CHAPTER 5

MATRIX STRUCTURAL ANALYSIS

5.1 INTRODUCTION

In this chapter, the analysis of structures by stiffness matrix method approach II which is also called the direct stiffness matrix method is given in detail. As was pointed out in Chapter 4, the stiffness matrix method approach II depends on the members, or elements stiffness matrices which can be augmented to obtain the structure stiffness matrix. The augmentation is done through applying the equilibrium and compatibility conditions at every joint (node) in the structure. To ease and automate the process of augmentation for all members, it is necessary to deal with unified coordinates which are called global coordinates. The transformation from member coordinates to global coordinates is given in section 5.3. The formulation and methods of solution are given in section 5.4. Numerical applications for all types of structures are given in Section 5.5. The equivalent joint loading due to direct member loading, temperature, settlements in supports and initial strains are given in section 5.6. Section 5.7 presents special problems to be solved by the stiffness method.

5.2 STIFFNESS MATRIX FOR A STRUCTURAL MEMBER IN SPACE

The direct stiffness matrix for a structural member was given in Chapter 2, section 2.24, neglecting shear deformations. The effect of shear deformations is considered in section 5.7. At present, we shall develop the overall stiffness matrix for a prismatic member using the unit displacement method. As was presented previously, the stiffness coefficient is an action developed due to a unit displacement. The unit displacements shall each be applied, in turn, in the directions of the member coordinates. The member (local) coordinates for a general member ij are considered in this text as shown in Figure 5.1. The coordinate x' passes through the centroid of the cross section, and the coordinates y' and z' coincide with the cross section principal axes. The advantage of using these member coordinates is that they do not depend on the numbering of joint i with respect to joint j or vice versa. They are unified, whatever the direction of looking at the member.

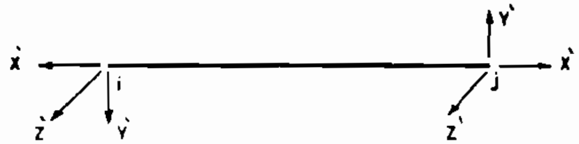


Figure 5.1

Let member ij be kinematically determinate as shown in Figure 5.2. By applying a unit displacement, in turn, at joint j, and by using the slope deflection equation and the equilibrium of actions, one obtains the stiffness coefficients as shown in Figure 5.3. The coefficients can be augmented in a direct stiffness matrix

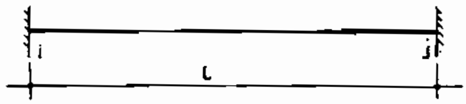


Figure 5.2

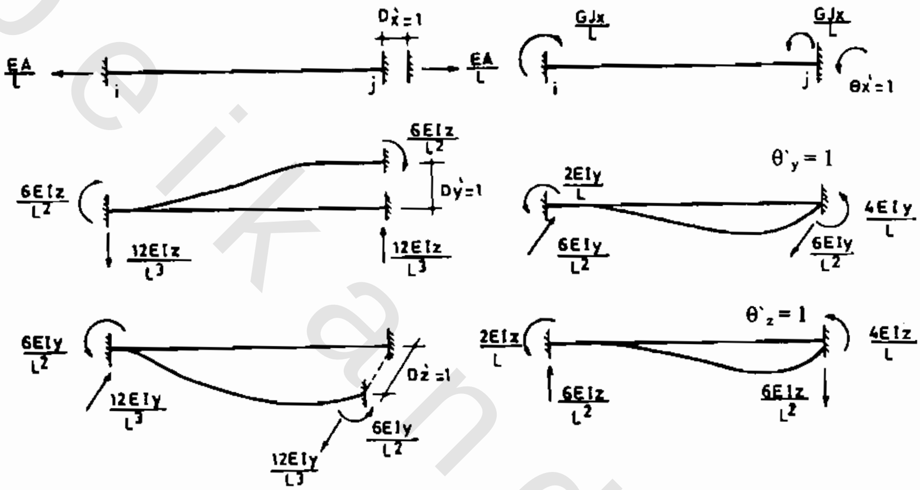


Figure 5.3

denoted by $\underline{S}'^{i}_{jj}$, and a cross stiffness matrix symbolized by $\underline{S}'^{i}_{ij}$. These two matrices are given, respectively, as follows:

$$\underline{S}'^{i}_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix} \qquad (5.1)$$

$$\underline{S}'_{ij} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{-2EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix} \quad (5.2)$$

By repeating the same procedure on joint i, as shown in Figure 5.4, one can develop the stiffness matrices $\underline{S}'^{j}_{ii}$ and $\underline{S}'_{ji}$. These matrices are obtained as

$$\underline{S}'^{j}_{ii} = \underline{S}'^{i}_{jj} \quad (5.3)$$

$$\underline{S}'_{ji} = \underline{S}'_{ij} \quad (5.4)$$

The overall stiffness matrix for member ij in the member coordinates is then given by

$$[\underline{S}']_{ij} = \left[\begin{array}{c|c} \underline{S}'^{j}_{ii} & \underline{S}'_{ij} \\ \hline \underline{S}'_{ji} & \underline{S}'^{i}_{jj} \end{array}\right] \quad (5.5)$$

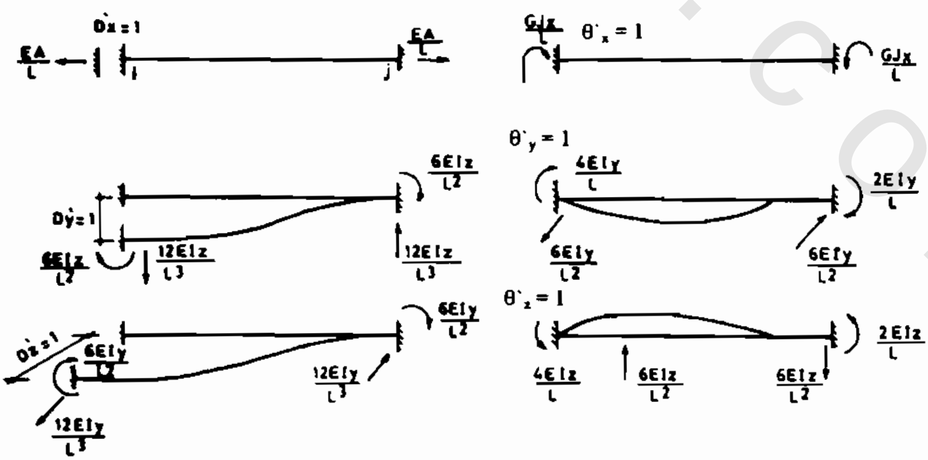


Figure 5.4

This matrix relates the member end actions to the end deformations at joints i and j in the member coordinates. We denote, in the member coordinates, the actions at joint i for member ij by $\underline{\mathbf{A}}'_{ij}$, and the deformation at joint i for member ij by $\underline{\mathbf{D}}'_{ij}$. Similarly $\underline{\mathbf{A}}'_{ji}$ and $\underline{\mathbf{D}}'_{ji}$ represent, respectively, the actions and deformation at j for member ji in the local coordinates. The actions-deformations relationships for member ij are thus given by

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{S}}'^{j}_{ii}\ \underline{\mathbf{D}}'_{ij} + \underline{\mathbf{S}}'_{ij}\ \underline{\mathbf{D}}'_{ji} \tag{5.6}$$

$$\underline{\mathbf{A}}'_{ji} = \underline{\mathbf{S}}'_{ji}\ \underline{\mathbf{D}}'_{ij} + \underline{\mathbf{S}}'^{i}_{jj}\ \underline{\mathbf{D}}'_{ji} \tag{5.7}$$

It is apparent that $\underline{\mathbf{S}}'^{j}_{ii}$ contains the end actions at i, $\underline{\mathbf{A}}'_{ij}$, for member ij due to the unit displacements or rotations at i, $\underline{\mathbf{D}}'_{ij}$. Similarly, $\underline{\mathbf{S}}'_{ij}$ contains the end actions, $\underline{\mathbf{A}}'_{ij}$, at i due to the unit displacements or rotations at j, $\underline{\mathbf{D}}'_{ji}$. The matrix $\underline{\mathbf{S}}'_{ji}$ contains the end actions at j, A'_{ji}, due to the unit displacements or rotations at i, $\underline{\mathbf{D}}'_{ij}$, and matrix $\underline{\mathbf{S}}'^{i}_{jj}$ contains the end actions at j, A'_{ji}, due to the unit displacements or rotations at j, $\underline{\mathbf{D}}'_{ji}$. The expressions given in Equations 5.1 to 5.4 can be applied for members in space frames when neglecting the shear deformations. Stiffness matrices for other types of structural members are derived in the following sections:

5.2.1 Stiffness Matrix for Truss Members

Truss members carry axial forces only, and are thus subjected to axial displacement. The force-displacement relationships are given by Equations 5.6 and 5.7, but the stiffness matrices $\underline{\mathbf{S}}'^{j}_{ii}$, $\underline{\mathbf{S}}'_{ij}$, $\underline{\mathbf{S}}'_{ji}$, and $\underline{\mathbf{S}}'^{i}_{jj}$ are here given by

$$\underline{\mathbf{S}}'^{j}_{ii} = \underline{\mathbf{S}}'^{i}_{jj} = \underline{\mathbf{S}}'_{ij} = \underline{\mathbf{S}}'_{ji} = \left[\frac{EA}{L}\right] \tag{5.8}$$

5.2.2 Stiffness Matrix for Plane Frame Members

For frames in x-y plane, the member end actions in the local coordinates consist of an axial force along x'-axis, a shear force along y'-axis, and bending moment along z'-axis. The stiffness matrix for member ij is thus given by

$$\underline{S}'^{j}_{ii} = \underline{S}'^{i}_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \tag{5.9}$$

$$\underline{S}'_{ij} = \underline{S}'_{ji} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \end{bmatrix} \tag{5.10}$$

5.2.3 Stiffness Matrix for Beams Members

A beam can be considered as a special case of plane frame member. Thus, one can use the stiffness matrix defined by Equations 5.9 and 5.10. However, in most beams and some types of plane frames, the axial deformations can be neglected. In this case, the stiffness matrices become

$$\underline{S}'^{j}_{ii} = \underline{S}'^{i}_{jj} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \tag{5.11}$$

$$\underline{S}'_{ij} = \underline{S}'_{ji} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \end{bmatrix} \tag{5.12}$$

5.2.4 Stiffness Matrix for Grids Members

A grid is a plane structure, and the loads are applied perpendicular to that plane. A member in a grid situated in x-y plane, as shown in Figure 5.5, is subjected to a twisting moment about x' axis, shear force along the z'-direction and bending moment about y' axis. The stiffness matrices which relate these actions with the corresponding deformation, θ'_x, D'_z, and θ'_y, are

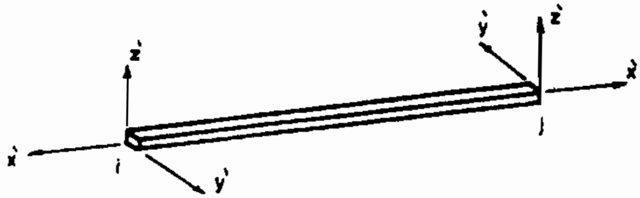


Figure 5.5

$$\underline{S}'^{j}_{ii} = \underline{S}'^{j}_{jj} = \begin{bmatrix} \dfrac{GJ_x}{L} & 0 & 0 \\ 0 & \dfrac{4EI_y}{L} & \dfrac{6EI_y}{L^2} \\ 0 & \dfrac{6EI_y}{L^2} & \dfrac{12EI_y}{L^3} \end{bmatrix} \tag{5.13}$$

$$\underline{S}'_{ij} = \underline{S}'_{ji} = \begin{bmatrix} \dfrac{GJ_x}{L} & 0 & 0 \\ 0 & \dfrac{-2EI_y}{L^3} & \dfrac{-6EI_y}{L^2} \\ 0 & \dfrac{-6EI_y}{L^2} & \dfrac{-12EI_y}{L^3} \end{bmatrix} \tag{5.14}$$

5.3 COORDINATES TRANSFORMATION

Coordinate axes are used as reference to define certain quantities in space. They could be cartesian, spherical, or cylindrical. We are here interested in cartesian orthogonal coordinates, those which have an angle 90° between any two of them, as shown in Figure 5.6.

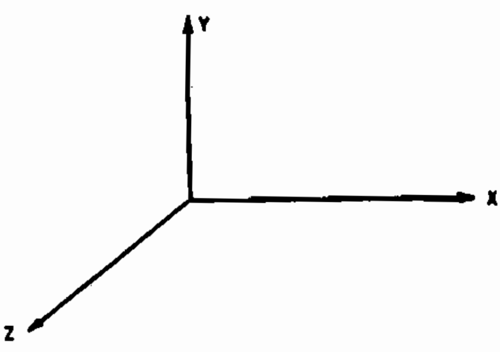


Figure 5.6

5.3.1 Transformation Matrices in 3-Dimensional Space

One basic step in matrix structural analysis is the transformation of actions and deformations which are defined in local coordinates into quantities defined in the global coordinates. This transformation is done through using the transformation matrices. In order to transfer quantities defined in global coordinates x-y-z into quantities defined in local coordinates x'−y'−z', the following transformation matrix is used:

$$\underline{\mathbf{R}} = \begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix} \tag{5.15}$$

in which (x'y) indicates the angle between the axes x' and y as shown in Figure 5.7.

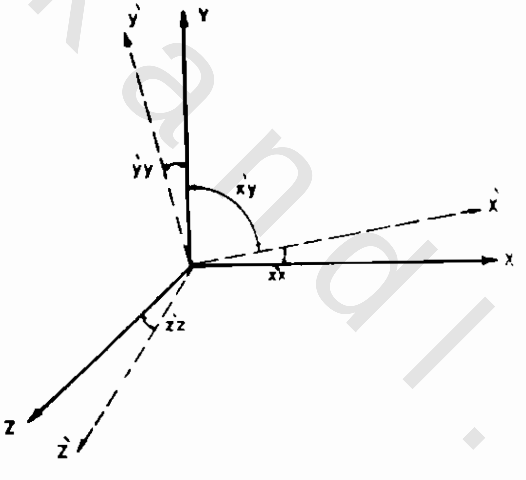


Figure 5.7

Equations 5.15 can also be written in another form as

$$\underline{\mathbf{R}} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \tag{5.16}$$

in which ℓ_1, m_1, and n_1 represent the direction cosines of the local axis x' with respect to the global axes x-y-z; ℓ_2, m_2, and n_2 represent the direction cosines of the local axis y' with respect to global axes x-y-z; ℓ_3, m_3, and n_3 represent the direction cosines of the local axis z' with respect to the global axes x-y-z.

For example, if the global coordinates of two points (1) and (2) lie on the local axis x' are given, respectively, by (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction cosines of x'-axis directed from point (1) to point (2) are obtained from

$$\cos(x'x) = \ell_1 = \frac{x_2 - x_1}{L} \tag{5.17}$$

$$\cos(x'y) = m_1 = \frac{y_2 - y_1}{L} \tag{5.18}$$

$$\cos(x'z) = n_1 = \frac{z_2 - z_1}{L} \tag{5.19}$$

in which $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

If certain quantities are expressed by a generalized vector $\underline{\mathbf{V}}$ defined in the global coordinates, then, one can define them in the local coordinates by $\underline{\mathbf{V}}'$ where

$$\underline{\mathbf{V}}' = \underline{\mathbf{R}}\,\underline{\mathbf{V}} \tag{5.20}$$

5.3.2 Transformation Matrices in 2-Dimensional Space

In a similar treatment, the transformation matrix between coordinates x'–y' and x-y is

$$\underline{\mathbf{R}} = \begin{bmatrix} \cos(x'x) & \cos(x'y) \\ \cos(y'x) & \cos(y'y) \end{bmatrix} \tag{5.21}$$

From Figure 5.8, one can write Equation 5.21 as

$$\underline{\mathbf{R}} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{5.22}$$

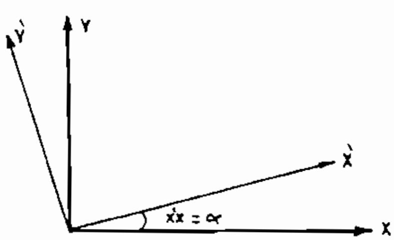


Figure 5 8

For this simple coordinate transformation, one can prove Equation 5.20, using Figure 5.9. The position vector $\underline{V}$ is defined with respect to x-y coordinates by

$$\underline{V} = V_x \underline{i} + V_y \underline{j}$$

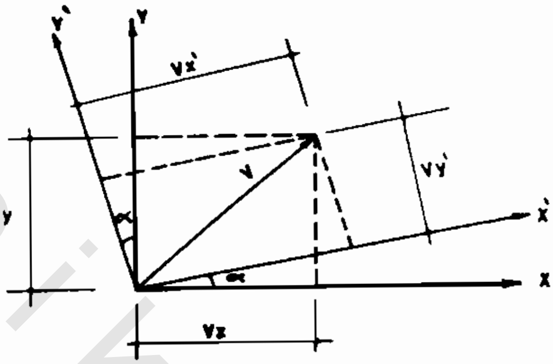


Figure 5.9

This vector can also be expressed in x'−y' coordinates as

$$\underline{V} = V'_x \underline{i}' + V'_y \underline{j}'$$

in which $\underline{i}$ and $\underline{j}$ are unit vectors in x-y directions and $\underline{i}'$, $\underline{j}'$ are unit vectors in x'−y' directions. From Figure 5.9 one has

$$V'_x = V_x \cos\alpha + V_y \sin\alpha$$

$$V'_y = -V_x \sin\alpha + V_y \cos\alpha$$

Therefore, one has

$$\begin{bmatrix} V'_x \\ V'_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} \tag{5.23}$$

which is the same relation as Equation 5.20 applied for x-y coordinates.

