CHAPTER 4

THE DISPLACEMENT METHODS

4.1 INTRODUCTION

The displacement methods solve the structural analysis problems in terms of deformations. These deformations are found using the equilibrium equations for the end actions at every free joint. That is why a method is also called the equilibrium method, or more generally, the stiffness method, because the coefficients of the equilibrium equations are stiffness coefficients. Thus, the displacement methods deal with the degree of kinematic indeterminacy which represents the degree of freedom in the structure. The obtained number of equilibrium equations must equal the degree of kinematic in in the structure. If the structure is kinematically determinate, it means that every member in the structure is fixed at both ends and that the end actions can easily be found from the available tables or using any of the flexibility methods. It means also that the structure which is kinematically determinate can not be analyzed using the displacement method. The situation is similar to the structures which are statically determinate. The analyst can determine the end actions in these statically determinate structures using only static principles.

Preference between the force method and the displacement method depends on the number of unknowns in each method. However, as it will be shown later at the end of this chapter, the displacement method can easily be programmed for a computer using the minimum input data. This is not the case for the flexibility method, which depends basically on developing the equilibrium matrix $\underline{\mathbf{E}}$ in order to solve the problem as was shown in section 3.9.

In this chapter, the classical methods will first be given. The transformation from the classical methods to the matrix approach without losing any of the fundamentals will also be shown. The methods given in this chapters are the following:

1) The slope-deflection equation method

2) The moment distribution method

3) The use of Castigliano's first theorem

4) The stiffness matrix method : Approach I

5) The stiffness matrix method : Approach II

4.2 DEGREE OF KINEMATIC INDETERMINACY

Determination of the degree of kinematic in is the first step in solving the structural analysis problem by the stiffness method. It indicates the number of equilibrium equations need to be solved for the problem. In the case of using the classical methods, and before using the computer, this was an important step in order to decide whether to use the flexibility or stiffness methods. The degree of kinematic in is directly related to the type of the free joints, and the unknown displacements which need to be considered for solving the problem. This is illustrated in the following subsections:

4.2.1 Free Joint in Space

A rigid joint in space should have six degrees of freedom, three translational and three rotational with respect to Cartesian Coordinates. However, the analyst could neglect, for example, axial deformations or torsional rotation from the degrees of freedom. This depends on the type of structures, loading, and the degree of accuracy required in the solution. A frictionless free joint in space should have three degrees of freedom which all are translational displacements with respect to Cartesian Coordinates.

4.2.2 Free Joint in Plane

A rigid joint in plane should have three degrees of freedom, two tanslational and one rotational, with respect to Cartesian Coordinates. The analyst, however, could neglect some of these displacements. For example, in beams, one may only consider the rotational displacements that are of importance. A frictionless free joint in plane should have two degrees of freedom which are translational with respect to the Cartesian Coordinates.

4.2.3 Numerical Examples

Example 4.1

Determine the degree of kinematic in (DKI) or the degree of freedom (DOF) in the space structures shown in Figure 4.1.

Solution

(a) The space frame has 4 space joints, two joints, at A and B, are completely fixed, and the third joint at D is restrained against translational displacements. Therefore, the degree of freedom of this structure is

$$\text{DKI} = 4 \times 6 - 2 \times 6 - 1 \times 3 = 9$$

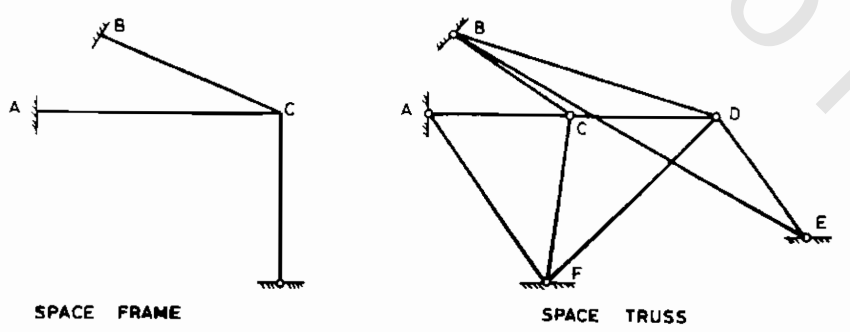

Figure 4.1

which represent six displacements at C and three displacements at D.

(b) The space truss has 6 space joints, four of them at A, B, E, and F are restrained against any translation. Therefore, DKI is given by

$$DKI = 6 \times 3 - 4 \times 3 = 6$$

which represents three displacements at each of C and D.

Example 4.2

Determine the degree of kinematic in in the structures shown in Figure 4.2.

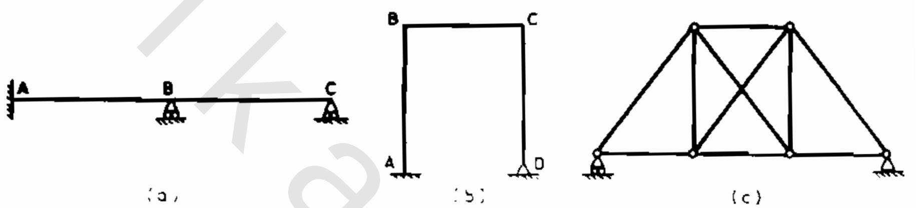

Figure 4.2

Solution

(a) The beam has three rigid joints. One is completely fixed at A, and the joints B and C are restrained against vertical motion. Therefore one has

$$DKI = 3 \times 3 - 3 - 1 - 1 = 4$$

which represents a rotation and horizontal displacement at each of B and C.

(b) The frame has four joints, one is fixed at A and one is restrained against horizontal and vertical displacements at D. Therefore,.

$$DKI = 6 \times 2 - 2 - 1 = 9$$

which represents two displacements at each of B, C, E, and F, and one horizontal displacement at A.

Example 4.3

Determine the degree of kinematic in for the structures shown in Figure 4.3.

Solution

(a) The frame has 5 joints, two are fixed at A and E. The hinge at C introduces relative rotation and DKI could be one of the following solutions:

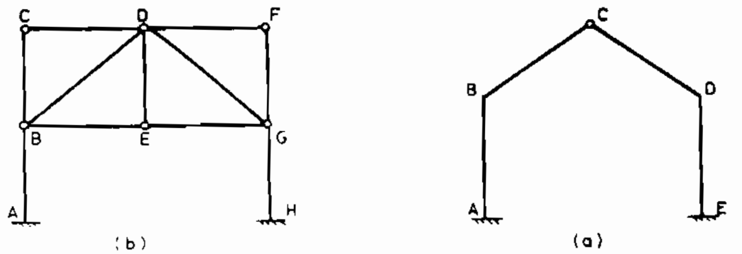

Figure 4.3

$$DKI = 5 \times 3 - 2 \times 3 = 9$$

If one considers the relative rotation at C is one unknown.

$$DKI = 9 + 1 = 10 \text{ (considering absolute rotations at C).}$$

If one considers the relative rotation at C consists of two unknowns, the left and right rotations at C.

(b) The structure has 8 joints, two clamped at A and H. Then,

$$DKI = 8 \times 3 - 2 \times 3 - 2 = 16$$

These include two displacement at each of E and D, three displacements and rotation at each of B, C, F, and G.

4.3 THE SLOPE-DEFLECTION EQUATION METHOD

The slope-deflection equation method deals only with rotational and relative translational displacements of the joints of every member. The method does not consider the axial deformation and therefore it can not be used in solving truss structures. The method is most suitable for beams and some frames where the axial deformations can be neglected without having significant errors.

Consider the plane frame shown in Figure 4.4. Member AB is subjected to

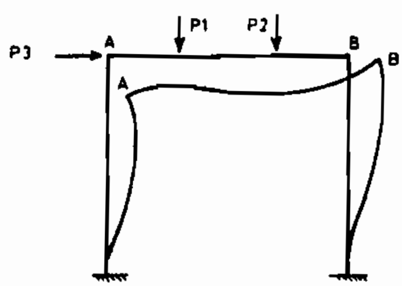

Figure 4.4

deformations due to the internal actions. The deformations consist of rotational displacements at A and B, and relative displacement between A and B. Neglecting the axial deformation, the horizontal displacements of Joints A and B are the same. Consider only member AB which is shown in Figure 4.5 and has constant stiffness EI. For simplicity, the end moments and rotations are considered positive when they are in the counterclockwise direction. Also, the right joint, B, when is displaced vertically upward with respect to the left joint A, is considered positive.

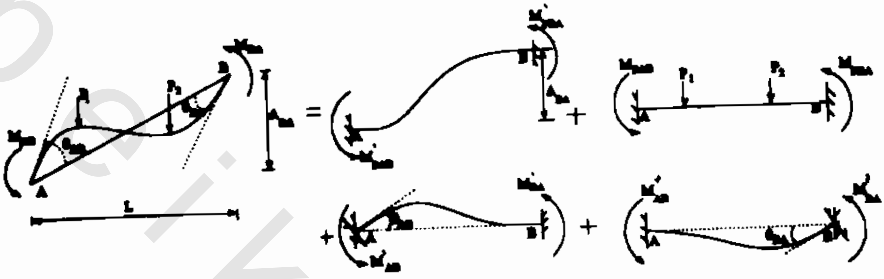

Figure 4.5

Using the superposition principle, the end moments M_{AB}, M_{BA} can be written as follows:

$$M_{AB} = M_{FAB} + M'_{FAB} + M'_{AB} + M''_{AB} \tag{4.1}$$

$$M_{BA} = M_{FBA} + M'_{FBA} + M'_{BA} + M''_{BA} \tag{4.2}$$

where M_{FAB} and M_{FBA} are the fixed end moments due to the applied loading. The M'_{FAB} and M'_{FBA} are the fixed end moments due to the deflection Δ_{BA}. The end moments which cause the angles of rotation θ_{AB} and θ_{BA}, are denoted by M'_{AB}, M'_{BA}, M''_{AB}, and M''_{BA} respectively.

The relationships between M'_{FAB}, M'_{FBA}, and Δ_{BA} can be found using any method in the previous chapter. According to the sign convention used in this chapter, these relationships are (refer to example 3.21):

$$M'_{FAB} = -\frac{6EI}{L^2}\Delta_{BA} \tag{4.3}$$

$$M'_{FBA} = -\frac{6EI}{L^2}\Delta_{BA} \tag{4.4}$$

The relationships between M'_{AB}, M'_{BA}, and θ_{AB} were also found in Example 3.20 of the previous chapter. They can also be found using the unit load method or the conjugate beam method. These relationships are

$$M'_{AB} = -\frac{4EI}{L}\theta_{BA} \tag{4.5}$$

$$M'_{BA} = -\frac{2EI}{L}\theta_{BA} \tag{4.6}$$

where the coefficient (4EI/L) is called direct stiffness coefficient, and the coefficient (2EI/L) is called cross-stiffness coefficient. Similar relations are obtained between M''_{BA}, M''_{AB}, and θ_{BA}. Substituting the values of M_{FAB}, M'_{AB} and M''_{AB} into Equation 4.1, one obtains the general slope-deflection equation for member AB as follows:

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_{AB} + \theta_{BA} - \frac{3\Delta_{BA}}{L}\right) \tag{4.7}$$

Similarly, substituting the expressions M_{FBA}, M'_{BA} and M''_{BA} into Equation 4.1 one obtains

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_{BA} + \theta_{AB} - \frac{3\Delta_{BA}}{L}\right) \tag{4.8}$$

Therefore, every member in the structure has two slope deflection equations like Equations 4.7 and 4.8. If one is able to determine the unknown joints displacements in the structure, then the end moments of each member can easily be found. Determination of the Joints displacements depends on satisfying the conditions of compatibility, equilibrium, and the boundaries. The compatibility conditions involve the relationships between the rotational or translational displacements of various members and the unknown displacements in the structure in order to insure the integrity of the structural elements. For example, in rigid joints, the angles of rotation of all members connected with the joint should be the same as the rotation of the joint. The equilibrium conditions satisfy the equilibrium between the internal actions and the external actions applied at a certain joint in the structure. The boundary conditions provide the necessary information for the kind of the boundary joints in order to solve the problem. These conditions, as explained in Chapter 2, are compulsory to solve any structural analysis problem.

Thus, the procedure to analyze a structure using the slope-deflection equation method is summarized as follows:

1. The structure is considered clamped at every joint in order to be kinematically determinate. In this case, the fixed end moments are determined for every member. The fixed end moments could be due to applied loads, rise in temperature, support, deformation, or any combination of them. Tables 3.2 could be used to determine the fixed end moments.

2. The two-slop-deflection equations are written for every member in the structure. These equations are used to form the equilibrium equations at the joints in order to determine the member end moments.

3. The compatibility conditions at every joint should be written. These conditions reduce the number of unknown displacements from the members displacements to the structure displacements. For example, if joint A is rigidly connected with members AB, AC, AD, then one can say that $\theta_{AB} = \theta_{AC} = \theta_{AD}$ and that the three angles of rotation can be denoted by only one variable which is the rotation of Joint A called θ_A.

4. The equilibrium conditions at every joint are applied. These conditions state that the sum of the end actions of all members connected with a joint must equal to the external actions applied at this joint. It is our interest to write these equations at the free joints since the external actions are usually known at these joints. Thus, one ends up with a number of equilibrium equations equal to the number of unknown displacements.

   In plane frames, the equilibrium equations at any joint subjected to the external actions P_x, P_y, and M_z can be written as

$$P_x = \sum_{i=1}^{k} A_{xi} \tag{4.9}$$

$$P_y = \sum_{i=1}^{k} A_{yi} \tag{4.10}$$

$$M_z = \sum_{i=1}^{k} M_{zi} \tag{4.11}$$

   where k indicates the total number of members connected to the joint, and i is any member of the set k.

   However, because of neglecting the axial deformations in the slope-deflection equation method, equations 4.9 and 4.10 are replaced by the equilibrium equations of forces in the direction of the translational displacements.

5. The boundary conditions help in using the known kinematic variables whether they are zero as in fixed supports or have values as in case of deformation of supports.

6. The obtained equilibrium equations are linear simultaneous equations. Their solution gives the unknown rotational and translational displacements. Substituting the displacements into the slope-deflection equations of each members, one can determine the end moments for each member.

4.3.1 Applications to Beams

Example 4.4

Draw the bending moment diagram for the beam shown in Figure 4.6 due to the applied loads, a rotation at support A, and a rise in temperature in member BC (EI = 10^5 kN.m^2, α = 10^{-5}/°c).

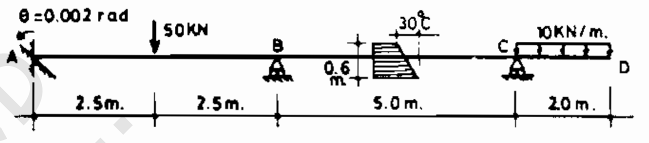

Figure 4.6

Solution

The beam has 7 degrees of freedom. However, due to neglecting axial deformations only θ_B, θ_C, θ_D, and Δ_D are significant. Since member CD is statically determinate, θ_D and Δ_D may be ignored if the loading on CD is applied at C. The slope deflection equations for the members AB and BC are:

$$M_{AB} = M_{FAB} + \frac{2EI}{L}(2\theta_{AB} + \theta_{BA})$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}(2\theta_{BA} + \theta_{AB})$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}(2\theta_{BC} + \theta_{CB})$$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}(2\theta_{CB} + \theta_{BC})$$

The fixed end moments of members AB and BC are determined from Table 3.2. The signs of the moments are transformed into the signs used in the slope deflection equation as shown in Figure 4.7.

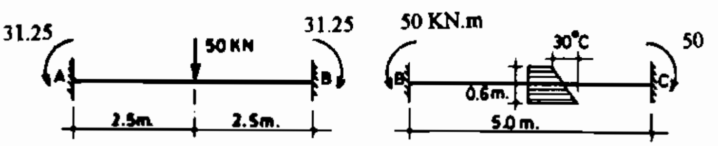

Figure 4.7

$M_{FAB} = +31.25$ kN.m ; $M_{FBA} = -31.25$ kN.m

$M_{FBC} = +40$ kN.m ; $M_{FCB} = -50$ kN.m

The compatibility conditions can be stated as:

$\theta_{AB} = \theta_A$; $\theta_{BA} = \theta_{BC} = \theta_B$; $\theta_{CB} = \theta_{CD} = \theta_C$

The equilibrium conditions state that:

$M_A = M_{AB}$; $M_B = M_{BA} + M_{BC} = 0$; $M_C = M_{CB} = -20$ kN.m

The boundary conditions state that $\theta_A = +0.002$ rad.

Therefore, substituting the slope-deflection equations, compatibility conditions, and the boundary conditions into the equilibrium equations, one obtains

$$M_B = 0 = -31.25 + \frac{2EI}{5}(2\theta_B + \theta_A) + 50 + \frac{2EI}{5}(2\theta_B + \theta_C)$$

$$M_C = -20 = -50 + \frac{2EI}{5}(2\theta_C + \theta_B)$$

These equations can be solved for θ_B and θ_C to have

$\theta_B = -81.25 \times 10^{-5}$ rad. ; $\theta_C = 78.125 \times 10^{-5}$ rad.

Substituting these values into the slope-deflection equations, one determines the end moments as follows:

$$M_{AB} = +31.25 + \frac{2EI}{5}\left(2 \times 200 \times 10^{-5} - 81.25 \times 10^{-5}\right) = 158.75 \text{ kN.m}$$

$$M_{BA} = -31.25 + \frac{2EI}{5}\left(2 \times 81.25 \times 10^{-5} + 200 \times 10^{-5}\right) = -16.25 \text{ kN.m}$$

$$M_{BC} = 50 + \frac{2EI}{5}\left(2 \times 81.25 \times 10^{-5} + 78.125 \times 10^{-5}\right) = 16.25 \text{ kN.m}$$

$$M_{CB} = -50 + \frac{2EI}{5}\left(2 \times 78.125 \times 10^{-5} - 81.25 \times 10^{-5}\right) = -20 \text{ kN.m}$$

The bending moment diagram can now be drawn as shown in Figure 4.8. The student should notice the matching between the signs of the moments and their directions in Figure 4.8.

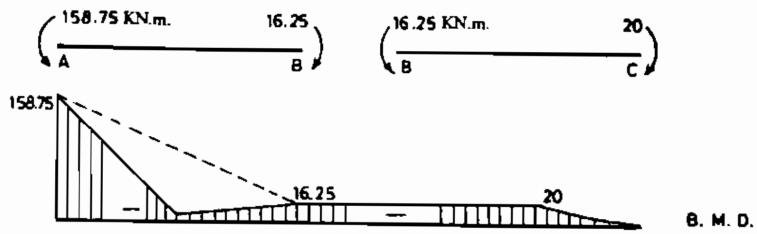

Figure 4.8

Example 4.5

Determine the moment and shear force diagrams for the beam shown in Figure 4.9 (EI = 10^5 kN.m^2, $\alpha = 10^{-5}/°c$, spring constant K = 10 kN/cm).

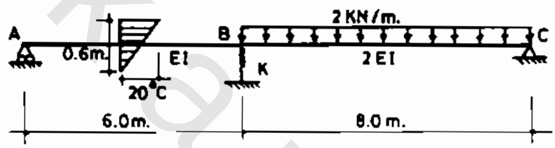

Figure 4.9

Solution

With neglecting axial deformations, the beam has 4 DOF $(\theta_A, \theta_B, \theta_C, \Delta_B)$. The settlement at B is denoted by Δ as shown in Figure 4.10. The slope-deflection equations are written for members AB and BC as follows:

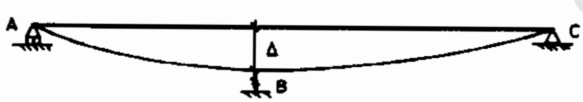

Figure 4.10

$$M_{AB} = M_{FAB} + \frac{2EI}{6}\left(2\theta_{AB} + \theta_{BA} - \frac{3(-\Delta)}{6}\right)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{6}\left(2\theta_{BA} + \theta_{AB} - \frac{3(-\Delta)}{6}\right)$$

$$M_{BC} = M_{FB} + \frac{4EI}{8}\left(2\theta_{BC} + \theta_{CB} - \frac{3(\Delta)}{8}\right)$$

$$M_{CB} = M_{FCB} + \frac{4EI}{8}\left(2\theta_{CB} + \theta_{BC} - \frac{3(\Delta)}{8}\right)$$

The student should notice the sign of Δ for members AB and BC.

The fixed end moments are determined from Table 3.2 as shown in Figure 4.11. They are written according to the sign of slope deflection equation as follows:

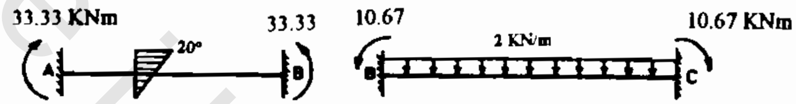

Figure 4.11

$M_{FAB} = -33.333\ \text{kN.m}$, $M_{FBA} = +33.333\ \text{kN.m}$

$M_{FBC} = +10.667\ \text{kN.m}$, $M_{FCB} = -10.667\ \text{kN.m}$

The compatibility conditions are given by

$\theta_{AB} = \theta_A$; $\theta_{BA} = \theta_{BC} = \theta_B$; $\theta_{CB} = \theta_C$

The unknown kinematic variables are $\theta_A, \theta_B, \theta_C$, and Δ. The equilibrium equations are then,

$M_{AB} = M_A = 0$; $M_{BA} = M_{BC} = M_B = 0$; $M_{CB} = M_C = 0$;

Thus sum of forces in direction of Δ at B = 0.

From Figure 4.12, the reaction at B is obtained from

$$R_B = \frac{M_{BC} + M_{CB}}{8} + 8 - \frac{M_{AB} + M_{BA}}{6} = K\Delta = 1000\Delta$$

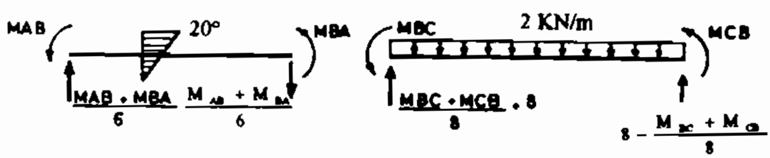

Figure 4.12

Therefore, the equilibrium equations need to be solved are

$$\theta_A + \theta_B + \frac{\Delta}{2} = 33.333 \times \frac{3}{EI} = \frac{100}{EI}$$

$$1.5\theta_C + \theta_A + 5\theta_B - 0.0625\Delta = \frac{-132}{EI}$$

$$2\theta_C + \theta_B - 0.375\Delta = \frac{21.333}{EI}$$

$$0.25\theta_B + 1.125\theta_C - \theta_A - 1.101875\Delta = \frac{-68}{EI}$$

Solving these equations one obtains

$$\theta_A = \frac{68.4098}{EI}\,\text{rad.}\ ,\ \theta_C = \frac{42.8298}{EI}\,\text{rad.}\ ,\ \theta_B = \frac{-52.5379}{EI}\,\text{rad.}\ ,\ \Delta = \frac{31.4366}{EI}\,\text{m}$$

Substituting into the slope-deflection equations one can determine the end moments as follows:

$$M_{BA} = 33.333 + \frac{EI}{3}\left(2\theta_B + \theta_A + \frac{\Delta}{2}\right) = 26.35\ \text{kN.m}$$

$$M_{BC} = 10.667 + \frac{EI}{2}\left(2\theta_B + \theta_C - \frac{3\Delta}{8}\right) = -26.35\ \text{kN.m}$$

It is obvious that the values of M_{BA} and M_{BC} satisfy the equilibrium at B. The bending moment and shear force diagrams can be developed as shown in Figure 4.13.

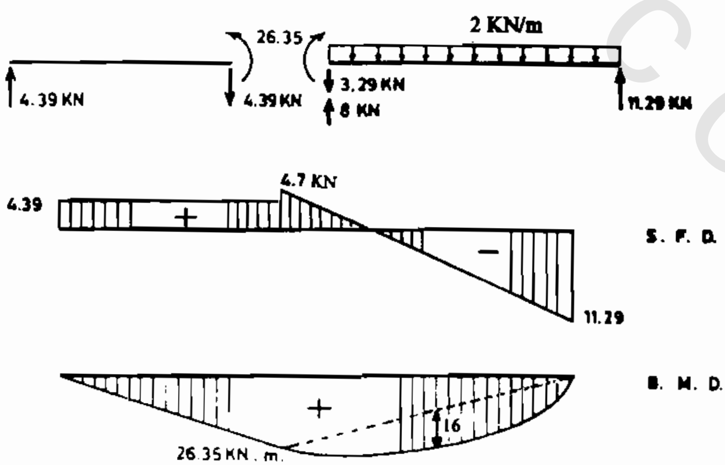

Figure 4.13

4.3.2 Applications to Frames

Example 4.6

Determine the bending moment and shear force diagrams for the frame shown in Figure 4.14 due to the loads shown and a rise in temperature in member BC ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}/°c$).

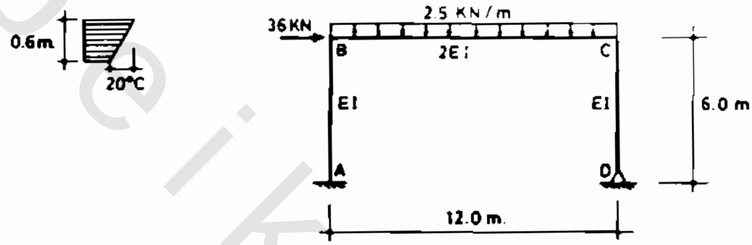

Figure 4.14

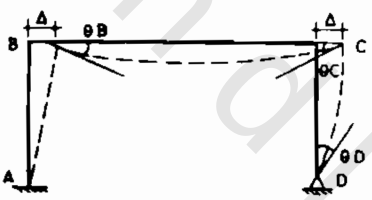

Figure 4.15

Solution

When axial deformations are neglected, the degree of kinematic indeterminacy is four which represents $\theta_B, \theta_C, \theta_D$, and Δ as shown in Figure 4.15.

The slope deflection equations for all members can be written considering the assumed direction for the sway of the frame as follows:

$$M_{AB} = M_{FAB} + \frac{2EI}{6}\left(2\theta_{AB} + \theta_{BA} - \frac{3(-\Delta)}{6}\right)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{6}\left(2\theta_{BA} + \theta_{AB} - \frac{3(-\Delta)}{6}\right)$$

$$M_{BC} = M_{FBC} + \frac{4EI}{12}\left(2\theta_{BC} + \theta_{CB}\right)$$

$$M_{CB} = M_{FCB} + \frac{4EI}{12}\left(2\theta_{CB} + \theta_{BC}\right)$$

$$M_{CD} = M_{FCD} + \frac{2EI}{6}\left(2\theta_{CD} + \theta_{DC} - \frac{3\Delta}{6}\right)$$

$$M_{DC} = M_{FDC} + \frac{2EI}{6}\left(2\theta_{DC} + \theta_{CD} - \frac{3\Delta}{6}\right)$$

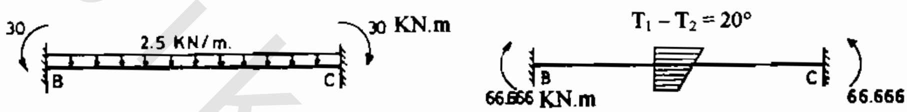

Figure 4.16

The fixed end moments for all members are determined, using Table 3.2 as shown in Figure 4.16, as follows:

$M_{FAB} = M_{FBA} = 0$; $M_{FCD} = M_{FDC} = 0$

$M_{FBC} = -36.666\,\text{kN.m}$; $M_{FCB} = +36.666\,\text{kN.m}$

The compatibility conditions are determined as follows:

$\theta_{AB} = \theta_A$; $\theta_{BA} = \theta_{BC} = \theta_B$; $\theta_{CB} = \theta_{CD} = \theta_C$; $\theta_{DC} = \theta_D$

The boundary conditions are determined as $\theta_A = 0$

The equilibrium equations are

$$M_{BA} + M_{BC} = M_B = 0 \quad \text{(a)}$$

$$M_{CB} + M_{CD} = M_C = 0 \quad \text{(b)}$$

$$M_{DC} = M_D = 0 \quad \text{(c)}$$

The sum of horizontal forces in the direction of $\Delta = 0$. (d)

Substituting the slope deflection equations, into the equilibrium equations one has

$$0+\frac{EI}{3}\left(2\theta_B+0.5\Delta\right)+(-36.666)+\frac{EI}{3}\left(2\theta_B+\theta_C\right)=0 \tag{a}$$

$$36.666+\frac{EI}{3}\left(2\theta_C+\theta_B\right)+0+\frac{EI}{3}\left(2\theta_C+\theta_D+\frac{\Delta}{2}\right)=0 \tag{b}$$

$$\frac{EI}{3}\left(2\theta_D+\theta_C+\frac{\Delta}{2}\right)=0 \tag{c}$$

The horizontal forces contain the horizontal reactions at A and D and the horizontal force at B of 36 kN. Thus one obtains

$$H_A+H_D=36$$

where H_A and H_D are obtained, using Figure 4.17, as follows:

$$H_A=\frac{M_{AB}+M_{BA}}{6} \quad ; \quad H_D=\frac{M_{CD}+M_{DC}}{6}$$

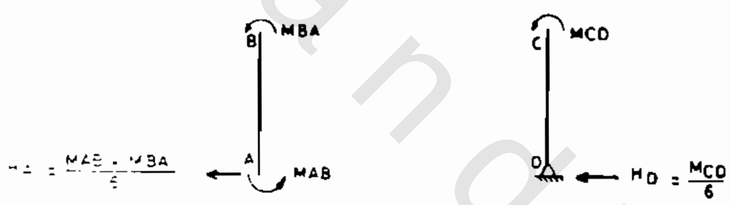

Figure 4.17

Substituting into equation (d), one obtains

$$\frac{1}{6}\times\frac{EI}{3}\left(3\theta_B+\Delta\right)+\frac{1}{6}\times\frac{EI}{3}\left(2\theta_B+\theta_c+0.5\Delta\right)=36\text{ kN} \tag{d}$$

Solving the equilibrium equations (a), (b), (c), and (d) one obtains

$\theta_B=-43.3636/EI$ rad. ; $\theta_D=-141.7272/EI$ rad.

