 0 & 0 & 0 & \frac{L}{GJ_x} & 0 & 0 \\ 0 & 0 & \frac{-L^2}{2EI_y} & 0 & \frac{L}{EI_y} & 0 \\ 0 & \frac{L^2}{2EI_z} & 0 & 0 & 0 & \frac{L}{EI_z} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \\ M_x \\ M_y \\ M_z \end{bmatrix} \tag{2.63}$$

This relationship can also be determined from the unit load method according to the principle of virtual work. This relation is the basis of the matrix flexibility method in structural analysis for framed structures.

2.24 ACTIONS – DEFORMATION RELATIONSHIP

For the beam element of Figure 2.63 the action-deformation relation can be found from Castigliano's first theorem by writing the strain energy in terms of the deformation. One can also use the unit displacement method according to the principle of virtual work. However, the inverse of Equation 2.63 gives directly the action-deformation relationship, which is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \\ M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \tag{2.64}$$

This matrix shall be used in Chapters 4 and 5 and is called the direct stiffness matrix as it relates the actions and deformation at the same joint j. A system of actions can also be developed at j due to unit displacements at i. In this case, the matrix is called cross-stiffness matrix. These matrices are the basis of the stiffness matrix method in structural analysis of skeletal structures.

2.25 THE BENDING MOMENT-CURVATURE RELATIONSHIP

The curvature of a deformation curve is defined as the rate of changing the curve direction. Considering a part of length ΔS from a deformation curve as shown in Figure 2.64, the curvature is defined as

$$K = \frac{1}{\rho} = \lim_{\Delta S \to 0} \frac{\Delta\phi}{\Delta S} = \frac{d\phi}{ds} \tag{2.65}$$

where ρ is the radius of the curvature, which is the inverse of the curvature.

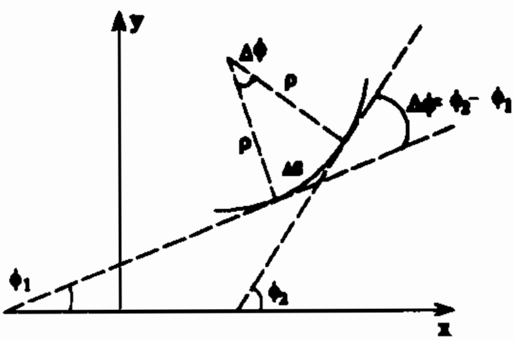


Figure 2.64

Since the slope at any point along the deformation curve is defined as tan ϕ = dy/dx, by taking the derivative of both terms, one has

$$\frac{d}{dx}(\tan\phi)=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

This leads to the relationship

$$\left(1+\tan^2\phi\right)\frac{d\phi}{dx}=\frac{d^2y}{dx^2} \tag{2.66}$$

which can be written as

$$\frac{d\phi}{dx}=\frac{d^2y/dx^2}{1+\left(dy/dx\right)^2} \tag{2.67}$$

The relationship between ds and dx is obtained as follows:

$$\begin{aligned} ds &= \sqrt{dx^2+dy^2} \\ &= dx\sqrt{1+(dy/dx)^2} \end{aligned} \tag{2.68}$$

Therefore, one has from Equation (2.68),

$$\frac{dx}{ds}=\frac{1}{\left[1+\left(dy/dx\right)^2\right]^{1/2}} \tag{2.69}$$

From Equations (2.67) and (2.69), the curvature is obtained as

$$\frac{d\phi}{ds}=\frac{d\phi}{dx}\times\frac{dx}{ds}=\frac{d^2y/dx^2}{\left[1+\left(dy/dx\right)^2\right]^{3/2}} \tag{2.70}$$