5.3.3 Properties of Transformation matrices

The transformation matrix of orthogonal cartesian coordinates is an orthogonal matrix. This means that one can obtain the inverse of the transformation matrix simply by finding its transpose. To prove this property consider the transformation matrix given by Equation 5.22. The inverse of this matrix is

$$\underline{\mathbf{R}}^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{5.24}$$

The matrix of Equation 5.24 represents also the transpose of Equation 5.21. Thus one has the following property for the transformation matrices:

$$\underline{\mathbf{R}}^{-1} = \underline{\mathbf{R}}^{T} \tag{5.25}$$

Other properties of the orthogonal transformation matrices are that the inner product of any row by itself gives unity, whereas the inner product of any two different rows is zero. This can be expressed as

$$\ell_i^2 + m_i^2 + n_i^2 = 1 \tag{5.26}$$

$$\ell_i \, \ell_j + m_i \, m_j + n_i \, n_j = 0 \qquad \text{for } i \neq j \tag{5.27}$$

where ℓ_i, m_i, and n_i for i = 1, 2, 3, are as defined in Equation 5.16.

5.3.4 Transformation of Linear Relationships

Suppose that one has some quantities related to other quantities by the following linear relationship, which is expressed in the local coordinates:

$$\underline{\mathbf{A}}' = \underline{\mathbf{S}}' \, \underline{\mathbf{D}}' \tag{5.28}$$

In order to express this relationship in global coordinates, one uses the transformations of Equation 5.20 on the quantities $\underline{\mathbf{A}}'$ and $\underline{\mathbf{D}}'$ as

$$\underline{\mathbf{A}}' = \underline{\mathbf{R}} \, \underline{\mathbf{A}} \tag{5.29}$$

$$\underline{\mathbf{D}}' = \underline{\mathbf{R}} \, \underline{\mathbf{D}} \tag{5.30}$$

From the orthogonality property, Equation 5.28 can be written as

$$\underline{\mathbf{R}} \, \underline{\mathbf{A}} = \underline{\mathbf{S}}' \, \underline{\mathbf{R}} \, \underline{\mathbf{D}} \tag{5.31}$$

Equation 5.31 can also be expressed as

$$\underline{\mathbf{A}} = \underline{\mathbf{R}}^{T} \, \underline{\mathbf{S}}' \, \underline{\mathbf{R}} \, \underline{\mathbf{D}} \tag{5.32}$$

which indicates that the term ($\underline{\mathbf{R}}^{T} \, \underline{\mathbf{S}}' \, \underline{\mathbf{R}}$) is the transformation of the matrix $\underline{\mathbf{S}}'$ from local coordinates into the global coordinates.

5.3.5 Actions–Deformations Relationship in Global Coordinates

From Equation 5.20 one can write the relationship between the end actions of member ij in the local coordinates and in the global coordinates as follows:

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{R}}_{ij}\, \underline{\mathbf{A}}_{ij} \tag{5.33}$$

in which $\underline{\mathbf{A}}_{ij}$ is the end actions at i for member ij in the global coordinates; and $\underline{\mathbf{R}}_{ij}$ is the transformation matrix for member ij at joint i.

Similarly, the relationship between local and global deformations at joint i for member ij is expressed as

$$\underline{\mathbf{D}}'_{ij} = \underline{\mathbf{R}}_{ij}\, \underline{\mathbf{D}}_{ij} \tag{5.34}$$

Similar expressions can be obtained for the end actions and deformation at joint j for member ij as follows:

$$\underline{\mathbf{A}}'_{ji} = \underline{\mathbf{R}}_{ji}\, \underline{\mathbf{A}}_{ji} \tag{5.35}$$

$$\underline{\mathbf{D}}'_{ji} = \underline{\mathbf{R}}_{ji}\, \underline{\mathbf{A}}_{ji} \tag{5.36}$$

in which $\underline{\mathbf{R}}_{ji}$ is the transformation matrix of member ji applied at joint j.

Substituting Equations 5.33 to 5.36 into Equation 5.6 one obtains

$$\underline{\mathbf{R}}_{ij}\, \underline{\mathbf{A}}_{ij} = \underline{\mathbf{S}}'^{i}_{ii}\, \underline{\mathbf{R}}_{ij}\, \underline{\mathbf{D}}_{ij} + \underline{\mathbf{S}}'^{i}_{ij}\, \underline{\mathbf{R}}_{ji}\, \underline{\mathbf{D}}_{ji} \tag{5.37}$$

Equation 5.37 can be written as

$$\underline{\mathbf{A}}_{ij} = \underline{\mathbf{R}}^{T}_{ij}\, \underline{\mathbf{S}}'^{j}_{ii}\, \underline{\mathbf{R}}_{ij}\, \underline{\mathbf{D}}_{ij} + \underline{\mathbf{R}}^{T}\, \underline{\mathbf{S}}'_{ij}\, \underline{\mathbf{R}}_{ji}\, \underline{\mathbf{D}}_{ji} \tag{5.38}$$

which can also be expressed as

$$\underline{\mathbf{A}}_{ij} = \underline{\mathbf{S}}^{j}_{ii}\, \underline{\mathbf{D}}_{ij} + \underline{\mathbf{S}}_{ij}\, \underline{\mathbf{D}}_{ji} \tag{5.39}$$

where $\underline{\mathbf{S}}^{j}_{ii}$ and $\underline{\mathbf{S}}_{ij}$ are the stiffness matrices expressed in global coordinates and obtained from

$$\underline{\mathbf{S}}^{j}_{ii} = \underline{\mathbf{R}}^{T}_{ij}\, \underline{\mathbf{S}}'^{j}_{ii}\, \underline{\mathbf{R}}_{ij} \tag{5.40}$$

$$\underline{\mathbf{S}}_{ij} = \underline{\mathbf{R}}^{T}_{ij}\, \underline{\mathbf{S}}'_{ij}\, \underline{\mathbf{R}}_{ji} \tag{5.41}$$

Similarly, Equation 5.7 can be expressed in global coordinates as

$$\underline{\mathbf{A}}_{ji} = \underline{\mathbf{S}}_{ji}\,\underline{\mathbf{D}}_{ij} + \underline{\mathbf{S}}^{i}_{jj}\,\underline{\mathbf{D}}_{ji} \tag{5.42}$$

where $\underline{\mathbf{S}}_{ji}$ and $\underline{\mathbf{S}}^{i}_{jj}$ are obtained from

$$\underline{\mathbf{S}}_{ji} = \underline{\mathbf{R}}^{T}_{ji}\,\underline{\mathbf{S}}'_{ji}\,\underline{\mathbf{R}}_{ij} \tag{5.43}$$

$$\underline{\mathbf{S}}^{i}_{jj} = \underline{\mathbf{R}}^{T}_{ji}\,\underline{\mathbf{S}}'^{i}_{jj}\,\underline{\mathbf{R}}_{ji} \tag{5.44}$$

5.4 THE STIFFNESS MATRIX METHOD : MATRIX APPROACH II

5.4.1 Formulation

It was shown in Chapter 4, section 4.7, that the stiffness coefficients of the structural stiffness matrix can be obtained by applying a unit displacement or rotation at every degree of freedom of the kinematically determinate structure. This approach becomes very complicated and lengthy for structures of high degrees of freedom. That is why the element approach is preferred in the general application of this method.

In the element (member) approach, the actions-deformations relationship for every member ij is first determined. This task can easily be done by the computer since the member stiffness matrices in the member coordinates ($\underline{\mathbf{S}}'^{j}_{ii}, \underline{\mathbf{S}}'_{ij}, \underline{\mathbf{S}}'_{ji}, \underline{\mathbf{S}}'^{i}_{jj}$) has fixed forms, as presented in section 5.2, which depend only on the member length, cross section area, material properties and moment of inertia. However, in order to assemble the member stiffness matrices for the whole structure, one has to deal with a unified reference. The unified reference is called global coordinates. The transformation of actions-deformations relationship for every member from the local coordinates to the global coordinates is performed according to Equations 5.33 to 5.44. This requires the determination of the rotation matrices $\underline{\mathbf{R}}_{ij}$ and $\underline{\mathbf{R}}_{ji}$ for every member ij. It is thus necessary to define the locations of all joints in the structure with respect to the chosen global coordinates.

By using the conditions of equilibrium and compatibility at every joint in the structure, the structure stiffness matrix is determined from the elements' stiffness matrices. The equilibrium conditions state that the sum of the end actions defined in the global coordinates must equal the external actions at that joint which are also defined in global coordinates. This can be expressed mathematically, for example, at joint i which connects members ij, ik, and $i\ell$ as follows:

$$\underline{\mathbf{A}}_{i} = \underline{\mathbf{A}}_{ij} + \underline{\mathbf{A}}_{ik} + \underline{\mathbf{A}}_{i\ell} \tag{5.45}$$

To illustrate this further, consider joint i in a plane truss which is connecting the two members ij and ik as shown in Figure 5.10. The equilibrium of this joint under the effect of an external force A_i at joint i gives the tension force T in member ij

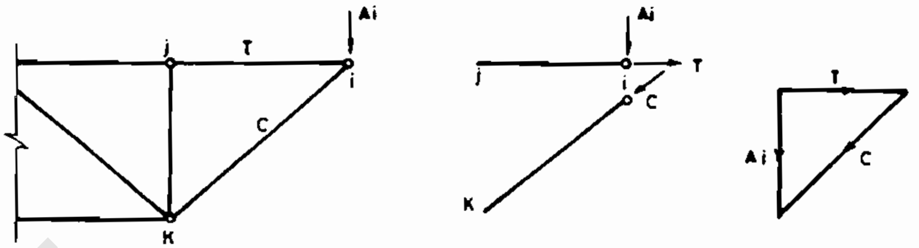


Figure 5.10

and the compression force C in member ik It is obvious that A_i is the resultant of the end actions T and C as illustrated in the force polygon, of Figure 5.10. In vector notation, one can write $\underline{\mathbf{A}}_i = \underline{\mathbf{T}} + \underline{\mathbf{C}}$.

The compatibility conditions insure the integrity of the structure after deformation. They state that if a rigid joint i is connecting members ij, ik, and $i\ell$ as shown in Figure 5.11, then, after loading, the deformations of joint i, in the global coordinates, for member ij, member ik, or member $i\ell$ must equal the deformations of joint i in the same coordinates. This can be expressed mathematically as

$$\underline{\mathbf{D}}_i = \underline{\mathbf{D}}_{ij} = \underline{\mathbf{D}}_{ik} = \mathbf{D}_{i\ell} \tag{5.46}$$

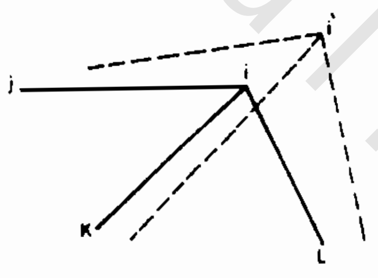


Figure 5.11

For nonrigid joints like trusses, as shown in Figure 5.10, the compatibility conditions still apply but do not include the rotation at the joint. In this case, Equation 5.46 is applied for the linear displacements only However, in trusses, the analyst does not deal with the rotations at the joints and thus Equation 5.46, can still be applied at every joint in the truss.

From equations like Equations 5.45 and 5.46 one can obtain the actions-deformations relationship at every joint in the structure. For simplicity, consider the structure shown in Figure 5.12. The equilibrium equations of this structure are:

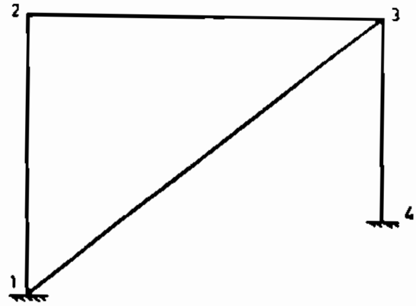


Figure 5.12

$$
\begin{aligned}
\underline{\mathbf{A}}_1 &= \underline{\mathbf{A}}_{12} + \underline{\mathbf{A}}_{13} \\
\underline{\mathbf{A}}_2 &= \underline{\mathbf{A}}_{23} + \underline{\mathbf{A}}_{21} \\
\underline{\mathbf{A}}_3 &= \underline{\mathbf{A}}_{31} + \underline{\mathbf{A}}_{32} + \underline{\mathbf{A}}_{34} \\
\underline{\mathbf{A}}_4 &= \underline{\mathbf{A}}_{43}
\end{aligned}
\tag{5.47}
$$

The compatibility conditions are

$$
\begin{aligned}
\underline{\mathbf{D}}_1 &= \underline{\mathbf{D}}_{12} = \underline{\mathbf{D}}_{13} \\
\underline{\mathbf{D}}_2 &= \underline{\mathbf{D}}_{21} = \underline{\mathbf{D}}_{23} \\
\underline{\mathbf{D}}_3 &= \underline{\mathbf{D}}_{31} = \underline{\mathbf{D}}_{32} = \underline{\mathbf{D}}_{34} \\
\underline{\mathbf{D}}_4 &= \underline{\mathbf{D}}_{43}
\end{aligned}
\tag{5.48}
$$

Now, by substituting the actions-deformations relationship for all members into Equation 5.47, and by considering the compatibility conditions of Equation 5.48, one obtains

$$
\begin{aligned}
\underline{\mathbf{A}}_1 &= \underline{\mathbf{S}}_{11}^2\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}_{12}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}_{11}^3\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}_{13}\,\underline{\mathbf{D}}_3 \\
\underline{\mathbf{A}}_2 &= \underline{\mathbf{S}}_{22}^2\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}_{23}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}_{22}^1\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}_{21}\,\underline{\mathbf{D}}_1 \\
\underline{\mathbf{A}}_3 &= \underline{\mathbf{S}}_{33}^1\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}_{31}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}_{33}^2\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}_{32}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}_{33}^4\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}_{34}\,\underline{\mathbf{D}}_4 \\
\underline{\mathbf{A}}_4 &= \underline{\mathbf{S}}_{44}^3\cdot\underline{\mathbf{D}}_4 + \underline{\mathbf{S}}_{43}\,\underline{\mathbf{D}}_3
\end{aligned}
\tag{5.49}
$$

Equations 5.49 can be written in a short matrix form as

$$
\begin{bmatrix} \underline{\mathbf{A}}_1 \\ \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \\ \underline{\mathbf{A}}_4 \end{bmatrix}
=
\begin{bmatrix}
\underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{\mathbf{S}}_{13} & \underline{\mathbf{0}} \\
\underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{\mathbf{0}} \\
\underline{\mathbf{S}}_{31} & \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} \\
\underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44}
\end{bmatrix}
\begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \\ \underline{\mathbf{D}}_4 \end{bmatrix}
\tag{5.50}
$$

where

$$
\begin{aligned}
\underline{S}_{11} &= \underline{S}^{2}_{11} + \underline{S}^{3}_{11} \\
\underline{S}_{22} &= \underline{S}^{2}_{22} + \underline{S}^{3}_{22} \\
\underline{S}_{33} &= \underline{S}^{1}_{33} + \underline{S}^{2}_{33} + \underline{S}^{4}_{33} \\
\underline{S}_{44} &= \underline{S}^{3}_{44}
\end{aligned}
\qquad (5.51)
$$

Each diagonal matrix $\underline{S}_{ii}$ in the overall structural stiffness matrix of Equation 5.50, represents the summation of the direct stiffness matrices for all members which are connected with joint i. Therefore, writing the overall stiffness relation like Equation 5.50 is not difficult, since the indices indicate the connectivity between the joints through the elements (members). One may then write directly the overall structural stiffness relation in the form of Equation 5.50 without the need of using explicity the equilibrium and compatibility conditions, like Equations 5.47, 5.48, or 5.49.

Example 5.1

Write the overall stiffness relations for the structure shown in Figure 5.13.

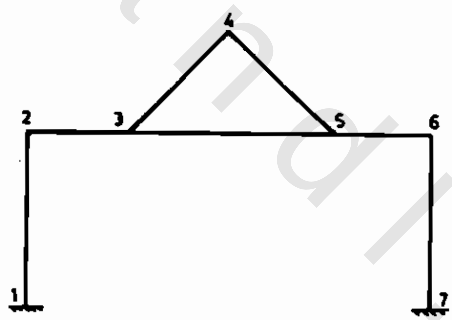


Figure 5.13

Solution

According to the connectivity between the elements, the actions-deformations relation for the structure can be expressed as follows:

$$\begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \\ \underline{A}_3 \\ \underline{A}_4 \\ \underline{A}_5 \\ \underline{A}_6 \\ \underline{A}_7 \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{S}_{21} & \underline{S}_{22} & \underline{S}_{23} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_{32} & \underline{S}_{33} & \underline{S}_{34} & \underline{S}_{35} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_{43} & \underline{S}_{44} & \underline{S}_{45} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_{53} & \underline{S}_{54} & \underline{S}_{55} & \underline{S}_{56} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{S}_{65} & \underline{S}_{66} & \underline{S}_{67} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{S}_{76} & \underline{S}_{77} \end{bmatrix} \begin{bmatrix} \underline{D}_1 \\ \underline{D}_2 \\ \underline{D}_3 \\ \underline{D}_4 \\ \underline{D}_5 \\ \underline{D}_6 \\ \underline{D}_7 \end{bmatrix}$$

The joint stiffness matrices are defined as follows:

$$\underline{S}_{11} = \underline{S}_{11}^2$$

$$\underline{S}_{22} = \underline{S}_{22}^2 + \underline{S}_{22}^3$$

$$\underline{S}_{33} = \underline{S}_{33}^2 + \underline{S}_{33}^4 + \underline{S}_{33}^5$$

$$\underline{S}_{44} = \underline{S}_{44}^2 + \underline{S}_{44}^5$$

$$\underline{S}_{55} = \underline{S}_{55}^3 + \underline{S}_{55}^4 + \underline{S}_{55}^6$$

$$\underline{S}_{66} = \underline{S}_{66}^5 + \underline{S}_{66}^7$$

$$\underline{S}_{77} = \underline{S}_{77}^6$$

5.4.2 Methods of Solution

In any structural actions-deformations relationship, one has actions and deformations which could be known or unknown quantities. The known actions represent the actual external actions applied on the free joints causing deformations in the structure which depend on the deformations at the free joints. The unknown actions represent the reactions at the nonmovable joints in order to keep these joints in place, in the presence of the external actions at the free joints. In structural analysis, one has to determine both the unknown reactions and the unknown deformations. Two methods are used to solve the structural analysis problem. The first method is based on rearranging the structural stiffness relationship such that the known deformations are gathered independently away from the unknown deformations. The known deformations are also divided into two sets. A set of zero deformations and a set of non zero deformations which happen due to deformations of supports. The structural actions-deformations relationship can thus be expressed, in general, as

$$\begin{bmatrix} \underline{\mathbf{A}}_{I} \\ \underline{\mathbf{A}}_{II} \\ \underline{\mathbf{A}}_{III} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{I,I} & \underline{\mathbf{S}}_{I,II} & \underline{\mathbf{S}}_{I,III} \\ \underline{\mathbf{S}}_{II,I} & \underline{\mathbf{S}}_{II,II} & \underline{\mathbf{S}}_{II,III} \\ \underline{\mathbf{S}}_{III,I} & \underline{\mathbf{S}}_{III,II} & \underline{\mathbf{S}}_{III,III} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_{I} \\ \underline{\mathbf{D}}_{II} \\ \underline{\mathbf{D}}_{III} \end{bmatrix} \tag{5.52}$$

in which $\underline{\mathbf{A}}_I$ represents the known actions at the free joints whose deformations are $\underline{\mathbf{D}}_I$; $\underline{\mathbf{A}}_{II}$ is the unknown reactions at supports of zero deformation $\underline{\mathbf{D}}_{II}$; and $\underline{\mathbf{A}}_{III}$ is the unknown reactions at supports of known deformation $\underline{\mathbf{D}}_{III}$.