$\theta_C=-69.4545/EI$ rad. ; $\Delta=705.818/EI$ m.

The positive value of Δ indicates that the sway is in the same assumed direction.

Substituting into the slope deflection equations one obtains the end moments of each member as follows:

$$M_{AB} = \frac{EI}{3}\left(\theta_B + \Delta\right) = 103.182 \quad kN.m$$

$$M_{BA} = \frac{EI}{3}\left(2\theta_B + \frac{\Delta}{2}\right) = 88.727 \quad kN.m$$

$$M_{BC} = -36.667 + \frac{EI}{3}\left(2\theta_B + \theta_C\right) = -88.727 \quad kN.m$$

$$M_{CB} = +36.667 + \frac{EI}{3}\left(2\theta_C + \theta_B\right) = -24.09 \quad kN.m$$

$$M_{CD} = 0 + \frac{EI}{3}\left(2\theta_C + \theta_D + \frac{\Delta}{2}\right) = 24.09 \quad kN.m$$

The bending moment and shear force diagrams can be developed as shown in Figure 4.18.

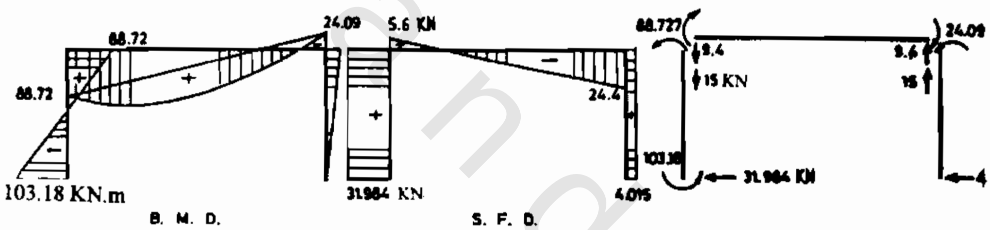

Figure 4.18

Example 4.7

Determine the bending moment and shear force diagrams for the frame given in Figure 4.19 due to the applied loads and a vertical settlement at B of 1 cm downward ($EI = 10^5$ kN.m^2).

Solution

This example shows an application when the relative displacement in a member is known. The expected deformed shape is shown in Figure 4.20. Due to neglecting axial deformations, the degree of kinematic Figure is two which represents θ_C and θ_B.

The slope deflection equations for members AC and CB are

$$M_{AC} = M_{FAC} + \frac{2EI}{6}\left(2\theta_{AC} + \theta_{CA}\right)$$

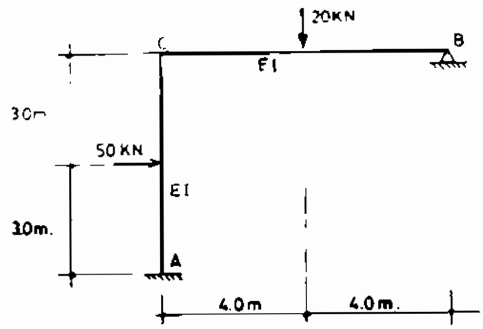

Figure 4.19

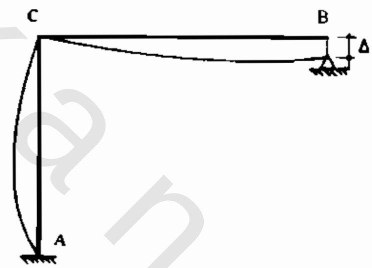

Figure 4.20

$$M_{CA} = M_{FAC} + \frac{2EI}{6}\left(2\theta_{CA} + \theta_{AC}\right)$$

$$M_{CB} = M_{FCB} + \frac{2EI}{8}\left(2\theta_{CB} + \theta_{BC} - \frac{3\times(-0.01)}{8}\right)$$

$$M_{BC} = M_{FBC} + \frac{2EI}{8}\left(2\theta_{BC} + \theta_{CB} - \frac{3\times(-0.010)}{8}\right)$$

in which the settlement Δ has been substituted for member CB.

The fixed end moments are found from the tables and shown in Figure 4.21, where

$M_{FAC} = +37.5$ kN.m , $M_{FCA} = -37.5$ kN.m

$M_{FCB} = +20$ kN.m , $M_{FBC} = -20$ kN.m

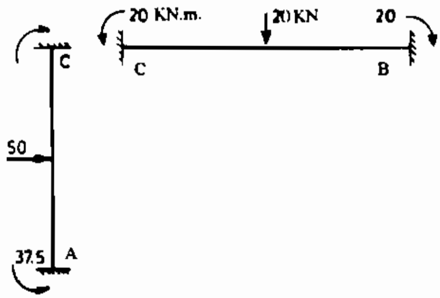

Figure 4.21

The equilibrium equations are

$$M_C = M_{CA} + M_{CB} = 0 \quad ; \quad M_B = M_{BC} = 0$$

Substituting the slope deflection equations into the equilibrium equations one obtains

$$-37.5 + \frac{EI}{3}\left(2\theta_C\right) + 20 + \frac{EI}{4}\left(2\theta_C + \theta_B + \frac{0.03}{8}\right) = 0 \qquad \text{(a)}$$

$$\frac{EI}{4}\left(2\theta_B + \theta_C + \frac{0.03}{8}\right) = 20 \qquad \text{(b)}$$

Solving equations (a) and (b) for θ_C and θ_B one obtains

$$\theta_C = -0.000378 \ \text{rad.} \quad ; \quad \theta_B = -0.001286 \ \text{rad.}$$

The end moments can be obtained by substituting the values of θ_C and θ_B into the slope-deflection equations as follows:

$$M_{AC} = 37.5 + \frac{EI}{3}(0.000378) = 24.9 \ \text{kN.m}$$

$$M_{CA} = -37.5 \times \frac{EI}{3}\left(2\theta_C\right) = -62.7 \ \text{kN.m}$$

$$M_{CB} = 20 + \frac{EI}{4}\left(2\theta_C + \theta_B + \frac{0.03}{8}\right) = 62.7 \ \text{kN.m}$$

$$M_{BC} = 0$$

The student should note that the results satisfy the equilibrium at joints C and B. The bending moment and shear force diagrams can be determined as shown in Figure 4.22.

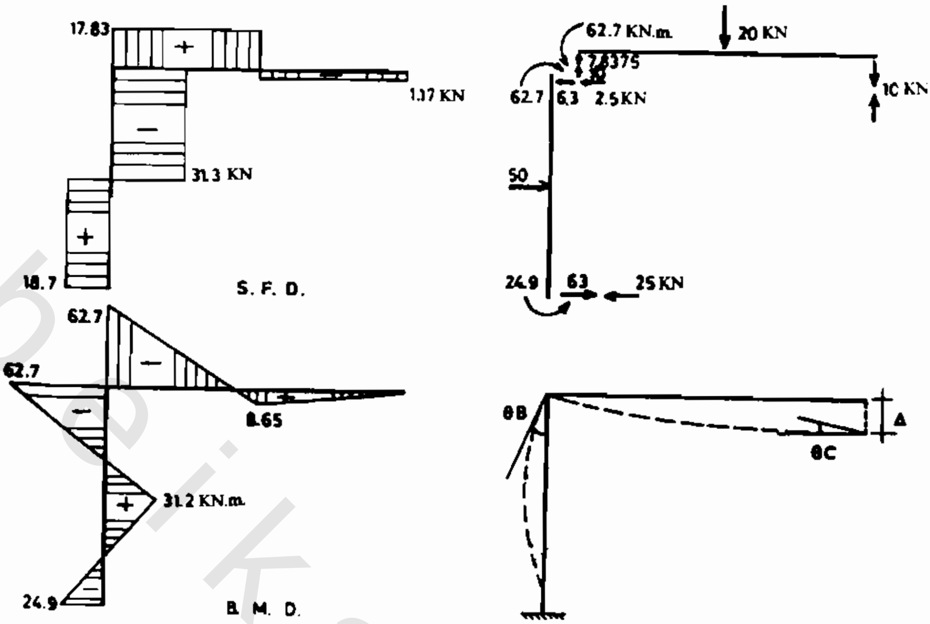

Figure 4.22

4.3.3 Applications to Members with Moments Releases

In many structural problems, the member can be hinged at one end and rigidly connected at the other end. The direct application of the slope deflection equation method, even when the joint is an external hinged support, was shown in the previous examples. One notices that the direct application approach deals with the angle of rotation of the hinged support, which is known to be of zero or known moment. The structural engineer is usually interested in determining the magnitudes of the end moments and not the deformations. Therefore, the direct slope-deflection equation can be modified to account for the fact that the moment at the hinged external support is zero and the angle of rotation at this hinge is not required for the solution. The modified slope-deflection equations can be derived using the superposition principle and Figure 4.23 as follows:

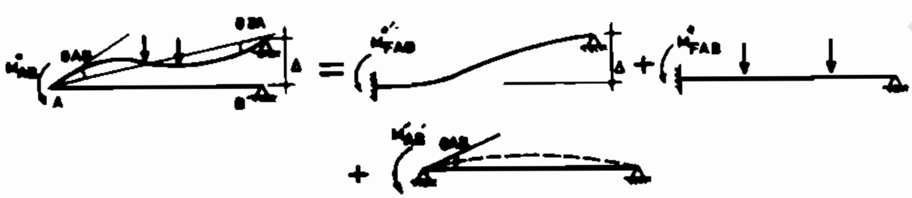

Figure 4.23

The superposition principle gives the end moment at A as

$$M^{*}_{AB} = M^{*'}_{FAB} + M^{*}_{FAB} + M^{*'}_{AB} \qquad (12)$$

where M^{*}_{AB} is the moment at the right end, M^{*}_{FAB} is the fixed end moment at A for the member which has a hinge at B due to the applied loading or temperature, $M^{*'}_{FAB}$ is the fixed end moment due to relative displacement and $M^{*'}_{AB}$ is the fixed end moment due to the rotation at A.

The fixed end moment $M^{*'}_{FAB}$ and $M^{*'}_{AB}$ can be found from Tables 3.2. The relationship between $M^{*'}_{FAB}$ and Δ is obtained as

$$M^{*'}_{FAB} = -\frac{3EI\Delta}{L^2} \qquad (13)$$

assuming that Δ at B is upward with respect to A.

The relationship between $M^{*'}_{AB}$ and θ_{AB} is given by

$$M^{*'}_{AB} = \frac{3EI}{L}\theta_{AB} \qquad (14)$$

Thus, the modified slope deflection equation is obtained as

$$M^{*}_{AB} = M^{*}_{FAB} + \frac{2EI}{L}\left(1.5\theta_{AB} - 1.5\frac{\Delta}{L}\right) \qquad (15)$$

Example 4.8

Solve the problem given in Example 4.4 using the modified slope-deflection equation method.

Solution

The fixed end moments are determined as shown in Figure 4.24, considering the members are with external hinges.

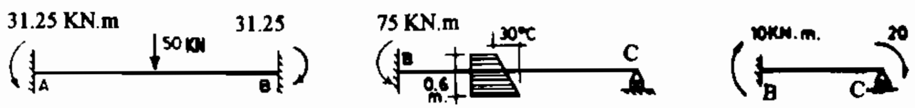

Figure 4.24

$$M_{FAB} = 31.25 \text{ kN.m} \quad ; \quad M_{FBA} = -31.25 \text{ kN.m}$$

The fixed end moment for member BC due to temperature change is obtained from Table 4.3 as

$$M^*_{FBC} = 1.5EI\frac{(T_1 - T_2)}{h} = 75 \text{ kN.m}$$

The fixed end moment for member BC due to the moment 20 kN.m applied at C is

$$M^*_{FBC} = -\frac{M}{2} = -10 \text{ kN.m}$$

Thus, the total $M^*_{FBC} = 75 - 10 = 65$ kN.m

The slope-deflection equations for the members are obtained as

$$M_{AB} = M_{FAB} + \frac{2EI}{L}(2\theta_{AB} + \theta_{BA})$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}(2\theta_{BA} + \theta_{AB})$$

$$M^*_{BC} = M^*_{FBC} + \frac{2EI}{L}(1.5\theta_{BC})$$

The equilibrium equation at B is $M_B = M_{BA} + M_{BC} = 0$ which leads to

$$-31.25 + \frac{2EI}{5}(2\theta_B + 0.002) + 65 + \frac{2EI}{5}(1.5\theta_B) = 0 \quad ; \quad \theta_B = -0.000813 \text{ rad.}$$

The end moments can now be determined from substitution as

$$M_{AB} = 31.25 + \frac{2EI}{5}(0.004 + \theta_B) = 158.73 \text{ kN.m}$$

$$M_{BA} = -31.25 + \frac{2EI}{5}(2\theta_B + 0.002) = -16.25 \text{ kN.m}$$

$$M^*_{BC} = 65 + \frac{2EI}{5}(1.5\theta_B) = 16.25 \text{ kN.m}$$

which are the same results as in Example 4.4.

Example 4.9

Solve the problem of Example 4.5 using the modified slope-deflection equation method.

Solution

The kinematic variable θ_A and θ_C here are not required since the moments at A and C are zero. The modified slope deflection equations are thus

$$M^*_{BA} = M^*_{FBA} + \frac{2EI}{6}\left(1.5\,\theta_{BA} - 1.5\,\frac{(-\Delta)}{6}\right)$$

$$M^*_{BC} = M^*_{FBC} + \frac{4EI}{8}\left(1.5\,\theta_{BC} - 1.5\,\frac{\Delta}{8}\right)$$

The fixed end moments are determined as in Figure 4.25 considering the hinges at A and C, to give

$$M^*_{FBA} = 1.5EI\alpha\frac{(T_1 - T_2)}{h} = 1.5\times\frac{20}{0.6} = 50\ \text{kN.m} \quad ; \quad M^*_{FBC} = \frac{2\times 8^2}{8} = 16\ \text{kN.m}$$

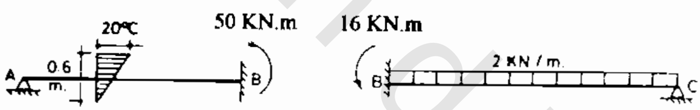

Figure 4.25

The equilibrium equations are thus

$$M^*_{BA} + M^*_{BC} = M_B = 0 \tag{a}$$

The reaction at B equals to $K\Delta$. Thus, equating the reaction at B one has

$$\frac{M^*_{BC}}{8} + 8 - \frac{M^*_{BA}}{6} = 1000\Delta \tag{b}$$

Substituting by the slope-deflection equations, one has

$$66 + \frac{EI}{3}\left(1.5\theta_B + \frac{\Delta}{4}\right) + \frac{EI}{2}\left(1.5\theta_B - \frac{1.5\Delta}{8}\right) = 0$$

$$2 + \frac{EI}{16}\left(1.5\theta_B - 1.5\frac{\Delta}{8}\right) + 8 - \frac{50}{6} - \frac{EI}{18}\left(1.5\theta_B - \frac{\Delta}{4}\right) = 1000\Delta$$

The solution of these equations is $\theta_B = -52.538 \times 10^{-5}$ rad ; $\Delta = 31.437 \times 10^{-5}$ m

which are the same results as in Example 4.5.

4.3.4 Slope-Deflection Equation with Relative Stiffness

It has been shown in the previous sections that the magnitude of the kinematic variables are usually very small. This may affect the accuracy of the results. The structural engineer rarely uses the magnitude of these variables. He is only interested, for design purposes, in the members bending moments. That is why one can deal with relative stiffnesses between the members instead of using the exact stiffness of each member. The relative stiffnesses are obtained by taking the ratios between (2EI/L) of all members. This eases solving the problem and provides more accurate results.

The slope deflection equations 4.7 and 4.15 can be written in terms of the relative stiffness as follows:

$$M_{AB} = M_{FAB} + K_{AB}\left(2\theta_{AB} + \theta_{BA} - \frac{3\Delta}{L}\right) \qquad (4.16)$$

$$M^{*}_{AB} = M^{*}_{FAB} + K_{AB}\left(1.5\theta_{AB} - 1.5\,\frac{\Delta}{L}\right) \qquad (4.17)$$

in which K_{AB} is the relative stiffness of member AB.

Example 4.10

Solve Example 4.6 by using the relative stiffness and the modified slope deflection method.

Solution

The relative stiffnesses are defined as shown in the following table:

Member / Stiffness	AB	BC	CD
$\frac{2EI}{L}$	$\frac{2EI}{6}$	$\frac{4EI}{12}$	$\frac{2EI}{6}$
Ratio = K	1	1	1

The slope-deflection equations can be written in terms of the relative stiffnesses as follows:

$$M_{AB} = 0 + 1\left(0 + \theta_{BA} + \frac{3\Delta}{6}\right) \quad ; \quad M_{BA} = 0 + 1\left(2\theta_{BA} + 0 + \frac{3\Delta}{6}\right)$$

$$M_{BC} = -36.666 + 1(2\theta_{BC} + \theta_{CB}) \quad ; \quad M_{CB} = 36.666 + 1(2\theta_{CB} + \theta_{BC})$$

$$M^{*}_{CD} = 0 + 1\left(1.5\theta_{CD} + \frac{1.5\Delta}{6}\right)$$

The equilibrium equations are obtained as

$$M_B = M_{BA} + M_{BC} = 0$$

$$= -36.666 + 4\theta_B + \theta_C + \frac{\Delta}{2} = 0 \qquad \text{(a)}$$

$$M_C = M_{CB} + M^{*}_{CD} = 0$$

$$= 36.666 + 3.5\theta_C + \theta_B + \frac{\Delta}{4} = 0 \qquad \text{(b)}$$

$$36 = H_A + H_D$$

$$= \frac{M_{AB} + M_{BA}}{6} + \frac{M^{*}_{CD}}{6} = \frac{1}{6}\left[3\theta_B + \Delta + 1.5\theta_C + \frac{\Delta}{4}\right] \qquad \text{(c)}$$

Rearranging and solving the three equilibrium equations (a), (b), and (c), one obtains

$$\theta_B = -14.4548 \quad ; \quad \theta_C = -23.1512 \quad ; \quad \Delta = 235.273$$

These values are relative quantities and not the real ones.

The end moments are obtained by substituting into the modified slope-deflection equations, as follows:

$$M_{AB} = \theta_B + \frac{\Delta}{2} = 103.18 \text{ kN.m}$$

$$M_{BA} = 2\theta_B + \frac{\Delta}{2} = 88.73 \text{ kN.m}$$

$$M_{BC} = -36.666 + 2\theta_B + \theta_C = -88.73 \text{ kN.m}$$

$$M_{CB} = 36.666 + 2\theta_C + \theta_B = -24.09 \text{ kN.m}$$

$$M_{CD} = 24.09 \text{ kN.m}$$

To determine the exact values of the deformations one can substitute into the original slope deflection equations using the exact stiffness factors.

Example 4.11

Solve Example 4.7 by using the relative stiffness, in the slope-deflection equations.

Solution

The fixed-end moments are determined, considering the hinge at B as shown in Figure 4.26.

$M_{FAC} = +37.5 \text{ kN.m} \quad ; \quad M_{FCA} = -37.5 \text{ kN.m}$

$$M^{*}_{FCB} = +\frac{3}{16}\times 8 + 3EI\frac{\Delta}{8^2} = 30 + 46.875 = 76.985 \text{ kN.m}$$

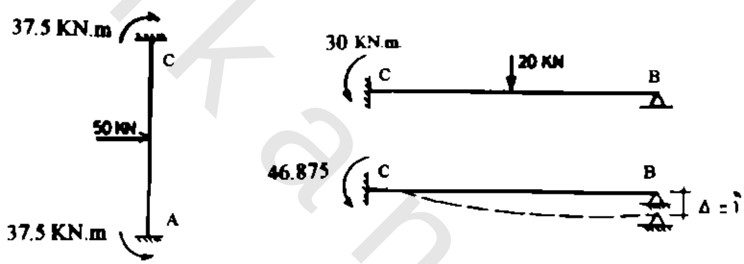

Figure 4.26

The relative stiffnesses are found as shown in table below:

Member / Stiffness	AC	CB
$\frac{2EI}{L}$	$\frac{2EI}{6}$	$\frac{4EI}{8}$
Ratio = K	4	3

The slope-deflection equations with the relative stiffnesses are obtained as

$$M_{AC} = 37.5 + 4(0 + \theta_C)$$

$$M_{CA} = -37.5 + 4(2\theta_C + 0)$$

$$M^{*}_{CB} = 76.875 + 3(1.5\theta_C)$$

Equilibrium equations are

$M_C = M_{CA} + M^*_{CB} = 0$

Substituting by M_{CA} and M^*_{CB} one obtains

$3.375 + 12.5\theta_C = 0$

The solution is $\theta_C = -3.15$. The end moments can then be found as follows:

$M_{AC} = 37.5 + 4(\theta_C) = 24.9$ kN.m

$M_{CA} = -37.5 + 4(\theta_C) = -62.7$ kN.m

$M_{CB} = 76.875 + 4.5\theta_C = 62.7$ kN.m

which are the same results as in Example 4.7.

4.3.5 Non-Prismatic Members

The slope deflection equations derived in the previous sections were dedicated to prismatic members only. For non-prismatic members with stepped variable moment of inertia as shown in Figure 4.27, one can still apply the previous equations but with considering a joint at each change in the moment of inertia. This process increases the number of equilibrium equations. However, one can deal with analogous slope-deflection equations dedicated to non-prismatic members. In this case, the equilibrium equations are obtained in the same routine as in the original approach.

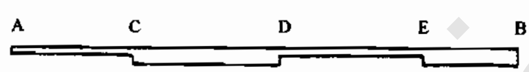

Figure 4.27

The derivation of the general slope deflection equations is the same as in the previous sections, except that one has to be careful in calculating the fixed-end moments and the angles of rotation for non-prismatic members. The fixed-end moments $M_{FAB}, M_{FBA}, M'_{FAB}$, and M'_{FBA} in Figure 4.28 can be found using any of the flexibility methods, for example, the unit load method, or the column analogy method.

In order to determine θ_{AB} and θ_{BA} as functions of M'_{AB} and M'_{BA} one applies, in turn, a unit moment at A and B, respectively, for a simple beam chosen as primary structure as shown in Figure 4.29.

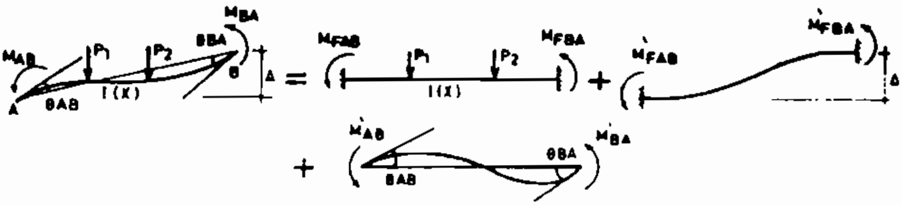

Figure 4.28

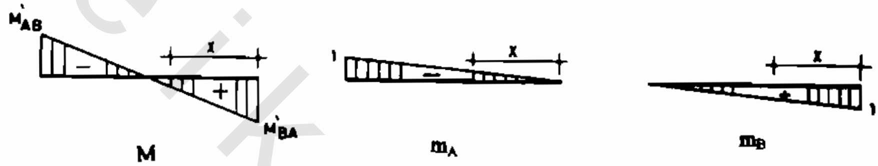

Figure 4.29

The angle of rotation θ_{AB} is determined as follows:

$$\theta_{AB} = \int_0^L \frac{M\, m_A\, dx}{EI(x)} = \int_0^L \frac{\left(-\frac{x}{L}\right)\left(-\frac{x}{L} M'_{AB} + \frac{L-x}{L} M'_{BA}\right)dx}{EI(x)}$$

$$= \int_0^L \frac{x^2}{EI(x)} \frac{M'_{AB}}{L^2}\, dx - \int_0^L \frac{x(L-x)}{EI(x)} \frac{M'_{BA}}{L^2}\, dx \qquad (4.18)$$

Similarly, the angle of rotation θ_{BA} is calculated as follows:

$$\theta_{BA} = \int_0^L \frac{M\, m_B\, dx}{EI} = \int_0^L \frac{\left(\frac{L-x}{L}\right)\left(-\frac{x}{L} M'_{AB} + \frac{L-x}{L} M'_{BA}\right)}{EI(x)}\, dx$$

$$= \int_0^L \frac{x(L-x)}{EI(x)} \frac{M'_{AB}}{L^2}\, dx + \int_0^L \frac{(L-x)^2}{EI(x)} \frac{M'_{BA}}{L^2}\, dx \qquad (4.19)$$

For simplicity, the three integrals in Equations 4.18 and 4.19 are denoted by C_1, C_2, and C_3 as follows:

$$C_1 = \int_0^L \frac{x^2}{L^3\, I(x)}\, dx \qquad (4.20)$$

$$C_2 = \int_0^L \frac{x(L-x)}{L^3\, I(x)}\, dx \tag{4.21}$$

$$C_3 = \int_0^L \frac{(L-x)^2}{L^3\, I(x)}\, dx \tag{4.22}$$

Equations 4.18 and 4.19 can thus be written as

$$\theta_{AB} = C_1 \frac{L \times M'_{AB}}{E} - C_2 \frac{L \times M'_{BA}}{E} \tag{4.23}$$

$$\theta_{BA} = C_3 \frac{L \times M'_{BA}}{E} - C_2 \frac{L \times M'_{BA}}{E} \tag{4.24}$$

The end moments M'_{AB} and M'_{BA} as functions of the angles of rotation are obtained as follows:

$$M'_{AB} = \frac{E}{L}\left(\frac{C_3}{C_1C_3 - C_2^2}\right)\theta_{AB} + \left(\frac{C_2}{C_1C_3 - C_2^2}\right)\theta_{BA} \tag{4.25}$$

$$M'_{BA} = \frac{E}{L}\left(-\frac{C_2}{C_1C_3 - C_2^2}\right)\theta_{AB} + \left(\frac{C_1}{C_1C_3 - C_2^2}\right)\theta_{BA} \tag{4.26}$$

Note that one can also derive these equations using the column analogy method.

In order to introduce the relative displacement Δ in the slope-deflection equations, the unit load method can be used, as shown in Figure 4.30.