For small deformations where dy/dx $\approx$ 0, this relation is approximated by

$$\frac{d\phi}{ds}\cong\frac{d^2y}{dx^2} \tag{2.71}$$

The relationship between bending moment and curvature is obtained by utilizing the stress strain relationship. For the beam element of length ds which is subjected to bending moment M at both ends as shown in Figure 2.65, it will be subjected to normal stresses due to the bending moment given by

$$\sigma=\frac{M}{I}c \tag{2.72}$$

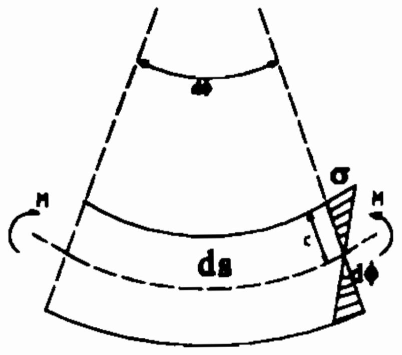


Figure 2.65

where σ is the normal stress, I is the moment of inertia of the section about its neutral axis, and c is the distance between any point and the neutral axis.

From the stress-strain relationship of linear elastic structure ($\sigma = E\,\varepsilon$), the total elongation Δ at the bottom fiber of the element ds is given by

$$\Delta = \varepsilon\, ds = \frac{\sigma}{E}\, ds \tag{2.73}$$

which must satisfy the relation $\Delta = c\ d\ \phi$.

From Equation 2.73 one obtains

$$\frac{M}{EI} - c \quad ds = c \ \ d\phi$$

$$\frac{M}{EI} = \frac{d\phi}{ds} \cong \frac{d^2y}{dx^2} \tag{2.74}$$

2.26 THE MOMENT AREA METHOD

The moment area method can be used to determine the deformation due to bending moment in beams and frames. The method is based on the bending moment curvature relationship given in Equation 2.74. From this equation, one can obtain the following identities:

$$\frac{dy}{dx} = \int \frac{M}{EI}\, dx \tag{2.75}$$

$$y = \int \left[\int \frac{M}{EI}\, dx \right] dx \tag{2.76}$$

Equation 2.75 can be used to determine the change in slope of the deformation curve between two points, and Equation 2.76 can be used to determine the distance between slopes of two points. Therefore, the moment area method depends on two theorems. The first theorem states that "the change in slope between two points on an elastic curve equals the area of (M/EI) diagram between the two points". The second theorem states that "the distance between the tangents of the elastic curve at two points measured at one of the points equals the moment of the area of (M/EI) diagram between the two points taken about that point".

Example 2.18

Determine the vertical deflection at mid-span and the maximum deflection for the beam shown in Figure 2.66. Consider EI = 10^5 kN.m^2.

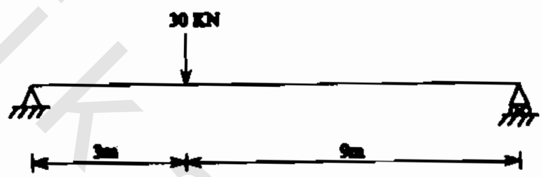


Figure 2.66

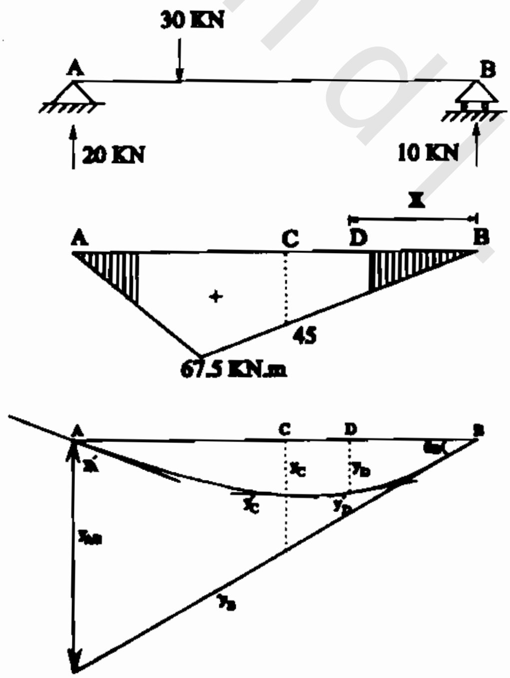


Figure 2.67

Solution

The bending moment for the beam and a sketch of the deformed shape must first be constructed as shown in Figure 2.67. To determine the deflection at point C one has to determine θ_B and y_{CB}, where:

$$\theta_B = y_{AB}/12 \quad , \quad y_c = 6\theta_B - y_{CB}$$

$$y_{AB} = \text{moment of } \frac{M}{EI} \text{diagram between A and B about A}$$

$$= \frac{1}{EI}\left[\frac{6.75 \times 3}{2} \times 2 + \frac{6.75 \times 9}{2} \times 6\right] = 2025 \times 10^{-5} \text{ m}$$

$$\theta_B = y_{AB}/12 = 168.75 \times 10^{-5} \text{ rad.}$$

$$y_{CB} = \text{A moment of } \frac{M}{EI} \text{ diagram between C and B about B}$$

$$= \frac{1}{EI}\left(\frac{45 \times 6}{2} \times 2\right) = 270 \times 10^{-5} \text{ m}$$

$$y_c = 6\theta_B - y_{CB}$$

$$= 6 \times 168.75 \times 10^{-5} - 270 \times 10^{-5} = 742.5 \times 10^{-5} \text{ m} = 0.7425 \text{ cm}$$

To determine the maximum deflection, one has to look for the location of zero slope on the elastic curve. This location is called point D. Since $\theta_D = 0$, and θ_B is known, one can use the first theorem to determine the location of point D as follows:

$$\theta_{BD} = \theta_B - \theta_D = \text{Area of } \frac{M}{EI} \text{diagram between B and D}$$

$$167.75 \times 10^{-5} - 0 = \frac{1}{EI}\left(\frac{67.5}{9} x\right)\frac{x}{2} = \frac{67.5\, x^2}{18} \times 10^{-5}$$

Solving for x one obtains $x = 6.708$ m.

The maximum deflection can now be determined from $y_D = 6.708\,\theta_B - y_{DB}$

$$y_D = 6.708 \times 168.75 \times 10^{-5} - 10^{-5}\left(\frac{67.5}{9} x\right)\frac{x}{2} \times \frac{x}{3}$$

$$= 754.67 \times 10^{-5} \text{ m} = 0.7546 \text{ cm.}$$

2.27 THE CONJUGATE BEAM METHOD

The conjugate beam method is also used to determine the deformation due to bending moment in beams and frames. It is considered as another version of the moment area method. As we have seen, the moment area method depends on taking the areas and the moment of (M/EI) diagram at certain points. If the (M/EI) diagram is considered as loading and the geometrical boundary conditions are applied, then one can determine the slope of the deformation curve at any point from the value of the equivalent shear force at that point. Similarly, the deflection at any point in reference to the tangent at another point can be calculated by taking the moment of the (M/EI) loading about that point. The transformation from the actual beam to the conjugate beam follows the rules summarized in Table 2.4.

Table 2.4 Transformation to Conjugate Beams

Actual Beam		Conjugate Beam	
Joint	Condition	Joint	Condition
Fixed Support	slope = 0 deflection = 0	Free Joint	shear = 0 moment = 0
Free Joint	slope ≠ 0 deflection ≠ 0	Fixed Support	shear ≠ 0 moment ≠ 0
Internal Hinge	slope ≠ 0 deflection ≠ 0	Interior Support	shear ≠ 0 moment ≠ 0
Simple Support	slope ≠ 0 deflection = 0	Simple Support	shear ≠ 0 moment = 0
Interior Support	slope ≠ 0 deflection = 0	Interior Hinge	shear ≠ 0 moment = 0

Example 2.19

Determine the deflection at point C and the slope at point D for the beam shown in Figure 2.68 using the conjugate beam method ($EI = 10^5$ kN.m^2).

The bending moment diagram and the conjugate beam are constructed as shown in Figure 2.69. The deflection at C is determined by taking the moment about C for the part AC.