The solution of Equation 5.53 is obtained from matrix algebra as follows:

$$\underline{\mathbf{A}}_I = \underline{\mathbf{S}}_{I,I}\, \underline{\mathbf{D}}_I + \underline{\mathbf{S}}_{I,III}\, \underline{\mathbf{D}}_{III} \tag{5.53}$$

$$\underline{\mathbf{D}}_I = \underline{\mathbf{S}}_{I,I}^{-1} \left(\underline{\mathbf{A}}_I - \underline{\mathbf{S}}_{I,III}\, \underline{\mathbf{D}}_{III}\right) \tag{5.54}$$

$$\underline{\mathbf{A}}_{II} = \underline{\mathbf{S}}_{II,I}\, \underline{\mathbf{D}}_I + \underline{\mathbf{S}}_{II,III}\, \underline{\mathbf{D}}_{III} \tag{5.55}$$

$$\underline{\mathbf{A}}_{III} = \underline{\mathbf{S}}_{III,I}\, \underline{\mathbf{D}}_I + \underline{\mathbf{S}}_{III,III}\, \underline{\mathbf{D}}_{III} \tag{5.56}$$

The members' end actions can then be determined using Equations 5.6 and 5.7.

The second method depends on finding the free deformations $\underline{\mathbf{D}}_I$ only. The reactions $\underline{\mathbf{A}}_{II}$ and $\underline{\mathbf{A}}_{III}$ are found later from the members end actions. In this case, the analyst does not need to determine the overall structural stiffness matrix and makes the rearrangement process. He can frequently substitute the boundary conditions in the equations in order to directly obtain the stiffness matrices $\underline{\mathbf{S}}_{I,I}$ and $\underline{\mathbf{S}}_{I,III}$, if needed. He may even replace the supports deformation by equivalent joint actions, as will be shown in section 5.6, and needs to determine $\underline{\mathbf{S}}_{I,I}$ only. In many problems, however, the boundary conditions may state that not all the deformations at a specific joint are zero. In this case, the analyst has to consider these joints as free joints and then delete the rows and columns, which are corresponding to the known boundary conditions. The members end actions can then be determined using Equation 5.6 and 5.7. When using the free joints deformations expressed in the global coordinates, the members end actions are determined from

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{S}}'^{j}_{ii}\, \underline{\mathbf{R}}_{ij}\, \underline{\mathbf{D}}_i + \underline{\mathbf{S}}'_{ij}\, \underline{\mathbf{R}}_{ji}\, \underline{\mathbf{D}}_j \tag{5.57}$$

$$\underline{\mathbf{A}}'_{ji} = \underline{\mathbf{S}}'_{ji}\, \mathbf{R}_{ij}\, \underline{\mathbf{D}}_i + \underline{\mathbf{S}}'^{i}_{jj}\, \underline{\mathbf{R}}_{ji}\, \mathbf{D}_j \tag{5.58}$$

Example 5.2

Show the two methods of solving the plane frame shown in Figure 5.14 by the stiffness matrix method.

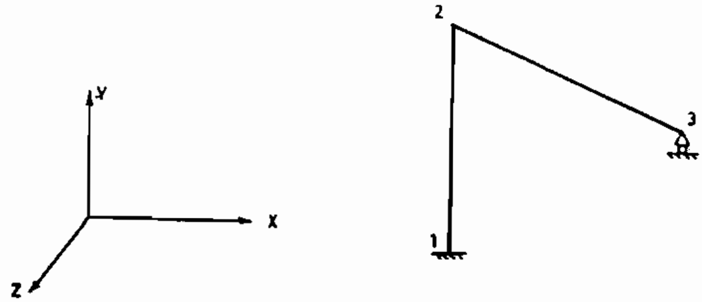


Figure 5.14

Solution

Method 1

The overall stiffness relation is given by

$$\begin{bmatrix} \underline{\mathbf{A}}_1 \\ \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{\mathbf{0}} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \end{bmatrix}$$

where $\underline{\mathbf{S}}_{11}$, $\underline{\mathbf{S}}_{22}$, and $\underline{\mathbf{S}}_{33}$ are

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}_{11}^{2}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3}$$

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}_{33}^{2}$$

For simplicity, the above relations can be expressed as

$$\begin{bmatrix} A_{1x} \\ A_{1y} \\ M_{1z} \\ A_{2x} \\ A_{2y} \\ M_{2z} \\ A_{3x} \\ A_{3y} \\ M_{3z} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} & 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} & 0 & 0 & 0 \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} & s_{47} & s_{48} & s_{49} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} & s_{57} & s_{58} & s_{59} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} & s_{67} & s_{68} & s_{69} \\ 0 & 0 & 0 & s_{74} & s_{75} & s_{76} & s_{77} & s_{78} & s_{79} \\ 0 & 0 & 0 & s_{84} & s_{85} & s_{86} & s_{87} & s_{88} & s_{89} \\ 0 & 0 & 0 & s_{94} & s_{95} & s_{96} & s_{97} & s_{98} & s_{99} \end{bmatrix} \begin{bmatrix} D_{1x} \\ D_{1y} \\ \theta_1 \\ D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ D_{3y} \\ \theta_3 \end{bmatrix}$$

in which s_{ij} is an element in the i^{th} row and the j^{th} column of the overall stiffness matrix.

The free unknown displacements and rotations include D_{2x}, D_{2y}, θ_2, D_{3x}, and θ_3. The known zero displacements and rotations include D_{1x}, D_{1y}, θ_1, and D_{3y}. The known actions are A_{2x}, A_{2y}, M_{2z}, and M_{3z}. The unknown reactions are A_{1x}, A_{1y}, M_{1z}, and A_{1y}. By rearranging the above relationship one obtains

$$\begin{bmatrix} A_{2x} \\ A_{2y} \\ M_{2z} \\ A_{3x} \\ M_{3z} \\ \hline A_{1x} \\ A_{1y} \\ M_{1z} \\ A_{3y} \end{bmatrix} = \left[\begin{array}{ccccc|cccc} s_{44} & s_{45} & s_{46} & s_{47} & s_{49} & s_{41} & s_{43} & s_{43} & s_{48} \\ s_{54} & s_{55} & s_{56} & s_{57} & s_{59} & s_{51} & s_{53} & s_{43} & s_{58} \\ s_{64} & s_{65} & s_{66} & s_{67} & s_{69} & s_{61} & s_{63} & s_{43} & s_{68} \\ s_{74} & s_{75} & s_{76} & s_{77} & s_{79} & 0 & 0 & 0 & s_{78} \\ s_{94} & s_{95} & s_{96} & s_{97} & s_{99} & 0 & 0 & 0 & s_{98} \\ \hline s_{14} & s_{15} & s_{16} & 0 & 0 & s_{11} & s_{12} & s_{13} & 0 \\ s_{24} & s_{25} & s_{26} & 0 & 0 & s_{21} & s_{22} & s_{23} & 0 \\ s_{34} & s_{35} & s_{36} & 0 & 0 & s_{31} & s_{32} & s_{33} & 0 \\ s_{84} & s_{85} & s_{86} & s_{87} & s_{89} & 0 & 0 & s_{12} & s_{88} \end{array}\right] \begin{bmatrix} D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ \theta_3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (5.60)$$

where the stiffness matrix is partitioned into $\underline{S}_{I,I}$, $\underline{S}_{I,II}$, $\underline{S}_{II,I}$, and $\underline{S}_{II,II}$ as in equation 5.52. Now, equation 5.60 can be solved for the unknown free joints displacements and rotations $\underline{D}_I$ and the unknown reactions $\underline{A}_{II}$ according to equations 5.54 and 5.55, respectively.

Method 2

The boundary conditions are $\underline{D}_1 = \underline{0}$, and $D_{3y} = 0$. The actions-deformations relationship needed to solve the problem is

$$\begin{bmatrix} \underline{A}_2 \\ \underline{A}_3 \end{bmatrix} = \begin{bmatrix} \underline{S}_{22} & \underline{S}_{23} \\ \underline{S}_{32} & \underline{S}_{33} \end{bmatrix} \begin{bmatrix} \underline{D}_2 \\ \underline{D}_3 \end{bmatrix}$$

where $\quad \underline{S}_{22} = \underline{S}_{22}^1 + \underline{S}_{22}^3$

$\qquad \underline{S}_{33} = \underline{S}_{33}^2$

After finding the above stiffness matrix, the 5^{th} row and 5^{th} column which are corresponding to $D_{3y} = 0$ should be deleted. The final relationship becomes

$$\begin{bmatrix} A_{2x} \\ A_{2y} \\ M_{2z} \\ A_{3x} \\ A_{3y} \\ M_{3z} \end{bmatrix} = \begin{bmatrix} s_{44} & s_{45} & s_{46} & s_{47} & s_{48} & s_{49} \\ s_{54} & s_{55} & s_{56} & s_{57} & s_{58} & s_{59} \\ s_{64} & s_{65} & s_{66} & s_{67} & s_{68} & s_{69} \\ s_{74} & s_{75} & s_{76} & s_{77} & s_{78} & s_{79} \\ s_{84} & s_{85} & s_{86} & s_{87} & s_{88} & s_{89} \\ s_{94} & s_{95} & s_{96} & s_{97} & s_{98} & s_{99} \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ 0 \\ \theta_3 \end{bmatrix}$$

which can be solved for the unknown deformations.

5.5 NUMERICAL APPLICATIONS

In this section, it is assumed that the external actions are only applied on the joints of the structure. Other types of loadings will be considered in section 5.6.

5.5.1 Applications to Plane Trusses

The force-displacement relationships for member ij of a plane truss can be expressed in the local coordinates as follows:

$$A'_{ij} = \left[\frac{EA}{L}\right] D'_{ij} + \left[\frac{EA}{L}\right] D'_{ji} \qquad (5.61)$$

$$A'_{ji} = \left[\frac{EA}{L}\right] D'_{ij} + \left[\frac{EA}{L}\right] D'_{ji} \qquad (5.62)$$

in which A'_{ij} and D'_{ij} are, respectively, the axial force and displacement at joint i along axis-x'; A'_{ji} and D'_{ji} are at joint j along axis-x' and [EA/L] is the stiffness matrix for each of $\underline{S}'^{j}_{ii}$, $\underline{S}'_{ij}$, $\underline{S}'_{ji}$ and $\underline{S}'^{i}_{jj}$, which is of dimension 1×1.

In order to express Equation 5.61 and 5.62 in the x-y global coordinates shown in Figure 5.15, the transformation matrices at joints i and j should be determined.

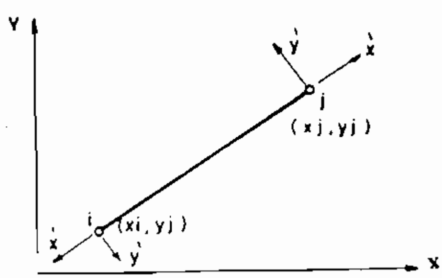


Figure 5.15

These matrices transform forces and displacements defined in x-y coordinates into corresponding values defined in x'–y' coordinates. As were derived before, they are obtained from

$$\underline{\mathbf{R}}_{ij} = \begin{bmatrix} \left(\dfrac{x_i - x_j}{L}\right) & \left(\dfrac{y_i - y_j}{L}\right) \\ -\left(\dfrac{y_i - y_j}{L}\right) & \left(\dfrac{x_i - x_j}{L}\right) \end{bmatrix} \tag{5.63}$$

$$\underline{\mathbf{R}}_{ji} = \begin{bmatrix} \left(\dfrac{x_j - x_i}{L}\right) & \left(\dfrac{y_j - y_i}{L}\right) \\ -\left(\dfrac{y_j - y_i}{L}\right) & \left(\dfrac{x_j - x_i}{L}\right) \end{bmatrix} \tag{5.64}$$

Since the axial force and displacement at the joint are only along x'–coordinate, it is assumed, for the sake of simplicity, that there are also zero force and displacement at the joint along y' coordinate. The relationship between $\underline{\mathbf{A}}'_{ij}$ and $\underline{\mathbf{A}}_{ij}$ can thus be applied as follows

$$\begin{bmatrix} A'_{ij} \\ 0 \end{bmatrix} = \underline{\mathbf{R}}_{ij} \begin{bmatrix} A_x \\ A_y \end{bmatrix}_{ij} \tag{5.65}$$

which leads to

$$\underline{\mathbf{A}}'_{ij} = \begin{bmatrix} \left(\dfrac{x_i - x_j}{L}\right) & \left(\dfrac{y_i - y_j}{L}\right) \end{bmatrix} \underline{\mathbf{A}}_{ij} \tag{5.66}$$

The relationship between $\underline{\mathbf{A}}_{ij}$ and A'_{ij} can be established as follows:

$$\underline{\mathbf{A}}_{ij} = \underline{\mathbf{R}}_{ij}^{T} \begin{bmatrix} A'_{ij} \\ 0 \end{bmatrix} = \begin{bmatrix} \left(\dfrac{x_i - x_j}{L}\right) \\ \left(\dfrac{y_i - y_j}{L}\right) \end{bmatrix} A'_{ij} \tag{5.67}$$

It is obvious that Equations 5.66 and 5.67 still satisfy analogous orthogonality relationshin althouah the transformation matrix is not a square matrix. Thus, from now on, the transformation matrix for plane truss members will be considered as follows:

$$\underline{\mathbf{R}}_{ij} = \left[\left(\frac{x_i - x_j}{L} \right) \quad \left(\frac{y_i - y_j}{L} \right) \right] \tag{5.68}$$

The relationship between $\underline{\mathbf{R}}_{ij}$ and $\underline{\mathbf{R}}_{ji}$ is given by

$$\underline{\mathbf{R}}_{ji} = - \underline{\mathbf{R}}_{ij} \tag{5.69}$$

Equations 5.61 and 5.62 can be expressed as usual in the global coordinates as follows:

$$\underline{\mathbf{A}}_{ij} = \underline{\mathbf{R}}_{ij}^T \underline{\mathbf{S}}'^{j}_{ii} \underline{\mathbf{R}}_{ij} \underline{\mathbf{D}}_i + \underline{\mathbf{R}}_{ij}^T \underline{\mathbf{S}}'_{ij} \underline{\mathbf{R}}_{ji} \underline{\mathbf{D}}_j \tag{5.70}$$

$$\underline{\mathbf{A}}_{ji} = \underline{\mathbf{R}}_{ji}^T \underline{\mathbf{S}}'_{ji} \underline{\mathbf{R}}_{ij} \underline{\mathbf{D}}_i + \underline{\mathbf{R}}_{ji}^T \underline{\mathbf{S}}'_{jj} \underline{\mathbf{R}}_{ji} \underline{\mathbf{D}}_j \tag{5.71}$$

where the dimension of each of $\underline{\mathbf{A}}_{ij}$, $\underline{\mathbf{D}}_i$, $\underline{\mathbf{D}}_j$, and $\underline{\mathbf{A}}_{ji}$ is 2 × 1.

Example 5.3

Analyze the plane truss shown in Figure 5.16 using the stiffness matrix method. (EA = 10^6 kN which is constant for all members).