The equations of consistent deformations at B are

$$\Delta_{10} + f_{11}\, x_1 + f_{12}\, x_2 = 0 \tag{4.27}$$

$$\Delta_{20} + f_{21}\, x_1 + f_{22}\, x_2 = \Delta \tag{4.28}$$

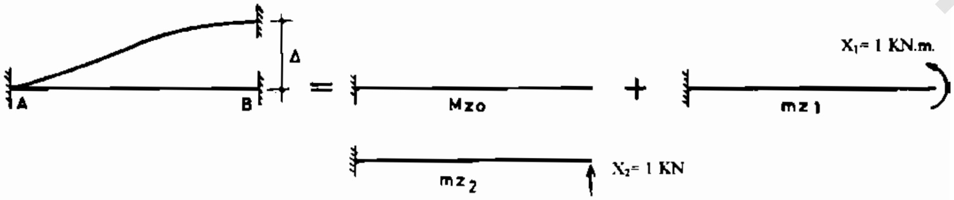

Figure 4.30

The flexibility coefficients f_{11}, f_{12} and f_{22} are determined using Figure 4.31 as follows:

$$f_{11} = \int_0^L \frac{m_{z1}^2\,dx}{EI(x)} = \int_0^L \frac{1\,dx}{EI(x)} = \frac{1}{E}\int_0^L \frac{dx}{I(x)} \tag{4.29}$$

$$f_{12} = \int_0^L \frac{1\,(x)\,dx}{EI(x)} = \frac{1}{E}\int_0^L \frac{x\,dx}{I(x)} \tag{4.30}$$

$$f_{22} = \int_0^L \frac{x^2\,dx}{EI(x)} = \frac{1}{E}\int_0^L \frac{x^2\,dx}{I(x)} \tag{4.31}$$

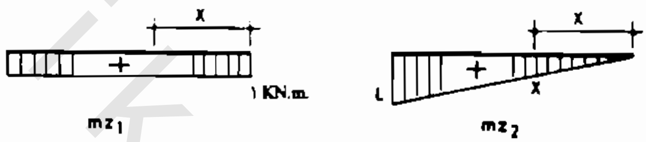

Figure 4.31

Substituting into Equations 4.27 and 4.28 one obtains

$$f_{11}\,x_1 + f_{12}\,x_2 = 0 \tag{4.32}$$

$$f_{21}\,x_1 + f_{22}\,x_2 = \Delta \tag{4.33}$$

The solutions of these equations are

$$x_1 = \frac{-f_{12}}{f_1\,f_{22} - f_{12}^2}\,\Delta \tag{4.34}$$

$$x_2 = \frac{f_{11}}{f_{11}\,f_{22} - f_{12}^2}\,\Delta \tag{4.35}$$

The fixed end moments at A and B according to the sign conventions are thus

$$M'_{FAB} = -\left(x_1 + L\,x_2\right) \tag{4.36}$$

$$M'_{FBA} = +\,x_1 \tag{4.37}$$

It is obvious that there are relationships between f_{11}, f_{12}, f_{22}, and the coefficients C_1, C_2, C_3, as follows:

$$C_1 = \frac{E}{L^3} f_{22} \tag{4.38}$$

$$C_2 = \frac{E}{L^3}\left(f_{12}L - f_{22}\right) \tag{4.39}$$

$$C_3 = \frac{E}{L^3}\left(f_{11}L^2 - 2f_{12}\,L + f_{22}\right) \tag{4.40}$$

Solving Equations 4.38 to 4.40 one has

$$f_{11} = \frac{L}{E}\left(C_1 + 2C_2 + C_3\right) \tag{4.41}$$

$$f_{12} = \frac{L^2}{E}\left(C_1 + C_2\right) \tag{4.42}$$

$$f_{22} = \frac{L^3}{E}\left(C_1\right) \tag{4.43}$$

Substituting into Equations 4.34 and 4.35 and using Equations 4.36 and 4.37 one has

$$M'_{FAB} = -\frac{E(C_2 + C_3)}{(C_1C_3 - C_2^2)}\,\frac{\Delta}{L^2} \tag{4.44}$$

$$M'_{FBA} = -\frac{E(C_1 + C_2)}{(C_1C_3 - C_2^2)}\,\frac{\Delta}{L^2} \tag{4.45}$$

The general slope-deflection equations are then given by

$$M_{AB} = M_{FAB} + \frac{E}{L}\left[S_{AA}\,\theta_{AB} + S_{AB}\,\theta_{BA} - \left(S_{AA} + S_{AB}\right)\frac{\Delta}{L}\right] \tag{4.46}$$

$$M_{BA} = M_{FBA} + \frac{E}{L}\left[S_{BA}\,\theta_{AB} + S_{BB}\,\theta_{BA} - \left(S_{BB} + S_{BA}\right)\frac{\Delta}{L}\right] \tag{4.47}$$

in which S_{AA}, S_{AB}, S_{BA}, and S_{BB} are, respectively, given by

$$S_{AA} = \frac{C_3}{C_1C_3 - C_2^2} \tag{4.48}$$

$$S_{AB} = \frac{C_2}{C_1C_3 - C_2^2} \tag{4.49}$$

$$S_{BA} = \frac{C_2}{C_1 C_3 - C_2^2} \tag{4.50}$$

$$S_{BB} = \frac{C_1}{C_1 C_3 - C_2^2} \tag{4.51}$$

4.3.6 Non-Prismatic Members with Moments Releases

If member AB is hinged, for example, at B, then the procedure is the same, as shown in Figure 4.32. The fixed end moments M^{*}_{FAB} and $M^{*'}_{FAB}$ are found using the flexibility methods. The angle of rotation θ_{AB} is calculated using Figure 4.33 as follows:

$$\begin{aligned} \theta_{AB} &= \int_0^L \left(-\frac{x}{L}\right)\left(-\frac{x}{L} M^{*'}_{AB}\right)\frac{dx}{EI(x)} \\ &= C_1 \frac{L \times M^{*'}_{AB}}{E} \end{aligned} \tag{4.52}$$

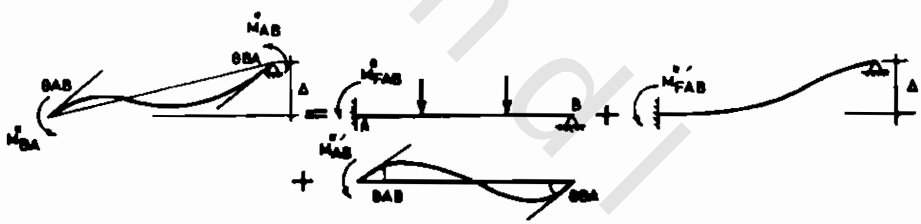

Figure 4.32

Figure 4.33

The moment $M^{*'}_{AB}$ is obtained from

$$M^{*'}_{AB} = \frac{E}{L}\frac{1}{C_1}\theta_{AB} \tag{4.53}$$

Similarly, from Figure 4.34, the fixed end moment $M^{*'}_{FAB}$ can be determined by using the consistent deformation at B as follows:

$$\Delta_{10} + f_{11}\, x_1 = \Delta \tag{4.54}$$

Figure 4.34

The flexibility coefficient f_{11} is obtained from

$$f_{11} = \int_0^L \frac{m^2\, dx}{EI(x)} = \int_0^L \frac{x^2\, dx}{EI(x)} = \frac{L^3}{E} C_1 \tag{4.55}$$

The fixed end moment is thus

$$\begin{aligned} M^{*'}_{FAB} &= - x_1 L \\ &= - \frac{\Delta}{f_{11}} \times L = - \frac{\Delta EL}{L^3 C_1} = - \frac{E}{LC_1}\frac{\Delta}{L} \end{aligned} \tag{4.56}$$

The modified slope-deflection equation is thus

$$M^{*}_{AB} = M^{*}_{FAB} + \frac{E}{L}\left(\frac{1}{C_1}\theta_{AB} - \frac{1}{C_1}\frac{\Delta}{L}\right) \tag{4.57}$$

If member AB is hinged at A, one has

$$M^{*}_{BA} = M^{*}_{FBA} + \frac{E}{L}\left(\frac{1}{C_3}\theta_{AB} - \frac{1}{C_3}\frac{\Delta}{L}\right) \tag{4.58}$$

These two equations are written in general as

$$M^{*}_{AB} = M^{*}_{FAB} + \frac{E}{L}\left(S'_{AA} \times \theta_{AB} - S'_{AA} \times \frac{\Delta}{L}\right) \tag{4.59}$$

$$M^{*}_{AB} = M^{*}_{FAB} + \frac{E}{L}\left(S'_{BB} \times \theta_{BA} - S'_{BB} \times \frac{\Delta}{L}\right) \tag{4.60}$$

where S'_{AA} and S'_{BB} are the modified stiffness factors given by

$$S'_{AA} = \left(\frac{C_1 C_3 - C_2^2}{C_1 C_3} \right) S_{AA} \tag{4.61}$$

$$S'_{BB} = \left(\frac{C_1 C_3 - C_2^2}{C_1 C_3} \right) S_{BB} \tag{4.62}$$

4.3.7 Numerical Applications to Non-Prismatic Members

Example 4.12

Determine the bending moment diagram for the beam of rectangular sections of width 0.30 m shown in Figure 4.35 using the slope-deflection equations.

Solution

One should first determine the fixed-end moments for members AB and BC. To save time in the calculation, one can determine the fixed-end moment using the column analogy method as shown in Figure 4.36.

The beam moment of inertia for rectangular section are calculated as follows:

$$I_1 = \frac{b \times 0.8^3}{12} = 0.012798 \text{ m}^4 \quad ; \quad I_2 = \frac{b \times 0.6^3}{12} = 0.0054 \text{ m}^4$$

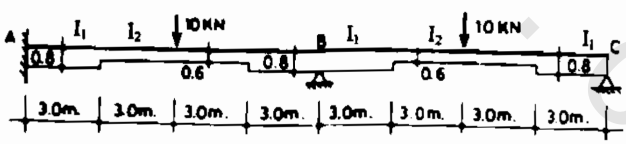

Figure 4.35

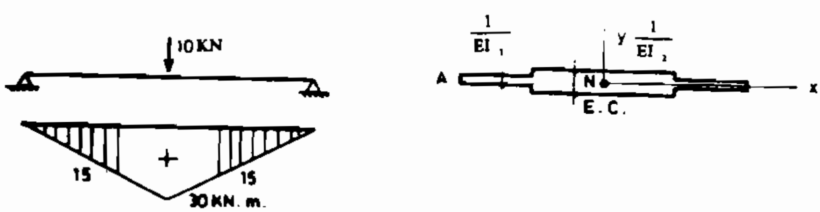

Figure 4.36

The properties of the column are calculated as follows:

$$A = \frac{6}{0.0054E} + \frac{3 \times 2}{0.012798E} = \frac{1579.93}{E}$$

$$I_y = \frac{6}{0.0054E} \times \frac{6^2}{12} + 2\left[3 \times \frac{1}{0.012798E} \times \frac{3^2}{12} + 3 \times \frac{1}{0.012798E} \times 4.5^2\right] = \frac{13178.62}{E}$$

$$N = 2 \times \frac{15 \times 3}{2} \times \frac{1}{0.0054E} + 15 \times 6 \times \frac{1}{0.0054E} + 2 \times \frac{15\ \times 3}{2} \times \frac{1}{0.012798E} = \frac{28516.17}{E}$$

$$M_y = 0$$

The end moments are thus

$$M = M_0 - \frac{N}{A} = 0 - \frac{28516.173}{1579.93} = -18.04 \ \text{kN.m}$$

$$M_A = -18.04 \ \text{kN.m} \qquad ; \qquad M_B = -18.04 \ \text{kN.m}$$

The fixed end moments according to the slope deflection sign conventions are then

$$M_{FAB} = +18.04 \ \text{kN.m} \qquad ; \qquad M_{FBA} = -18.04 \ \text{kN.m}$$

The stiffness factors $S_{AA}, S_{AB}, \text{and}\ S_{BB}$ are determined using the coefficients $C_1, C_2, \text{and}\ C_3$ as given in Equations 4.48 to 4.51 and the relative stiffness.

$$C_1 = \int_0^L \frac{x^2\,dx}{L^3\ I(x)}$$
$$= \frac{1}{12^3}\left[\int_0^3 \frac{1}{0.012798}\ x^2\,dx + \int_3^9 \frac{1}{0.0054}\ x^2\,dx + \int_9^{12} \frac{1}{0.012798}\ x^2\,dx\right]$$
$$= \frac{1}{12^3}\left[\frac{1}{0.012798}\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{0.0054}\left[\frac{x^3}{3}\right]_3^9 + \frac{1}{0.012798}\left[\frac{x^3}{3}\right]_9^{12}\right] = 40.5418$$

$$C_2 = \int_0^L \frac{x(L-x)\,dx}{L^3\ I(x)}$$
$$= \frac{12}{12^3}\left[\frac{1}{0.012798}\left(\frac{x^2}{2}\right)_0^3 + \frac{1}{0.0054}\left(\frac{x^2}{2}\right)_3^9 + \frac{1}{0.01278}\left(\frac{x^2}{2}\right)_9^{12}\right] - \frac{70056.247}{12^3}$$
$$= 25.288$$

$$C_3 = \int_0^L \frac{(L-x)^2\,dx}{L^3\,I(x)} = \frac{1}{12^3}\left[\int_0^L \frac{12^2\,dx}{I(x)} - 2\int_0^L \frac{12x\,dx}{I(x)} + \int_0^L \frac{x^2\,dx}{I(x)}\right]$$

$$= \frac{1}{12^3}\left[144\left(\frac{3}{0.012798} + \frac{6}{0.0054} + \frac{3}{0.012798}\right) - 2\times 12\,(9479.597) + 70056.247\right]$$

$$= 40.5418$$

The stiffness coefficients are thus

$$S_{AA} = \frac{C_3}{C_1C_3 - C_2^2} = \frac{40.5418}{(40.5418)(40.5418) - (25.288)^2} = 0.04037$$

$$S_{AB} = S_{BA} = \frac{C_2}{C_1C_3 - C_2^2} = 0.0251833$$

$$S_{BB} = S_{AA} = 0.040374$$

Substituting into the slope deflection equations one has

$$M_{AB} = 18.04 + \frac{E}{12}\left[S_{AA}\,\theta_{AB} + S_{AB}\,\theta_{BA}\right]$$
$$= 18.04 + \frac{E}{12}\left[0.0251833\,\theta_B\right]$$

$$M_{BA} = -18.04 + \frac{E}{12}\left[S_{BB}\,\theta_{BA} + S_{BA}\,\theta_{AB}\right]$$
$$= -18.04 + \frac{E}{12}\left[0.04074\,\theta_B\right]$$

$$M_{BC} = 18.04 + \frac{E}{12}\left[0.040374\,\theta_B + 0.0251833\,\theta_C\right]$$

$$M_{CB} = -18.04 + \frac{E}{12}\left[0.040374\,\theta_C + 0.0251833\,\theta_B\right]$$

The equilibrium equations are

$$M_{CB} = 0 \quad , \quad \text{and} \quad M_B = M_{BA} + M_{BC} = 0$$

Substituting, one obtains

$$\frac{12\times 18.04}{E} = 0.040374\,\theta_C + 0.0251833\,\theta_B \qquad \text{(a)}$$

$$0 = 0.080748\,\theta_B + 0.0251833\,\theta_C \qquad \text{(b)}$$

The solution of these equation is

$$\theta_C = \frac{6656.836}{E} \text{ rad.} \quad , \quad \theta_B = \frac{-2076.1}{E} \text{ rad.}$$

The end moments can now be determined from the slope deflection equations as follows:

$$M_{AB} = 18.04 + \frac{E}{12}\left[0.0251833\ \theta_B\right] = 13.683 \ \text{ kN.m}$$

$$M_{BA} = -18.04 + \frac{E}{12}\left[0.040374\,\theta_B\right] = -25.025 \ \text{kN.m}$$

$$M_{BC} = 18.04 + \frac{E}{12}\left[0.040374\,\theta_B + 0.0251833\,\theta_C\right] = 25.025 \ \text{ kN.m}$$

$$M_{CB} = 0$$

The bending moment diagram is plotted in Figure 4.37. Notice that one can use the modified slope deflection equations considering the hinge at C, and also the relative stiffnesses which are the ratios between S_{AA}, S_{AB}, and S_{BB}.

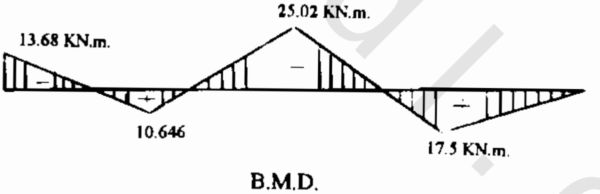

Figure 4.37

Example 4.13

Solve the problem of the previous example considering the hinge at C using the relative stiffnesses.

Solution

In this case, one needs to determine the fixed end moment for member BC considering C as a hinge. This is shown in Figure 4.38 using the column analogy method.

The elastic center in this case is at C. Using the properties of the column in the previous example, one has

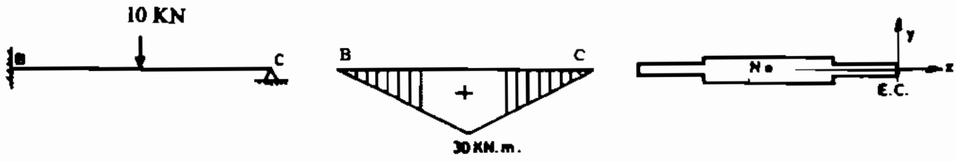

Figure 4.38

$$M_y = \frac{28516.17}{E} \times (-6) = -\frac{171097.02}{E}$$

$A = \infty$

$$I_y = \frac{3}{0.012798E}\left(\frac{3^2}{3}\right) + \frac{6}{0.0054E}\left(\frac{6^2}{12} + 6^2\right) + \frac{3}{0.012798E}\left(\frac{3^2}{12} + 10.5^2\right) = \frac{70056.2588}{E}$$

$$M_B = +\frac{171097.02}{70056.2588} \times (-12) = -29.29 \text{ kN.m}$$

The fixed end moment at B is thus $M^*_{FBC} = 29.29$ kN.m.

The stiffness coefficients are the same as in the previous example, except for S'_{BB}, where

$S_{AA} = 0.04037$, $S_{AB} = 0.0251833$, $S_{BB} = 0.04037$

$$S'_{BB} = \left(\frac{C_1C_3 - C_2^2}{C_1C_3}\right) \times S_{BB} = 0.610934 \times S_{BB} = 0.024663$$

The slope deflection equation for member BC is then

$$M^*_{BC} = 29.29 + \frac{E}{12}\left(0.024663\,\theta_B\right)$$

The equilibrium equation is only at B, where $M_B = M_{BA} + M^*_{BC} = 0$. This gives

$$-18.04 + \frac{E}{12}\left[0.04037\,\theta_B\right] + 29.29 + \frac{E}{12}\left[0.024663\,\theta_B\right] = 0$$

The solution is $\theta_B = -\dfrac{2079}{E}$ rad.

The moments at B and A are obtained from the slope deflection equations as follows:

$$M^*_{BC} = 29.29 + \frac{E}{12}\left(0.024520\,\theta_B\right) = 25.023 \text{ kN.m}$$

$$M_{AB} = 18.04 + \frac{E}{12}\left(0.0251833\,\theta_B\right) = 13.684 \text{ kN.m}$$

which are the same results as in the previous exammple.

4.3.8 Applications to Gable Frames

Gable frames are special types of structures which have inclined members as shown in Figure 4.39, instead of having only horizontal and vertical members. The geometry of the frame imposes a kind of relative deformation even when the frame is subjected to gravity loads only. That is why gable frames should be treated carefully in order to determine the effect of the distortion in the frame geometry on the various members.

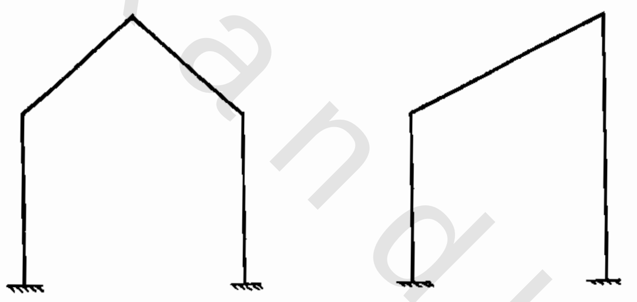

Figure 4.39

Consider, for example, the symmetrical gable frame, shown in Figure 4.40, which is subjected to symmetrical gravity loads. The frame geometry imposes the kind of deformation shown by a dotted line in Figure 4.40. By this deformation, each member in the frame is subjected to a relative displacement between its ends. Determination of these relative displacements needs geometrical analysis as shown in Figure 4.41. From Figure 4.41 and neglecting the axial deformations, one obtains the following relationships:

$$\Delta_D = \Delta_B = \Delta_C \tan\alpha \tag{4.63}$$

$$\Delta_{CD} = \Delta_{CB} = \frac{\Delta_D}{\sin\alpha} \tag{4.64}$$

The slope deflection equations for member CD, for example, are thus

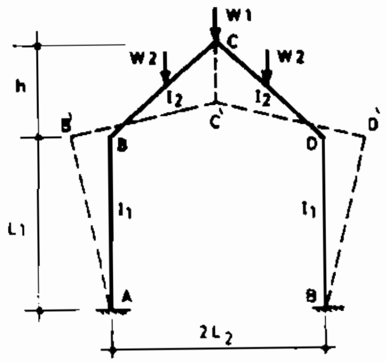

Figure 4.40

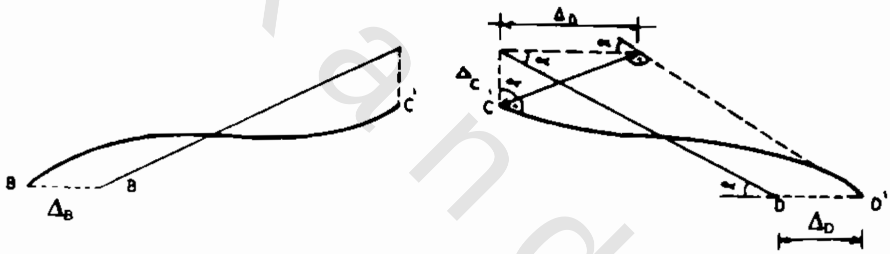

Figure 4.41

$$M_{CD} = M_{FCD} + \frac{2EI_2}{L_{CD}}\left(2\theta_{CD} + \theta_{DC} - \frac{3\Delta_{CD}}{L_{CD}}\right) \tag{4.65}$$

$$M_{DC} = M_{FDC} + \frac{2EI_2}{L_{CD}}\left(2\theta_{DC} + \theta_{CD} - \frac{3\Delta_{CD}}{L_{CD}}\right) \tag{4.66}$$

The slope deflection equations for member DE, are

$$M_{DE} = M_{FDE} + \frac{2EI_1}{L_1}\left(2\theta_{DE} + \theta_{ED} - \frac{3(-\Delta_D)}{L_1}\right) \tag{4.67}$$

$$M_{ED} = M_{FED} + \frac{2EI_1}{L_1}\left(2\theta_{ED} + \theta_{DE} - \frac{3(-\Delta_D)}{L_1}\right) \tag{4.68}$$

The equilibrium equations for this frame are

$$M_B = M_{BA} + M_{BC} \tag{4.69}$$

$$M_C = M_{CB} + M_{CD} \tag{4.70}$$

$$M_D = M_{DC} + M_{DE} \tag{4.71}$$

The summation of the horizontal forces in direction of $\Delta_D = 0$ (4.72)

Equation 4.72 can be found either by taking the horizontal reactions at A and E, or by calculating the horizontal reactions at B and D for the frame BCD. The solution of Equations 4.59 to 4.72 provides the kinematic variables $\theta_B, \theta_C, \theta_D$, and Δ_D.

There are also more difficult situations for asymmetrical frames or symmetrical frames subjected to horizontal forces as shown in Figure 4.42. The difficulties encountered in developing the geometrical relationships for these kinds of frames are obvious. The solution, however, can easily be obtained by using the stiffness matrix method approach II, as shall be shown in Chapter 5.

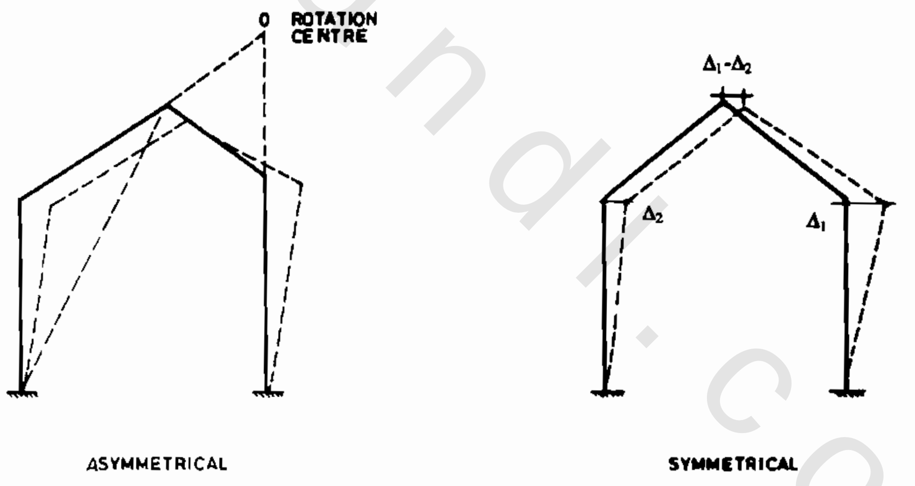

Figure 4.42

4.4 THE MOMENT DISTRIBUTION METHOD

The moment distribution method is basically the slope deflection equation method but the equilibrium equations are solved by successive iterations, instead of solving the linear simultaneous equilibrium equations by the exact mathematical methods.

The basic concept in the moment distribution method starts by making the structure kinematically determinate by fixing all joints. This results in fixed end moments at the fixed joints due to the applied loading and other effects on the members. In order to satisfy the equilibrium equations at the free joint, one has to balance the moment at these joints by introducing an external balance moment. The application of the balance moment must, however, satisfy the equilibrium and obeys the stiffness in each member. In order to clarify this point further, consider the beam which is given a balance moment M_B at joint B, as shown in Figure 4.43.

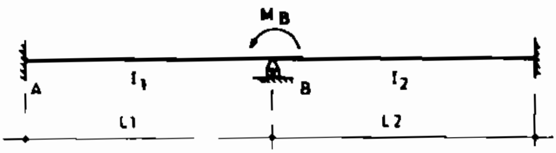

Figure 4.43

The slope deflection equations for members BA and BC at B are, respectively,

$$M_{BA} = 0 + \frac{2EI_1}{L_1}\left(2\theta_B + 0\right) \tag{4.73}$$

$$M_{BC} = 0 + \frac{2EI_2}{L_2}\left(2\theta_B + 0\right) \tag{4.74}$$

The equilibrium at the free joint B requires that

$$M_B = M_{BA} + M_{BC} \tag{4.75}$$

Solving equations 4.73 to 4.75 one obtains

$$M_{BA} = \frac{(I_1/L_1)}{(I_1/L_1)+(I_2/L_2)} M_B \tag{4.76}$$

$$M_{BC} = \frac{(I_2/L_2)}{(I_1/L_1)+(I_2/L_2)} M_B \tag{4.77}$$

which indicates that the balance moment at the free joint is distributed to the members connected to this joint according to their stiffness factors. The stiffness ratios in Equations 4.76 and 4.77 are called distribution factors, and their sum at every free joint must be unity.

The external balance moment applied to the members at the free joints should, however, obey the equilibrium conditions. For example, the end moments in member AB of Figure 4.43 are obtained as follows:

$$M_{BA} = \frac{4EI_1}{L_1}\theta_B = \frac{(I_1/L_1)}{(I_1/L_1)+(I_2/L_2)} M_B \tag{4.78}$$

$$M_{AB} = \frac{2EI_1}{L_1}\theta_B = \frac{1}{2} M_{BA} \tag{4.79}$$

which indicates that half of the end moment obtained from the balance moment at the free joint of a member is given to the other joint of the member. The ratio (M_{AB}/M_{BA}) is called the carry-over factor.

The procedure in the moment distribution method can now be summarized in the following steps:

1. Determine the fixed end moments in all members after fixing all joints in the structure.

2. Determine the stiffness factors, the distribution factors, and the carry-over factors for all members.

3. Remove the fixation of the free joints by applying external balance moments in order to satisfy the equilibrium of moments at these joints.

4. Distribute the balance moments to the members connected with the free joint according to their distribution factors and the carry-over factors.

5. Repeat steps 3 and 4 until the values of the balance moments at all free joints are very small and can be neglected.

6. The sum of the fixed end moments and the balance moments iterations of steps 4 and 5 gives the final end moment at every joint. As a check of the solution, the sum of the end moments at every free joint must equal the external moment applied on that joint. Another check is to calculate the angle of rotation of the free joint, which must be the same when calculated for the individual members connected with the free joints.

Example 4.14

Determine the bending moment diagram for the beam of Example 4.4 using the moment distribution method ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}$/°c).