$y_c = +180 \times 10^{-5} \times 4 = +720 \times 10^{-5}$ m (downward)

The slope at D is the shear force at D. By calculating the reactions one obtains:

$$M_B = 160 \times 10^{-5} \times \frac{4}{3} - 40 \times 10^{-5} \times 2 + 180 \times 10^{-5} \times 8 - R_c \times 4 = 0 \quad ; \quad R_c = 393 \times 10^{-5}$$

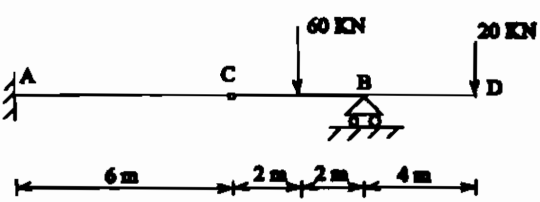


Figure 2.68

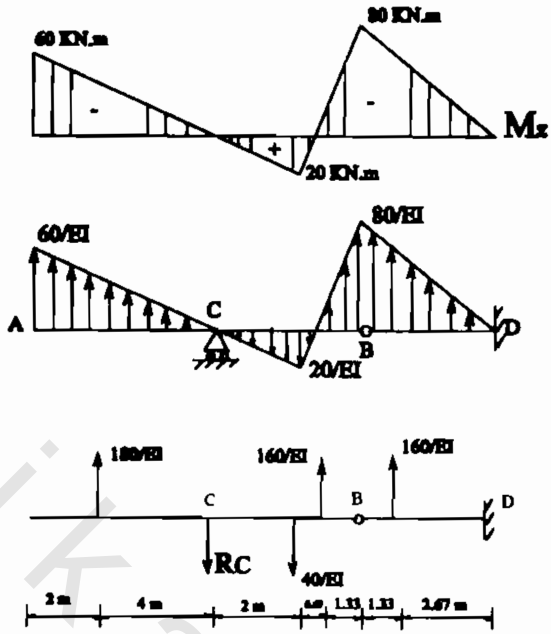


Figure 2.69

The slope at D is thus

$V_D = (180 + 160 + 160 - 393.33 - 40)\ 10^{-5} = 66.67 \times 10^{-5}$ rad.

2.28 DEFORMATION OF STRUCTURES DUE TO SUPPORTS SETTLEMENTS

For statically determinate unloaded structures, the supports settlements do not generate any internal forces. Therefore, the work done due to a virtual load must be zero. From this principle, one can determine the deformed shape of the statically determinate structures due to supports settlements.

Example 2.20

Determine the horizontal deflection at B due to a vertical settlement at A of 1 cm downward for the frame shown in Figure 2.70.

Solution

Apply a virtual horizontal unit load at B and determine the reactions at the supports. The total work done by the reactions on supports movements must be zero. The reactions are calculated and shown in Figure 2.71.

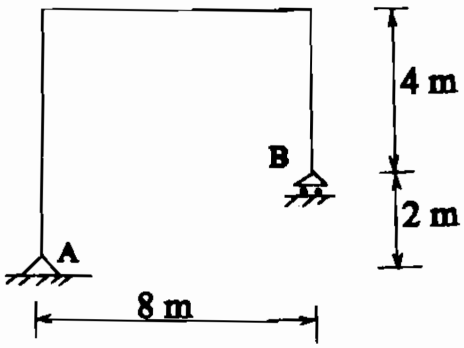


Figure 2.70

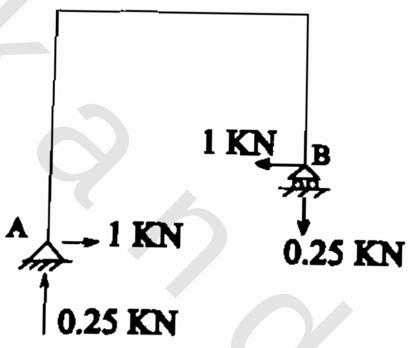


Figure 2.71

The total work done = $1 \times \Delta_{Bh} + (-0.25)(\Delta_{AV}) = 0$

$\Delta_{Bh} = 0.25 \times 1 = 0.25$ cm   (←)

which is in the same assumed direction.