Solution

The joints are numbered as shown in Figure 5.17. The coordinates of each joint with respect to the chosen global coordinates are determined. The boundary conditions in the global coordinates are $\underline{\mathbf{D}}_1 = \underline{\mathbf{D}}_4 = \underline{0}$. The final force-displacement matrix relationship after applying the boundary conditions is

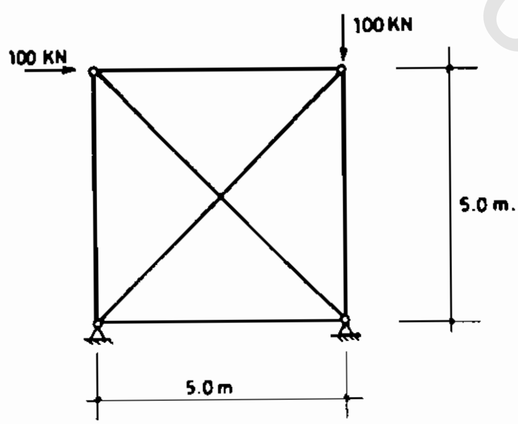


Figure 5.16

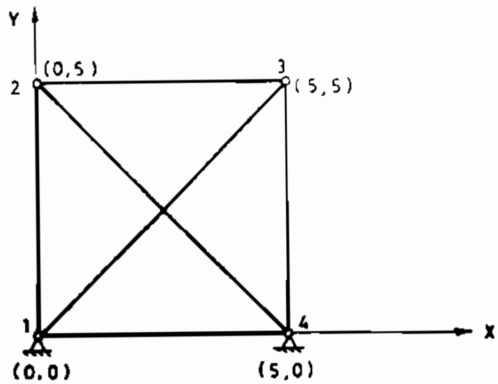


Figure 5.17

$$\begin{bmatrix} \underline{A}_2 \\ \underline{A}_3 \end{bmatrix} = \begin{bmatrix} \underline{S}_{22} & \underline{S}_{23} \\ \underline{S}_{32} & \underline{S}_{33} \end{bmatrix} \begin{bmatrix} \underline{D}_2 \\ \underline{D}_3 \end{bmatrix}$$

where the stiffness matrices $\underline{S}_{22}$ and $\underline{S}_{33}$ are obtained from

$$\underline{S}_{22} = \underline{S}^1_{22} + \underline{S}^3_{22} + S^4_{22}$$

$$\underline{S}_{33} = \underline{S}^1_{33} + \underline{S}^2_{33} + \underline{S}^4_{33}$$

Therefore, one has to determine the stiffness matrices $\underline{S}^1_{22}, \underline{S}^3_{22}, \underline{S}^4_{22}, \underline{S}^1_{33}, \underline{S}^2_{33}, \underline{S}^4_{33}, \underline{S}_{23}, \underline{S}_{32}$, in order to solve this problem. The calculations for each matrix are shown as follows:

$$\underline{R}_{21} = \begin{bmatrix} \frac{0-0}{5} & \frac{5-0}{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\underline{S}^1_{22} = \underline{R}^T_{21}\ \underline{S}'^1_{22}\ \underline{R}_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{EA}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \left(\frac{EA}{5} \right) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{R}_{23} = \begin{bmatrix} \frac{0-5}{5} & \frac{5-5}{5} \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$\underline{S}^3_{22} = \underline{R}^T_{23}\ \underline{S}'^3_{22}\ \underline{R}_{23} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{EA}{5} \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} = \left(\frac{EA}{5} \right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{R}_{24} = \begin{bmatrix} \frac{0-5}{5\sqrt{2}} & \frac{5-0}{5\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -0.707 & 0.707 \end{bmatrix}$$

$$\underline{S}_{22}^{4} = \underline{R}_{24}^{T}\ \underline{S}'^{4}_{22}\ \underline{R}_{24} = \begin{bmatrix}-0.707\\0.707\end{bmatrix}\left[\frac{EA}{7.07}\right]\begin{bmatrix}-0.707 & 0.707\end{bmatrix} = \left(\frac{EA}{7.07}\right)\begin{bmatrix}0.5 & -0.5\\-0.5 & 0.5\end{bmatrix}$$

Therefore, $\underline{S}_{22} = \underline{S}_{22}^{1} + \underline{S}_{22}^{3} + \underline{S}_{22}^{4} = \dfrac{EA}{35.35}\begin{bmatrix}9.57 & -2.5\\-2.5 & 9.57\end{bmatrix}$

$$\underline{R}_{31} = \left[\frac{5-0}{5\sqrt{2}}\quad \frac{5-0}{5\sqrt{2}}\right] = \begin{bmatrix}0.707 & 0.707\end{bmatrix}$$

$$\underline{S}_{33}^{1} = \underline{R}_{31}^{T}\ \underline{S}'^{1}_{33}\ \underline{R}_{31} = \begin{bmatrix}0.707\\0.707\end{bmatrix}\left[\frac{EA}{7.07}\right]\begin{bmatrix}0.707 & 0.707\end{bmatrix} = \left(\frac{EA}{7.07}\right)\begin{bmatrix}0.5 & 0.5\\0.5 & 0.5\end{bmatrix}$$

$$\underline{R}_{32} = \left[\frac{5-0}{5}\quad \frac{5-0}{5}\right] = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\underline{S}_{33}^{2} = \underline{R}_{32}^{T}\ \underline{S}'^{2}_{33}\ \underline{R}_{32} = \begin{bmatrix}1\\0\end{bmatrix}\left[\frac{EA}{5}\right]\begin{bmatrix}1 & 0\end{bmatrix} = \left(\frac{EA}{5}\right)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$\underline{R}_{34} = \left[\frac{5-5}{0}\quad \frac{5-0}{5}\right] = \begin{bmatrix}0 & 1\end{bmatrix}$$

$$\underline{S}_{33}^{4} = \underline{R}_{34}^{T}\ \underline{S}'^{4}_{33}\ \underline{R}_{34} = \begin{bmatrix}0\\1\end{bmatrix}\left[\frac{EA}{5}\right]\begin{bmatrix}0 & 1\end{bmatrix} = \left(\frac{EA}{5}\right)\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$\underline{R}_{32} = -\underline{R}_{23} = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\underline{S}_{23} = \underline{R}_{23}^{T}\ \underline{S}'_{23}\ \underline{R}_{32} = \begin{bmatrix}-1\\0\end{bmatrix}\left[\frac{EA}{5}\right]\begin{bmatrix}1 & 0\end{bmatrix} = \left(\frac{EA}{5}\right)\begin{bmatrix}-1 & 0\\0 & 0\end{bmatrix}$$

$$\underline{S}_{32} = \underline{S}_{32}^{T} = \frac{EA}{5}\begin{bmatrix}-1 & 0\\0 & 0\end{bmatrix}$$

$$\underline{S}_{33} = \underline{S}_{33}^{1} + \underline{S}_{33}^{2} + \underline{S}_{33}^{4} = \frac{EA}{35.35}\begin{bmatrix}9.57 & 2.5\\2.5 & 9.57\end{bmatrix}$$

$$\underline{A}_{2} = \begin{bmatrix}100\\0\end{bmatrix} \quad ; \quad \underline{A}_{3} = \begin{bmatrix}0\\-100\end{bmatrix}$$

Substituting these matrices into the final force-displacement stiffness matrix relationship one obtains:

$$\begin{bmatrix} 100 \\ 0 \\ 0 \\ -100 \end{bmatrix} = \left(\frac{EA}{35.35}\right) \begin{bmatrix} 9.57 & -2.5 & -7.07 & 0 \\ -2.5 & 9.57 & 0 & 0 \\ -7.07 & 0 & 9.57 & 2.5 \\ 0 & 0 & 2.5 & 9.57 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ D_{3x} \\ D_{3y} \end{bmatrix}$$

The solution for $\underline{\mathbf{D}}_2$ and $\underline{\mathbf{D}}_3$ is

$$\begin{bmatrix} D_{2x} & D_{2y} & D_{3x} & D_{3y} \end{bmatrix}^T = 10^{-6} \begin{bmatrix} 1288.68 & 336.638 & 1125.296 & -663.343 \end{bmatrix} \text{ m.}$$

The members forces can thus be determined as follows:

$$A'_{12} = \underline{\mathbf{S}}'^{2}_{11}\, \underline{\mathbf{R}}_{12}\, \underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{12}\, \underline{\mathbf{R}}_{21}\, \underline{\mathbf{D}}_2$$
$$= 0 + \left[\frac{EA}{5}\right] \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \end{bmatrix} = +67.327 \text{ kN}$$

$$A'_{23} = \underline{\mathbf{S}}'^{3}_{22}\, \underline{\mathbf{R}}_{23}\, \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\, \underline{\mathbf{R}}_{32}\, \underline{\mathbf{D}}_3$$
$$= \left[\frac{EA}{5}\right] \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \end{bmatrix} + \left[\frac{EA}{5}\right] \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} D_{3x} \\ D_{3y} \end{bmatrix} = -32.677 \text{ kN}$$

$$A'_{34} = \underline{\mathbf{S}}'^{4}_{33}\, \underline{\mathbf{R}}_{34}\, \underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{34}\, \underline{\mathbf{R}}_{43}\, \underline{\mathbf{D}}_4$$
$$= \left[\frac{EA}{5}\right] \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} D_{3x} \\ D_{3y} \end{bmatrix} + 0 = -132.668 \text{ kN}$$

$$A'_{41} = \underline{\mathbf{S}}'^{1}_{44}\, \underline{\mathbf{R}}_{41}\, \underline{\mathbf{D}}_4 + \underline{\mathbf{S}}'^{1}_{41}\, \underline{\mathbf{R}}_{14}\, \underline{\mathbf{D}}_1 = 0$$

$$A'_{13} = \underline{\mathbf{S}}'^{3}_{11}\, \underline{\mathbf{R}}_{13}\, \underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{13}\, \underline{\mathbf{R}}_{31}\, \underline{\mathbf{D}}_3$$
$$= 0 + \left[\frac{EA}{7.07}\right] \begin{bmatrix} 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} D_{3x} \\ D_{3y} \end{bmatrix} = +46.193 \text{ kN}$$

$$A'_{24} = \underline{\mathbf{S}}'^{4}_{22}\, \underline{\mathbf{R}}_{24}\, \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{24}\, \underline{\mathbf{R}}_{42}\, \underline{\mathbf{D}}_4$$
$$= \left[\frac{EA}{7.07}\right] \begin{bmatrix} -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \end{bmatrix} + 0 = -95.2 \text{ kN}$$

The reactions at the supports can be determined in the global coordinates as follows:

$\underline{\mathbf{A}}_1 = \underline{\mathbf{A}}_{12} + \underline{\mathbf{A}}_{13} + \underline{\mathbf{A}}_{14}$, where

$$\underline{\mathbf{A}}_{12} = \underline{\mathbf{R}}_{12}^{T} \, A'_{12} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} (67.327) = \begin{bmatrix} 0 \\ -67.327 \end{bmatrix} \text{ kN}$$

$$\underline{\mathbf{A}}_{13} = \underline{\mathbf{R}}_{13}^{T} \, A'_{13} = \begin{bmatrix} -0.701 \\ -0.707 \end{bmatrix} (46.193) = \begin{bmatrix} -32.658 \\ -32.658 \end{bmatrix} \text{ kN}$$

$$\underline{\mathbf{A}}_{14} = \underline{\mathbf{R}}_{14}^{T} \, A'_{14} = 0$$

This gives

$$\underline{\mathbf{A}}_{1} = \begin{bmatrix} -32.658 \\ -99.985 \end{bmatrix} \text{ kN}$$

Similarly, at joint 4 one has

$$\underline{\mathbf{A}}_{4} = \underline{\mathbf{A}}_{41} + \underline{\mathbf{A}}_{42} + \underline{\mathbf{A}}_{43} \quad , \quad \text{where}$$

$$\underline{\mathbf{A}}_{41} = \underline{\mathbf{R}}_{41}^{T} \, A'_{41} = 0$$

$$\underline{\mathbf{A}}_{42} = \underline{\mathbf{R}}_{42}^{T} \, A'_{42} = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix} (-95.2) = \begin{bmatrix} -67.306 \\ 67.306 \end{bmatrix} \text{ kN}$$

$$\underline{\mathbf{A}}_{43} = \mathbf{R}_{43}^{T} \, A'_{43} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} (-132.668) = \begin{bmatrix} 0 \\ 132.668 \end{bmatrix} \text{ kN}$$

which gives

$$\underline{\mathbf{A}}_{4} = \begin{bmatrix} 67.342 \\ 199.985 \end{bmatrix} \text{ kN}$$

It is obvious that $\underline{\mathbf{A}}_1 + \underline{\mathbf{A}}_2 + \underline{\mathbf{A}}_3 + \underline{\mathbf{A}}_4 = \underline{\mathbf{0}}$ which is used as a check of the obtained solution. The member forces and reactions are shown in Figure 5.18.

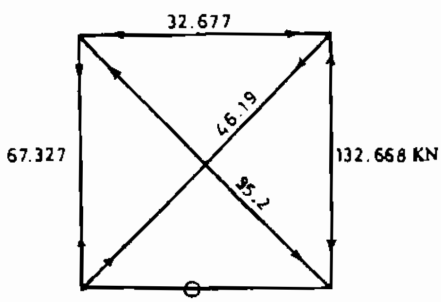


Figure 5.18

5.5.2 Applications to Space Trusses

The force-displacement relationships for member ij in a space truss expressed in the local coordinates are the same as Equation 5.61 and 5.62. The transformation matrix, however, consists of the direction cosines of the local axis x' with respect to the chosen three global cartesian axes x-y-z. With the same reasoning which enabled driving Equations 5.65 and 5.67, the transformation matrices for member ij in space truss are

$$\underline{\mathbf{R}}_{ij} = \left[\left(\frac{x_i - x_j}{L}\right)\left(\frac{y_i - y_j}{L}\right)\left(\frac{z_i - z_j}{L}\right)\right] \tag{5.72}$$

$$\underline{\mathbf{R}}_{ji} = -\underline{\mathbf{R}}_{ij} \tag{5.73}$$

The force-displacement relationship in the global coordinates are the same as Equation 5.70 and 5.71.

Example 5.4

The space truss shown in Figure 5.19 has three hinged supports at joints (1), (3), and (4). Analyze the truss due to the applied loads and a vertical settlement at support (1) of 0.1 cm downward (EA = 10^6 kN for all members).

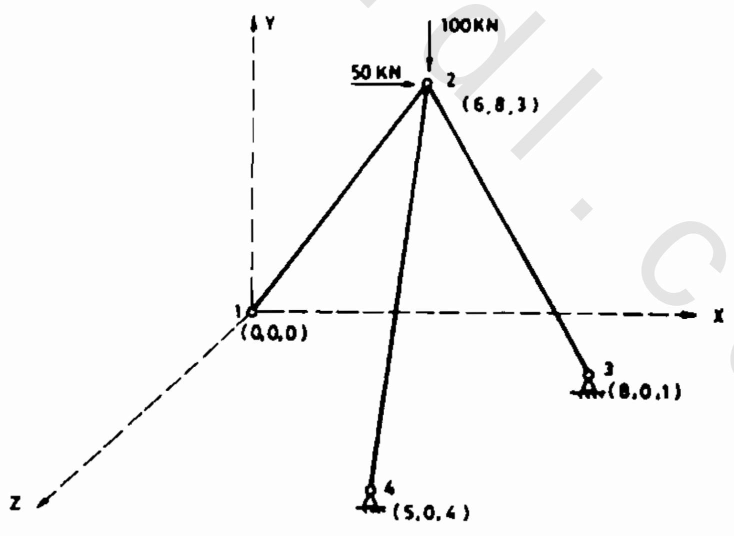


Figure 5.19

Solution

The boundary conditions in the truss are $\underline{\mathbf{D}}_3 = \underline{\mathbf{0}}, \underline{\mathbf{D}}_4 = \underline{\mathbf{0}}, D_{1x} = 0, D_{1y} = -0.01\,\text{m}$, and $D_{1z} = 0$. The force-displacement relationship which needs to be determined in order to solve the problem is

$$\begin{bmatrix} \underline{\mathbf{A}}_1 \\ \underline{\mathbf{A}}_2 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \end{bmatrix}$$

where

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}_{11}^2 \quad ; \quad \underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^1 + \underline{\mathbf{S}}_{22}^3 + \underline{\mathbf{S}}_{22}^4.$$

The calculations for these matrices are as follow:

$$\underline{\mathbf{R}}_{12} = \begin{bmatrix} \dfrac{0-6}{10.44} & \dfrac{0-8}{10.44} & \dfrac{0-3}{10.44} \end{bmatrix} = [-0.5747 \quad -0.7663 \quad -0.2873]$$

$$\underline{\mathbf{S}}_{11}^2 = \underline{\mathbf{R}}_{12}^T \ \underline{\mathbf{S}}'^2_{11} \ \underline{\mathbf{R}}_{12} = \begin{bmatrix} -0.5747 \\ -0.7663 \\ -0.2873 \end{bmatrix} \begin{bmatrix} \dfrac{EA}{10.44} \end{bmatrix} [-0.5757 \quad -0.7663 \quad -0.2873]$$

$$= \left(\frac{EA}{10.44}\right) \begin{bmatrix} 0.3303 & 0.4404 & 0.1651 \\ 0.4404 & 0.5872 & 0.2202 \\ 0.1651 & 0.2202 & 0.0825 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{21} = -\underline{\mathbf{R}}_{12}$$

$$\underline{\mathbf{S}}_{22}^1 = \underline{\mathbf{R}}_{21}^T \ \underline{\mathbf{S}}'^1_{22} \ \underline{\mathbf{R}}_{21} = \left(\frac{EA}{10.44}\right) \begin{bmatrix} 0.3303 & 0.4404 & 0.1651 \\ 0.4404 & 0.5872 & 0.2202 \\ 0.1651 & 0.2202 & 0.0825 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} \dfrac{6-8}{8.485} & \dfrac{8-0}{8.485} & \dfrac{3-1}{8.485} \end{bmatrix} = [-0.2357 \quad 0.9428 \quad 0.2357]$$

$$\underline{\mathbf{S}}_{22}^3 = \underline{\mathbf{R}}_{23}^T \ \underline{\mathbf{S}}'^3_{22} \ \underline{\mathbf{R}}_{23} = \begin{bmatrix} -0.2357 \\ 0.9428 \\ 0.2357 \end{bmatrix} \begin{bmatrix} \dfrac{EA}{8.485} \end{bmatrix} [-0.2357 \quad 0.9428 \quad 0.2351]$$

$$= \left(\frac{EA}{8.485}\right) \begin{bmatrix} 0.0555 & -0.222 & -0.0555 \\ 0.2222 & 0.8888 & 0.2222 \\ 0.0555 & 0.2222 & 0.0555 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{24} = \begin{bmatrix} \dfrac{6-5}{8.124} & \dfrac{8-0}{8.124} & \dfrac{3-4}{8.124} \end{bmatrix} = \begin{bmatrix} 0.1231 & 0.9847 & -0.1231 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22}^{4} = \underline{\mathbf{R}}_{24}^{T}\ \underline{\mathbf{S}}'^{4}_{22}\ \underline{\mathbf{R}}_{24} = \begin{bmatrix} 0.1231 \\ 0.9847 \\ -0.1231 \end{bmatrix} \begin{bmatrix} \dfrac{EA}{8.124} \end{bmatrix} \begin{bmatrix} 0.1231 & 0.9847 & -0.1231 \end{bmatrix}$$

$$= \left(\frac{EA}{8.124}\right) \begin{bmatrix} 0.0152 & 0.1212 & -0.0152 \\ 0.1212 & 0.9696 & -0.1212 \\ -0.0152 & -0.1212 & 0.0152 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{12} = \underline{\mathbf{R}}_{12}^{T}\ \underline{\mathbf{S}}'_{12}\ \underline{\mathbf{R}}_{21} = \begin{bmatrix} -0.5747 \\ -0.7663 \\ -0.2873 \end{bmatrix} \begin{bmatrix} \dfrac{EA}{10.44} \end{bmatrix} \begin{bmatrix} 0.5747 & 0.7663 & 0.2873 \end{bmatrix}$$

$$= \left(\frac{EA}{10.44}\right) \begin{bmatrix} -0.3303 & -0.4404 & -0.1651 \\ -0.4404 & -0.5872 & -0.2202 \\ -0.1651 & -0.2202 & -0.0825 \end{bmatrix}$$