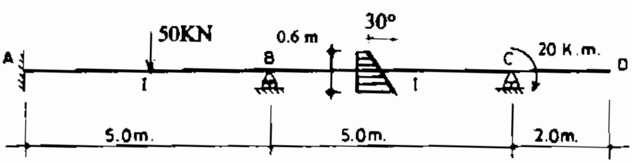

Figure 4.44

Solution

The stiffness factors, distribution factors, and carry-over factors are determined using the data shown in Figure 4.44, as in the table below:

Member	AB	BA	BC	CB
Stiffness coefficient	4EI/5	4EI/5	4EI/5	4EI/5
Stiffness ratio	1	1	1	1
Distribution factors	1	0.5	0.5	1
Carry-over factor	0.5	0.5	0.5	0.5

The fixed end moments are determined from the tables for the cases shown in Figure 4.45 using the signs of the slope deflection equation as follows:

$$M_{FAB} = \frac{50 \times 5}{8} + \frac{4EI}{5}\theta_A = +191.25 \text{ kN.m}$$

$$M_{FBA} = -\frac{50 \times 5}{8} + \frac{2EI}{5}\theta_A = +48.75 \text{ kN.m}$$

$$M_{FBC} = EI\alpha\frac{(T_1 - T_2)}{h} = +50 \text{ kN.m} \qquad M_{FCB} = -50 \text{ kN.m}$$

The steps of the moment distribution method can be organized as in the following table. As indicated in the table, after determining the fixed end moments, joint B which is a free joint is offered a balance moment of (-98.75) which is distributed between members BA and BC according to their distribution factors. One half of the balance moment is given to the other end of the member at A and C. Similar step is

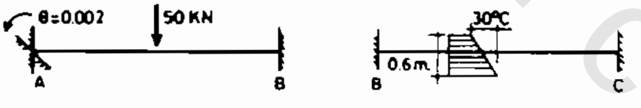

Figure 4.45

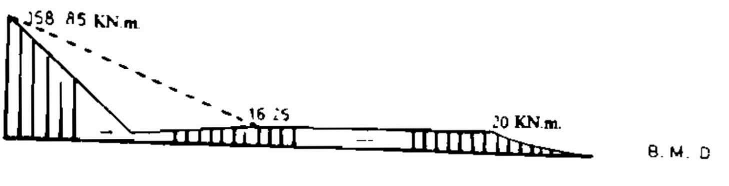

Figure 4.46

done for joint C which is given a balance moment of (+30) and one half of that moment is given to joint B. The steps are repeated until the balance is very small.

Joint	A	B		C	
Member	AB	BA	BC	CB	CD
Distribution Factor	1	0.5	0.5	1.0	
Fixed End Moment	+191.25	+48.75	+50	-50	+20
Balance Moment	-	-49.375	-49.375	+30	
Carry-over Moment	-24.687	0	+15	-24.687	
Balance	-	-7.5	-7.5	+24.687	
Carry-over	-3.75	0	12.343	-3.75	
Balance	-	-6.171	-6.171	3.75	
Carry-over	-3.08	0	1.875	-3.085	
Balance	-	-0.937	-0.937	3.085	
Carry-over	-0.468	0	1.542	-0.468	
Balance	0	-0.771	-0.771	0.468	
Carry-over	-0.385	0	0.234	-0.385	
Balance	0	-0.117	-0.117	0.385	
Carry-over	-0.058	0	0.192	-0.058	
Balance	0	-0.096	-0.096	0.058	
Carry-over	-0.048	0	+0.029	-0.048	
Balance	-	-0.0145	-0.0145	+0.048	
Final Moment	158.774	-16.232	16.232	-20	+20

The bending moment diagram is given in Figure 4.46, which is approximately the same as Figure 4.8.

Example 4.15

Determine the bending moment diagram for the frame shown in Figure 4.47 due to the applied loads and a vertical settlement at B of 1 cm downward ($EI = 10^5$ kN.m^2).

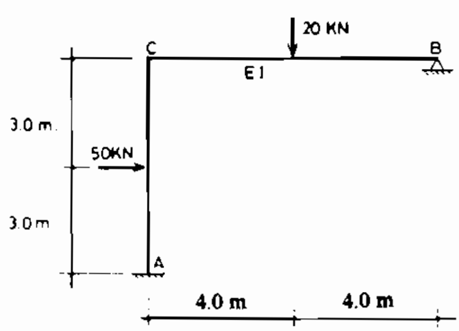

Figure 4.47

Solution

The fixed end moments are determined from the tables for the loading cases shown in Figure 4.48.

$M_{FAC} = +37.5$ kN.m ; $M_{FCA} = -37.5$ kN.m

$M_{FCB} = +20 + 93.75 = +113.75$ kN.m ; $M_{FBC} = +93.75 - 20 = +73.75$ kN.m

The steps of the moment distribution method ca be organized in the following table:

Joint	A	C		B
Member	AC	CA	CB	BC
Stiffness Coefficient	4EI/6	4EI/6	4EI/8	4EI/8
Stiffness Ratio	4	4	3	3
Distribution Factor	1	0.57143	0.42857	1
Fixed End Moment	+37.5	-37.5	+113.75	+73.75
Balance Moment	-	-43.571	-32.678	-73.75
Carry-over Moment	-21.785	0	-36.875	-16.339
Balance	-	21.071	15.803	16.339
Carry-over	10.535	0	8.169	7.901
Balance	-	-4.668	-3.501	-7.901
Carry-over	-2.334	0	-3.950	-1.750
Balance	-	2.257	1.693	1.75
Carry-over	1.128	0	0.875	0.846
Balance	-	-0.5	-0.375	-0.846
Carry-over	-0.25	0	-0.423	-0.187
Balance	-	0.242	0.181	0.187
Carry-over	0.121	0	0.093	0.090
Balance	-	-0.053	-0.04	-0.09
Carry-over	-0.0026	0	-0.045	-0.02
Balance	-	-0.0257	+0.0193	+0.02
Final Moment	24.8885	-62.6963	62.6963	0

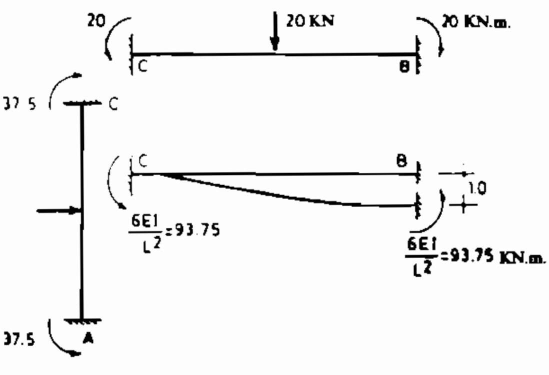

Figure 4.48

The bending moment diagram for the frame is shown in Figure 4.49, which is the same as solved by the slope deflection equations.

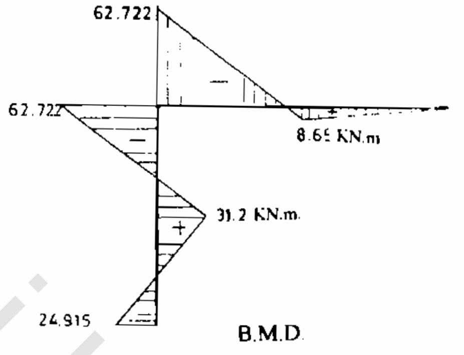

Figure 4.49

4.4.1 Angles of Rotation by Moment Distribution Method

It has been pointed out that to check the solution obtained by the moment distribution method, the equilibrium condition of the final moment applied at each free joint must be satisfied. This means that the summation of end moments at the free joint must equal the external moment applied at that joint. However, the process of distribution may hid an error which is eliminated at the end of the iterations. Therefore, one needs another tool to verify the solutions obtained by the moment distribution method.

Since we are dealing with linear elastic structures, the compatibility conditions should also be verified at every free joint. This means that the angles of rotation calculated at the end of all members connected with a free joint must be the same. In order to calculate the angle of rotation, one may use the slope deflection equations. Member AB, for example, has the following slope deflection equations:

$$M_{AB} = M_{FAB} + K_{AB}\left(2\theta_{AB} + \theta_{BA}\right) \tag{4.80}$$

$$M_{BA} = M_{FBA} + K_{AB}\left(2\theta_{BA} + \theta_{AB}\right) \tag{4.81}$$

in which K_{AB} is the relative stiffness for member AB.

Solving Equations 4.80 and 4.81 for θ_{AB} and θ_{BA}, one obtains

$$\theta_{AB} = \left[\left(M_{AB} - M_{FAB}\right) - 0.5\left(M_{BA} - M_{FBA}\right)\right] \Big/ \left(1.5\,K_{AB}\right) \tag{4.82}$$

$$\theta_{BA} = \left[\left(M_{BA} - M_{FBA}\right) - 0.5\left(M_{AB} - M_{FAB}\right)\right] \Big/ \left(1.5\,K_{AB}\right) \tag{4.83}$$

It is obvious that one can obtain the angles of rotation from the final results of the moment distribution method. This tool is very powerful in detecting errors in the distribution process. It can be augmented at the end of the moment distribution table as illustrated in the following examples.

Example 4.16

Check the compatibility conditions at the free joints of Example 4.13.

Solution

The check can be organized at the bottom of the same table used to solve Example 4.13, as follows:

Member	AB	BA	BC	CB
Stiffness Ratio (K)	1	1	1	1
Fixed end moment (FEM)	191.25	48.75	50	-50
Final Moment (FM)	158.774	-16.232	16.232	-20
(FM–FEM)	-32.476	-64.982	-33.768	+30
- 0.5 (FM – FEM) of other end	+32.491	+16.238	-15	+16.884
Sum = (FM–FEM) - 0.5 (FM–FEM)	0.015	-48.744	-48.768	+46.884
θ = sum/1.5K	0.01	-32.496	-32.512	31.256

The results indicate that the angle of rotation at A is approximately zero, and the angle θ_{BA} equals the angle θ_{BC}. The little difference is coming from not carrying out the iteration to full convergence. One can thus sketch the deformed shape of the beam as shown in Figure 4.50. Notice that if the exact stiffness (2EI/L) has been used instead of the relative stiffness (K), one obtains the exact values for the angle of rotations.

Example 4.17

Determine the angles of rotation in Example 4.14 and draw a sketch for the deformed shape.

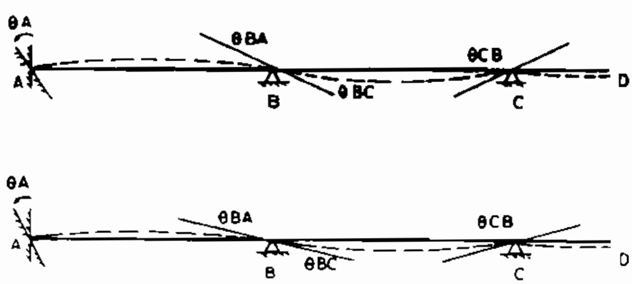

Figure 4.50

Solution

Member	AB	BA	BC	CB
Stiffness Ratio (K)	4	4	4	4
Fixed end moment (FEM)	37.5	-37.5	113.75	73.75
Final Moment (FM)	24.8885	-62.6963	62.6963	0
(FM–FEM)	-12.6115	-25.1963	-51.0537	-73.75
- 0.5(FM – FEM) of other end	12.59815	6.3057	36.875	25.52685
Sum = (FM–FEM) – 0.5 (FM–FEM)	-0.01335	-18.89	-14.1787	-48.223
θ = sum/1.5K	-0.00222	-3.1484	-3.1508	-10.716

The results indicate that the angle of rotation at A is approximately zero since it is a fixed support, and the angle θ_{CA} is approximately equal θ_{CB}. The deformed shape of the frame can be sketched as shown in Figure 4.51.

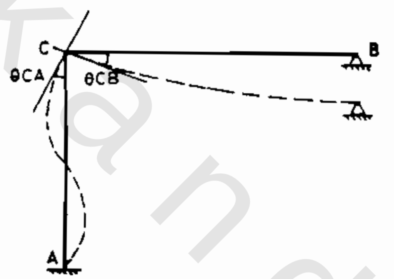

Figure 4.51

4.4.2 Applications to Members with Moments Releases

It was shown in section 4.3.3 that the modified slope deflection equation 4.15 is used when the end of the member is a hinge with zero moment. In the case that the relative displacement is zero, the equation can be written as:

$$M^{*}_{BA} = M^{*}_{FBA} + \frac{3EI}{L}\theta_{BA} \tag{4.84}$$

where member AB is hinged at joint A which has zero moment.

Equation 4.84 indicates that the stiffness factor for a member with a moment release is (3EI/L), not (4EI/L) as was shown before for ordinary members. Thus, one can use this modification in the moment distribution method whenever a member has a moment release. The modification can be done by multiplying the stiffness factor (4EI/L) by the fraction 0.75. In this case the distribution of moment to the hinged support is always zero.

Example 4.18

Solve Example 4.13 using the modified stiffness factor for member BC.

Solution

One proceeds as in Example 4.13 but the stiffness factor of member BC is multiplied by 0.75. The steps of the moment are organized in the following table:

Joint	A	B		C	
Member	AB	BA	BC	CB	CD
Stiffness Ratio (K)	1	1	1	1	
Modified Stiffness (K*)	1	1	0.75	0.75	
Distribution Factor	1	0.57143	0.42857	1	
FEM	+191.25	+48.75	+50	-50	+20
Balance Moment	-	-56.428	-42.321	+30	
Carry-over Moment	-28.214	0	+15	0	
Balance	0	-8.571	-6.428	0	
Carry-over	-4.285	0	0	0	
Final moment (FM)	158.75	-16.25	16.25	-20	+20
(FM – FEM)	-32.5	-65	-33.75	+30	
-0.5 (FM – FEM)	+32.5	+16.25	-15	+16.875	
Sum	0	-48.75	-48.75	46.875	
θ = Sum/1.5K	0	-32.5	-32.5	31.25	

One notices the fast convergence of the iterations leading to the exact solution, which gives the rotation at A is exactly zero, and $\theta_{BA} = \theta_{BC}$.

One can solve this problem by using the fixed end moment M^{*}_{FBC} for member BC which is fixed at A and hinged at C. This sometimes provides a faster convergence for the distribution process. The solution according to this method is shown in the next table:

Joint	A	B		C	
Member	AB	BA	BC	CB	CD
Stiffness Ratio (K)	1	1	1	1	
Modified Stiffness (K*)	1	1	0.75	0.75	
Distribution Factor	1	0.57143	0.42857	1	
FEM	191.25	48.75	75	0	+20
Balance Moment	-	-70.715	-54.035	-20	
Carry-over Moment	-35.357	0	-10	0	
Balance	0	5.714	4.285	0	
Carry-over	2.857	0	0	0	
Final moment (FM)	158.75	-16.25	16.25	-20	+20

Example 4.19

Solve Example 4.14 using the modified stiffness factor method.

Solution

The steps of the moment distribution in this case are given in the following table:

Joint	A	C		B
Member	AC	CA	CB	BC
Stiffness Ratio (K)	4	4	3	3
Modified Stiffness (K*)	4	4	2.25	2.25
Distribution Factor	1	0.64	0.36	1
FEM	37.5	-37.5	113.75	73.75
Balance Moment	0	-48.8	-27.45	-73.75
Carry-over Moment	-24.4	0	-36.875	0
Balance	0	23.6	13.275	0
Carry-over	11.8	0	0	0
Final moment (FM)	24.9	-62.7	62.7	0

One can also use M^*_{FCB} for member CB which is hinged at B instead of M_{FCB}. The value of M^*_{FCB} is calculated using the fixed end moments tables as follows:

$$M^*_{FCB} = +\frac{3PL}{16} + \frac{3EI\Delta}{L^2} = \frac{3\times 20\times 8}{16} + \frac{3\times 10^5 \times \frac{1}{100}}{8^2} = 76.875\ \text{k.ft.}$$

The steps of the moment distribution in this case can be organized in the next table. One notices the fast convergence leading to the exact solution of the problem.

Joint	A	C		B
Member	AB	CA	CB	BC
Stiffness Ratio (K)	4	4	3	3
Modified Stiffness (K*)	4	4	2.25	2.25
Distribution Factor	1	0.64	0.36	1
FEM	37.5	-37.5	76.875	0
Balance Moment	-	-25.2	-14.175	0
Carry-over Moment	-12.6	0	0	0
Final moment (FM)	24.9	-62.7	62.7	0

.4.3 Consideration of Relative Displacements

It has been pointed out that the moment distribution method deals basically ith the moments at the joints, and the distribution process depends on satisfying the quilibrium of moments at every free joint. In the presence of relative displacement etween the joints of a member of known quantities like settlement, one uses the fixed end moment due to the settlement as was shown in Example 4.14. However, if the structure is subjected to sidesway or deformation, which also need to be determined, the problem can be solved by the moment distribution method if the sidesway is assumed to have a specific value in order to carry out the distribution process. By using the equilibrium of forces in the direction of the sway, one can determine the exact moment and the exact relative displacement, in the members.

To illustrate the procedure, consider, the portal frame shown in Figure 4.52. The applied loads impose the deformation shown by the dotted curves. In order to solve the problem by the moment distribution method, the frame is prevented from the sidesway by applying the force H_1 as shown. The value of H_1 is obtained after determining the final moment M_1 and from studying the equilibrium of the forces in the direction of the sway. Since the sidesway Δ is unknown and it is difficult to determine its magnitude when applying the force H_1 in the opposite direction, one usually assumes a certain value for the sidesway which is denoted by Δ'. This assumption assists in calculating the fixed end moments due to the assumed sway for the structure. The moment distribution of this structure provides the final moment M_2. The magnitude of the force H_2 which has caused the assumed sway Δ' can be calculated from the equilibrium of forces in the direction of the sway. If one is fortunate, the force H_1 should be equal and opposite to the force H_1. However, one can correct any deviation by linear interpolation since the structure is linear elastic. This means that the value of Δ and the final moment M of the actual structure can be determined by the following relationships:

$$\Delta = \Delta' \frac{H_1}{H_2} \tag{4.85}$$

$$M = M_1 + \frac{H_1}{H_2} M_2 \tag{4.86}$$

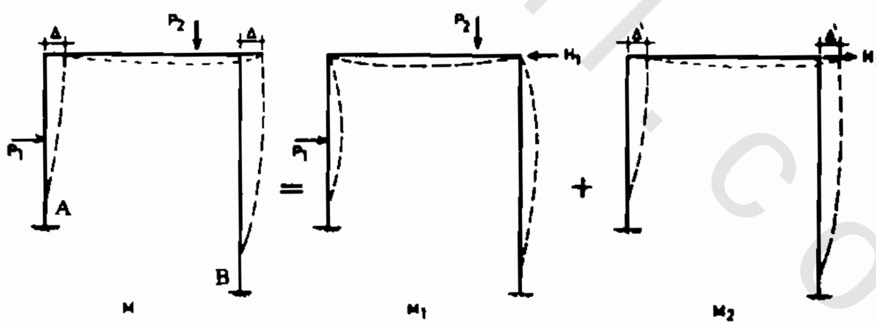

Figure 4.52

One should, however, realize that the above process should be repeated for every unknown relative displacement. For example, if one has the three story frame shown in Figure 4.53, in this case one has to solve the problem by the moment distribution method four times as illustrated in the figure in order to obtain the final moment diagram. The interpolation factors β_1, β_2, and β_3 which are applied for the assumed deformations Δ_1', Δ_2', and Δ_3' can be obtained by writing the equilibrium equation of forces in the direction of each assumed sway as follows:

$$\begin{aligned} &H_1 - \beta_1 H_1' + \beta_2 H_1'' + \beta_3 H_1''' = 0 \\ &H_2 + \beta_1 H_2' - \beta_2 H_2'' + \beta_3 H_2''' = 0 \\ &H_3 + \beta_1 H_3' + \beta_2 H_3'' - \beta_3 H_3''' = 0 \end{aligned} \qquad (4.87)$$

Values of the forces H_i, H_i', H_i'', and H_i''' for i = 1, 2, and 3 are obtained from the equilibrium of moment distributions M_1, M_2, M_3, and M_4.

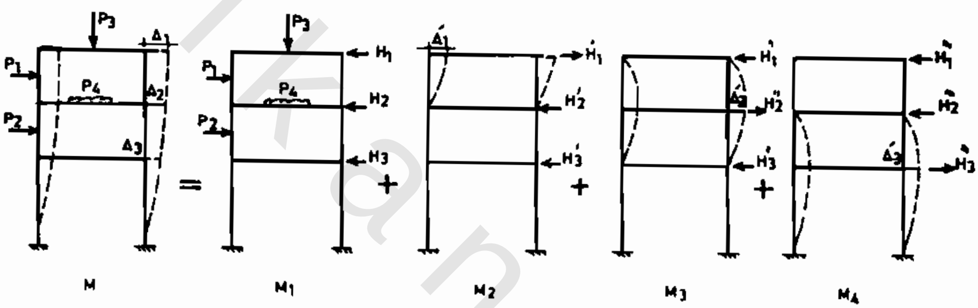

Figure 4.53

The final moment M and the sidesways Δ_1, Δ_2, and Δ_3 can now be determined from:

$$M = M_1 + \beta_1 M_2 + \beta_2 M_3 + \beta_3 M_4 \qquad (4.88)$$

$$\Delta_1 = \beta_1 \Delta_1' \qquad (4.89)$$

$$\Delta_2 = \beta_2 \Delta_2' \qquad (4.90)$$

$$\Delta_3 = \beta_3 \Delta_3' \qquad (4.91)$$

For example, the value of H_1 and H_2 in Figure 4.52 are determined by studying the equilibrium of forces in the horizontal direction as shown in Figure 4.54. The horizontal forces H_{1A}, H_{1B}, H_{2A}, and H_{2B} are determined as follows:

$$H_{1A} = \frac{M_{1AB} + M_{1AB}}{L_{AB}} + \frac{P_1 \ell}{L_{AB}}$$
$$H_{1B} = \frac{M_{1CD} + M_{1DC}}{L_{CD}} \qquad (4.92)$$
$$H_{2A} = \frac{M_{2AB} + M_{2BA}}{L_{AB}}$$
$$H_{2B} = \frac{M_{2DC} + M_{2CD}}{L_{CD}}$$

Values of H_1 and H_2 are thus obtained from solving the following equations:

$$H_1 + H_{1A} + H_{1B} - P_1 = 0$$
$$H_{2A} + H_{2B} - H_2 = 0 \qquad (4.93)$$

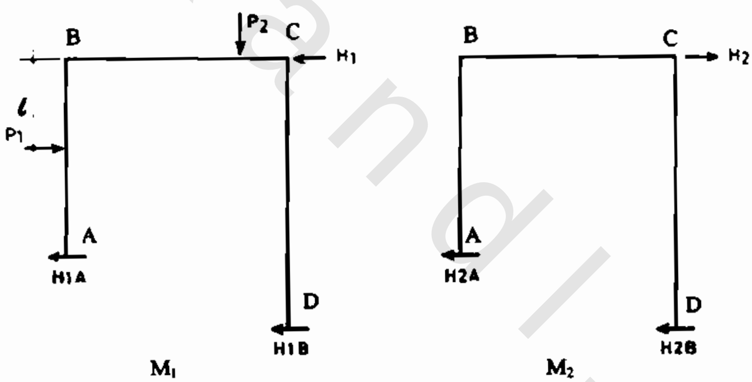

Figure 4.54

Example 4.20

Determine the bending moment diagram for the beam shown in Figure 4.55 (Example 4.5) using the moment distribution method ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}/°c$, K = 10 kN.cm).

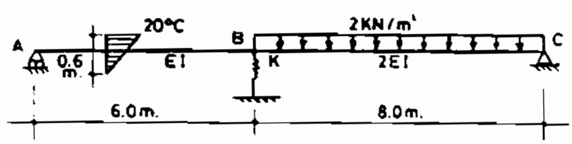

Figure 4.55

Solution

In this problem the settlement at B is unknown. One thus solves the problem in two stages. The first stage is to assume B a rigid support without settlement as shown in Figure 4.56. The second stage is to assume a certain amount of settlement at B, as shown in Figure 4.58. The final bending moment is obtained using the superposition and equilibrium principles.

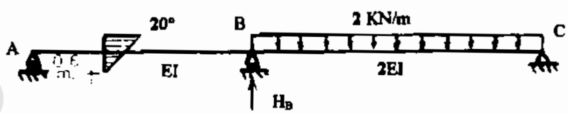

Figure 4.56

Stage 1

The fixed end moments are determined from the tables according to the signs of the slope deflection equation as follows:

$$M_{FAB} = -\alpha\, EI \frac{(T_1 - T_2)}{h} = -33.333 \text{ kN.m} \quad ; \quad M_{FBA} = +33.333 \text{ kN.m} \quad ,$$
$$M_{FCB} = -10.667 \text{ kN.m}$$

The steps of moment distribution are organized in the next table.

The reaction force H_B which prevented the settlement at B can thus be determined from the equilibrium of forces in the vertical direction. Using Figure 4.57, one has

$$H_B = 8 - 2.95 - 3.933 = 1.117 \text{ kN (upward)}.$$

Joint	A	C		B
Member	AB	BA	BC	CB
Stiffness Coefficient	4EI/6	4EI/6	8EI/8	8EI/8
Stiffness Ratio (K)	2	2	3	3
Modified Stiffness (K*)	1.5	1.5	2.25	2.25
Distribution Factor	1	0.4	0.6	1
FEM	-33.333	33.333	10.667	-10.667
Balance Moment	33.333	-17.6	-26.4	10.667
Carry-over Moment	0	16.667	5.334	0
Balance	0	-8.8	-13.2	0
Carry-over	0	0	0	0
Final Moment (M_1)	0	23.6	-23.6	0

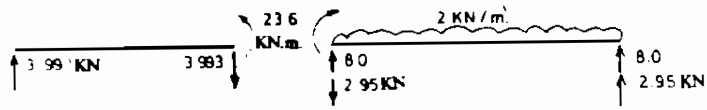

Figure 4.57

Stage 2

Assuming a certain deflection at B, say 1 cm, the fixed end moments due to this settlement according to the signs of the slope deflection equation are calculated using Figure 4.58 as follows:

$$M_{FAB} = \frac{6\,EI\,\Delta}{L^2} = \frac{6\,EI\,(0.01)}{6^2} = +166.667 \text{ kN.m} \quad ; \quad M_{FBA} = +166.667 \text{ kN.m}$$

$$M_{FBC} = -\frac{6\,(2EI)\,\Delta}{L^2} = \frac{-12\,EI\,(0.01)}{8^2} = -187.5 \text{ kN.m} \; ; \quad M_{FCB} = -187.5 \text{ kN.m}$$

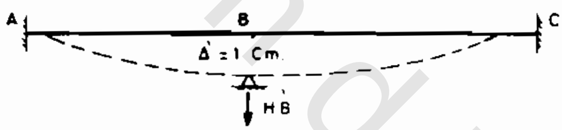

Figure 4.58

The steps of moment distribution of this stage are given in the following table:

Member	AB	BA	BC	CB
Distribution Factor	1	0.4	0.6	1
FEM	166.667	166.667	-187.5	-187.5
Balance Moment	-166.667	8.333	12.500	187.5
Carry-over Moment	0	-83.334	93.75	0
Balance	0	-4.166	-6.249	0
Carry-over	0	0	0	0
Final Moment (M_2)	0	87.5	-87.5	0

The forces $H_B^{'}$ which has caused the assumed settlement $\Delta' = 1$ cm is calculated from the equilibrium of forces in the vertical direction. Using Figure 4.59 one has

$$H_B^{'} = 14.5833 + 10.9375 = 25.52 \text{ kN} \quad \text{(downward)}$$

In order to determine the final moment diagram, the following equilibrium equation is developed using Figure 4.60:

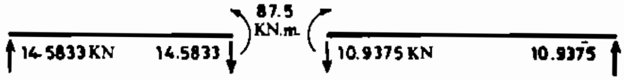

Figure 4.59

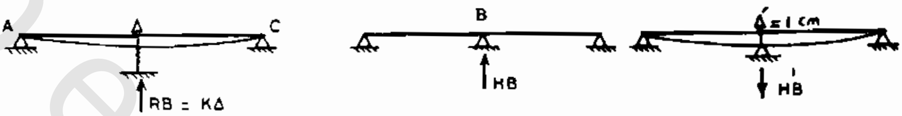

Figure 4.60

$R_B = H_B - \beta_1 H_B'$

Since $R_B = K\,\Delta = K\,\beta_1\,\Delta'$, one has

$10\beta_1 = 1.117 - 25.52\beta_1$, which leads to $\beta_1 = \dfrac{1.117}{35.52} = 0.03144$

The final moment at B is thus obtained from

$M_{BA} = M_1 + \beta_1 M_2 = +23.6 + 0.03144\,(87.5) = 26.352$ kN.m

The deflection at B is also obtained from

$\Delta = \beta_1\,\Delta' = 0.03144$ cm

which are the same results obtained in Example 4.5. The bending moment diagram is given again in Figure 4.61.

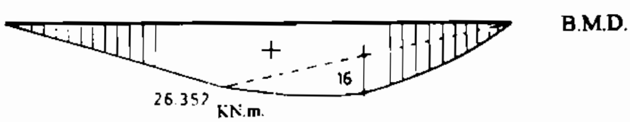

Figure 4.61

Example 4.21

Determine the bending moment diagram for the frame shown in Figure 4.62 (Example 4.6) using the moment distribution method ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}$/ °c).

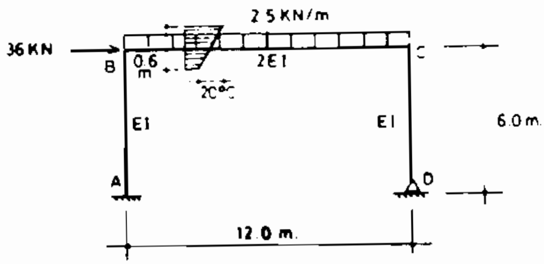

Figure 4.62

Solution

The frame is subjected to a sidesway in the right direction. The solution is obtained from the sum of two stages. In the first stage the frame is prevented from the sidesway, and in the second stage a certain value to the sway is assumed, as shown in Figure 4.63.