Exercises

1. Calculate the strain energy for the structures shown in Figure 1. (EA = 10^5 kN, EI = 10^5 kN.m^2).

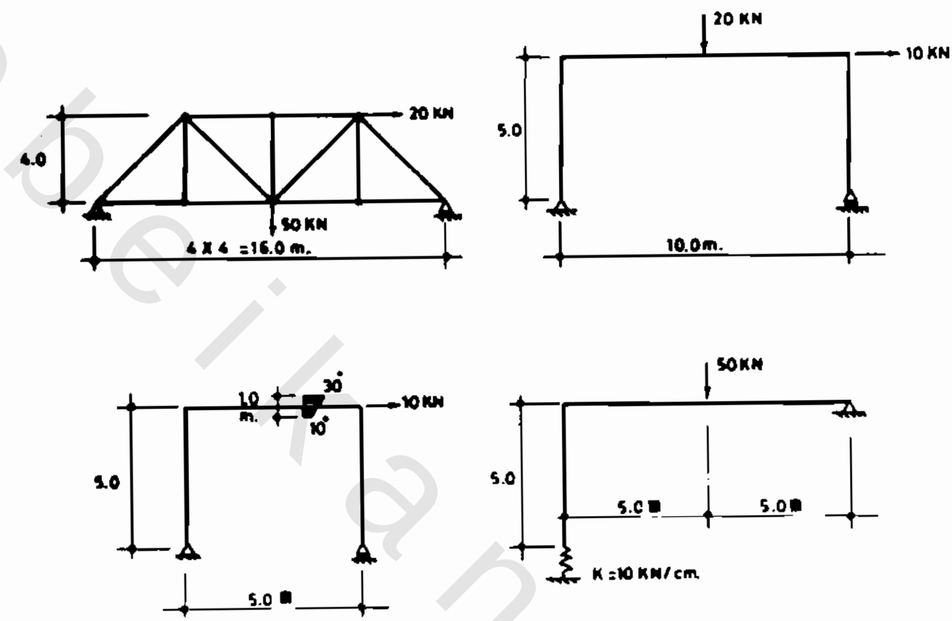


Figure 1

2. Calculate the vertical deflection at point C for the trusses shown in Figure 2 using the conservation of energy principle, and verify the results by the unit load method. (EA = 10^5 kN).