Therefore, $\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3} + \underline{\mathbf{S}}_{22}^{4}$ gives

$$\underline{\mathbf{S}}_{22} = \left(\frac{EA}{100}\right) \begin{bmatrix} 4.005 & 3.092 & 0.740 \\ 3.092 & 28.034 & 3.236 \\ 0.740 & 3.236 & 1.631 \end{bmatrix}$$

$$\underline{\mathbf{A}}_{2}^{T} = \begin{bmatrix} +50 & -100 & 0 \end{bmatrix}$$

The substitution into the final force-displacement relationship gives

$$\begin{bmatrix} A_{1x} \\ A_{1y} \\ A_{1z} \\ 50 \\ -100 \\ 0 \end{bmatrix} = \left(\frac{EA}{100}\right) \begin{bmatrix} 3.1638 & 4.128 & 1.581 & -3.1638 & -4.218 & -1.581 \\ 4.218 & 5.624 & 2.109 & -4.218 & -5.624 & -2.109 \\ 1.581 & 2.109 & 0.790 & -1.581 & -2.109 & -0.79 \\ -3.1638 & -4.218 & -1.581 & 4.005 & 3.092 & 0.742 \\ -4.218 & -5.624 & -2.109 & 3.092 & 28.034 & 3.236 \\ -1.581 & -2.109 & -0.79 & 0.724 & 3.236 & 1.631 \end{bmatrix} \begin{bmatrix} 0 \\ -0.001 \\ 0 \\ D_{2x} \\ D_{2y} \\ D_{2z} \end{bmatrix}$$

which can be partitioned to give

$$\begin{bmatrix} 50 \\ -100 \\ 0 \end{bmatrix} = \left(\frac{EA}{100}\right) \begin{bmatrix} -4.218 \\ -5.624 \\ -2.109 \end{bmatrix} (-0.001) + \left(\frac{EA}{100}\right) \begin{bmatrix} 4.005 & 3.092 & 0.742 \\ 3.092 & 28.034 & 3.236 \\ 0.742 & 3.236 & 1.632 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ D_{2z} \end{bmatrix}$$

The solution for the free displacements at joint 2 is

$$\underline{\mathbf{D}}_2^T = 10^{-4}\begin{bmatrix}7.3 & -5.84 & -4.665\end{bmatrix} \text{ m.}$$

The member axial force is obtained as follows:

$$A'_{12} = \underline{\mathbf{S}}'^{2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 = 57.88 \text{ kN}$$

$$A'_{42} = \underline{\mathbf{S}}'_{42}\,\underline{\mathbf{R}}_{24}\,\underline{\mathbf{D}}_2 = -52.655 \text{ kN}$$

$$A'_{32} = \underline{\mathbf{S}}'_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 = -98.127 \text{ kN}$$

The reactions are obtained from

$$\underline{\mathbf{A}}_1 = \underline{\mathbf{R}}_{12}^T\ \underline{A}'_{12} = \begin{bmatrix}-0.5747\\ -0.7663\\ -.0.2873\end{bmatrix}\begin{bmatrix}57.88\end{bmatrix} = \begin{bmatrix}-33.2636\\ -44.353\\ -16.628\end{bmatrix} \text{ kN}$$

$$\underline{\mathbf{A}}_3 = \underline{\mathbf{R}}_{32}^T\ \underline{A}'_{32} = \begin{bmatrix}-0.2357\\ -0.9428\\ -.2357\end{bmatrix}\begin{bmatrix}-98.127\end{bmatrix} = \begin{bmatrix}-23.128\\ 92.514\\ 23.128\end{bmatrix} \text{ kN}$$

$$\underline{\mathbf{A}}_4 = \underline{\mathbf{R}}_{42}^T\ \underline{\mathbf{A}}'_{42} = \begin{bmatrix}-0.1231\\ -0.9847\\ +0.1231\end{bmatrix}\begin{bmatrix}-52.655\end{bmatrix} = \begin{bmatrix}6.482\\ 51.849\\ -6.482\end{bmatrix} \text{ kN}$$

The equilibrium can be checked from

$$\underline{\mathbf{A}}_1 + \underline{\mathbf{A}}_2 + \underline{\mathbf{A}}_3 + \underline{\mathbf{A}}_4 = \underline{\mathbf{0}}$$

The member forces are shown in Figure 5.20.

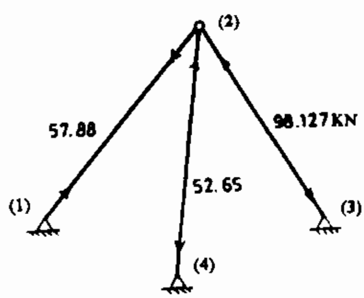


Figure 5.20

5.5.3 Applications to Plane Frames

The stiffness matrices for member ij in a plane frame described in the local coordinates have been given in Equation 5.9 and 5.10. These stiffness matrices relate the end actions, forces and moments, with the end deformations, displacements and rotations, along the local coordinates $x'-y'-z'$. In order to transform the actions-deformations relationship into global coordinates x-y-z, the following transformation matrix is used:

$$\underline{\mathbf{R}} = \begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix} \tag{5.74}$$

Considering the global coordinates x-y-z shown in Figure 5.21, one may write the transformation matrices at i and j for member ij as follows:

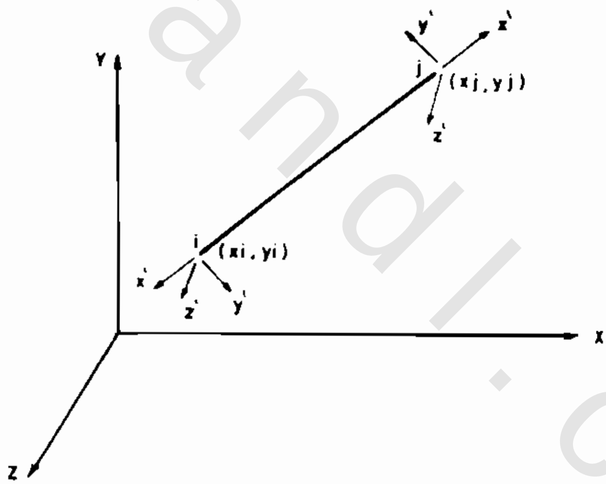


Figure 5.21

$$\underline{\mathbf{R}}_{ji} = \begin{bmatrix} (\frac{x_j - x_i}{L}) & (\frac{y_j - y_i}{L}) & 0 \\ -(\frac{y_j - y_i}{L}) & (\frac{x_j - x_i}{L}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.75}$$

$$\underline{\mathbf{R}}_{ij} = \begin{bmatrix} (\frac{x_i - x_j}{L}) & (\frac{y_i - y_j}{L}) & 0 \\ -(\frac{y_i - y_j}{L}) & (\frac{x_i - x_j}{L}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.76)$$

The actions-deformations relationships can be formulated in the global coordinates according to Equations 5.70 and 5.71.

Example 5.5

Determine the bending moment, shear force, and axial force diagrams for the frame shown in Figure 5.22 using the stiffness matrix method – approach II. ($EA = 10^6$ kN, $EI = 10^7$ kN.m^2).

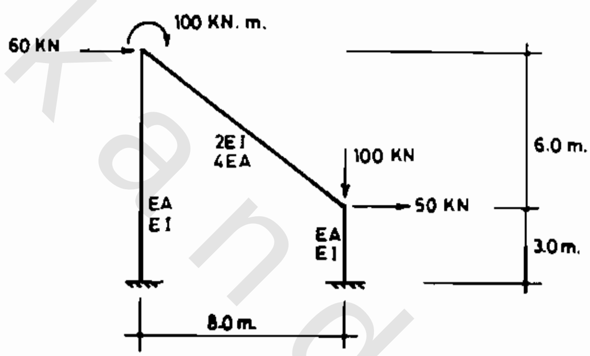


Figure 5.22

Solution

The global coordinates are selected and the locations of the joints with respect to these coordinates are determined as shown in Figure 5.23.

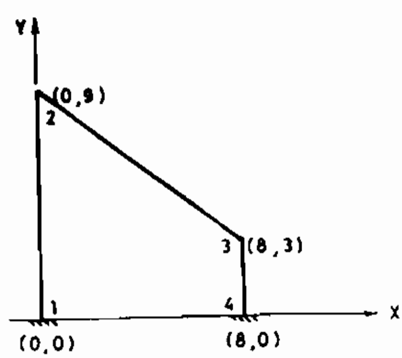


Figure 5.23

The boundary conditions in the global coordinates are $\underline{D}_1 = \underline{D}_4 = 0$. Thus the final actions-deformations relationship which is needed to solve the problem is

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} \\ \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \end{bmatrix} \quad , \quad \text{where}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^1_{21} + \underline{\mathbf{S}}^3_{22}$$

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}^2_{33} + \underline{\mathbf{S}}^4_{33}$$

These matrices are determined as follows:

To determine $\underline{\mathbf{S}}^1_{22}$:

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} (\frac{0-0}{9}) & (\frac{9-0}{9}) & 0 \\ -(\frac{9-0}{9}) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{S}}^1_{22} = \begin{bmatrix} \frac{EA}{9} & 0 & 0 \\ 0 & \frac{12EI}{9^3} & \frac{-6EI}{9^2} \\ 0 & \frac{-6EI}{9^2} & \frac{4EI}{9} \end{bmatrix} = 10^5 \begin{bmatrix} 1.111 & 0 & 0 \\ 0 & 1.646 & -7.407 \\ 0 & -7.407 & 44.444 \end{bmatrix}$$

$$\underline{\mathbf{S}}^1_{22} = \underline{\mathbf{R}}^T_{21} \; \underline{\mathbf{S}}'^1_{22} \; \underline{\mathbf{R}}_{21} = 10^5 \begin{bmatrix} 1.648 & 0 & 7.407 \\ 0 & 1.111 & 0 \\ 7.407 & 0 & 44.444 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}^3_{22}$:

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} (\frac{0-8}{10}) & (\frac{9-3}{10}) & 0 \\ (\frac{9-3}{10}) & (\frac{0-8}{10}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.8 & 0.6 & 0 \\ -0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{S}}'^{3}_{22} = 10^5 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.4 & -12 \\ 0 & -12 & 80 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{3}_{22} = \underline{\mathbf{R}}^{T}_{23}\ \underline{\mathbf{S}}'^{3}_{22}\ \underline{\mathbf{R}}_{23} = 10^5 \begin{bmatrix} 3.424 & -0.768 & 7.2 \\ -0.768 & 2.976 & 9.6 \\ 7.2 & 9.6 & 80 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^{1}_{22} + \underline{\mathbf{S}}^{3}_{22} = 10^5 \begin{bmatrix} 5.07 & -0.768 & 14.607 \\ -0.768 & 4.087 & 9.6 \\ 14.607 & 9.6 & 124.444 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}^{2}_{33}$:

$$\underline{\mathbf{R}}_{32} = \begin{bmatrix} (\frac{8-0}{10}) & (\frac{3-9}{10}) & 0 \\ -(\frac{3-9}{10}) & (\frac{8-0}{10}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{2}_{33} = \underline{\mathbf{R}}^{T}_{32}\ \underline{\mathbf{S}}'^{2}_{33}\ \underline{\mathbf{R}}_{32} = 10^5 \begin{bmatrix} 3.424 & -0.768 & -7.2 \\ -0.768 & 2.976 & -9.6 \\ -7.2 & -9.6 & 80 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}^{4}_{33}$

$$\underline{\mathbf{R}}_{34} = \begin{bmatrix} (\frac{0-0}{3}) & (\frac{3-0}{3}) & 0 \\ -(\frac{3-0}{3}) & (\frac{0-0}{3}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{S}}'^{4}_{33} = 10^5 \begin{bmatrix} 3.333 & 0 & 0 \\ 0 & 44.444 & -66.667 \\ 0 & -66.667 & 133.333 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{4}_{33} = \underline{\mathbf{R}}^{T}_{34}\ \underline{\mathbf{S}}'^{4}_{33}\ \underline{\mathbf{R}}_{34} = 10^5 \begin{bmatrix} 44.444 & 0 & 66.667 \\ 0 & 3.333 & 0 \\ 66.667 & 0 & 133.333 \end{bmatrix}$$

$$\underline{S}_{33} = \underline{S}_{33}^{2} + \underline{S}_{33}^{4} = 10^5 \begin{bmatrix} 47.868 & -0.768 & 59.467 \\ -0.768 & 6.309 & -9.6 \\ 59.467 & -9.6 & 213.333 \end{bmatrix}$$

To determine $\underline{S}_{23}$, one has

$$\underline{S}_{23} = \underline{R}_{23}^{T}\ \underline{S}'_{23}\ \underline{R}_{32} = 10^5 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.4 & -12 \\ 0 & -12 & 40 \end{bmatrix}$$

$$\underline{S}_{23} = \underline{S}_{32}^{T} = 10^5 \begin{bmatrix} 3.424 & 0.768 & 7.2 \\ 0.768 & -2.976 & 9.6 \\ -7.2 & -9.6 & 40 \end{bmatrix}$$

$$\underline{A}_2 = \begin{bmatrix} +60 \\ 0 \\ -100 \end{bmatrix} \quad ; \quad \underline{A}_3 = \begin{bmatrix} +50 \\ -100 \\ 0 \end{bmatrix}$$

The final actions-deformations relationship can thus be formed as follows:

$$\begin{bmatrix} 60 \\ 0 \\ -100 \\ 50 \\ -100 \\ 0 \end{bmatrix} = 10^5 \begin{bmatrix} 5.07 & -0.768 & 14.607 & -3.424 & 0.768 & 7.2 \\ -0.768 & 4.087 & 9.6 & 0.768 & -2.976 & 9.7 \\ 14.607 & 9.6 & 124.444 & -7.2 & -9.6 & 40 \\ -3.424 & 0.768 & -7.2 & 47.868 & -0.768 & 59.467 \\ 0.768 & -2.976 & -9.6 & -0.768 & 6.309 & -9.6 \\ 7.2 & 9.6 & 40 & 59.467 & -9.6 & 213.333 \end{bmatrix} \begin{bmatrix} \underline{D}_2 \\ \underline{D}_3 \end{bmatrix}$$

The solution is for $\underline{D}_2$ and $\underline{D}_3$ is obtained as

$$\begin{bmatrix} \underline{D}_2^T & \underline{D}_3^T \end{bmatrix} = 10^{-5} \begin{bmatrix} 68.713 & 23.395 & -11.034 & 10.785 & -37.805 & -6.010 \end{bmatrix}$$

which are in meter and radian units.

The member end actions are determined as follows:

$$\underline{A}'_{12} = \underline{S}'^{2}_{11}\ \underline{R}_{12}\ \underline{D}_1 + \underline{S}'_{12}\ \underline{R}_{21}\ \underline{D}_2$$

$$= 10^5 \begin{bmatrix} 1.111 & 0 & 0 \\ 0 & 1.646 & -7.407 \\ 0 & -7.407 & 22.222 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{D}_2 = \begin{bmatrix} 25.992 \\ -31.373 \\ 263.760 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{1}_{22}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{21}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1$$

$$= 10^5 \begin{bmatrix} 1.111 & 0 & 0 \\ 0 & 1.646 & -7.407 \\ 0 & -7.407 & 44.444 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{D}}_2 = \begin{bmatrix} 25.992 \\ -31.373 \\ 18.562 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{43} = \underline{\mathbf{S}}'^{2}_{3}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 + \underline{\mathbf{S}}'_{43}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3$$

$$= 10^5 \begin{bmatrix} 3.333 & 0 & 0 \\ 0 & 44.444 & -66.667 \\ 0 & -66.667 & 66.667 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{D}}_3 = \begin{bmatrix} -126.004 \\ -78.660 \\ 318.335 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{34} = \underline{\mathbf{S}}'^{4}_{33}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3$$

$$= \begin{bmatrix} 3.333 & 0 & 0 \\ 0 & 44.444 & -66.667 \\ 0 & -66.667 & 133.333 \end{bmatrix} \begin{bmatrix} -37.805 \\ -10.785 \\ -6.01 \end{bmatrix} = \begin{bmatrix} -126.004 \\ -78.660 \\ -82.33 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.4 & -12 \\ 0 & -12 & 80 \end{bmatrix} \begin{bmatrix} -40.933 \\ -59.944 \\ -11.034 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.4 & -12 \\ 0 & -12 & 40 \end{bmatrix} \begin{bmatrix} 31.311 \\ -23.773 \\ -6.01 \end{bmatrix} = \begin{bmatrix} -38.488 \\ 3.6074 \\ -118.516 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

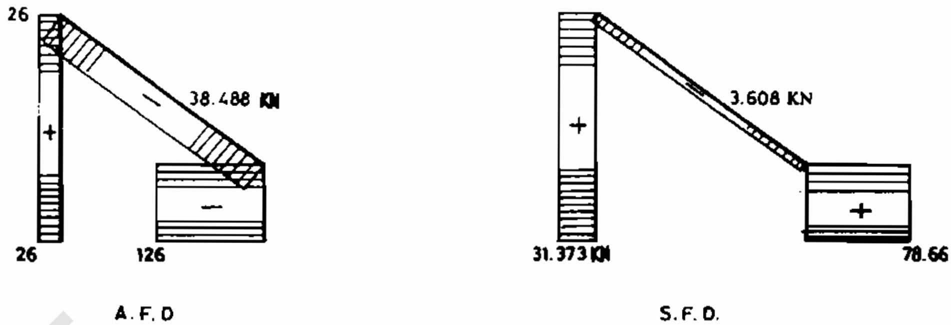
$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.4 & -12 \\ 0 & -12 & 80 \end{bmatrix} \begin{bmatrix} 3.311 \\ -23.773 \\ -6.01 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2.4 & -12 \\ 0 & -12 & 40 \end{bmatrix} \begin{bmatrix} -40.933 \\ -59.94 \\ -11.034 \end{bmatrix} = \begin{bmatrix} -38.486 \\ 3.608 \\ 82.444 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The axial force, shear force and bending moment diagrams are shown in Figure 5.24.