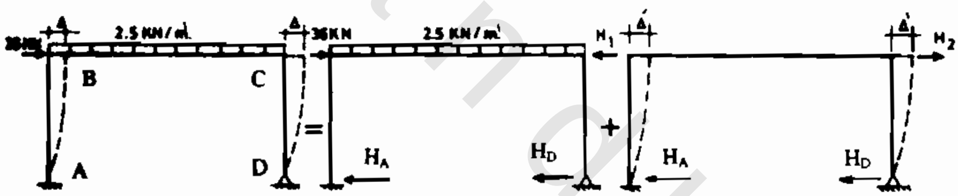

Figure 4.63

Stage 1

The fixed end moments of member BC are determined from the tables according to the signs of the slope deflection equation as follows:

$$M_{FBC} = \frac{qL^2}{12} - \alpha\,(2EI)\frac{(T_1 - T_2)}{h} = \frac{2.5\times 12^2}{12} - \frac{20\times 2}{0.6} = 36.667\ \text{kN.m}$$

$$M_{FCB} = +36.667\ \text{kN.m}$$

The steps of moment distribution of this stage are organized in the next table. The value of H_1 is determined from the equilibrium of forces in the horizontal direction, as shown in Figure 4.63 as follows:

Joint	A	B		C		B
Member	AB	BA	BC	CB	CD	CB
Stiffness Coefficient	4EI/6	4EI/6	8EI/12	8EI/12	4EI/6	4EI/6
Stiffness Ratio (K)	1	1	1	1	1	1
Modified Stiffness (K*)	1	1	1	1	0.75	0.75
Distribution Factor	1	0.5	0.5	0.57143	0.42857	1
FEM	0	0	-36.667	36.667	0	0
Balance Moment	0	18.334	18.334	-20.953	-15.7144	0
Carry-over Moment	9.167	0	-10.476	9.167	0	0
Balance	0	5.238	5.238	-5.238	-3.928	0
Carry-over	2.619	0	-2.619	2.619	0	0
Balance	0	1.309	1.309	-1.496	-1.122	0
Carry-over	0.654	0	-0.748	0.654	0	0
Balance	0	0.374	0.374	-0.374	-0.28	0
Carry-over	0.187	0	-0.187	0.187	0	0
Balance	0	0.093	0.093	-0.017	-0.08	0
Carry-over	0.046	0	-0.053	0.046	0	0
Final Moment (M_1)	12.686	25.38	-25.38	21.144	-21.144	0

$H_1 + H_A + H_D - 36 = 0$, where

$$H_A = \frac{M_{AB} + M_{BA}}{6} = 6.343 \text{ kN} \quad , \quad \text{and} \quad H_D = \frac{M_{CD}}{6} = -3.524 \text{ kN}$$

$H_1 = 36 - H_A - H_D = 33.181$ kN (in the assumed direction).

Stage 2

In this stage, the frame is subjected to an assumed sidesway, say Δ' = 1 cm. The fixed end moments due to this sway are determined from the tables according to the signs of the slope deflection equation, as shown in Figure 4.64, as follows:

$$M_{FAB} = M_{FBA} = M_{FDC} = M_{FCD} = \frac{+6EI\Delta'}{L^2} = \frac{6(10^5)\frac{1}{100}}{36} = +166.667 \text{ kN.m}$$

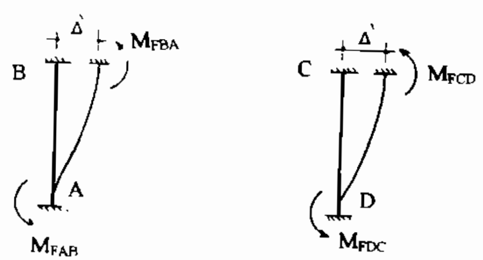

Figure 4.64

The steps of moment distribution of this stage are given in the following table:

Member	AB	BA	BC	CB	CD	DC
Modified Stiffness (K*)	1	1	1	1	0.75	0.75
Distribution Factor	1	0.5	0.5	0.57143	0.42857	1
FEM	166.667	166.667	0	0	166.667	166.667
Balance Moment	-	-83.334	-83.334	-95.238	-71.428	-166.667
Carry-over Moment	-41.667	0	-47.619	-41.667	-83.334	0
Balance	0	23.809	23.809	71.43	53.57	0
Carry-over	11.904	0	35.715	11.904	0	0
Balance	0	-17.858	-17.858	-6.802	-5.102	0
Carry-over	-8.929	0	-3.401	-8.929	0	0
Balance	-	1.7	1.7	5.102	3.827	0
Carry-over	0.85	0	2.551	0.85	0	0
Balance	-	-1.275	-1.275	-0.486	-0.364	0
Carry-over	-0.638	0	-0.243	-0.638	0	0
Balance	-	0.121	0.121	0.365	0.323	0
Carry-over	0.06	0	0.182	0.06	0	0
Balance	-	-0.091	-0.091	-0.035	-0.025	0
Carry-over	-0.048	0	-0.017	-0.046	0	0
Final Moment (M_2)	128.199	89.75	-89.75	-64.134	64.134	0

The force H_2 which has caused the assumed sway $\Delta' = 1$ cm is obtained from

$$H_2 = H_A + H_B$$

$$H_2 = \frac{M_{AB} + M_{BA}}{6} + \frac{M_{CD}}{6} = 47.012 \text{ kN}$$

The interpolation factor β_1 is obtained from

$$\beta_1 = \frac{H_1}{H_2} = \frac{33.181}{47.012} = 0.7057$$

The actual sway Δ is obtained from $\Delta = \beta_1 \Delta' = 0.7057$ cm.

The final bending moment M is obtained by the superposition of M_1 and $\beta_1 M_2$ as given in the following table:

Member	AB	BA	BC	CB	CD	DC
M_1	12.686	25.38	-25.38	21.144	-21.144	0
$\beta_1 M_2$	90.47	63.329	-63.329	-45.256	45.256	0
M	103.15	88.7	-88.7	-24.11	24.11	0

The blending moment diagram is given in Figure 4.65, which is approximately the same as the results of Example 4.6.

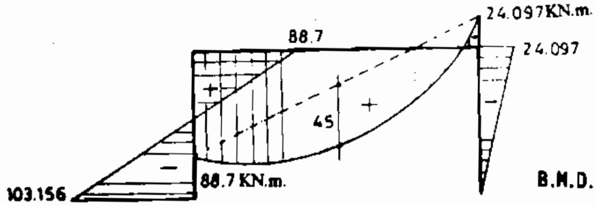

Figure 4.65

4.4.4 Non-Prismatic Members

In order to use the moment distribution method for non-prismatic members, one needs to determine the fixed end moments and the stiffness factors for these members. It has been shown in section 4.3.5 that the stiffness factors for a non-prismatic member AB are obtained from

$$K_{AA} = \frac{E}{L} S_{AA} = \frac{C_3}{C_1 C_3 - C_2^2} \frac{E}{L} \tag{4.94}$$

$$K_{BA} = \frac{E}{L} S_{BB} = \frac{C_1}{C_1 C_3 - C_2^2} \frac{E}{L} \tag{4.95}$$

$$K_{AB} = \frac{E}{L} S_{AB} = \frac{C_2}{C_1 C_3 - C_2^2} \frac{E}{L} \tag{4.96}$$

where the coefficients C_1, C_2, and C_3 are as defined in equations 4.20 to 4.22 and given here again for completeness as follows:

$$C_1 = \int_0^L \frac{x^2}{L^3\, I(x)} dx \tag{4.97}$$

$$C_2 = \int_0^L \frac{x(L-x)^2}{L^3\, I(x)} dx \tag{4.98}$$

$$C_3 = \int_0^L \frac{(L-x)^2}{L^3\, I(x)} dx \tag{4.99}$$

Equations 4.94 to 4.96 indicate that the carry-over fa[illegible]rom A and B which is denoted by C_{AB} is given by

$$C_{AB} = \frac{K_{AB}}{K_{AA}} = \frac{C_2}{C_3} \tag{4.100}$$

Similarly, the carry-over factor from B to A which is denoted by C_{BA} is given by

$$C_{BA} = \frac{K_{BA}}{K_{BB}} = \frac{C_2}{C_1} \tag{4.101}$$

It is obvious that symmetric non-prismatic members have S_{AA} the same as S_{BB}, and thus $C_{AB} = C_{BA}$. Once the stiffness factors, carry-over factors, and the fixed end moments have been determined for each member, the moment distribution process can be carried out in the usual manner.

Example 4.22

Determine the bending moment diagram for the beam shown in Figure 4.66 (same as Example 4.11) using the moment distribution method.

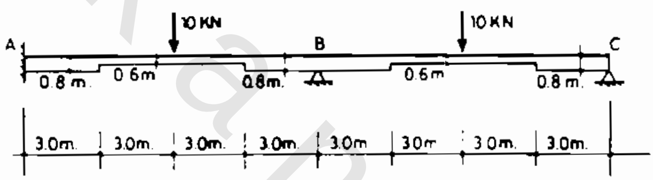

Figure 4.66

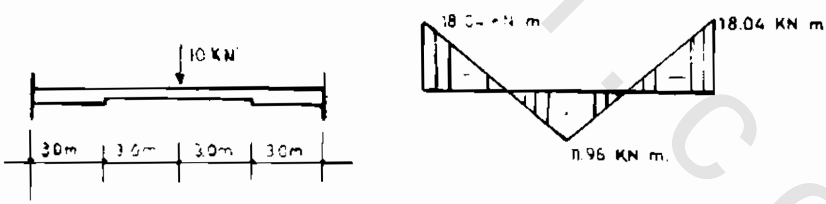

Figure 4.67

Solution

The fixed end moments of members AB and BC can be obtained by using the unit load method or the column analogy method for the beam shown in Figure 4.67. The solution by the column analogy method was obtained in Example 4.11 and is given in Figure 4.67. The fixed end moments according to the signs of the slope deflection equation are

$M_{FAB} = 18.04$ kN.m , $M_{FBA} = -18.04$ kN.m

$M_{FBC} = 18.04$ kN.m , $M_{FCB} = -18.04$ kN.m

The stiffness and carry-over factors can be obtained by calculating the coefficients C_1, C_2, and C_3. According to Example 4.11, the values of these have been determined as follows:

$$C_1 = 40.5418 \quad ; \quad C_2 = 2.288 \quad ; \quad C_3 = 40.5418$$

The stiffness coefficients are thus obtained as

$$K_{AA} = K_{BB} = K_{CC} = \frac{E}{L}\left(\frac{C_1}{C_1 C_3 - C_2^2}\right) = 0.04037\frac{E}{L} \text{ kN/m}$$

$$K_{AB} = K_{BA} = K_{BC} = K_{CB} = \frac{E}{L}\left(\frac{C_2}{C_1 C_3 - C_2^2}\right) = 0.025183\ \frac{E}{L} \text{ kN/m}$$

The carry-over factors are thus

$$C_{AB} = C_{BA} = C_{BC} = C_{CB} = \frac{S_{AB}}{S_{AA}} = \left(\frac{0.0251833}{0.04037}\right) = 0.62366$$

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness Factor	0.0403 E/12	0.0403 E/12	0.0403 E/12	0.0403 E/12
Stiffness Ratio (K)	1	1	1	1
Distribution Factor	1	0.5	0.5	1
Carry-over Factor	0.62366	0.62366	0.62366	0.62366
Fixed End Moment	18.04	-18.04	18.04	-18.04
Balance Moment	-	0	0	18.04
Carry-over Moment	0	0	11.2508	0
Balance	-	-5.6254	-5.6254	0
Carry-over	-3.5083	0	0	-3.5083
Balance	-	0	0	3.5083
Carry-over	0	0	2.188	0
Balance	0	-1.094	-1.094	0
Carry-over	-0.6823	0	0	-0.6823
Balance	-	-	-	0.6823
Carry-over	0	0	0.4255	0
Balance	0	-0.2127	-0.2127	0
Carry-over	-0.1326	0	0	-0.1326
Balance	-	0	0	0.1326
Carry-over	-	-	0.0827	-
Balance	-	-0.0413	-0.0413	-
Carry-over	-0.0258	0	0	-0.0258
Balance	0	0	0	0.0258
Final Moment	13.691	-25.0136	25.0136	0

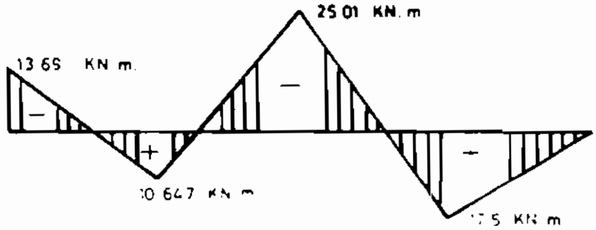

Figure 4.68

The bending moment diagram is given in Figure 4.68 which is almost the same as Figure 4.37 of Example 4.11.

4.4.5 Non-prismatic Members with Moment Releases

The derivation of the modified stiffness factors accounting for a moment release at the end of a nonprismatic member was shown in Section 4.3.6. For member AB which is hinged at the right end B, the modified stiffness factor is

$$K^*_{AA} = \frac{1}{C_1}\frac{E}{L} \tag{4.102}$$

in which A represents the unreleased end of member AB.

If member AB is hinged at A, the modified stiffness factor at B becomes

$$K^*_{BB} = \frac{1}{C_3}\frac{E}{L} \tag{4.103}$$

One can also prove that Equation 4.102 may be replaced by

$$K^*_{AA} = K_{AA}\left(1 - C_{AB}\,C_{BA}\right) \tag{4.104}$$

where C_{AB} and C_{BA} are the carry over factors given by Equations 4.100 and 4.101.

Similarly, Equation 4.103 may be replaced by

$$K^*_{BB} = K_{BB}\left(1 - C_{AB}\,C_{BA}\right) \tag{4.105}$$

In using K^*_{AA} or K^*_{BB} for members with a hinged end, one can disregard the carry-over moment to this hinged end. This prompts the process of convergence during the moment distribution.

Example 4.23

Solve Example 4.21 by using the modified stiffness considering the hinge at C.

Solution

Since the support at C is hinge at the right end of member BC, the modified stiffness factor for member BC is

$$K^*_{BB} = \frac{1}{C_1}\frac{E}{L} = 0.024666\frac{E}{L} \text{ kN/m}$$

One can use the fixed end moments M_{FBC} and M_{FCB}, or use the modified fixed end moment M^*_{FBC}. The steps of moment distribution considering M_{FBC} and M_{FCB} are given in the following table:

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness Factor (K)	0.04037 E/12	0.04037 E/12	0.04037 E/12	0.04037 E/12
Modified Stiffness (K*)	0.04037	0.04037	0.024666	-
Stiffness Ratio	1	1	0.61100	-
Distribution Factor	1	0.6207	0.3793	1
Carry-over Factor	0.62366	0.62366	0	0.62366
Fixed End Moment	18.04	-18.04	18.04	-18.04
Balance Moment	-	0	0	18.04
Carry-over Moment	0	0	11.2508	0
Balance	-	-6.9834	-4.2674	0
Carry-over	-4.3553	0	0	0
Final Moment (M_1)	13.684	-25.023	25.023	0

The moment distribution considering M^*_{FBC} which was determined in example 4.12 is shown in the next table.

Member	AB	BA	BC	CB
Distribution Factor	1	0.6207	0.3793	1
Carry-over Factor	0.62366	0.62366	0	0.62366
Fixed End Moment	18.04	-18.04	+29.29	0
Balance Moment	0	-6.9833	-4.2671	0
Carry-over Moment	-4.3552	0	0	0
Final Moment (M_2)	13.684	-25.023	25.023	0

The results are the same as example 4.21.

Example 4.24

Determine the stiffness factors and carry-over factors for the beam shown in Figure 4.69.

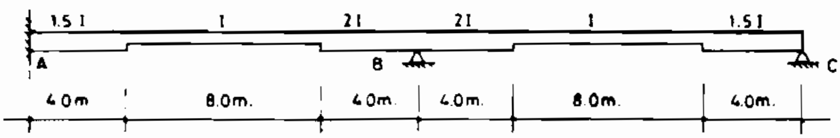

Figure 4.69

Solution

The coefficients C_1, C_2, and C_3 are calculated for member AB considering point A as an origin, as shown in Figure 4.70.

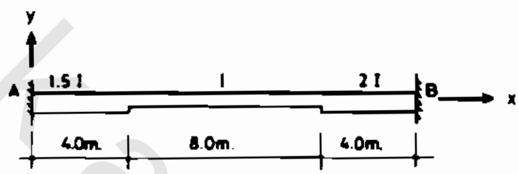

Figure 4.70

$$C_3 = \frac{1}{L^3}\int_0^L \frac{(L-x)^2}{I(x)}dx$$

$$= \frac{1}{(16)^3}\left[\int_0^4 \frac{(16-x)^2}{1.5I}dx + \int_4^{12} \frac{(16-x)^2}{I}dx + \int_{12}^{16} \frac{(16-x)^2}{2I}dx\right] = \frac{1}{16^3}\left(\frac{1091.555}{I}\right)$$

$$C_2 = \frac{1}{L^3}\int_0^L \frac{x(L-x)}{I(x)}dx$$

$$= \frac{1}{(16)^3}\left[\int_0^4 \frac{x(16-x)}{1.5I}dx + \int_4^{12} \frac{x(16-x)}{I}dx + \int_{12}^{16} \frac{x(16-x)}{2I}dx\right] = \frac{1}{16^3}\left(\frac{493.7775}{I}\right)$$

$$C_1 = \frac{1}{L^3}\int_0^L \frac{x^2}{I(x)}dx$$

$$= \frac{1}{(16)^3}\left[\int_0^4 \frac{x^2}{1.5I}dx + \int_4^{12} \frac{x^2\,dx}{I}dx + \int_{12}^{16} \frac{x^2\,d}{2I}\right] = \frac{1}{16^3}\left(\frac{963.555}{I}\right)$$

From equations 4.94 to 4.101 one has for member AB

$$K_{AA} = \left(\frac{C_3}{C_1C_3 - C_2^2}\right)\frac{E}{L} = \left(\frac{963.555}{699201.5}\right)\frac{16^3\ EI}{16} = 5.6443\frac{EI}{16}\ \text{kN/m}$$

$$K_{BB} = \left(\frac{C_1}{C_1C_3 - C_2^2}\right)\frac{E}{L} = \left(\frac{1091.555}{699201.5}\right)\frac{16^3\ EI}{16} = 6.3943\frac{EI}{16}\ \text{kN/m}$$

$$C_{AB} = \frac{C_2}{C_3} = \frac{593.7775}{963.555} = 0.6162$$

$$C_{BA} = \frac{C_2}{C_1} = \frac{593.7775}{1091.555} = 0.5439$$

The coefficients C_1, C_2, and C_3 are calculated for member BC, considering the origin is at B, as shown in Figure 4.71.

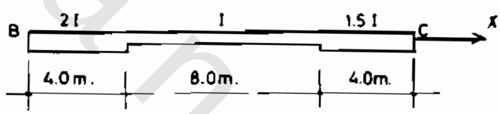

Figure 4.71

$$C_3 = \frac{1}{L^3}\int_0^L \frac{(L-x)^2}{I(x)}\,dx$$

$$= \frac{1}{(16)^3}\left[\int_0^4 \frac{(16-x)^2}{2I}\,dx + \int_4^{12} \frac{(16-x)^2}{I}\,dx + \int_{12}^{16} \frac{(16-x)^2}{1.5I}\,dx\right] = \frac{1}{16^3}\left(\frac{963.555}{I}\right)$$

$$C_2 = \frac{1}{16^3}\left(\frac{593.7775}{I}\right) \qquad C_1 = \frac{1}{16^3}\left(\frac{1091.555}{I}\right)$$

For member BC, the stiffness coefficients are

$$K_{BB} = \left(\frac{C_3}{C_1C_3 - C_2^2}\right)\frac{E}{L} = 6.3943\frac{EI}{16}\ \text{kN/m}$$

$$K_{CC} = \left(\frac{C_1}{C_1C_3 - C_2^2}\right)\frac{E}{L} = 5.6443\frac{EI}{16}\ \text{kN/m}$$

$$K^{*}_{BB} = \frac{1}{C_1}\frac{E}{L} = \frac{1}{963.555}\;\frac{16^3\,EI}{L} = 4.2509\frac{EI}{L}\ \text{kN/N}$$

or, $$K^{*}_{BB} = K_{BB}\left(1 - C_{AB}\,C_{BA}\right) = 4.2509\frac{EI}{L}\ \text{kN/N}$$

4.4.6 Gable Frames

The difficulties in analyzing gable frames by the slope deflection equation were pointed out in Section 4.3.8. These difficulties are encountered from the geometrical analysis of the deformed shape of the frame. One will have more difficulties if he intends to use the moment distribution method to solve gable frames. The reason is that the moment distribution must be applied for undistorted frames or for frame of known distortion. For the symmetrical gable frame shown in Figure 4.72, the moment distribution has to be carried out two times. The first time is to restrain the frame against any distortion. The second time is to assume a certain value for the distortion Δ' in order to carry out the moment distribution. The exact values of Δ' and the final moment M are obtained by interpolation after determining the imposed forces H_1 and H_2 from the equilibrium conditions.

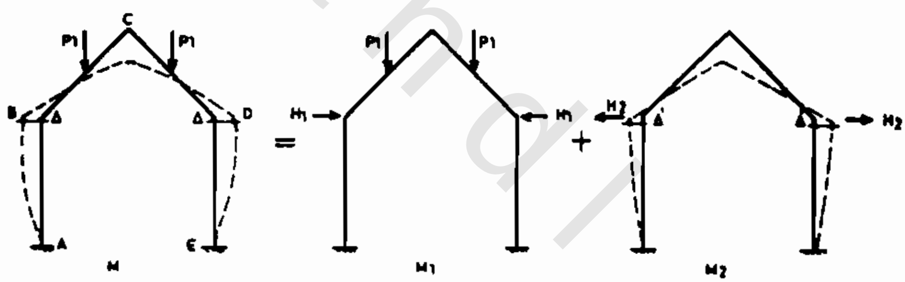

Figure 4.72

The problem may even be more difficult if the symmetrical frame is subjected to unsymmetrical loading, or the frame is asymmetric in geometry. For example, if the frame is subjected to asymmetric loading as shown in Figure 4.73, one has two

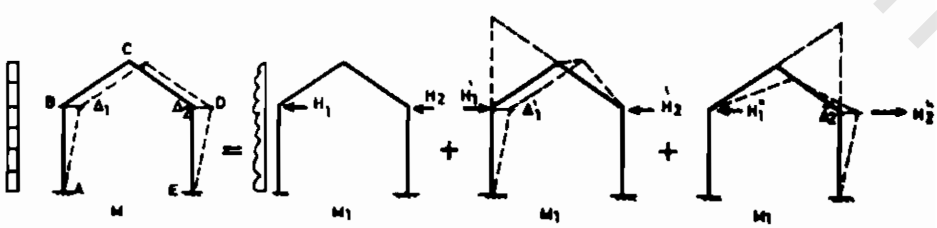

Figure 4.73

unknown sways, Δ_1, Δ_2, and the problem has to be solved three times by the moment distribution method. Therefore, it is recommended to not solve these kinds of frames by the moment distribution method and use the stiffness matrix method which is a more general method.

4.5 THE USE OF CASTIGLIANO'S FIRST THEOREM

In order to apply Castigliano's first theorem, the strain energy must be written as a function of the displacements. The general form of strain energy is given by

$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dv \tag{4.106}$$

Castigliano's First Theorem, as given in Chapter 2, states that

$$\frac{\partial U}{\partial D_i} = A_i \tag{4.107}$$

where A_i is the action associated with the deformation D_i.

The applications of Castigliano's First Theorem are very limited and usually fit the small kinematic indeterminate problems. By applying Equation 4.107 at the free joints, one ends up with a number of equations in terms of the unknown displacements. The following simple example illustrates the procedure:

Example 4.25

Determine the displacements of joint C in the truss shown in Figure 4.74. (EA = 10^4 kN).

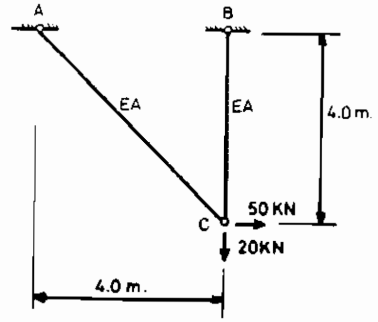

Figure 4.74

Solution

The truss has two degrees of freedom at C. Let the kinematic variables be expressed by Δ_h and Δ_v, which denote, respectively, horizontal and vertical displacements at C. Assuming that the internal forces in members CA and CB are in tension and are denoted by P_A and P_B, respectively, the strain energy is thus given by

$$U = \frac{1}{2}\int_v \underline{\sigma}^T\,\underline{\varepsilon}\,dv = \frac{1}{2}\Sigma\int_0^L \underline{\sigma}^T\,\underline{\varepsilon}\,A\,dl$$

$$= \frac{1}{2}\frac{P_A^2\sqrt{2L}}{EA} + \frac{1}{2}\frac{P_B^2\,L}{EA}$$

Using the relationship between the axial deformation and axial forces one may write the strain energy as

$$U = \frac{1}{2}\frac{EA}{\sqrt{2L}}\Delta_A^2 + \frac{1}{2}\frac{EA}{L}\Delta_B^2$$

where $P = EA\,\Delta/L$.

Now, one has to find the relationships between $\Delta_h = \sqrt{2}\,\Delta_A - \Delta_v$. Using Figure 4.75 and assuming small deformations one can write

$$\Delta_v = \Delta_B$$

$$\Delta_h = \sqrt{2}\,\Delta_A - \Delta_v$$

$$U = \frac{1}{2}\frac{EA}{\sqrt{2}\,L}\left(\frac{\Delta_h + \Delta_v}{\sqrt{2}}\right)^2 + \frac{1}{2}\frac{EA}{L}\Delta_v^2$$

One can now apply Castigliano's First Theorem at the free joint C as follows:

$$\frac{\partial U}{\partial \Delta_h} = 50 = \frac{EA}{2\sqrt{2}\,L}\left(\Delta_h + \Delta_v\right) \tag{a}$$

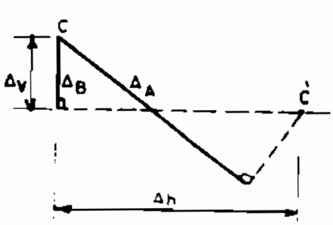

Figure 4.75

$$\frac{\partial U}{\partial \Delta_v} = 20 = \frac{EA}{2\sqrt{2}\,L}\left(\Delta_h + \Delta_v\right) + \frac{EA}{L}\Delta_v \qquad \text{(b)}$$

Solution of equations (a) and (b) gives

$$\Delta_v = -\frac{30L}{EA} = -1.2\,\text{cm}\ (\uparrow) \quad , \qquad \Delta_h = 0.0685\,\text{m} = 6.85\,\text{cm}\ (\rightarrow)$$

4.6 THE STIFFNESS MATRIX METHOD : APPROACH I

In this method, the slope deflection equations method is formulated using matrix operations. The method can be formulated for whether the members have relative displacements or not.

4.6.1 Members Without Relative Displacements

In this approach, the slope deflection equations for each member are collected in a matrix form. For example, the slope deflection equations for member AB in a matrix form become:

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} M_{FAB} \\ M_{FBA} \end{bmatrix} + \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_{AB} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \end{bmatrix} \qquad (4.108)$$

In a short form, one may write Equation 4.108 as

$$\underline{\mathbf{M}}_{AB} = \underline{\mathbf{M}}_{FAB} + \underline{\mathbf{S}}_{AB}\,\underline{\mathbf{D}}_{AB} \qquad (4.109)$$

For a structure consists of a set of members, Equation 4.109 can be repeated for each member. The structure shown in Figure 4.76, for example, the slope deflection equations are collected in a matrix form as

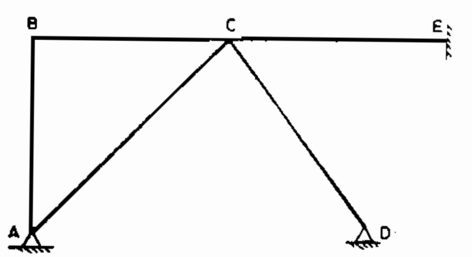

Figure 4.76

$$\begin{bmatrix} \underline{\mathbf{M}}_{AB} \\ \underline{\mathbf{M}}_{BC} \\ \underline{\mathbf{M}}_{AC} \\ \underline{\mathbf{M}}_{CD} \\ \underline{\mathbf{M}}_{CD} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{M}}_{FAB} \\ \underline{\mathbf{M}}_{FBC} \\ \underline{\mathbf{M}}_{FAC} \\ \underline{\mathbf{M}}_{FCD} \\ \underline{\mathbf{M}}_{FCD} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{S}}_{AB} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{BC} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{AC} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{CD} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{CE} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_{AB} \\ \underline{\mathbf{D}}_{BC} \\ \underline{\mathbf{D}}_{AC} \\ \underline{\mathbf{D}}_{CD} \\ \underline{\mathbf{D}}_{CD} \end{bmatrix} \tag{4.110}$$

Equations 4.110 can be written for any structure whose members are not subjected to relative displacements. It can be expressed in a general form as

$$\underline{\mathbf{M}}_m = \underline{\mathbf{M}}_{Fm} + \left[\mathbf{S}_m\right]\underline{\mathbf{D}}_m \tag{4.111}$$

in which $\underline{\mathbf{M}}_m$ members end moments, $\underline{\mathbf{M}}_{Fm}$ contains the members fixed end moments, and $\left[\mathbf{S}_m\right]$ is the augmented members stiffness matrices.