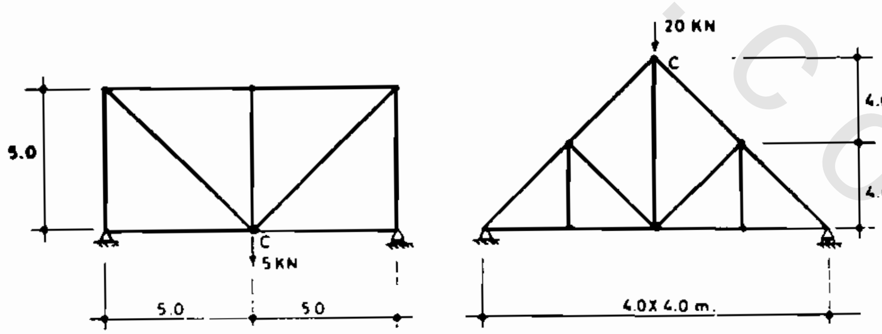


Figure 2

3. Calculate the vertical deflection at point C for the structures shown in Figure 3 using the unit load method.

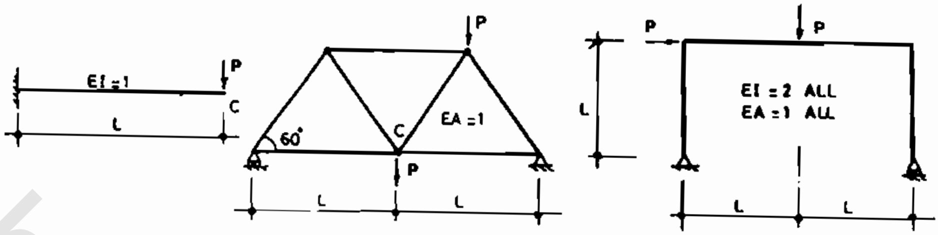


Figure 3

4. Determine the flexibility matrix using the unit load method for the structures shown in Figure 4.

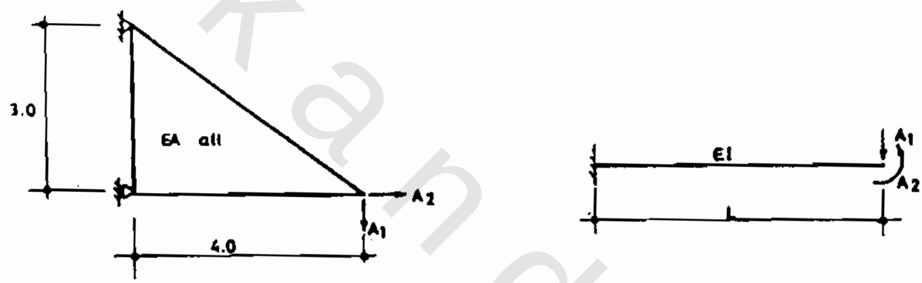


Figure 4

5. Determine the stiffness matrix using the unit displacement method for the structures shown in Figure 5.

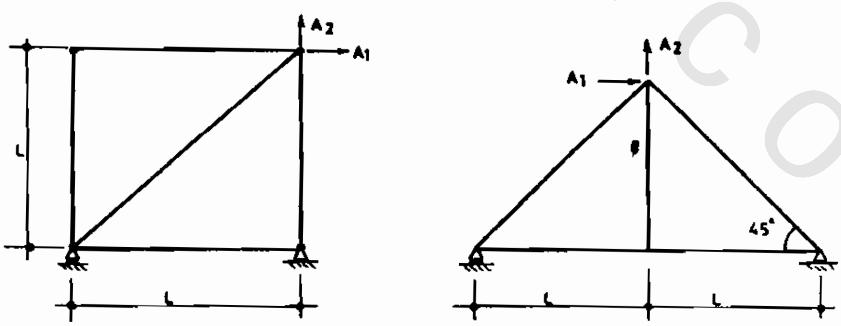


Figure 5

6. Solve problem 3 using Castigliano's theorems.

7. Solve problem 4 using Castigliano's theorems.

8. Solve problem 5 using Castigliano's theorems.

9. Use the unit load method to determine the vertical deflection at C due to the loads applied on the frame shown in Figure 6. ($EA = 10^5$ kN, $EI = 10^4$ kN.m^2).

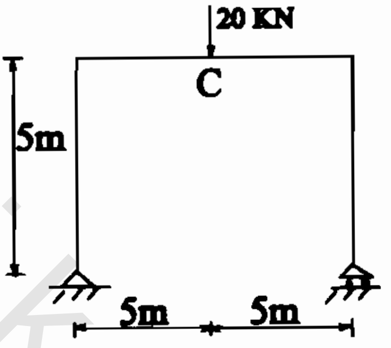


Figure 6

10. Use the moment area method and verify the results by the conjugate beam method to determine the vertical deflection at C and the slope at D for the beam shown in Figure 7 ($EI = 10^{-5}$ kN.m^2).

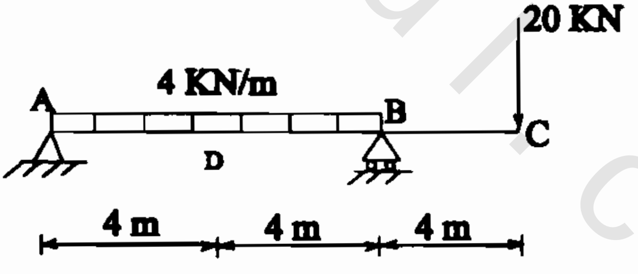


Figure 7