In some types of frames, the axial deformations can be neglected. In this case, the stiffness matrices in the local coordinates are given by Equations 5.11 and 5.12. The stiffness coefficients in these matrices represent shear force along axis y' and bending moment about axis z'. In order to transform these matrices into the global coordinates shown in Figure 5.25, the following relations are established:

$$\begin{bmatrix} 0 \\ A'_y \\ M'_z \end{bmatrix}_{ij} = \begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix}_{ij} \begin{bmatrix} A_x \\ A_y \\ M_z \end{bmatrix}_{ij}$$



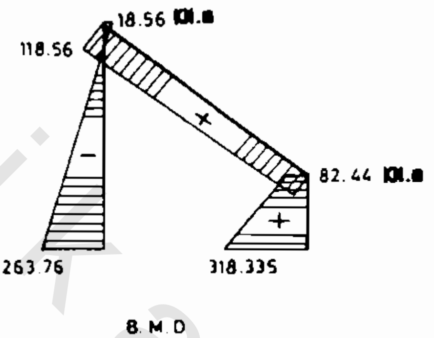


Figure 5.24

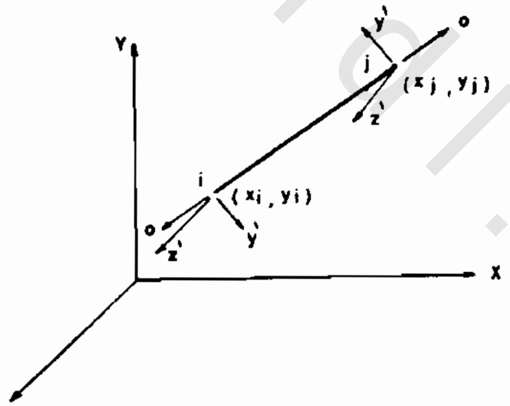


Figure 5.25

which leads to

$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{R}}_{ij}\, \underline{\mathbf{A}}_{ij}$, where

$$\underline{\mathbf{R}}_{ij} = \begin{bmatrix} \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix}$$

$$= -\begin{bmatrix} \left(\dfrac{y_i - y_j}{L}\right) & \left(\dfrac{x_i - x_j}{L}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.77}$$

The orthogonality properties to the above transformation matrix is analogous to if one considers the end actions and deformations are vectors each of dimension (3×1). One can also solve these kinds of frames in the usual manner but assuming EA is very large value.

Example 5.6

Determine the axial force, shear force and bending moment diagrams for the frame of Figure 5.23 (Example 5.5) neglecting the axial deformations, ($EI = 10^7$ kN.m^2).

Solution

Using the matrices calculated in example 5.5, the stiffness matrices in the global coordinates are determined as follows:

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; \underline{\mathbf{S}}'^{1}_{22} = 10^5 \begin{bmatrix} 1.646 & -7.407 \\ -7.407 & 44.444 \end{bmatrix}$$

$$\underline{\mathbf{S}}'_{22} = \underline{\mathbf{R}}^{T}_{21}\, \underline{\mathbf{S}}'^{1}_{22}\, \underline{\mathbf{R}}_{21} = 10^5 \begin{bmatrix} 1.646 & 0 & 7.407 \\ 0 & 0 & 0 \\ 7.407 & 0 & 44.444 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} 0.8 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; \underline{\mathbf{S}}'^{3}_{22} = 10^5 \begin{bmatrix} 2.4 & -12 \\ -12 & 80 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{3}_{22} = \underline{\mathbf{R}}^{T}_{23}\, \underline{\mathbf{S}}'^{3}_{22}\, \underline{\mathbf{R}}_{23} = 10^5 \begin{bmatrix} 0.864 & 1.152 & 7.2 \\ 1.152 & 1.536 & 9.6 \\ 7.2 & 9.6 & 80 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^{1}_{22} + \underline{\mathbf{S}}^{3}_{22} = 10^6 \begin{bmatrix} 2.51 & 1.152 & 14.607 \\ 1.152 & 1.536 & 9.6 \\ 14.607 & 9.6 & 124.444 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{2}_{33} = \underline{\mathbf{R}}^{T}_{32}\, \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32} = 10^5 \begin{bmatrix} 0.864 & 1.152 & -7.2 \\ 1.152 & 1.536 & -9.6 \\ -7.2 & -9.6 & 80 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{34} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; \underline{\mathbf{S}}'^{4}_{33} = 10^5 \begin{bmatrix} 44.44 & -66.667 \\ -66.667 & 133.333 \end{bmatrix}$$

$$\underline{S}_{33}^{4} = \underline{R}_{34}^{T}\,\underline{S}'^{4}_{33}\,\underline{R}_{34} = 10^5 \begin{bmatrix} 44.444 & 0 & 66.667 \\ 0 & 0 & 0 \\ 66.667 & 0 & 133.333 \end{bmatrix}$$

$$\underline{S}_{33} = \underline{S}_{33}^{2} + \underline{S}_{33}^{4} = 10^5 \begin{bmatrix} 45.308 & 1.152 & 59.467 \\ 1.152 & 1.536 & -9.6 \\ 59.467 & -9.6 & 213.333 \end{bmatrix}$$

$$\underline{R}_{32} = \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{S}'_{23} = 10^5 \begin{bmatrix} 2.4 & -12 \\ -12 & 40 \end{bmatrix}$$

$$\underline{S}_{23} = \underline{R}_{23}^{T}\,\underline{S}'_{23}\,\underline{R}_{32} = 10^5 \begin{bmatrix} -0.864 & -1.152 & 7.2 \\ -1.152 & -1.536 & 9.6 \\ -7.2 & -9.6 & 40 \end{bmatrix}$$

The final actions-deformations relationship can thus be formed as follows:

$$\begin{bmatrix} 60 \\ 0 \\ -100 \\ 50 \\ -100 \\ 0 \end{bmatrix} = 10^5 \begin{bmatrix} 2.51 & 1.152 & 14.607 & -0.864 & -1.152 & 7.2 \\ 1.152 & 1.536 & 9.6 & -1.152 & -1.536 & 9.6 \\ 14.607 & 9.6 & 124.444 & -7.2 & -9.6 & 40 \\ -0.864 & -1.152 & -7.2 & 45.308 & 1.152 & 59.467 \\ -1.152 & -1.536 & -9.6 & 1.152 & 1.536 & -9.6 \\ 7.2 & 9.6 & 40 & 59.467 & -9.6 & 213.333 \end{bmatrix} \begin{bmatrix} \underline{D}_2 \\ \underline{D}_3 \end{bmatrix}$$

The above matrix is singular, which does not give a unique solution. To obtain a unique solution one specifies the boundary conditions due to neglecting the axial deformations as $D_{2y} = D_{3y} = 0$, and $D_{2x} = D_{3x}$. Therefore, by eliminating the second and fourth rows and columns, and expressing D_{3x} as D_{2x} one obtains

$$\begin{bmatrix} 110 \\ -100 \\ 0 \end{bmatrix} = 10^5 \begin{bmatrix} 46.09 & 7.407 & 66.667 \\ 7.407 & 124.44 & 40 \\ 66.667 & 40 & 213.333 \end{bmatrix} \begin{bmatrix} D_{2x} \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

The solution for the deformations is

$$\begin{bmatrix} D_{2x} & \theta_2 & \theta_3 \end{bmatrix} = 10^{-5} \begin{bmatrix} 4.2215 & -0.671 & -1.1934 \end{bmatrix}$$

Therefore, $\underline{D}_2$ and $\underline{D}_3$ are expressed as

$$\underline{D}_2^T = 10^{-5}\begin{bmatrix} 4.2215 & 0 & -0.671 \end{bmatrix} \quad ; \quad \underline{D}_3^T = 10^{-5}\begin{bmatrix} 4.2215 & 0 & -1.1934 \end{bmatrix}$$

The member end actions are obtained as usual, which provide the shear force and bending moment at the end joint. The axial force in the members can be determined from calculating the reactions, or applying static principles.

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'_{12}\, \underline{\mathbf{R}}_{21}\, \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'^{2}_{11}\, \underline{\mathbf{R}}_{12}\, \underline{\mathbf{D}}_1 = \begin{bmatrix} 1.647 & -7.407 \\ -7.407 & 22.222 \end{bmatrix} \begin{bmatrix} -4.2215 \\ -0.671 \end{bmatrix} = \begin{bmatrix} -1.983 \\ 16.357 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m} \end{matrix}$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{1}_{22}\, \underline{\mathbf{R}}_{21}\, \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{21}\, \underline{\mathbf{R}}_{12}\, \underline{\mathbf{D}}_1 = \begin{bmatrix} 1.647 & -7.407 \\ -7.407 & 44.444 \end{bmatrix} \begin{bmatrix} -4.2215 \\ -0.671 \end{bmatrix} = \begin{bmatrix} -1.983 \\ 1.446 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{43} = \underline{\mathbf{S}}'^{3}_{44}\, \underline{\mathbf{R}}_{43}\, \underline{\mathbf{D}}_4 + \underline{\mathbf{S}}'_{43}\, \underline{\mathbf{R}}_{34}\, \underline{\mathbf{D}}_3 = \begin{bmatrix} 44.444 & -66.667 \\ -66.667 & 66.667 \end{bmatrix} \begin{bmatrix} -4.2215 \\ -1.1934 \end{bmatrix} = \begin{bmatrix} -108.06 \\ 201.874 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{34} = \underline{\mathbf{S}}'^{4}_{33}\, \underline{\mathbf{R}}_{34}\, \underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{34}\, \underline{\mathbf{R}}_{43}\, \underline{\mathbf{D}}_4 = \begin{bmatrix} 44.444 & -66.667 \\ -66.667 & 133.333 \end{bmatrix} \begin{bmatrix} -4.2215 \\ -1.1934 \end{bmatrix} = \begin{bmatrix} -108.06 \\ 122.315 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\, \underline{\mathbf{R}}_{23}\, \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\, \underline{\mathbf{R}}_{32}\, \underline{\mathbf{D}}_3$$

$$= \begin{bmatrix} 2.4 & -12 \\ -12 & 80 \end{bmatrix} \begin{bmatrix} -2.5339 \\ -0.671 \end{bmatrix} + \begin{bmatrix} 2.4 & -12 \\ -12 & 40 \end{bmatrix} \begin{bmatrix} 2.5329 \\ -1.1934 \end{bmatrix} = \begin{bmatrix} 22.373 \\ -101.44 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32}\, \mathbf{D}_3 + \underline{\mathbf{S}}'_{32}\, \underline{\mathbf{R}}_{23}\, \underline{\mathbf{D}}_2$$

$$= \begin{bmatrix} 2.4 & -12 \\ -12 & 80 \end{bmatrix} \begin{bmatrix} 2.5339 \\ -1.1934 \end{bmatrix} + \begin{bmatrix} 2.4 & -12 \\ -12 & 40 \end{bmatrix} \begin{bmatrix} -2.5329 \\ -0.671 \end{bmatrix} = \begin{bmatrix} 22.373 \\ -122.312 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

The end actions and corresponding diagrams are given in Figure 5.26. Comparing these results with Figure 5.24, one observes the significance of neglecting the axial deformations in some frame structures.

5.5.4 Applications to Beams

Beams can be treated like plane frames if axial deformations are taken into consideration. However, when axial deformations are ignored, we deal with the transverse displacement and rotation at every joint. In this case, the stiffness matrices are as given by Equations 5.11 and 5.12, and the transformation matrices using the global coordinates shown in Figure 5.27 are

$$\underline{\mathbf{R}}_{ji} = \begin{bmatrix} \cos(y'y) & \cos(y'z) \\ \cos(z'y) & \cos(z'z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.78}$$

$$\underline{\mathbf{R}}_{ij} = \begin{bmatrix} \cos(y'y) & \cos(y'z) \\ \cos(z'y) & \cos(z'z) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.79}$$

Example 5.7

Determine the shear force and bending moment diagrams for the beam shown in Figure 5.28 (EI = 10^6 kN.m^2).

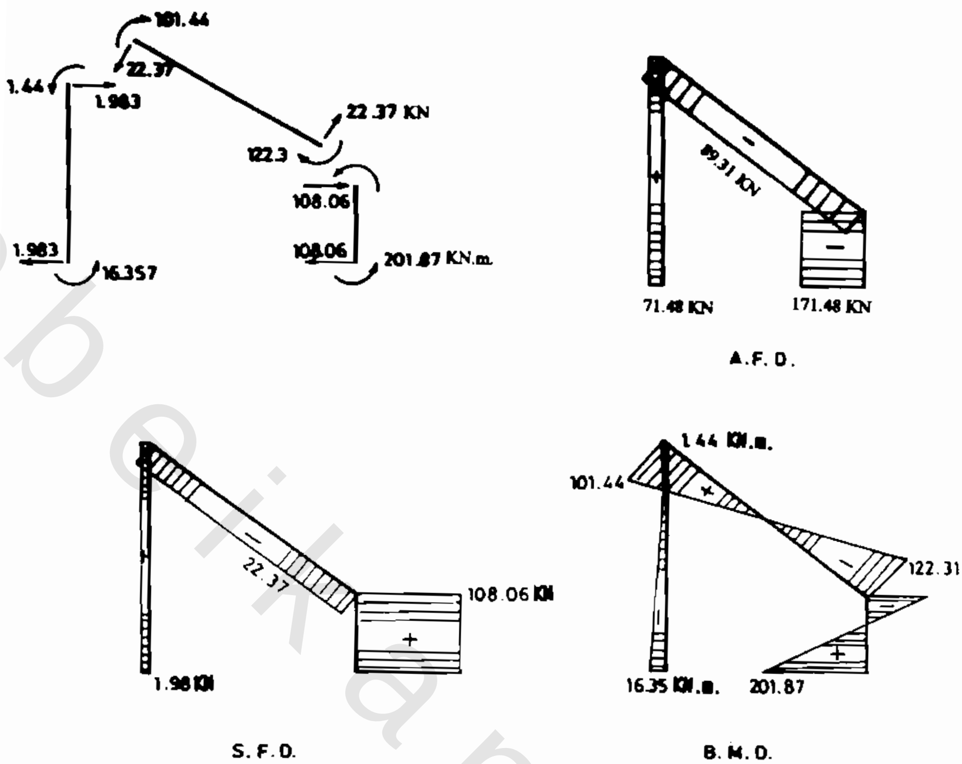


Figure 5.26

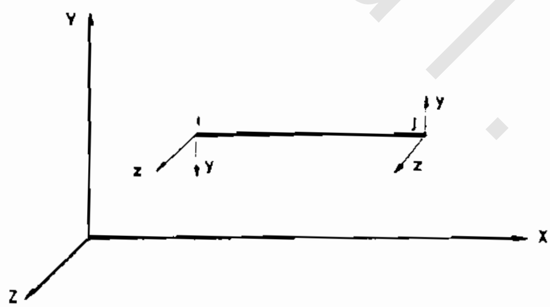


Figure 5.27

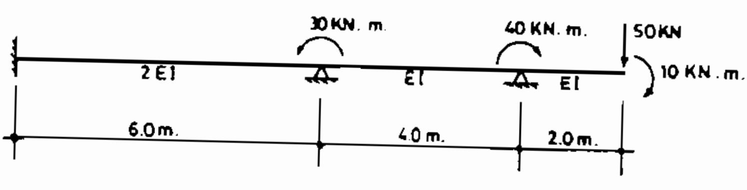


Figure 5.28

Solution

Global coordinates are selected as shown in Figure 5.29. The actions at joint 4 can be transmitted to joint 3, but in this case we shall not obtain the deformations of joint 4. To obtain the deformations at joint 4 it is necessary to consider joint 4 as a free joint.