The compatibility conditions for the frame of Figure 4.76 can be stated as follows:

$$\begin{aligned} &\theta_{AB} = \theta_{AC} = \theta_A \\ &\theta_{BA} = \theta_{BC} = \theta_B \\ &\theta_{CA} = \theta_{CB} = \theta_{CD} = \theta_{CE} = \theta_C \\ &\theta_{DC} = \theta_D \end{aligned} \tag{4.112}$$

where θ_A, θ_B, θ_C, and θ_D are the kinematic variables at the joints of this structure. Equation 4.112 can be expressed in a matrix form as

$$\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \\ \theta_{AC} \\ \theta_{CA} \\ \theta_{DC} \\ \theta_{CD} \\ \theta_{CE} \\ \theta_{EC} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{bmatrix} \tag{4.113}$$

Equation 4.113 can be expressed in a short form by

$$\underline{\mathbf{D}}_m = \underline{\mathbf{C}}\,\underline{\mathbf{D}} \tag{4.114}$$

where $\underline{\mathbf{C}}$ is called the compatibility or connectivity matrix.

In order to solve the problem one has to apply the equilibrium conditions at the free joints. For the structure of Figure 4.76 one has the following equilibrium conditions:

$$\begin{aligned} M_A &= M_{AB} + M_{AC} \\ M_B &= M_{BC} + M_{BA} \\ M_C &= M_{CB} + M_{CA} + M_{CD} + M_{CE} \\ M_D &= M_{DC} \end{aligned} \tag{4.115}$$

Equation 4.115 can also be expressed in a matrix form as

$$\begin{bmatrix} M_A \\ M_B \\ M_C \\ M_D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{AC} \\ M_{CA} \\ M_{DC} \\ M_{CD} \\ M_{CE} \\ M_{EC} \end{bmatrix} \tag{4.116}$$

Comparing Equation 4.116 with Equation 4.113, one may write Equation 4.116 as

$$\underline{\mathbf{M}} = \underline{\mathbf{C}}^T \underline{\mathbf{M}}_m \tag{4.117}$$

where $\underline{\mathbf{M}}$ is a vector which contains the external moments at the free joints.

Substituting Equation 4.111 and 4.114 into Equation 4.117, one obtains

$$\underline{\mathbf{M}} = \underline{\mathbf{C}}^T \underline{\mathbf{M}}_{Fm} + \underline{\mathbf{C}}^T [S_m] \underline{\mathbf{C}}\, \underline{\mathbf{D}} \tag{4.118}$$

Equation 4.118 can also be written in a shorter form as

$$\underline{\mathbf{M}} = \underline{\mathbf{M}}_F + [S] \underline{\mathbf{D}} \tag{4.119}$$

where $\underline{\mathbf{M}}_F$ and [S] are, respectively, obtained from

$$\underline{\mathbf{M}}_F = \underline{\mathbf{C}}^T \underline{\mathbf{M}}_{Fm} \tag{4.120}$$

$$[S] = \underline{\mathbf{C}}^T [S_m] \underline{\mathbf{C}} \tag{4.121}$$

Equation 4.119 is now in a suitable form to be solved for the unknown displacements $\underline{\mathbf{D}}$ as follows:

$$\underline{\mathbf{D}} = [S]^{-1} (\underline{\mathbf{M}} - \underline{\mathbf{M}}_F) \qquad (4.122)$$

It is a common practice to call the terms ($-\underline{\mathbf{M}}_F$) in Equation 4.122 by the equivalent joint moments due to the direct loadings on the members. It is obvious that this matrix approach is suitable for computer applications. The analyst stores the structural data in the form of $[S_m]$, the loading data in the form of $\underline{\mathbf{M}}$ and $\underline{\mathbf{M}}_{Fm}$, and finally the compatibility matrix $\underline{\mathbf{C}}$ which is obtained from boundary and connectivity conditions. The matrices $\underline{\mathbf{M}}_F$ and S are then calculated by matrix multiplication according to Equations 4.120 and 4.121. The free displacements $\underline{\mathbf{D}}$ are obtained according to Equation 4.122. The member end moments are determined from Equation 4.111 using Equation 4.114.

Example 4.26

Determine the bending moment diagram for the beam shown in Figure 4.77 using the stiffness matrix method – approach I. ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}/°c$).

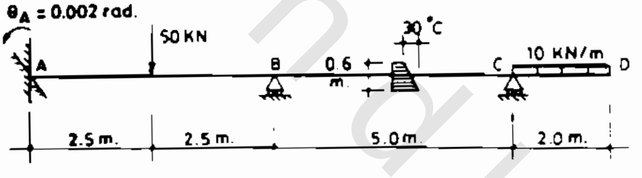

Figure 4.77

Solution

The stiffness matrices $\underline{\mathbf{S}}_{AB}$ and $\underline{\mathbf{S}}_{BC}$, are determined as follows:

$$\underline{\mathbf{S}}_{AB} = \underline{\mathbf{S}}_{BC} = \begin{bmatrix} \frac{4EI}{5} & \frac{2EI}{5} \\ \frac{2EI}{5} & \frac{4EI}{5} \end{bmatrix} = EI \begin{bmatrix} 0.80 & 0.40 \\ 0.40 & 0.80 \end{bmatrix}$$

The members stiffness matrix $[S_m]$ is composed as follows:

$$
[S_m] = EI \begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0.4 & 0.8 \end{bmatrix}
$$

The degree of freedom in this structure is two, which represents the angles of rotation at B and C. The compatibility equations are put in a matrix form as

$$
\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \underline{\mathbf{C}}\,\underline{\mathbf{D}}
$$

Figure 4.78

The fixed end moments are determined for the cases shown in Figure 4.78 using the tables and according to the signs of the slope deflection equation as follows:

$$
M_{FAB} = \frac{4EI}{5}\theta_A + \frac{50 \times 5}{8} = 191.25 \ \text{kN.m}
$$

$$
M_{FBA} = \frac{2EI}{5}\theta_A - \frac{50 \times 5}{8} = 48.75 \ \text{kN.m}
$$

$$
M_{FBC} = EI\,\alpha \cdot \frac{(T_1 - T_2)}{h} = 50 \, \text{kN.m} \quad ; \qquad M_{FCB} = -50 \ \text{kN.m}
$$

The fixed end moments are collected to form matrix $\underline{\mathbf{M}}_{Fm}$ as follows:

$$
\underline{\mathbf{M}}_{Fm}^{T} = [191.25 \quad 48.75 \quad 50 \quad -50]
$$

The matrix $\underline{\mathbf{M}}_F$ and $[\mathbf{S}]$ are obtained according to Equations 4.120 and 4.121 as follows:

$$
\underline{\mathbf{M}}_F = \underline{\mathbf{C}}^T \, \underline{\mathbf{M}}_{Fm} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{M}}_{Fm} = \begin{bmatrix} 98.75 \\ -50 \end{bmatrix}
$$

$$[\underline{S}] = \underline{C}^T [\underline{S}_m] \underline{C} = EI \begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$$

Substituting into the equilibrium equation one obtains

$$\underline{\mathbf{M}} = \underline{\mathbf{M}}_F + [\underline{S}]\underline{\mathbf{D}}$$

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \begin{bmatrix} 98.75 \\ -50 \end{bmatrix} + E \begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}$$

The deformation $\underline{\mathbf{D}}$ is solved to have

$$\underline{\mathbf{D}} = \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} -81.25 \\ 78.125 \end{bmatrix}$$

The members end moments are calculated from Equation 4.111 as follows:

$$\underline{\mathbf{M}}_m = \underline{\mathbf{M}}_{Fm} + [\mathbf{S}_m]\underline{\mathbf{D}}_m$$

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} 191.25 \\ 48.75 \\ 50 \\ -50 \end{bmatrix} + EI \begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ -81.25/EI \\ -81.25/EI \\ 78.125/EI \end{bmatrix} = \begin{bmatrix} 158.75 \\ -16.25 \\ 16.25 \\ -20 \end{bmatrix} \text{kN.m}$$

which gives the same results obtained previously. The bending moment diagram is given again in Figure 4.79.

Example 4.27

Determine the bending moment diagram for the frame shown in Figure 4.80 using the stiffness matrix method approach I, where Support B has displaced down 1 cm, and $EI = 10^5$ kN.m^2.

Solution

The stiffness matrices $\underline{S}_{AC}$ and $\underline{S}_{CB}$ are determined as follows:

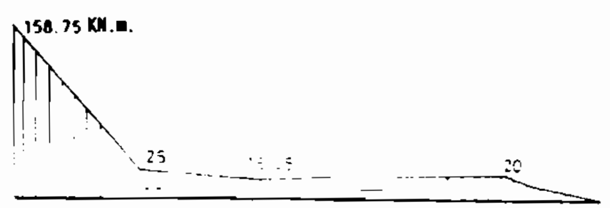

Figure 4.79

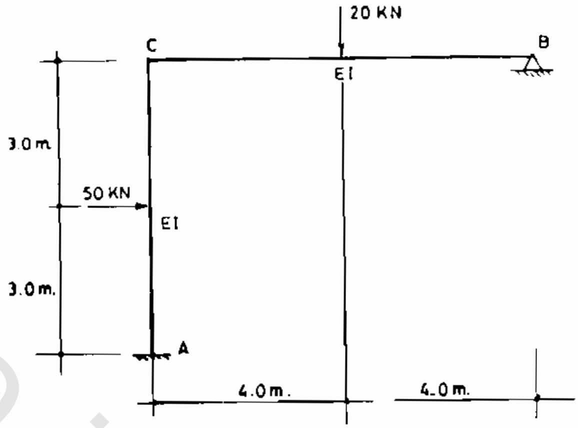

Figure 4.80

$$\underline{S}_{AC} = \begin{bmatrix} \frac{4EI}{6} & \frac{2EI}{6} \\ \frac{2EI}{6} & \frac{4EI}{6} \end{bmatrix}, \quad \underline{S}_{CB} = \begin{bmatrix} \frac{4EI}{8} & \frac{2EI}{8} \\ \frac{2EI}{8} & \frac{4EI}{8} \end{bmatrix}$$

The members stiffness matrix $[\mathbf{S_m}]$ is obtained from

$$[\mathbf{S}_m] = \begin{bmatrix} \underline{S}_{AC} & \underline{0} \\ \underline{0} & \underline{S}_{CB} \end{bmatrix} = EI \begin{bmatrix} 0.667 & 0.333 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

The degree of freedom in this structure is two, which represents θ_C and θ_B. The compatibility equations are $\theta_{AC} = 0$, $\theta_{CA} = \theta_C$, and $\theta_{BC} = \theta_B$. These relations are put in a matrix form as follows:

$$\begin{bmatrix} \theta_{AC} \\ \theta_{CA} \\ \theta_{CB} \\ \theta_{BC} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_C \\ \theta_B \end{bmatrix} = \underline{\mathbf{C}}\,\underline{\mathbf{D}}$$

The fixed end moments are determined for each member using the tables for the cases shown in Figure 4.81, and in the signs of the slope deflection equation.

$$M_{FAC} = \frac{50 \times 6}{8} = 37.5 \text{ kN.m} \quad ; \quad M_{FCA} = -37.5 \text{ kN.m}$$

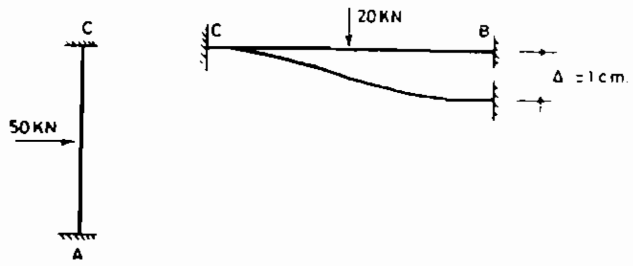

Figure 4.81

$$M_{FCB} = + \frac{20 \times 8}{8} + \frac{6EI}{8^2}\left(\frac{1}{100}\right) = 113.75 \text{ kN.m} \;;\; M_{FBC} = -20 + 9375 = 73.75 \text{ kN.m}$$

The calculations of $\underline{\mathbf{M}}_F$ and $\underline{\mathbf{S}}$ follow equations 4.120 and 4.121 as follows:

$$\underline{\mathbf{M}}_{Fm}^{T} = \begin{bmatrix} 37.5 & -37.5 & 113.75 & 73.75 \end{bmatrix}$$

$$\underline{\mathbf{M}}_F = \underline{\mathbf{C}}^T \, \underline{\mathbf{M}}_{Fm} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{M}}_{Fm} = \begin{bmatrix} 76.25 \\ 73.75 \end{bmatrix} \text{ kN.m}$$

$$[S] = \underline{\mathbf{C}}^T [S_m] \underline{\mathbf{C}} = EI \begin{bmatrix} 0.333 & 0 \\ 0.667 & 0 \\ 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix} = EI \begin{bmatrix} 1.167 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}$$

Substituting into Equation 4.119, one obtains

$$\begin{bmatrix} M_C \\ M_B \end{bmatrix} = \begin{bmatrix} 76.25 \\ 73.75 \end{bmatrix} + EI \begin{bmatrix} 1.167 & 0.25 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} \theta_C \\ \theta_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution for θ_C and θ_B gives

$$\begin{bmatrix} \theta_C \\ \theta_B \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} -37.788 \\ -128.606 \end{bmatrix} \text{ rad.}$$

The end moments are now calculated using Equation 4.111 as follows:

$$\underline{\mathbf{M}}_m = \underline{\mathbf{M}}_{Fm} + [S_m] \underline{\mathbf{D}}_m$$

$$\begin{bmatrix} M_{AC} \\ M_{CA} \\ M_{CB} \\ M_{BC} \end{bmatrix} = \begin{bmatrix} 37.5 \\ -37.5 \\ 113.75 \\ 73.75 \end{bmatrix} + EI \begin{bmatrix} 0.667 & 0.333 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ -37.788/EI \\ -37.788/EI \\ -128.606/EI \end{bmatrix} = \begin{bmatrix} 24.916 \\ -62.704 \\ 62.704 \\ 0 \end{bmatrix} kN.m$$

The bending moment diagram is given in Figure 4.82, which is the same as the results of Example 4.7.

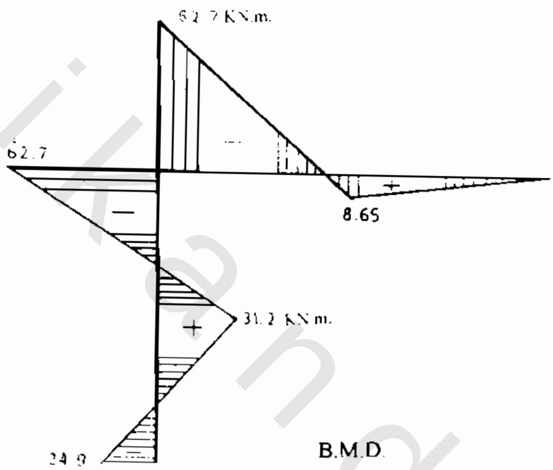

Figure 4.82

4.6.2 Effect of Members with Moments Releases

If the moment in a member is released at one end, one may use the modified slope deflection equation, given in Equation 4.15. For any member AB which has a hinge at B, one may write Equation 4.15 in a matrix form as follows:

$$\left[M^*_{AB}\right] = \left[M^*_{FAB}\right] + \left[\frac{3EI}{L}\right] \theta_{AB} \tag{4.123}$$

where the stiffness matrix $\underline{\mathbf{S}}_{AB}$ is in this case of dimension 1×1.

The procedure for obtaining the solution is the same as in the previous section. The following example illustrates the procedure:

Example 4.28

Solve Example 4.26 by using the moment released at joint B for members BC.

Solution

The stiffness matrices $\underline{S}_{AC}$ and $\underline{S}_{CB}$ are respectively given by

$$\underline{S}_{AC} = \begin{bmatrix} \frac{4EI}{6} & \frac{2EI}{6} \\ \frac{2EI}{6} & \frac{4EI}{6} \end{bmatrix} \quad , \quad \underline{S}_{CB} = \begin{bmatrix} \frac{3EI}{8} \end{bmatrix}$$

The members stiffness matrix $[S_m]$ is thus given by

$$[S_m] = EI \begin{bmatrix} 0.667 & 0.333 & 0 \\ 0.33 & 0.667 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

The compatibility relation between $\theta_{AC}, \theta_{CA}, \theta_{CB}$, and θ_C is

$$\begin{bmatrix} \theta_{AC} \\ \theta_{CA} \\ \theta_{CB} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \theta_C = \underline{C}\,\underline{D}$$

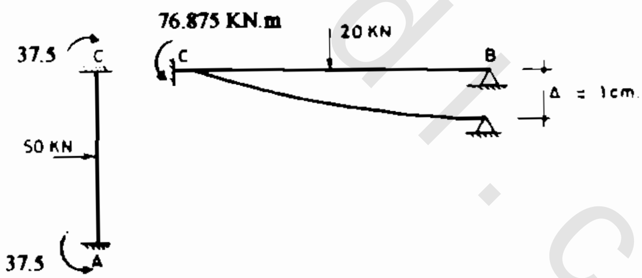

Figure 4.83

The fixed end moments are obtained for the cases shown in Figure 4.83 using the tables and put according to the signs of the slope deflection equations as follows:

$$M_{FAC} = \frac{50 \times 6}{8} = 37.5 \text{ kN.m} \qquad M_{FCA} = -37.5 \text{ kN.m}$$

$$M^{*}_{FCB} = \frac{3 \times 20 \times 8}{16} + \frac{3EI(0.01)}{8^2} = 76.875 \text{ kN.m}$$

One can now form $\underline{M}_{Fm}, \underline{M}_F$ and $[\underline{S}]$ as follows:

$$\underline{\mathbf{M}}_{Fm}^{T} = \begin{bmatrix} 37.5 & -37.5 & 78.875 \end{bmatrix}$$

$$\underline{\mathbf{M}}_{F} = \underline{\mathbf{C}}^{T}\ \underline{\mathbf{M}}_{Fm} = 39.375 \text{ kN.m}$$

$$[\mathbf{S}] = \underline{\mathbf{C}}^{T}\left[\mathbf{S}_{m}\right]\underline{\mathbf{C}} = \text{EI}\,(1.042) \text{ kN/m}$$

The equilibrium equation is $M_C = 0$, which gives

$$\underline{\mathbf{M}} = \underline{\mathbf{M}}_{F} + [\mathbf{S}]\mathbf{D}$$

$$0 = 39.375 + 1.042\,\text{EI}\,\theta_{C}$$

The solution for θ_C gives $\theta_{C} = -37.788/\text{EI}$ rad.

The member end moments are obtained from

$$\underline{\mathbf{M}} = \underline{\mathbf{M}}_{F} = \underline{\mathbf{M}}_{Fm} + \left[\mathbf{S}_{m}\right]\underline{\mathbf{C}}\,\underline{\mathbf{D}}$$

$$\begin{bmatrix} M_{AC} \\ M_{CA} \\ M^{*}_{CB} \end{bmatrix} = \begin{bmatrix} 37.5 \\ -37.5 \\ 76.875 \end{bmatrix} + \text{EI}\begin{bmatrix} 0.667 & 0.333 & 0 \\ 0.333 & 0.667 & 0 \\ 0 & 0 & 0.3751 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\left(\frac{-37.788}{\text{EI}}\right) = \begin{bmatrix} 24.916 \\ -62.7 \\ 62.7 \end{bmatrix} \text{kN.m}$$

which are the same results as in Example 4.26.

4.6.3 Members with Relative Displacement

In order to consider relative displacement, or sidesway, the slope deflection equations (4.7) and (4.8) must be used. According to Figure 4.84, the end moments and shear force in member AB are given by

$$M_{AB} = M_{FAB} + \frac{2\text{EI}}{L}\left(2\theta_{AB} + \theta_{BA} - \frac{3\Delta_{AB}}{L}\right) \qquad (4.124)$$

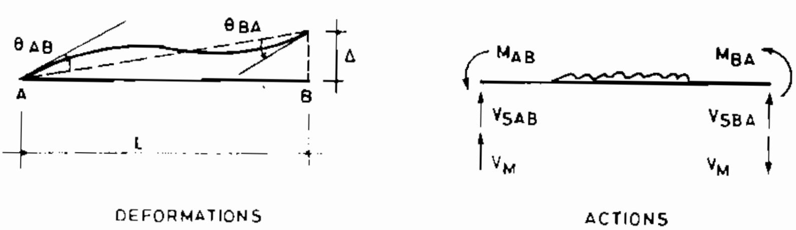

Figure 4.84

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_{BA} + \theta_{AB} - \frac{3\Delta_{BA}}{L}\right) \tag{4.125}$$

$$V_{BA} = V_{SBA} - \left(\frac{M_{AB} + M_{BA}}{L}\right) \tag{4.126}$$

where V_{SBA} is the shear force due to the member loading on simply supported beam AB; and V_M is defined as the shear force due to end moments, which equal $(M_{AB} + M_{BA})/L$.

Substituting Equations 4.124 and 4.125 into Equation 4.126 one obtains

$$V_{BA} = \left[V_{SBA} - \left(\frac{M_{FAB} + M_{FBA}}{L}\right)\right] - \frac{2EI}{L^2}\left(3\theta_{AB} + 3\theta_{BA} - \frac{6\Delta_{AB}}{L}\right)$$

$$= V_{FBA} - \frac{6EI}{L^2}\theta_{AB} - \frac{6EI}{L^2}\theta_{BA} + \frac{12\,EI\,\Delta_{AB}}{L^3} \tag{4.127}$$

One observes that the direction of the shear force V_M due to end moments is in opposite direction to the sign convention of Δ. In order to unify the sign conventions between the actions and deformations, the shear force at B is taken with negative sign to be in the same direction as Δ. Equations 4.124, 4.126, and 4.127 can thus be expressed in a matrix form as

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{BA} \end{bmatrix} = \begin{bmatrix} M_{FAB} \\ M_{FBA} \\ V_{FBA} \end{bmatrix} + \begin{bmatrix} \dfrac{4EI}{L} & \dfrac{2EI}{L} & \dfrac{-6EI}{L^2} \\ \dfrac{2EI}{L} & \dfrac{4EI}{L} & \dfrac{-6EI}{L^2} \\ \dfrac{-6EI}{L^2} & \dfrac{-6EI}{L^2} & \dfrac{12EI}{L^3} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_{BA} \end{bmatrix} \tag{4.128}$$

where $V_{FBA} = V_{SBA} - (M_{FAB} + M_{FBA})/L$.

One should observe that the sign conventions here are the same as in the slope deflection equation method. Positive moments and rotations are anticlockwise. Positive shear force, V, and relative displacement, Δ, are obtained when the right hand side of a member is displaced upward with respect to the left hand side. These signs are again given in Figure 4.85. Therefore, one may use the shear at A instead of that at B where $V_{FAB} = -V_{SAB} - (M_{FAB} + M_{FBA})/L$.

Equation 4.128 is expressed in a matrix form as

$$[\underline{A}_m]_{AB} = [\underline{A}_{Fm}]_{AB} + \underline{S}_{AB}[\underline{D}_m]_{AB} \tag{4.129}$$

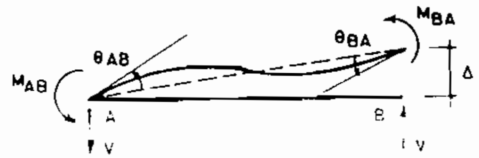

Figure 4.85

in which $\left[\underline{\mathbf{D}}_m\right]_{AB} = \left[\theta_{AB} \quad \theta_{BA} \ \Delta_{BA}\right]^T$; and $\left[\underline{\mathbf{A}}_{Fm}^T\right]_{AB} = \left[M_{FAB} \ M_{FBA} \ \Delta_{FBA}\right]^T$.

Repeating Equation 4.129 for each member in the structure, one obtains a matrix equation as

$$\underline{\mathbf{A}}_m = \underline{\mathbf{A}}_{Fm} + \left[\mathbf{S}_m\right]\underline{\mathbf{D}}_m \tag{4.130}$$

This equation is in a similar form as Equation 4.111 of the previous section. In steps similar to these of section 4.6.1, one can develop the compatibility matrix $\underline{\mathbf{C}}$ and generate an equation similar to Equation 4.119 as follows:

$$\underline{\mathbf{A}} = \underline{\mathbf{A}}_F + \left[\mathbf{S}\right]\underline{\mathbf{D}} \tag{4.131}$$

where $\underline{\mathbf{A}}_F$ and $\left[\mathbf{S}\right]$ are generated from

$$\underline{\mathbf{A}}_F = \underline{\mathbf{C}}^T \, \underline{\mathbf{A}}_{Fm} \tag{4.132}$$

$$\underline{\mathbf{S}} = \underline{\mathbf{C}}^T \left[\mathbf{S}_m\right] \underline{\mathbf{C}} \tag{4.133}$$

The following examples illustrate the application in the case of relative displacement.

Example 4.29

Determine the bending moment and shear force diagrams for the frame shown in Figure 4.86, where member BC is subjected to rise in temperature as shown (EI = 10^5, kN.m^2, $\alpha = 10^{-5}/^{o}$c).

Solution

The members stiffness matrices considering the relative displacements are determined as follows:

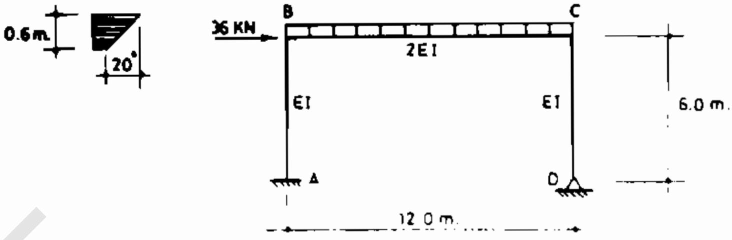

Figure 4.86

$$\underline{S}_{AB} = \underline{S}_{CD} = \begin{bmatrix} \frac{4EI}{6} & \frac{2EI}{6} & \frac{-6EI}{6^2} \\ \frac{2EI}{6} & \frac{4EI}{6} & \frac{-6EI}{6^2} \\ \frac{-6EI}{6^2} & \frac{-6EI}{6^2} & \frac{12EI}{6^3} \end{bmatrix}$$

$$\underline{S}_{BC} = \begin{bmatrix} \frac{8EI}{12} & \frac{4EI}{12} & \frac{-12EI}{12^2} \\ \frac{4EI}{12} & \frac{8EI}{12} & \frac{-12EI}{12^2} \\ \frac{-12EI}{12^2} & \frac{-12EI}{12^2} & \frac{24EI}{12^3} \end{bmatrix}$$

$$\left[S_m\right] = \begin{bmatrix} \underline{S}_{AB} & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_{BC} & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_{CD} \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.667 & 0.333 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.333 & 0.667 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.167 & -0.167 & 0.056 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.667 & 0.333 & -0.083 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.333 & 0.667 & -0.083 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.083 & -0.083 & 0.0139 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.667 & 0.333 & -0.167 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.333 & 0.667 & -0.167 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.167 & -0.167 & 0.055 \end{bmatrix}$$

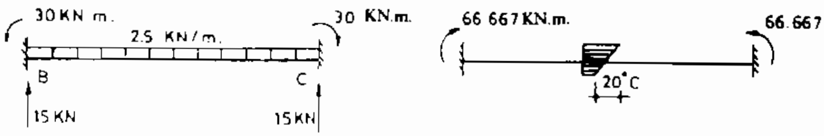

Figure 4.87

The fixed end actions due to the member loading and temperature are determined as shown in Figure 4.87 according to the sign convention of slope deflection equation as follows:

$$M_{FBC} = +30 - 66.667 = -36.667 \text{ kN.m} \quad ; \quad M_{FCB} = +30 + 66.667 = +36.667 \text{ kN.m}$$

$$V_{SCB} = +15 \text{ kN} \quad ; \quad V_{FCB} = V_{SCB} - (M_{FBC} + M_{FCB})/12 = +15 \text{ kN}$$

$$\underline{\mathbf{A}}_{Fm}^{T} = [M_{FAB} \quad M_{FBA} \quad V_{FBA} \quad M_{FBC} \quad M_{FCB} \quad V_{FCB} \quad M_{FCD} \quad M_{FDC} \quad V_{FDC}]$$
$$= [0 \quad 0 \quad 0 \quad -36.667 \quad +36.667 \quad +15 \quad 0 \quad 0 \quad 0\;]$$

In order to develop the compatibility matrix, it is assumed that the frame is deformed as shown in Figure 4.88, indicating the frame has four degrees of freedom. Considering the member coordinates shown in the figure, the compatibility relation is

$$\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_{BA} \\ \theta_{BC} \\ \theta_{CB} \\ \Delta_{CB} \\ \theta_{CD} \\ \theta_{DC} \\ \Delta_{CD} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_D \\ \Delta \end{bmatrix} = \underline{\mathbf{C}}\,\underline{\mathbf{D}}$$

The fixed end action $\underline{\mathbf{A}}_F$ is obtained using Equation 4.132 as follows:

$$\underline{\mathbf{A}}_F = \underline{\mathbf{C}}_T\,\underline{\mathbf{A}}_{Fm} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \underline{\mathbf{A}}_{Fm} = \begin{bmatrix} -36.667 \\ 36.667 \\ 0 \\ 0 \end{bmatrix}$$

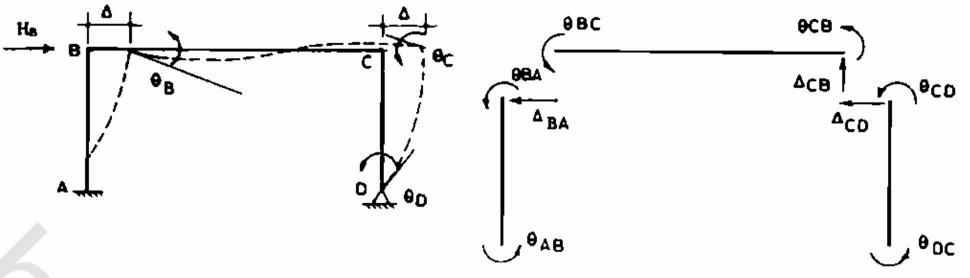

Figure 4.88

The structure stiffness matrix [S] is obtained using Equation 4.133 as follows:

$$[S] = \underline{C}^T [S_m] \underline{C} = EI \begin{bmatrix} 1.333 & 0.333 & 0 & 0.167 \\ 0.333 & 1.333 & 0.333 & 0.167 \\ 0 & 0.333 & 0.167 & 0.167 \\ 0.167 & 0.167 & 0.167 & 0.112 \end{bmatrix}$$

Substituting into Equation 4.131, one obtains

$$\begin{bmatrix} M_B \\ M_C \\ M_D \\ H_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -36.667 \\ 36.667 \\ 0 \\ 0 \end{bmatrix} + EI \begin{bmatrix} 1.333 & 0.333 & 0 & 0.167 \\ 0.333 & 1.333 & 0.333 & 0.167 \\ 0 & 0.333 & 0.667 & 0.167 \\ 0.167 & 0.167 & 0.167 & 0.1112 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_D \\ \Delta \end{bmatrix}$$

The solution gives $\quad \underline{D}^T = \dfrac{1}{EI}[-43.364 \quad -69.455 \quad -141.727 \quad 705.818]$

The member end actions can then be obtained using Equation 4.130 as follows:

$$\underline{A}_m = \underline{A}_{Fm} + [S_m] \, \underline{C} \, \underline{D}$$

The substitution gives

$$\begin{aligned} \underline{A}_m^T &= [M_{AB} \quad M_{BA} \quad V_{BA} \quad M_{BC} \quad M_{CB} \quad V_{CB} \quad M_{CD} \quad M_{DC} \quad V_{DC}\;] \\ &= [103.18 \quad 88.72 \quad -31.984 \quad -88.72 \quad -24.09 \quad 24.4 \quad 24.09 \quad 0 \quad -4.015] \end{aligned}$$

The bending moment and shear force diagrams are given in Figure 4.89 which are the same as the results of Example 4.6.

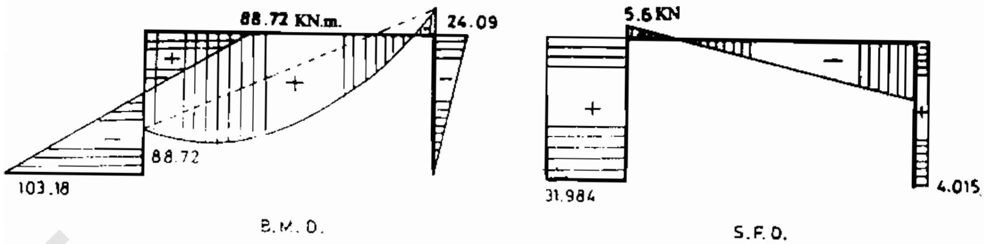

Figure 4.89

Example 4.30

Determine the bending moment and shear force diagrams, and the deformations at B for the beam shown in Figure 4.90 due to the loading on member BC, and the rise in temperature in member AB. (EI = 10^5 kN.m^2, α = 10^{-5}/°c, spring constant K = 10 kN.cm).

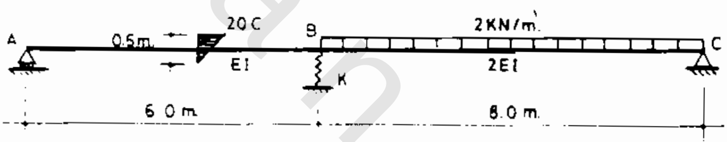

Figure 4.90

Solution

Because of the spring at B, the deformed shape is expected to be as shown in Figure 4.91. The degree of freedom is thus four, which represents θ_A, θ_B, θ_C, and Δ_B. The member coordinates are also shown in Figure 4.91.

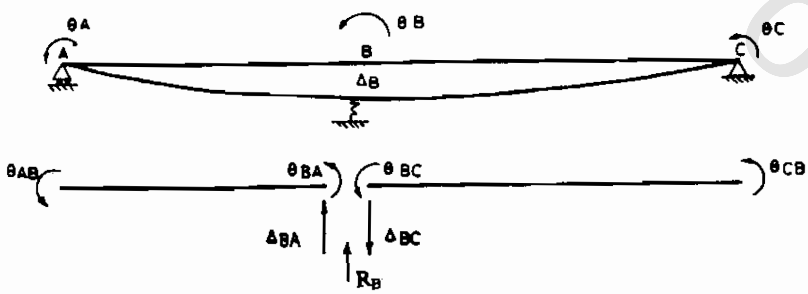

Figure 4.91

The members stiffness matrices are

$$\underline{S}_{AB} = \begin{bmatrix} \frac{4EI}{6} & \frac{2EI}{6} & \frac{-6EI}{6^2} \\ \frac{2EI}{6} & \frac{4EI}{6} & \frac{-6EI}{6^2} \\ \frac{-6EI}{6^2} & \frac{-6EI}{6^2} & \frac{12EI}{6^3} \end{bmatrix} \qquad \underline{S}_{BC} = \begin{bmatrix} \frac{8EI}{8} & \frac{4EI}{8} & \frac{-12EI}{8^2} \\ \frac{4EI}{8} & \frac{8EI}{8} & \frac{-12EI}{8^2} \\ \frac{-12EI}{8^2} & \frac{-12EI}{8^2} & \frac{24EI}{8^3} \end{bmatrix}$$

$$\left[S_m\right] = EI \begin{bmatrix} 0.667 & 0.333 & -0.167 & 0 & 0 & 0 \\ 0.333 & 0.667 & -0.167 & 0 & 0 & 0 \\ -0.167 & -0.167 & 0.0555 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.5 & -0.1875 \\ 0 & 0 & 0 & 0.5 & 1 & -0.1875 \\ 0 & 0 & 0 & -0.1875 & -0.1875 & 0.046875 \end{bmatrix}$$

The members fixed end actions are obtained from the tables for the cases shown in Figure 4.92 and expressed according to the signs of the slope deflection equation as follows:

$$\underline{A}_m^T = [M_{FAB} \quad M_{FBA} \quad V_{FBA} \quad M_{FBC} \quad M_{FCB} \quad V_{FBC}]$$
$$= [-33.333 \quad 33.333 \quad 0 \quad 10.667 \quad -10.667 \quad -8 \;]$$

where $V_{FBC} = -V_{SBC} - \left(M_{FBC} + M_{FCB}\right)/8 = -8$ kN.

The compatibility relation can be developed according to Figure 4.91 as follows:

$$\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta_{BA} \\ \theta_{BC} \\ \theta_{CB} \\ \Delta_{BC} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_D \\ \Delta \end{bmatrix} = \underline{C}\,\underline{D}$$

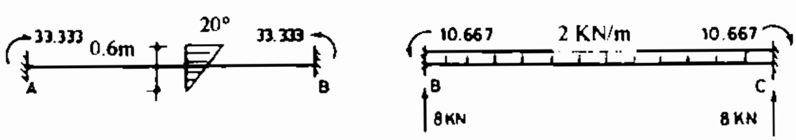

Figure 4.92

The fixed end action $\underline{\mathbf{A}}_F$ can be obtained using Equation 4.132 as follows:

$$\underline{\mathbf{A}}_F = \underline{\mathbf{C}}^T \, \underline{\mathbf{A}}_{Fm} = [-33.333 \quad 44 \quad -10.667 \quad -8]^T$$

The structural stiffness matrix [S] is determined using Equation 4.133 as follows:

$$[S] = \underline{\mathbf{C}}^T [S_m] \underline{\mathbf{C}} = EI \begin{bmatrix} 0.667 & 0.333 & 0 & 0.167 \\ 0.333 & 1.667 & 0.5 & -0.0208 \\ 0 & 0.5 & 0.1 & -0.1875 \\ 0.167 & -0.0208 & -0.1875 & 0.10242 \end{bmatrix}$$

Substituting into Equation 4.131, one obtains

$$\begin{bmatrix} M_A \\ M_B \\ M_C \\ -R_B \end{bmatrix} = \begin{bmatrix} -33.333 \\ 44 \\ -10.667 \\ -8 \end{bmatrix} + EI \begin{bmatrix} 0.667 & 0.333 & 0 & 0.167 \\ 0.333 & 1.667 & 0.5 & -0.0208 \\ 0 & 0.5 & 1 & -0.1875 \\ 0.167 & -0.0208 & -0.1875 & 0.102425 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \Delta_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1000\Delta_B \end{bmatrix}$$

The negative sign of R_B is due to its opposite direction to Δ_B. Rearranging the equations one obtains

$$EI = \begin{bmatrix} 0.667 & 0.333 & 0 & 0.167 \\ 0.333 & 1.667 & 0.5 & -0.0208 \\ 0 & 0.5 & 1 & -0.1875 \\ 0.167 & -0.0208 & -0.1875 & 0.112425 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \Delta_B \end{bmatrix} = \begin{bmatrix} 33.333 \\ -44 \\ 10.667 \\ +8 \end{bmatrix}$$

The solution gives : $\underline{\mathbf{D}}^T = \dfrac{1}{EI}[68.41 \quad -52.538 \quad 42.829 \quad 31.4366]$

The member end actions are thus determined from Equation 4.130 as follows:

$$\underline{\mathbf{A}}_m = \underline{\mathbf{A}}_{Fm} + [S_m] \, \underline{\mathbf{C}} \, \underline{\mathbf{D}}$$
$$= [M_{AB} \quad M_{BA} \quad V_{BA} \quad M_{BC} \quad M_{CB} \quad V_{BC}]$$

$$\underline{\mathbf{A}}_m^T = [0 \quad 26.35 \quad -4.39 \quad -26.35 \quad 0 \quad -4.71]$$

The bending moment and shear force diagrams are given in Figure 4.93, which are the same as in Example 4.5.

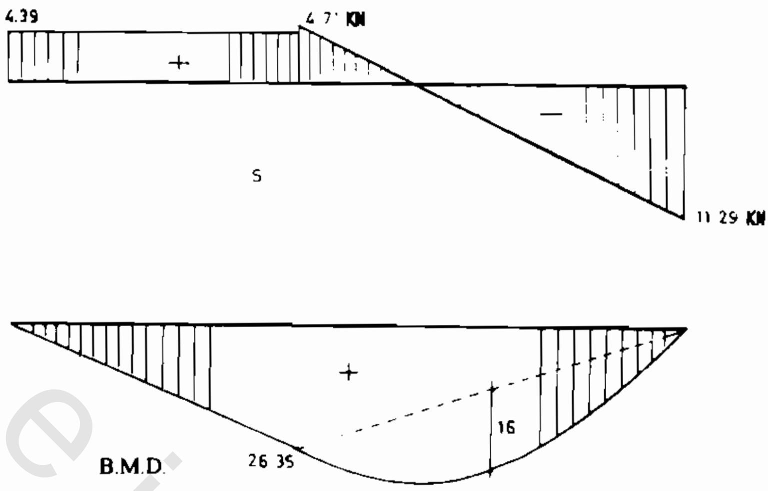

Figure 4.93

4.6.4 Relative Displacements with Moments Releases

If member AB, for example, is hinged at the end support B and is subjected to a relative displacement between A and B, the slope deflection equation is given by Equation 4.15 as

$$M^{*}_{AB} = M^{*}_{FAB} - \frac{3EI}{L}\theta_{AB} - \frac{3EI}{L^{2}}\Delta_{BA} \tag{4.134}$$

The shear force V_{BA} can be calculated using Figure 4.94, as follows:

$$\begin{aligned} V^{*}_{BA} &= V_{SBA} - V_{M} \\ &= V_{SBA} - M^{*}_{AB}/L \\ &= \left(V_{SBA} - M^{*}_{FAB}/L\right) - \frac{3EI}{L^{2}}\theta_{AB} + \frac{3EI}{L^{3}}\Delta_{BA} \end{aligned} \tag{4.135}$$

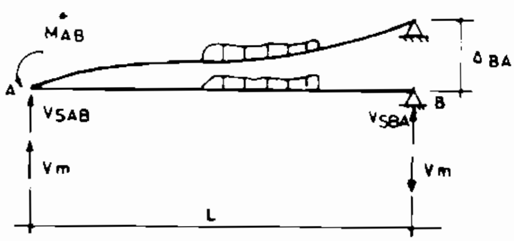

Figure 4.94

Equations 4.134 and 4.135 may be combined in a matrix form as

$$\begin{bmatrix} M^{*}_{AB} \\ V^{*}_{BA} \end{bmatrix} = \begin{bmatrix} M^{*}_{FAB} \\ V^{*}_{FBA} \end{bmatrix} \begin{bmatrix} \dfrac{3EI}{L} & \dfrac{-3EI}{L^2} \\ \dfrac{-3EI}{L^2} & \dfrac{3EI}{L^3} \end{bmatrix} \begin{bmatrix} \theta_{AB} \\ \Delta_{BA} \end{bmatrix} \qquad (4.136)$$

where $V^{*}_{FBA} = V_{SBA} - M^{*}_{FAB} / L$. One can use $V^{*}_{FAB} = -V_{SAB} - M^{*}_{FAB} / L$ instead of V^{*}_{FBA}.

By using Equation 4.136 one can save the determination of the rotation at the hinged support. The steps of solution are the same as in the previous sections and are illustrated in the following example:

Example 4.31

Solve Example 4.29 by using the modified stiffness matrix method due to the moments releases at supports A and C.

Solution

In this example, only θ_B and Δ_B need to be determined, if the modified stiffness matrices are used. The directions of these variables are assumed in Figure 4.91. The modified member stiffness matrices for members AB and BC are given by

$$\underline{S}_{AB} = \begin{bmatrix} \dfrac{3EI}{6} & \dfrac{-3EI}{6^2} \\ \dfrac{-3EI}{6^2} & \dfrac{3EI}{6^3} \end{bmatrix} \quad ; \quad \underline{S}_{BC} = \begin{bmatrix} \dfrac{EI}{8} & \dfrac{-6EI}{8^2} \\ \dfrac{-6EI}{8^2} & \dfrac{6EI}{8^3} \end{bmatrix}$$

$$[S_m] = EI \begin{bmatrix} 0.5 & -0.0833 & 0 & 0 \\ -0.0833 & 0.0138 & 0 & 0 \\ 0 & 0 & 0.75 & -0.09385 \\ 0 & 0 & -0.09375 & 0.011718 \end{bmatrix}$$

The compatibility relation is obtained, according to Figure 4.91, as follows:

$$\begin{bmatrix} \theta_{BA} \\ \Delta_{BA} \\ \theta_{BC} \\ \Delta_{BC} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_B \\ \Delta_B \end{bmatrix} = \underline{C}\,\underline{D}$$

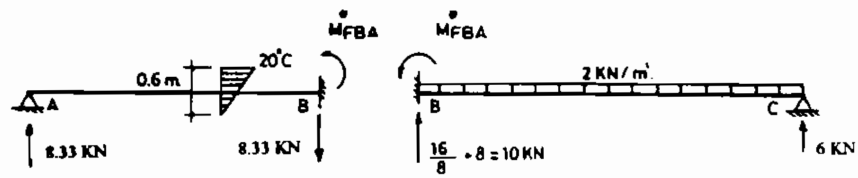

Figure 4.95

The fixed end moments are determined for the cases of loading shown in Figure 4.95, using the tables and are put according to the signs of the slope deflection equation as follows:

$$M^{*}_{FBC} = \frac{2 \times 8^{2}}{8} = 16 \text{ kN.m}$$

The fixed end actions are then obtained using Figure 4.95 as follows:

$$\begin{aligned}\underline{\mathbf{A}}^{T}_{Fm} &= [M^{*}_{FBA} \quad V^{*}_{FBA} \quad M^{*}_{FBC} \quad V^{*}_{FBC}] \\ &= [\ 50 \quad -8.333 \quad 16 \quad -10\ ]\end{aligned}$$

$$\underline{\mathbf{A}}^{T}_{F} = \left[\underline{\mathbf{C}}^{T}\underline{\mathbf{A}}_{Fm}\right]^{T} = [66 \quad -1.667]$$

The structural stiffness matrix is determined from Equation 4.133 as follows:

$$[\mathbf{S}] = \underline{\mathbf{C}}^{T}\left[\mathbf{S}_{m}\right]\underline{\mathbf{C}} = EI\begin{bmatrix} 1.25 & -0.01042 \\ -0.01042 & 0.025598 \end{bmatrix}$$

Substituting into Equation 4.131, one obtain

$$\begin{bmatrix} M_{B} \\ -R_{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -1000\Delta_{B} \end{bmatrix} = \begin{bmatrix} 66 \\ -1.667 \end{bmatrix} + EI\begin{bmatrix} 1.25 & -0.01042 \\ 0.01042 & 0.025598 \end{bmatrix}\begin{bmatrix} \theta_{B} \\ \Delta_{B} \end{bmatrix}$$

The solution for $\underline{\mathbf{D}}$ is obtained as

$$\begin{bmatrix} \theta_{B} \\ \Delta_{B} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -52.538 \\ 31.4366 \end{bmatrix}$$

The member end actions are determined from

$$\begin{aligned}\underline{\mathbf{A}}_{m} &= \underline{\mathbf{A}}_{Fm} + \left[\mathbf{S}_{m}\right]\underline{\mathbf{C}}\,\underline{\mathbf{D}} \\ &= \left[M^{*}_{BA} \quad V^{*}_{BA} \quad M^{*}_{BC} \quad M^{*}_{BC}\right] = [26.35 \quad -4.39 \quad -26.35 \quad -4.71]\end{aligned}$$

which are the same results as in Example 4.29.

4.6.5 Nonprismatic Members

In steps similar to those in the previous sections, and according to Equations 4.46 and 4.47 of section 4.3.5, one may write the slope-deflection equations considering moment only in a general matrix form as follows:

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} M_{FAB} \\ M_{FBA} \end{bmatrix} + \frac{E}{L}\begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix}\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \end{bmatrix} \tag{4.137}$$

In the presence of relative displacements, one uses the matrix equation

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ V_{BA} \end{bmatrix} = \begin{bmatrix} M_{FAB} \\ M_{FBA} \\ V_{FBA} \end{bmatrix} + \frac{E}{L}\begin{bmatrix} K_{AA} & K_{AB} & -\left(\frac{K_{AA}+K_{AB}}{L}\right) \\ K_{BA} & K_{BB} & -\left(\frac{K_{BA}+K_{BB}}{L}\right) \\ -\left(\frac{K_{AA}+K_{AB}}{L}\right) & -\left(\frac{K_{BA}+K_{BB}}{L}\right) & \left(\frac{K_{AA}+K_{BB}+K_{AB}+K_{BA}}{L^2}\right) \end{bmatrix}\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BA} \end{bmatrix} \tag{4.138}$$

In case of a non-prismatic member AB with a moment released at B, Equation 4.137 and 4.138 become, respectively,

$$M^*_{AB} = M^*_{FAB} + \frac{E}{L}\left[K^*_{AA}\right]\theta_{AB} \tag{4.139}$$

$$\begin{bmatrix} M^*_{AB} \\ V^*_{BA} \end{bmatrix} = \begin{bmatrix} M^*_{FAB} \\ V^*_{FBA} \end{bmatrix} + \frac{E}{L}\begin{bmatrix} K^*_{AA} & -\frac{K^*_{AA}}{L} \\ -\frac{K^*_{AA}}{L} & \frac{K^*_{AA}}{L^2} \end{bmatrix}\begin{bmatrix} \theta_{AB} \\ \Delta_{AB} \end{bmatrix} \tag{4.140}$$

where $K_{AA}, K_{AB}, K_{BA}, K_{BB}$, and K^*_{AA} are as defined in sections 4.3.5 and 4.3.6.

The procedure here is exactly the same as in the previous sections. The extra efforts in the calculations consist of the determination of stiffness coefficients and fixed end moments for the nonprismatic members. Example 4.11 is solved again next by using the matrix approach.

Example 4.32

Determine the bending moment diagram for the beam of Example 4.11 by using the stiffness matrix method, approach I.

Solution

The stiffness coefficients are determined as in Example 4.11 and given by

$$[K_{AA}]_{AB} = [K_{BB}]_{BC} = 0.04037$$

$$K_{AB} = K_{BA} = K_{BC} = K_{CB} = 0.0251833$$

$$[K_{BB}]_{BA} = K_{CC} = 0.04037$$

The stiffness matrices for members AB, and BC are obtained as

$$\underline{S}_{AB} = \frac{E}{12}\begin{bmatrix} 0.04037 & 0.02518 \\ 0.02518 & 0.04037 \end{bmatrix} = \underline{S}_{BC}$$

$$[S_m] = \frac{E}{12} = \begin{bmatrix} 0.04037 & 0.02518 & 0 & \\ 0.02518 & 0.04037 & 0 & \\ 0 & 0 & 0.04037 & 0.02518 \\ 0 & 0 & 0.02518 & 0.04037 \end{bmatrix}$$

The fixed end moments have previously been determined as follows:

$$\underline{M}_{Fm}^{T} = [18.04 \quad -18.04 \quad 18.04 \quad -18.04]$$

The compatibility relations is

$$\begin{bmatrix} \theta_{AB} \\ \theta_{BA} \\ \theta_{BC} \\ \theta_{CB} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \underline{C}\,\underline{D}$$

The fixed end moment $\underline{M}_F$ is obtained from

$$\underline{M}_F = \underline{C}^T\,\underline{M}_{Fm} = \begin{bmatrix} 0 \\ -18.04 \end{bmatrix}$$

The structural stiffness matrix [S] is obtained from

$$[S] = \underline{C}^T [S_m] \underline{C}$$

$$[S] = \frac{E}{12}\begin{bmatrix} 0.08074 & 0.02518 \\ 0.02518 & 0.04037 \end{bmatrix}$$

Substituting into Equation 4.131 one obtains

$$\underline{M} = \underline{M}_{Fm} + [S]\,\underline{D}$$

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -18.04 \end{bmatrix} + \frac{E}{12}\begin{bmatrix} 0.08074 & 0.02518 \\ 0.02518 & 0.04037 \end{bmatrix}\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

The solution of this equation gives

$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \frac{1}{E}\begin{bmatrix} -2076.1 \\ 6656.836 \end{bmatrix}$$

The end moments are thus obtained from $\underline{M}_m = M_{Fm} + [S_m]\,\underline{C}\,\underline{D}$ which gives

$$[M_{AB} \quad M_{BA} \quad M_{BC} \quad M_{CB}] = [13.688 \quad -25.02 \quad 25.02 \quad 0]$$

which are the same results as in Example 4.11. One can also solve this problem by the modified stiffness matrix for member BC using Equation 4.139. In this case, only θ_B need to be determined.

4.7 THE STIFFNESS MATRIX METHOD : APPROACH II

Stiffness matrix method approach II is a more general approach which depends basically on finding the relationship between external actions and the free deformations using the unit displacement method. For simple types of structures, this approach can easily be used, where the structure is made to be kinematically determinate and each degree of freedom is subjected, in turn, to a unit displacement. The restraining action's obtained are the stiffness coefficients which are arranged to form the stiffness matrix. This approach has briefly been mentioned in Chapter 2, section 2.16, when talking about the applications of the unit displacement method.

For complicated structures like space frames, or multistorey frames, it is very difficult to use the above-mentioned method in developing the relationships between the external actions and the free deformations. In such cases, the element method is more general and can be used. This method shall be given in more detail in Chapter 5. The philosophy of this method is to develop the stiffness relationship for every member and then augment these relationships using the compatibility and equilibrium conditions at the free joints to obtain finally the structure stiffness matrix. This method is very systematic and suitable for computer programming.

In the following are few examples to show the applications of the unit displacement method for simple types structures.

Example 4.33

Determine the free deformations at the joints for the frame shown in Figure 4.96 (Example 4.27) using the stiffness matrix method approach II (EI = 10^5 kN.m^2, α = 10^{-5}/°c).

Solution

The degree of freedom of the structure is 4, which represents the joints rotations θ_B, θ_C, θ_D, and the horizontal sway Δ. The frame is made to be kinematically determinate, by fixing all joints. Every free deformation is given a unit value as shown in Figure 4.97. The corresponding restraining forces and moments are calculated and arranged to form the structure stiffness matrix as shown below.

$$\begin{bmatrix} M_B \\ M_C \\ M_D \\ M_B \end{bmatrix} = \begin{bmatrix} \left(\frac{8EI}{12}+\frac{4EI}{6}\right) & \frac{4EI}{12} & 0 & \frac{6EI}{6^2} \\ \frac{4EI}{12} & \left(\frac{8EI}{12}+\frac{4EI}{6}\right) & \frac{2EI}{6} & \frac{6EI}{6^2} \\ 0 & \frac{2EI}{6} & \frac{4EI}{6} & \frac{6EI}{6^2} \\ \frac{6EI}{6^2} & \frac{6EI}{6^2} & \frac{6EI}{6^2} & \frac{24EI}{6^3} \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_D \\ \Delta \end{bmatrix}$$

$$= EI \begin{bmatrix} 1.333 & 0.333 & 0 & 0.167 \\ 0.333 & 1.333 & 0.333 & 0.167 \\ 0 & 0.333 & 0.667 & 0.167 \\ 0.167 & 0.167 & 0.167 & 0.1112 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \theta_D \\ \Delta \end{bmatrix}$$

which is the same matrix determined in Example 4.27.