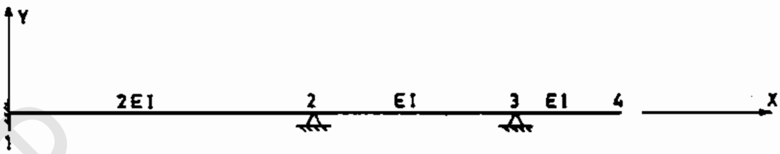


Figure 5.29

The boundary conditions are $\underline{\mathbf{D}}_1 = \underline{0}$, $D_{2y} = 0$. The final actions-deformations relation which needs to be determined to solve this problem is

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \\ \underline{\mathbf{A}}_4 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{0} \\ \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} \\ \underline{0} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \\ \underline{\mathbf{D}}_4 \end{bmatrix}, \quad \text{where}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3}$$

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}_{33}^{2} + \underline{\mathbf{S}}_{33}^{4}$$

$$\underline{\mathbf{S}}_{44} = \underline{\mathbf{S}}_{44}^{3}$$

These matrices can be determined as follows:

$$\underline{\mathbf{S}}'^{1}_{22} = 10^5 \begin{bmatrix} 1.111 & -3.333 \\ -3.33 & 13.333 \end{bmatrix} \quad ; \quad \mathbf{R}_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad \underline{\mathbf{S}}^{1}_{22} = \underline{\mathbf{R}}^{T}_{21}\, \underline{\mathbf{S}}'^{1}_{22}\, \underline{\mathbf{R}}_{21} = \underline{\mathbf{S}}'^{1}_{22}$$

$$\underline{\mathbf{S}}'^{3}_{22} = 10^5 \begin{bmatrix} 1.875 & -3.75 \\ -3.75 & 10 \end{bmatrix} ; \quad \underline{\mathbf{R}}_{23} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad \underline{\mathbf{S}}^{3}_{22} = \underline{\mathbf{R}}^{T}_{23}\, \underline{\mathbf{S}}'^{3}_{22}\, \underline{\mathbf{R}}_{23} = 10^5 \begin{bmatrix} 1.875 & 3.75 \\ 3.75 & 10 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{2}_{33} = \underline{\mathbf{R}}^{T}_{32}\, \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32} = 10^5 \begin{bmatrix} 1.875 & -3.75 \\ -3.75 & 10 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}_{23} = \underline{\mathbf{R}}^{T}_{23}\, \underline{\mathbf{S}}'_{23}\, \underline{\mathbf{R}}_{32} = 10^5 \begin{bmatrix} 1.875 & 3.75 \\ -3.75 & 5 \end{bmatrix}$$

$$\underline{S}^3_{44} = \underline{R}^T_{43}\,\underline{S}'^3_{44}\,\underline{R}_{43} = 10^5\begin{bmatrix} 15 & -15 \\ -15 & 20 \end{bmatrix} \quad ; \quad \underline{S}^4_{33} = \underline{R}^T_{34}\,\underline{S}'^4_{33}\,\underline{R}_{34} = 10^5\begin{bmatrix} 15 & 15 \\ 15 & 20 \end{bmatrix}$$

$$\underline{S}_{34} = \underline{R}^T_{34}\,\underline{S}'_{34}\,\underline{R}_{43} = 10^5\begin{bmatrix} -15 & 15 \\ -15 & 10 \end{bmatrix}$$

Substituting into the final actions-deformations relation, one obtains

$$\begin{bmatrix} R_{2y} \\ 30 \\ R_{3y} \\ -40 \\ -50 \\ -10 \end{bmatrix} = 10^5 \begin{bmatrix} 2.986 & 0.417 & -1.875 & 3.75 & 0 & 0 \\ 0.417 & 23.333 & -3.75 & 5 & 0 & 0 \\ -1.875 & -3.75 & 16.875 & 11.25 & -15 & 15 \\ 3.75 & 5 & 11.25 & 30 & -15 & 10 \\ 0 & 0 & -15 & -15 & 15 & -15 \\ 0 & 0 & 15 & 10 & -15 & 20 \end{bmatrix} \begin{bmatrix} 0 \\ \theta_2 \\ 0 \\ \theta_3 \\ D_{4y} \\ \theta_4 \end{bmatrix}$$

which can be reduced to

$$\begin{bmatrix} 30 \\ -40 \\ -50 \\ -10 \end{bmatrix} = 10^5 \begin{bmatrix} 23.333 & 5 & 0 & 0 \\ 5 & 30 & -15 & 10 \\ 0 & -15 & 15 & -15 \\ 0 & 10 & -15 & 20 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ D_{4y} \\ \theta_4 \end{bmatrix}$$

The solution gives

$$\begin{bmatrix} \theta_2 & \theta_3 & D_{4y} & \theta_4 \end{bmatrix} = 10^{-5}\,[5.04 \quad -17.52 \quad -50.373 \quad -29.52] \quad \text{(meter and radian units)}$$

The member end actions are determined as follows:

$$\underline{A}'_{12} = \underline{S}'^2_{11}\,\underline{R}_{12}\,\underline{D}_1 + \underline{S}'_{12}\,\underline{R}_{21}\,\underline{D}_2 = \begin{bmatrix} 1.111 & -3.333 \\ -3.333 & 6.667 \end{bmatrix}\begin{bmatrix} 0 \\ 5.04 \end{bmatrix} = \begin{bmatrix} -16.798 \\ 33.599 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{A}'_{21} = \underline{S}'^1_{22}\,\underline{R}_{21}\,\underline{D}_2 + \underline{S}'_{21}\,\underline{R}_{12}\,\underline{D}_1 = \begin{bmatrix} 1.111 & -3.33 \\ -3.333 & 13.333 \end{bmatrix}\begin{bmatrix} 0 \\ 5.04 \end{bmatrix} = \begin{bmatrix} -16.798 \\ 67.198 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{A}'_{23} = \underline{S}'^3_{22}\,\underline{R}_{23}\,\underline{D}_2 + \underline{S}'_{23}\,\underline{R}_{32}\,\underline{D}_3$$

$$= \begin{bmatrix} 1.875 & -3.75 \\ -3.75 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 5.04 \end{bmatrix} + \begin{bmatrix} 1.875 & -3.75 \\ -3.75 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ -17.52 \end{bmatrix} = \begin{bmatrix} 46.8 \\ -37.2 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32}\, \underline{\mathbf{D}}_{3} + \underline{\mathbf{S}}'_{32}\, \underline{\mathbf{R}}_{23}\, \underline{\mathbf{D}}_{2}$$

$$= \begin{bmatrix} 1.875 & -3.75 \\ -3.75 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ -17.52 \end{bmatrix} + \begin{bmatrix} 1.875 & -3.75 \\ -3.75 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 5.04 \end{bmatrix} = \begin{bmatrix} 46.8 \\ -150 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{34} = \underline{\mathbf{S}}'^{4}_{33}\, \underline{\mathbf{R}}_{34}\, \underline{\mathbf{D}}_{3} + \mathbf{S}'_{34}\, \underline{\mathbf{R}}_{43}\, \underline{\mathbf{D}}_{4}$$

$$= \begin{bmatrix} 15 & -15 \\ -15 & 20 \end{bmatrix} \begin{bmatrix} 0 \\ -17.52 \end{bmatrix} + \begin{bmatrix} 15 & -15 \\ -15 & 10 \end{bmatrix} \cdot \begin{bmatrix} -50.373 \\ -29.52 \end{bmatrix} = \begin{bmatrix} -50 \\ 110 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{43} = \underline{\mathbf{S}}'^{3}_{44}\, \underline{\mathbf{R}}_{43}\, \underline{\mathbf{D}}_{4} + \underline{\mathbf{S}}'_{43}\, \underline{\mathbf{R}}_{34}\, \underline{\mathbf{D}}_{3}$$

$$= \begin{bmatrix} 15 & -15 \\ -15 & 20 \end{bmatrix} \begin{bmatrix} -50.373 \\ -29.52 \end{bmatrix} + \begin{bmatrix} 15 & -15 \\ -15 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ -17.52 \end{bmatrix} = \begin{bmatrix} -50 \\ -10 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

The bending moment and shear force diagrams are shown in Figure 5.30.

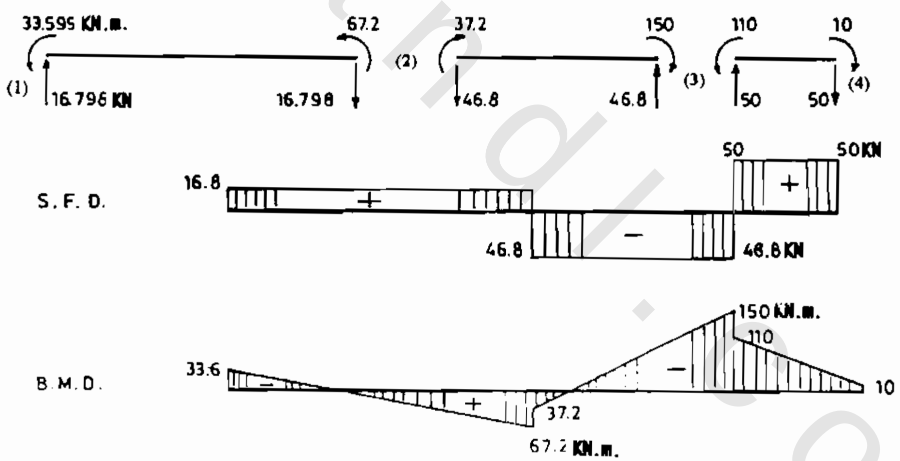


Figure 5.30

5.5.5 Applications to Grids

Stiffness matrices for grid members defined in the local coordinates were presented in Equations 5.13 and 5.14. These matrices relate the end actions M'_x, M'_y, and A'_z with the corresponding deformations θ'_x, θ'_y, and D'_z. In order to transform the action-deformations relationships into the global coordinates x-y-z shown in Figure 5.31, one has to establish the transformation matrices as follows:

$$\underline{\mathbf{R}}_{ji} = \begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'x) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix} = \begin{bmatrix} (\frac{x_j - x_i}{L}) & (\frac{y_j - y_i}{L}) & 0 \\ -(\frac{y_j - y_i}{L}) & (\frac{x_j - x_i}{L}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.80)$$

which is similar to the transformation matrix in plane frames.

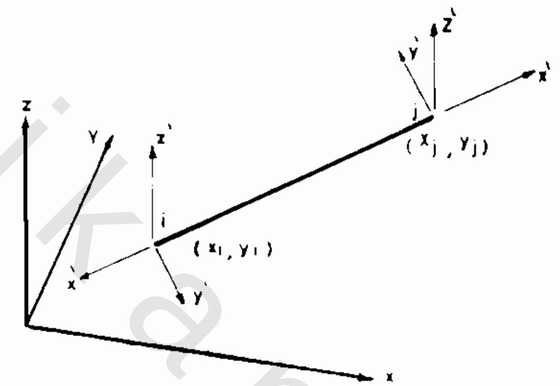


Figure 5.31

Example 5.8

Determine the end actions and deformations for the grid shown in Figure 5.32 (EI_y = 10^6 kN.m^2, $GJ_x = 10^5$ kN.m^2).

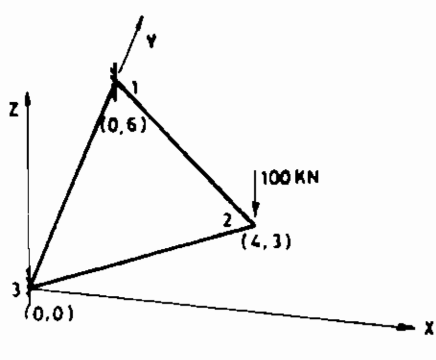


Figure 5.32

Solution

The boundary conditions are $\underline{D}_1 = \underline{D}_3 = \underline{0}$. The final actions-deformations relation which need to be determined to solve the problem is

$$\underline{A}_2 = \underline{S}_{22}\,\underline{D}_2 \quad ; \quad \text{where} \quad \underline{S}_{22} = \underline{S}^1_{22} + \underline{S}^3_{22}$$

The calculations for these matrices are shown below:

$$\underline{S}'^1_{22} = 10^5 \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 8 & 2.4 \\ 0 & 2.4 & 0.96 \end{bmatrix} \quad ; \quad \underline{R}_{21} = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{S}^1_{22} = \underline{R}^T_{21}\,\underline{S}'^1_{22}\,\underline{R}_{21} = 10^5 \begin{bmatrix} 3.008 & 3.744 & 1.44 \\ 3.744 & 5.192 & 1.92 \\ 1.44 & 1.92 & 0.96 \end{bmatrix}$$

$$\underline{S}'^3_{22} = \underline{S}'^1_{22} \quad ; \quad \underline{R}_{23} = \begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ .0 & 0 & 1 \end{bmatrix}$$

$$\underline{S}^3_{22} = \underline{R}^T_{23}\,\underline{S}'^3_{22}\,\underline{R}_{23} = 10^5 \begin{bmatrix} 3.008 & -3.744 & -1.44 \\ -3.744 & 5.192 & 1.92 \\ -1.44 & 1.92 & 0.96 \end{bmatrix}$$

$$\underline{S}_{22} = \underline{S}^1_{22} + \underline{S}^3_{22} = 10^5 \begin{bmatrix} 6.016 & 0 & 0 \\ 0 & 10.384 & 3.84 \\ 0 & 3.84 & 1.92 \end{bmatrix}$$

$$\underline{A}^T_2 = [0 \quad 0 \quad -100]$$

Solving for $\underline{D}_2$ one obtains

$$\underline{D}^T_2 = 10^{-5}\ [0 \quad 73.964 \quad -200.012] \ \text{(meter, radian units)}$$

The member end actions are determined from

$$\underline{A}'_{21} = \underline{S}'^1_{22}\,\underline{R}_{21}\,\underline{D}_2 + \underline{S}'_{21}\,\underline{R}_{12}\,\underline{D}_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 8 & 2.4 \\ 0 & 2.4 & 0.96 \end{bmatrix} \begin{bmatrix} -44.378 \\ 59.171 \\ -200.012 \end{bmatrix} = \begin{bmatrix} -8.8756 \\ -6.661 \\ -50 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{A}'_{23} = \underline{S}'^3_{22}\,\underline{R}_{23}\,\underline{D}_2 + \underline{S}'_{23}\,\underline{R}_{32}\,\underline{D}_3 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 8 & 2.4 \\ 0 & 2.4 & 0.96 \end{bmatrix} \begin{bmatrix} 44.378 \\ 59.171 \\ -200.012 \end{bmatrix} = \begin{bmatrix} 8.8756 \\ -6.561 \\ -50 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\, \mathbf{R}_{12}\, \underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{12}\, \underline{\mathbf{R}}_{21}\, \underline{\mathbf{D}}_2 = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -4 & -2.4 \\ 0 & -2.4 & -0.96 \end{bmatrix} \underline{\mathbf{R}}_{21}\, \underline{\mathbf{D}}_2 = \begin{bmatrix} +8.8756 \\ 243.345 \\ 50 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32}\, \underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{32}\, \underline{\mathbf{R}}_{23}\, \underline{\mathbf{D}}_2 = \underline{\mathbf{S}}'_{32} \begin{bmatrix} 44.378 \\ 59.171 \\ -200.012 \end{bmatrix} = \begin{bmatrix} -8.8756 \\ 243.345 \\ 50 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The internal actions in the grid members are shown in Figure 5.33.

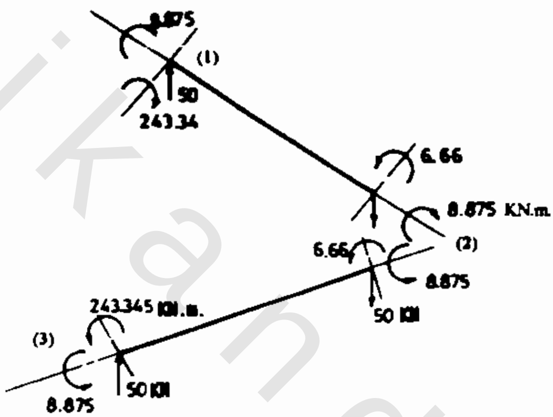


Figure 5.33

5.5.6 Applications to Space Frames

The stiffness matrices for member ij in a space frame defined in the local coordinates were given in Equations 5.1 and 5.2. These matrices relate the end actions ($A'_x, A'_y, A'_z, M'_x, M'_y, M'_z$) to the end deformations ($D'_x, D'_y, D'_z, \theta'_x, \theta'_y, \theta'_z$) which are applied in the directions of the principal axes of the cross section. Therefore, in the process of transformation into the global coordinates, one first has to define the directions of the local coordinates y' and z', since only x' axis can be located using the coordinates of the joints with respect to the chosen global coordinates. The next step is to transfer the local axes y' and z' into the cross section principal axes if they are not coincident.

The direction cosines of the local axis x' are determined from the coordinates of the two joints i and j of the member ij with respect to the global axes x-y-z as follows:

$$\ell_{ji} = \frac{x_j - x_i}{L} \quad , \quad m_{ji} = \frac{y_j - y_i}{L} \quad , \quad n_{ji} = \frac{z_j - z_i}{L} \qquad (5.81)$$

in which L is the length from i to j.

In order to determine the directions of other local axes y' and z', it is assumed that y' is perpendicular to the plane of $x'z$. Thus, y' can be obtained from

$$\underline{\mathbf{y}}'_{ji} = \frac{1}{D_{ji}}\left(\underline{\mathbf{z}} \times \underline{\mathbf{x}}'_{ji}\right) = \frac{1}{D_{ji}}\begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 0 & 0 & 1 \\ \ell_{ji} & m_{ji} & n_{ji} \end{bmatrix} = -\left(\frac{m}{D}\right)_{ji}\underline{\mathbf{i}} + \left(\frac{\ell}{D}\right)_{ji}\underline{\mathbf{j}} \tag{5.82}$$

where D_{ji} is obtained from

$$D_{ji} = \sqrt{\left(m_{ji}\right)^2 + \left(\ell_{ji}\right)^2} \tag{5.83}$$

Now, the axis z' can be determined from

$$\underline{\mathbf{z}}'_{ji} = \underline{\mathbf{x}}'_{ji} \times \underline{\mathbf{y}}'_{ji} = \begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \ell_{ji} & m_{ji} & n_{ji} \\ \dfrac{m_{ji}}{D_{ji}} & \dfrac{\ell_{ji}}{D_{ji}} & 0 \end{bmatrix} = -\left(\frac{\ell n}{D}\right)_{ji}\underline{\mathbf{i}} - \left(\frac{mn}{D}\right)_{ji}\underline{\mathbf{j}} + D_{ji}\,\underline{\mathbf{k}} \tag{5.84}$$

The transformation from the local coordinates $x'-y'-z'$ to the global coordinates x-y-z can thus be done using the following transformation matrix:

$$\underline{\mathbf{R}} = \begin{bmatrix} \ell & m & n \\ -\left(\dfrac{m}{D}\right) & \left(\dfrac{\ell}{D}\right) & 0 \\ -\left(\dfrac{\ell n}{D}\right) & -\left(\dfrac{mn}{D}\right) & D \end{bmatrix} \tag{5.85}$$

One may, however, face a problem when the value of D is zero. This means that the axis x' is parallel to z axis. In this case, the transformation matrix is usually taken, according to Figure 5.34, as follows:

$$\underline{\mathbf{R}}_{ji} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{\mathbf{R}}_{ij} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{5.86}$$

Therefore, in order to transform actions or deformations from local coordinates into global coordinates, the transformation matrix for member ij in space frames is given by

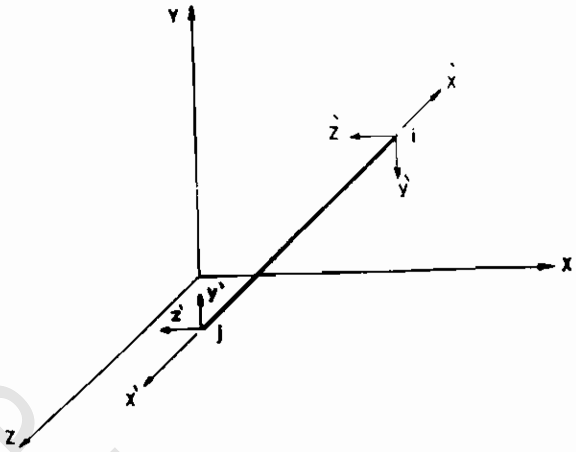


Figure 5.34

$$
\underline{\mathbf{R}}_{ji} = \begin{bmatrix}
\ell_{ji} & m_{ji} & n_{ji} & 0 & 0 & 0 \\
-(\frac{m}{D})_{ji} & -(\frac{\ell}{D})_{ji} & 0 & 0 & 0 & 0 \\
-(\frac{\ell n}{D})_{ji} & -(\frac{mn}{D})_{ji} & D_{ji} & 0 & 0 & 0 \\
0 & 0 & 0 & \ell_{ji} & m_{ji} & n_{ji} \\
0 & 0 & 0 & -(\frac{m}{D})_{ji} & -(\frac{\ell}{D})_{ji} & 0 \\
0 & 0 & 0 & -(\frac{\ell n}{D})_{ji} & -(\frac{mn}{D})_{ji} & D_{ji}
\end{bmatrix} \tag{5.87}
$$