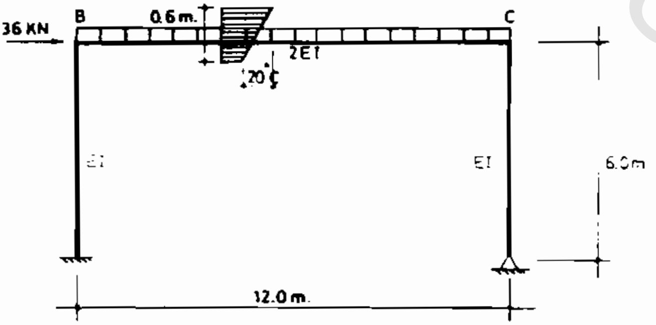

Figure 4.96

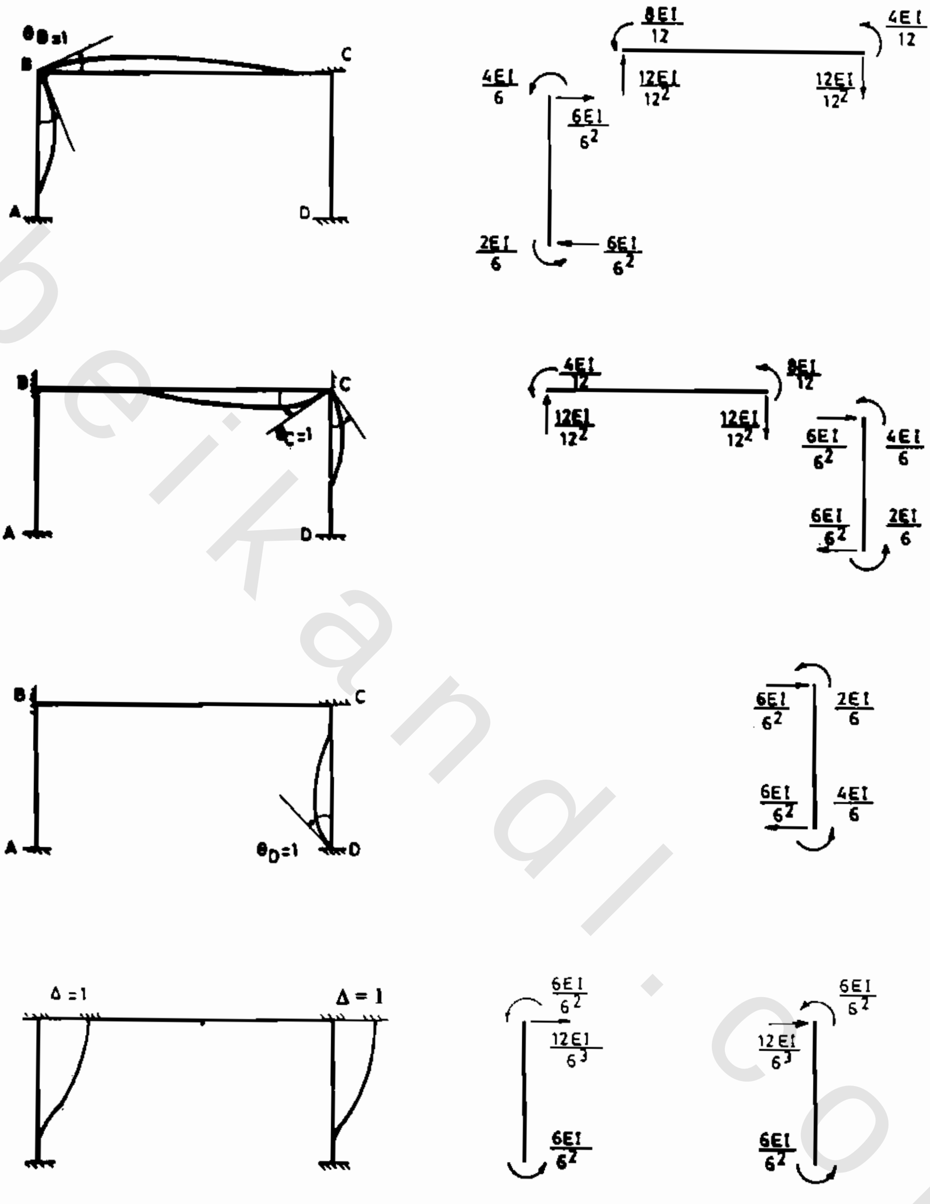

Figure 4.97

The external actions at the free joints include the actions applied at these joints and the equivalent joint actions due to the member loadings. The negative of the fixed end actions provides the equivalent joint actions. In this example, the actions applied at the free joints in the direction of the free deformations are

$$\begin{bmatrix} M_B \\ M_C \\ M_D \\ H_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 36 \end{bmatrix} - \begin{bmatrix} M_{FB} \\ M_{FC} \\ M_{FD} \\ H_{FB} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 36 \end{bmatrix} - \begin{bmatrix} -36.667 \\ 36.667 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} +36.667 \\ -36.667 \\ 0 \\ 36 \end{bmatrix}$$

Solving the above equation for the free displacements one obtains

$$\begin{bmatrix} \theta_B & \theta_C & \theta_D & \Delta \end{bmatrix} = 10^{-5}\begin{bmatrix} -43.364 & -69.455 & -141.727 & 705.818 \end{bmatrix}$$

which are the same results obtained in Example 4.27.

The members end actions are obtained from the substitution into the actions – deformations relationship for each member. In this example, the slope deflection equations can be used.

Example 4.34

Develop the stiffness relationship between the external actions and the free deformations for the beam shown in Figure 4.98 (EI = 10^5 kN.m^2, K = 10 kN/cm).

Solution

The beam has four degrees of freedom which are represented by θ_A, θ_B, θ_C and Δ_B. The stiffness matrix relation can be obtained by applying a unit deformation at every

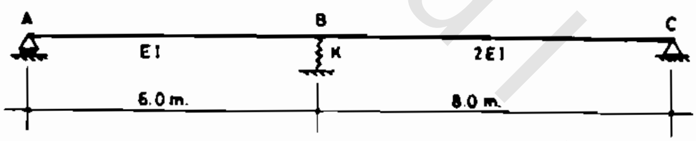

Figure 4.98

degree of freedom, in turn, for the kinematic determinate structure as shown in Figure 4.99. The stiffness matrix relation can be arranged as follows:

$$\begin{bmatrix} MA \\ M_B \\ M_C \\ M_B \end{bmatrix} = \begin{bmatrix} \frac{4EI}{6} & \frac{2EI}{6} & 0 & \frac{6EI}{6^2} \\ \frac{2EI}{6} & \left(\frac{4EI}{6}+\frac{8EI}{8}\right) & \frac{4EI}{8} & \left(\frac{6EI}{6^2}-\frac{12EI}{8^2}\right) \\ 0 & \frac{4EI}{8} & \frac{8EI}{8} & \frac{-12EI}{8^2} \\ \frac{6EI}{6^2} & \left(\frac{6EI}{6^2}-\frac{12EI}{8^2}\right) & \frac{-12EI}{8^2} & \left(\frac{12EI}{6^3}+\frac{24EI}{8^3}\right) \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \Delta_B \end{bmatrix}$$

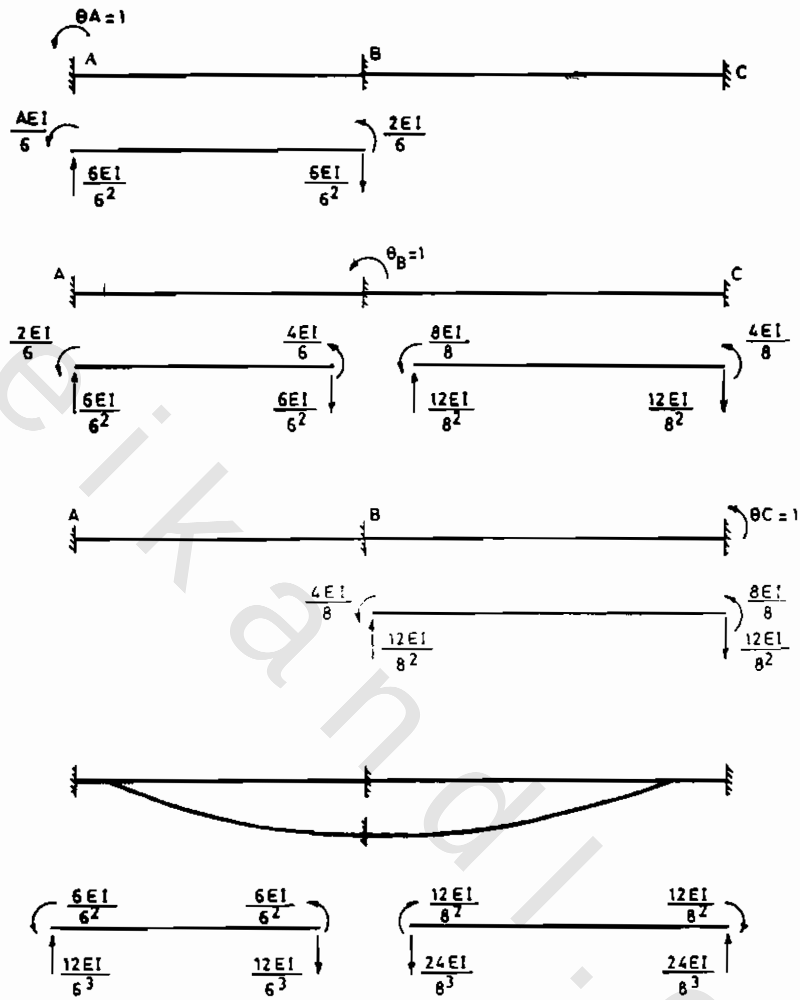

Figure 4.99

which can be expressed as

$$\begin{bmatrix} M_A \\ M_B \\ M_C \\ H_B \end{bmatrix} = EI \begin{bmatrix} 0.667 & 0.333 & 0 & 0.167 \\ 0.333 & 1.667 & 0.5 & -0.0208 \\ 0 & 0.5 & 1.0 & -0.1875 \\ 0.167 & -0.0208 & -0.1875 & 0.102425 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \Delta_B \end{bmatrix}$$

which is the same relation obtained in Example 4.28. The equivalent actions are the joint actions minus the fixed end actions, which give

$$\begin{bmatrix} M_A \\ M_B \\ M_C \\ H_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1000\Delta \end{bmatrix} - \begin{bmatrix} M_{FA} \\ M_{FB} \\ M_{FC} \\ R_{FB} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1000\Delta \end{bmatrix} - \begin{bmatrix} -33.333 \\ 44 \\ -10.667 \\ -8 \end{bmatrix} = \begin{bmatrix} 33.333 \\ -44 \\ 10.667 \\ 8.1000\Delta \end{bmatrix}$$

Exercises

1. Determine the bending moment and shear force diagrams for the beam shown in Figure 1, using the Moment-Distribution Method. ($E = 25 \times 10^6$ kN/m^2, $I = 0.0054$ m^4).

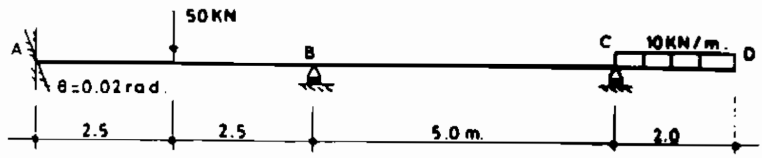

Figure 1

2. Draw the bending moment diagram for the frame shown in Figure 2. Use the Moment-Distribution Method. ($EI = 10 \times 10^4$ kN.m^2).

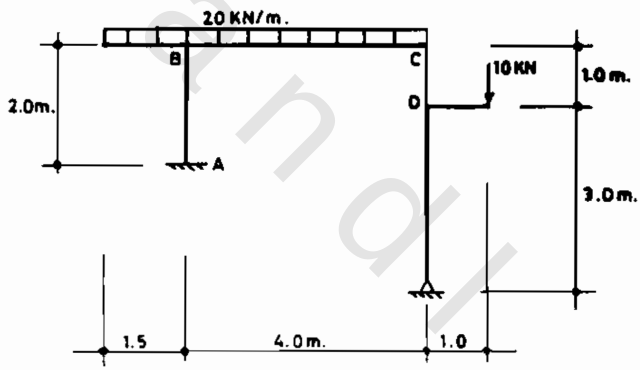

Figure 2

3. Establish the slope deflection equations, and the equilibrium conditions, for the frame shown in Figure 3. ($EI = 1.32 \times 10^5$ kN.m^2).

4. Draw the bending moment diagram for the frame shown in Figure 4. Use the slope deflection equation method. ($E = 30 \times 10^6$ kN/m^2; $I_1 = 5.4 \times 10^{-3}$ m^4; $I_2 = 1.6 \times 10^{-3}$ m^4).

5. Use the slope deflection equation method to determine the bending moment diagram for the frame shown in Figure 5. (EI = constant).

6. Determine the bending moment and shear force diagrams for the frame shown in Figure 6 using the moment distribution method. ($E = 30 \times 10^{-6}$ kN/m^2, I = constant).

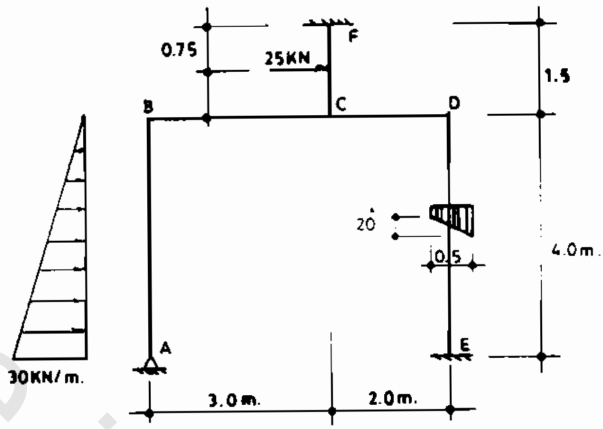

Figure 3

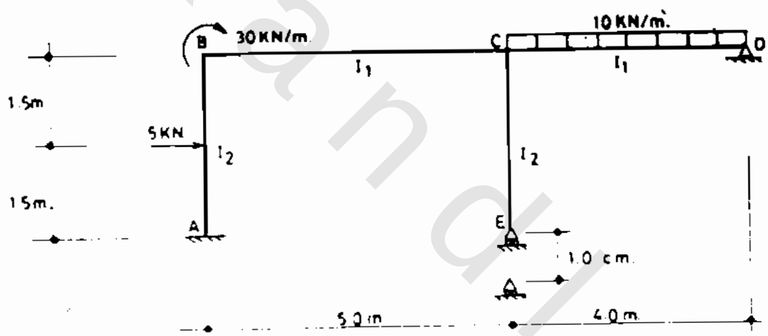

Figure 4

7. Analyze the rigid frame shown in Figure 7 by the moment distribution method. Draw the shear force and bending moment diagrams. (E = constant).

8. Solve Problem 2 by the moment distribution method.

9. Solve Problem 3 by the slope deflection equation method.

10. (a) Find the bending moment diagram for the beam shown in Figure 8 due to a settlement of B 2" and at C 1" down. (Consider EI = 3000 Kft^2 and use slope deflection equation).

    (b) Considering a concentrated load at mid-span of AB of magnitude 10 kips, determine the bending moment and shear force diagrams.

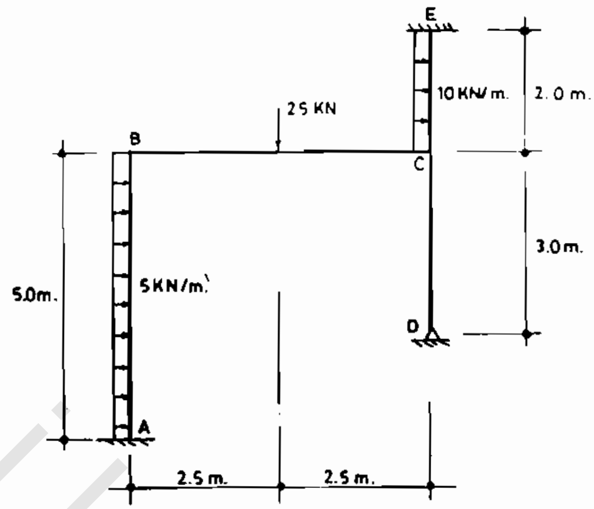

Figure 5

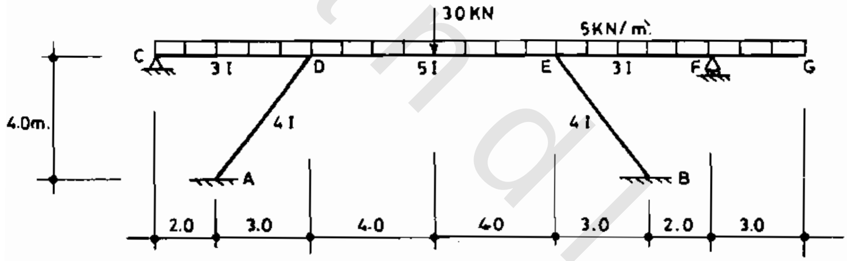

Figure 6

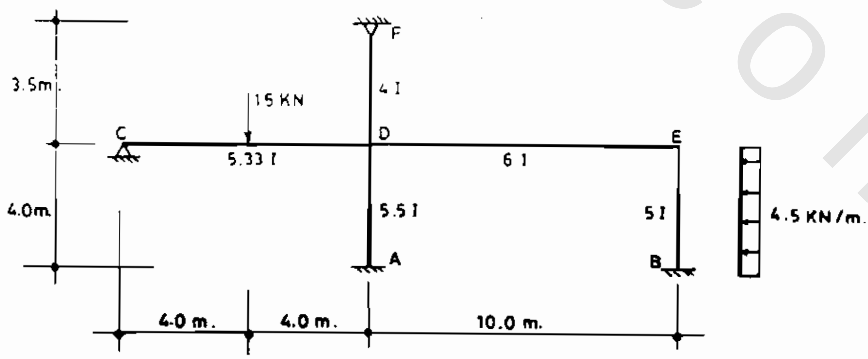

Figure 7

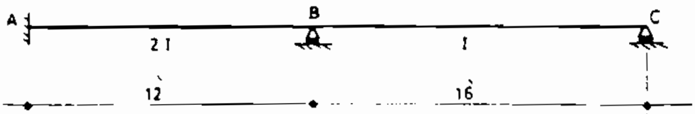

Figure 8

11. Determine the bending moment and shear force diagrams for the frame shown in Figure 9 using the method of slope deflection equation method. (EI = constant).

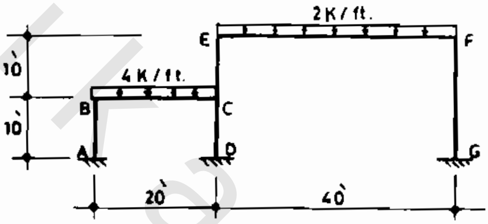

Figure 9

12. Using the slope deflection equation method, write the necessary equations to analyze the frame shown in Figure 10. (EI = constant for all members).

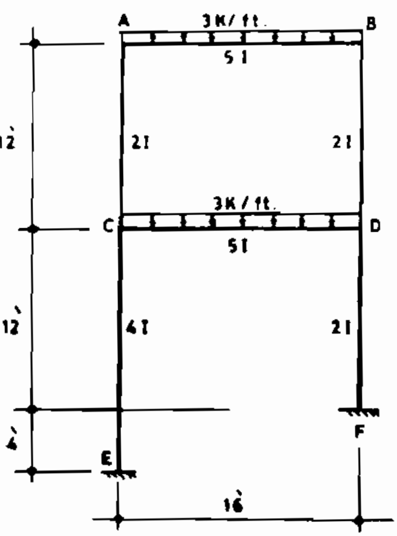

Figure 10

12. Determine the bending moment diagram and sketch the deformed shape for the frame shown in Figure 11 using any of the stiffness methods.

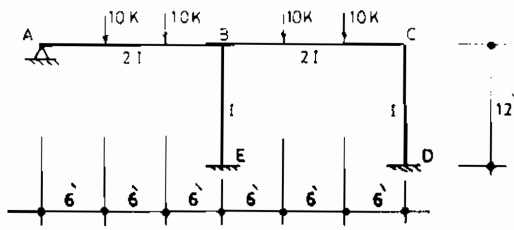

Figure 11

14. Determine the stiffness factors and carry over factors for the beams shown in Figure 12.

Figure 12

15. Using any of the stiffness methods, determine the bending moment diagram for the beam shown in Figure 13, where all members have a rise in temperature as shown. (EI = 30000 K.ft^2, $\alpha = 6.5 \times 10^{-6}$/° F).

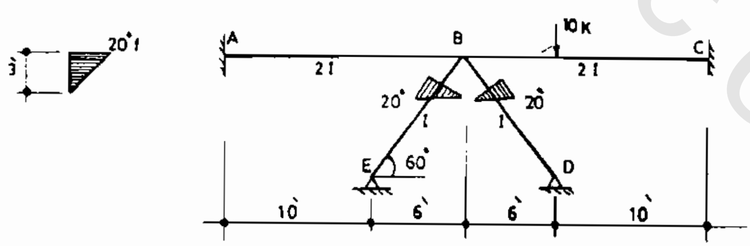

Figure 13

16. Using the method of moment distribution, determine the bending moment and shear force diagrams for the beam shown in Figure 14. (EI = constant).

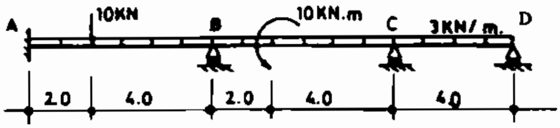

Figure 14

17. Using the moment distribution method, determine the bending moment diagram for the frame shown, in Figure 15.

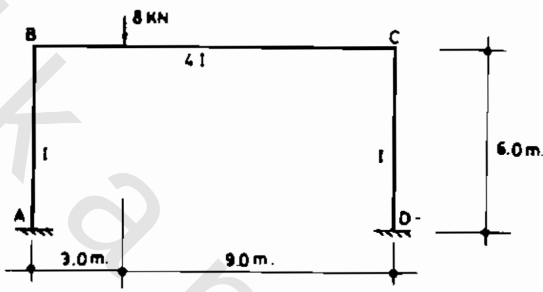

Figure 15

18. The continuous beam ABCD is subjected to the loads shown in Figure 16 and settlement at B of 0.5 inch downward. Determine the bending moment diagram using the slope deflection equations. (EI = 10000 K.ft^2).

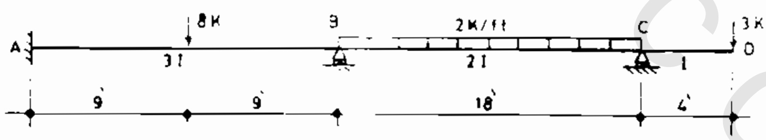

Figure 16

19. The frame ABCD shown in Figure 17 was subjected to settlement at D of 0.5 inch down. Use the slope deflection equations to determine the bending moment diagram. (EI = 10000 K.ft^2).

20. Use the slope deflection equation method or the moment distribution method to determine the bending moment diagram for the beam shown in Figure 18. (EI = 3000 K.ft^2).

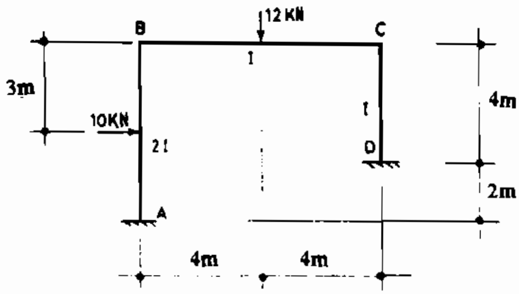

Figure 17

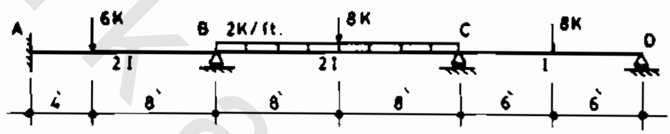

Figure 18

21. Using the slope deflection equation method, determine the bending moment diagram and the deformed shape for the frame shown in Figure 19. (EI = 3000 Kft^2).

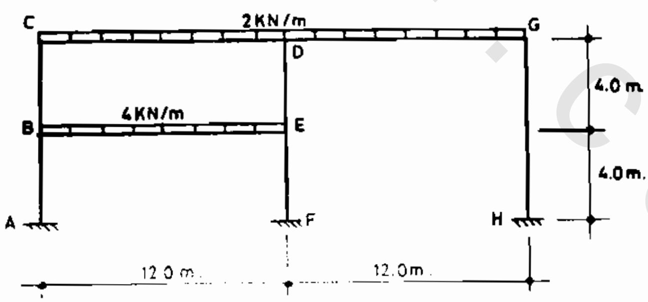

Figure 19

22. (a) Determine the number of kinematic unknowns in the frame shown in Figure 20.

(b) Write the equilibrium equations needed to determine these unknowns.

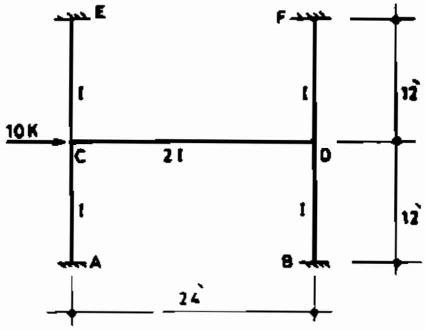

Figure 20

23. (a) Determine the fixed end moments, stiffness factors, and the carry-over factors for the beam shown in Figure 21.

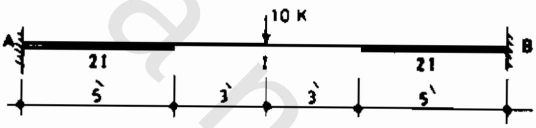

Figure 21

(b) Use the moment distribution to determine the bending moment diagram shown in Figure 22.

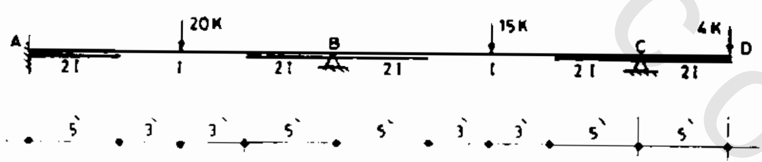

Figure 22

24. The continuous beam ABCD shown in Figure 23 is subjected to the given loading, clockwise rotation at A of 0.002 radians, rise in temperature in member AB, and settlement at B of 0.50 inch down. Use the slope deflection equations or moment distribution to determine the bending moment diagrams for the beam. ($EI = 10{,}000\ K.ft^2$, $\alpha = 6.5 \times 10^{-6}/°F$).

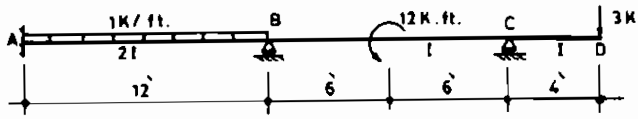

Figure 23

25. (a) Using the method of column analogy, determine the stiffness factors, carry-over factors, and the fixed end moments for the beam AB shown in Figure 24.

(b) Determine the bending moment diagram for the beam ABC shown using the moment distribution method (EI = 30000 K.ft^2).

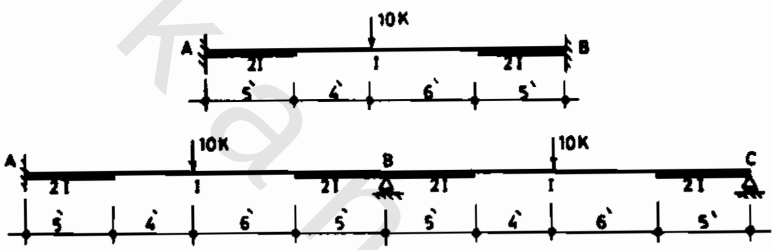

Figure 24

26. Using the moment distribution method, determine the bending moment diagram for the beam shown in Figure 25.

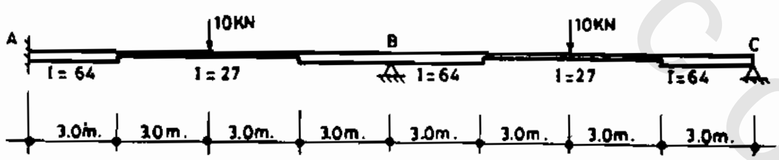

Figure 25

27. Using the stiffness matrix method, determine the bending moment and shear force diagrams for the beam shown in Figure 26. (EA = 6×10^{-6} kN, EI = 10^5 kN.m^2).

28. Using the stiffness matrix method, determine all member forces in the truss shown in Figure 27. (EA = 10^6 kN).

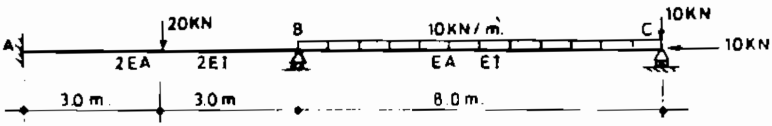

Figure 26

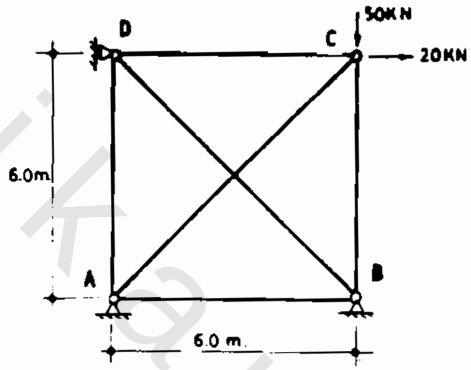

Figure 27

29. Using the stiffness matrix method, determine the bending moment and axial force diagrams in the frame shown in Figure 28. (EI = 10^5 kN.m^2, EA = 10^6 kN).

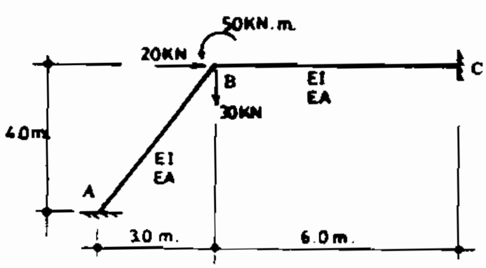

Figure 28