After the determination of the directions of the local axes $x'-y'-z'$, one must check the orientation of the cross section principal axes with respect to these local axes. In case of any deviation between $y-z'$ axis and the principal axes, one must modify the actions-deformations relations of Equations 5.1 and 5.2 to be defined in the directions of the determined local axes $x'-y'-z'$. Suppose that, in general, the axes y' and z' are oriented with an angle β with respect to the principal axes y'_p and z'_p as shown in Figure 5.35. In order to transform the end actions or end deformations from x'_p-y'_p-z'_p axes into $x'-y'-z'$ axes, the following transformation matrix is used:

$$\underline{\mathbf{R}}_{jip} = \begin{bmatrix} \cos(x'_p\, x') & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(y'_p\, y') & \cos(y'_p\, z') & 0 & 0 & 0 \\ 0 & \cos(z'_p\, y') & \cos(z'_p\, z') & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(x'_p\, x') & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(y'_p\, y') & \cos(y'_p\, z') \\ 0 & 0 & 0 & 0 & \cos(z'_p\, y') & \cos(z'_p\, z') \end{bmatrix} \tag{5.88}$$

For the orientation of angle β shown in Figure 5.35, Equation 5.88 becomes

$$\underline{\mathbf{R}}_{jip} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\beta & \sin\beta & 0 & 0 & 0 \\ 0 & -\sin\beta & \cos\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix} \tag{5.89}$$

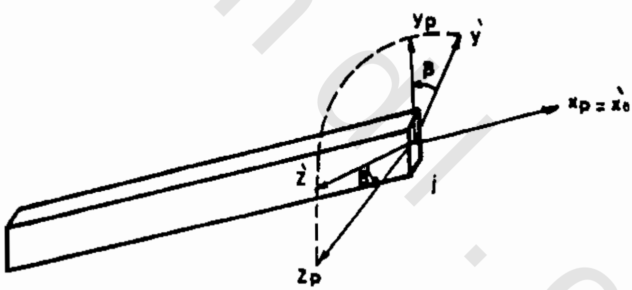


Figure 5.35

Thus, the actions-deformations relationship for member ij in the local coordinates is given by

$$\underline{\mathbf{A}}'_{ij} = \left(\underline{\mathbf{R}}^T_{ijp}\;\; \underline{\mathbf{S}}'^{j}_{iip}\;\; \underline{\mathbf{R}}_{ijp}\right)\underline{\mathbf{D}}'_{ij} + \left(\underline{\mathbf{R}}^T_{ijp}\;\; \underline{\mathbf{S}}'_{ijp}\;\; \underline{\mathbf{R}}_{jip}\right)\underline{\mathbf{D}}'_{ji} \tag{5.90}$$

in which $\underline{\mathbf{S}}'^{j}_{iip}$ and $\underline{\mathbf{S}}'_{ijp}$ are the stiffness matrices defined in Equations 5.1 and 5.2.

Example 5.9

Analyze the space frame shown in Figure 5.36 considering that EA = 8000 kN, EI_y = EI_z = 2000 kN.m^2; and GJ_x = 200 kN.m^2 for all members. Consider the angle β = 0°.

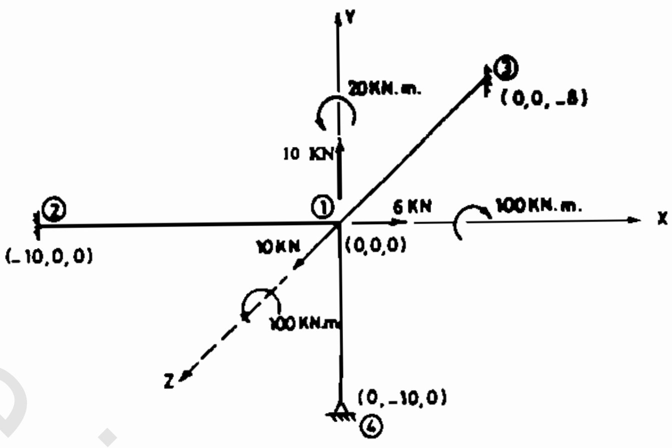


Figure 5.36

The boundary conditions are $\underline{D}_3 = \underline{0}$, $\underline{D}_4 = \underline{0}$, $D_{2x} = D_{2z} = 0$. The actions-deformations relationship which needs to be determined in order to solve this problem is

$$\begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{S}_{21} & \underline{S}_{22} \end{bmatrix} \begin{bmatrix} \underline{D}_1 \\ \underline{D}_2 \end{bmatrix}$$

where the rows and columns numbers 7, 8, and 9 shall be deleted, due to the boundary conditions at joint 2; and

$$\underline{S}_{11} = \underline{S}_{11}^2 + \underline{S}_{11}^3 + \underline{S}_{11}^4 \quad ; \quad S_{22} = S_{22}^1 \quad , \quad \underline{R}_{ijp} = \underline{I}.$$

The calculations for the stiffness matrices follows:

$$\underline{S}_{11}^2 = \underline{R}_{12}^T \; S'^2_{11} \; \underline{R}_{12}$$

$$\ell_{12} = \frac{0+10}{10} = 1 \quad ; \quad m_{12} = 0 \quad ; \quad n_{12} = 0 \quad ; \quad D_{12} = 1.0.$$

$$\underline{R}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \underline{I}$$

$$\underline{S}_{11}^{2}=\underline{S}_{11}^{'2}=\begin{bmatrix}800 & 0 & 0 & 0 & 0 & 0\\ 0 & 24 & 0 & 0 & 0 & -120\\ 0 & 0 & 24 & 0 & 120 & 0\\ 0 & 0 & 0 & 20 & 0 & 0\\ 0 & 0 & 120 & 0 & 800 & 0\\ 0 & -120 & 0 & 0 & 0 & 800\end{bmatrix}$$

For $\underline{S}_{22}^{1}$ one has

$\ell_{21}=-1$; $m_{21}=0$; $n_{21}=0$; $D_{21}=1.$

$$\underline{R}_{21}=\begin{bmatrix}-1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$\underline{S}_{22}^{1}=\underline{R}_{21}^{T}\ \underline{S}_{22}^{'1}\ \underline{R}_{21}=\begin{bmatrix}800 & 0 & 0 & 0 & 0 & 0\\ 0 & 24 & 0 & 0 & 0 & 120\\ 0 & 0 & 24 & 0 & -120 & 0\\ 0 & 0 & 0 & 20 & 0 & 0\\ 0 & 0 & -120 & 0 & 800 & 0\\ 0 & 120 & 0 & 0 & 0 & 800\end{bmatrix}$$

For $\underline{S}_{11}^{3}$ one has

$\ell_{13}=0$; $m_{13}=0$; $n_{13}=\dfrac{0+8}{8}=+1$; $D=0$

$$\underline{R}_{13}=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\end{bmatrix}$$

$$\underline{S}_{11}^{3}=\underline{R}_{13}^{T}\ \underline{S}_{11}^{'3}\ \underline{R}_{13}=\begin{bmatrix}23.43 & 0 & 0 & 0 & -93.75 & 0\\ 0 & 95.75 & 0 & 375 & 0 & 0\\ 0 & 0 & 1000 & 0 & 0 & 0\\ 0 & 975 & 0 & 2000 & 0 & 0\\ -93.75 & 0 & 0 & 0 & 500 & 0\\ 0 & 0 & 0 & 0 & 0 & 50\end{bmatrix}$$

For $\underline{S}_{11}^{4}$ one has

$\ell_{14} = 0 \quad ; \quad m_{14} = 0 \quad ; \quad n_{14} = 0 \quad ; \quad D_{14} = 1$

$$\underline{\mathbf{R}}_{14} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{11}^{4} = \underline{\mathbf{R}}_{14}^{T}\ \underline{\mathbf{S}}'^{4}_{11}\ \underline{\mathbf{R}}_{14} = \begin{bmatrix} 36 & 0 & 0 & 0 & 0 & 180 \\ 0 & 800 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & -120 & 0 & 0 \\ 0 & 0 & -120 & 800 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 \\ 180 & 0 & 0 & 0 & 0 & 1200 \end{bmatrix}$$

For $\underline{\mathbf{S}}_{12}$ one has

$$\underline{\mathbf{S}}_{12} = \underline{\mathbf{R}}_{12}^{T}\ \underline{\mathbf{S}}'_{12}\ \underline{\mathbf{R}}_{21} = \begin{bmatrix} -800 & 0 & 0 & 0 & 0 & 0 \\ 0 & -24 & 0 & 0 & 0 & -120 \\ 0 & 0 & -24 & 0 & 120 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 \\ 0 & 0 & -120 & 0 & 400 & 0 \\ 0 & 120 & 0 & 0 & 0 & 400 \end{bmatrix}$$

The final actions-deformations relationship after deleting the rows and columns numbers 7, 8, and 9 is

$$\begin{bmatrix} 0 \\ 10 \\ 10 \\ -100 \\ 20 \\ 100 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 859.43 & 0 & 0 & 0 & -93.75 & 180 & 0 & 0 & 0 \\ 0 & 917.75 & 0 & 375 & 0 & -120 & 0 & 0 & -120 \\ 0 & 0 & 1048 & -120 & 120 & 0 & 0 & 120 & 0 \\ 0 & 375 & -120 & 2820 & 0 & 0 & -20 & 0 & 0 \\ -93.750 & 0 & 120 & 0 & 1350 & 0 & 0 & 400 & 0 \\ 180 & -120 & 0 & 0 & 0 & 2050 & 0 & 0 & 400 \\ 0 & 0 & 0 & -20 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 120 & 0 & 400 & 0 & 0 & 800 & 0 \\ 0 & -120 & 0 & 0 & 0 & 400 & 0 & 0 & 800 \end{bmatrix} \begin{bmatrix} D_{1x} \\ D_{1y} \\ D_{1z} \\ \theta_{1x} \\ \theta_{1y} \\ \theta_{1z} \\ \theta_{2x} \\ \theta 2_{y} \\ \theta_{2z} \end{bmatrix}$$

The solution for the unknown deformations is

$$\begin{bmatrix} D_{1x}\ D_{1y}\ D_{1z}\ \theta_{1x}\ \theta_{1y}\ \theta_{1z}\ \theta_{2x}\ \theta \quad \theta_{2z} \end{bmatrix} = [-0.0028 \quad 0.0314 \quad 0.0041 \quad -0.0397 \quad 0.017 \quad 0.0553 \quad -0.0397 \quad -0.0091 \quad -0.023]$$

The reactions at supports are determined as follows:

$$\underline{\mathbf{A}}_{21} = \underline{\mathbf{S}}_{22}^{1}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}_{21}\,\underline{\mathbf{D}}_1 = [2.24 \;\; 3.12 \;\; -1.046 \;\; 0 \;\; 0 \;\; 0]^T$$

$$\underline{\mathbf{A}}_{31} = \underline{\mathbf{S}}_{33}^{1}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}_{31}\,\underline{\mathbf{D}}_1 = [1.66 \;\; 11.94 \;\; -4.1 \;\; -27.925 \;\; 4.512 \;\; 2.765]^T$$

$$\underline{\mathbf{A}}_{41} = \underline{\mathbf{S}}_{44}^{1}\,\underline{\mathbf{D}}_4 + \underline{\mathbf{S}}_{41}\,\underline{\mathbf{D}}_1 = [-9.85 \;\; -25.12 \;\; -4.88 \;\; -16.37 \;\; -0.85 \;\; 32.675]^T$$

The members internal actions can be determined as usual from substituting into the actions-deformations relationships in local coordinates of each member.

5.6 EQUIVALENT JOINT ACTIONS

In practice, the structure is not only subjected to joint loads, but also to direct member loading, initial strains, settlements in supports, or rise in temperature. In order to analyze the structure due to these actions, it is necessary to transfer their effect to equivalent joint actions. This section shows how to determine the equivalent joint actions due to each kind of loading.

5.6.1 Equivalent Joint Actions due to Direct Loading

This type of loading was treated in Chapter 3, section 3.8.3. From Maxwell's or Betti's theorems one decomposes the member loading into fixed end actions and equivalent joint actions. The fixed end actions for a particular member are equal in magnitude and opposite in directions to the equivalent joint actions, as illustrated in Figure 5.37.

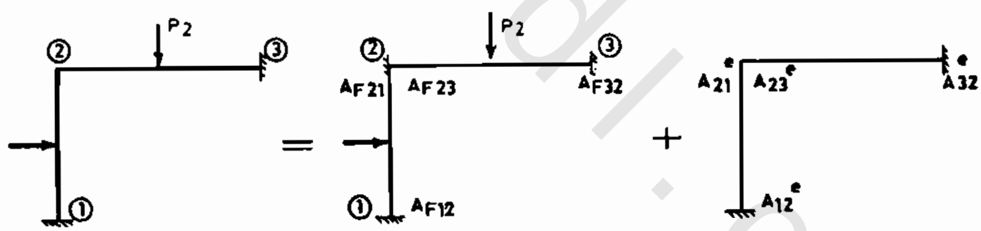


Figure 5.37

At joint 2, for example, the equivalent joint actions are the sum of the equivalent joint actions of the members connected to this joint. These actions are written as

$$\underline{\mathbf{A}}_2^e = \underline{\mathbf{A}}_{21}^e + \underline{\mathbf{A}}_{23}^e \tag{5.91}$$

Meanwhile, the equivalent joint actions for member ij are obtained from

$$\underline{\mathbf{A}}_{ij}^e = -\,\underline{\mathbf{A}}_{Fij} \tag{5.92}$$

Since the fixed end actions, $\underline{\mathbf{A}}_{Fij}$, are determined in the member coordinates, it is then necessary to transform these actions to the global coordinates. The relationship between fixed end actions in the local coordinates, $\underline{\mathbf{A}}'_{Fij}$, and those in the global coordinates, $\underline{\mathbf{A}}_{Fij}$, is given by

$$\underline{\mathbf{A}}'_{Fij} = \underline{\mathbf{R}}_{ij}\,\underline{\mathbf{A}}_{Fij} \tag{5.93}$$

Therefore, for a joint i which is connecting member ij, ik and iℓ, the equivalent joint actions are calculated as follows:

$$\begin{aligned}\underline{\mathbf{A}}_i^e &= \underline{\mathbf{A}}_{ij}^e + \underline{\mathbf{A}}_{ik}^e + \underline{\mathbf{A}}_{i\ell}^e \\ &= -\underline{\mathbf{A}}_{Fij} - \underline{\mathbf{A}}_{Fik} - \underline{\mathbf{A}}_{Fi\ell} \\ &= -\underline{\mathbf{R}}_{ij}^T\,\underline{\mathbf{A}}'_{Fij} - \underline{\mathbf{R}}_{ik}^T\,\underline{\mathbf{A}}'_{Fik} - \underline{\mathbf{R}}_{i\ell}^T\,\underline{\mathbf{A}}'_{Fi\ell}\end{aligned} \tag{5.94}$$

It is obvious from Figure 5.37 that the final end actions in a member are obtained from the superposition with the fixed end actions as follows:

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{S}}'^{j}_{ii}\,\underline{\mathbf{R}}_{ij}\,\underline{\mathbf{D}}_i + \underline{\mathbf{S}}'_{ij}\,\underline{\mathbf{R}}_{ji}\,\underline{\mathbf{D}}_j + \underline{\mathbf{A}}'_{Fij} \tag{5.95}$$

$$\underline{\mathbf{A}}'_{ji} = \underline{\mathbf{S}}'^{j}_{ji}\,\underline{\mathbf{R}}_{ij}\,\underline{\mathbf{D}}_i + \underline{\mathbf{S}}'^{i}_{jj}\,\underline{\mathbf{R}}_{ji}\,\underline{\mathbf{D}}_j + \underline{\mathbf{A}}'_{Fji} \tag{5.96}$$

5.6.2 Equivalent Joint Actions due to Supports Deformations

It has been pointed out in section 5.4.2 that in case of supports settlements or rotations, one may substitute these deformations into the boundary conditions. Another method presented here is to replace the supports deformations by equivalent joint actions using Betti's law.

The frame shown in Figure 5.38, for example, has deformations at support 4. In order to determine the equivalent joint actions, the frame is made to be kinematically determinate by fixing all joints. The fixed end actions due to support deformations are determined from the actions-deformations relationships for the members affected by the deformations. The fixed end actions in members 2-4 and 3-4 are calculated as follows:

$$\begin{aligned}\underline{\mathbf{A}}'_{F24} &= \underline{\mathbf{S}}'^{4}_{22}\,\underline{\mathbf{R}}_{24}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{24}\,\underline{\mathbf{R}}_{42}\,\underline{\mathbf{D}}_4 = \underline{\mathbf{S}}'_{24}\,\underline{\mathbf{R}}_{42}\,\underline{\mathbf{D}}_4 \\ \underline{\mathbf{A}}'_{F42} &= \underline{\mathbf{S}}'_{42}\,\underline{\mathbf{R}}_{24}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'^{2}_{44}\,\underline{\mathbf{R}}_{42}\,\underline{\mathbf{D}}_4 = \underline{\mathbf{S}}'^{2}_{44}\,\underline{\mathbf{R}}_{42}\,\underline{\mathbf{D}}_4 \\ \underline{\mathbf{A}}'_{F34} &= \underline{\mathbf{S}}'^{4}_{33}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{34}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 = \underline{\mathbf{S}}'_{34}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 \\ \underline{\mathbf{A}}'_{F43} &= \underline{\mathbf{S}}'_{43}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'^{3}_{44}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 = \underline{\mathbf{S}}'^{4}_{33}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4\end{aligned} \tag{5.97}$$

Once all fixed end actions have been determined, as in Equations 5.97, the equivalent joint actions in the global coordinates are calculated as was shown in

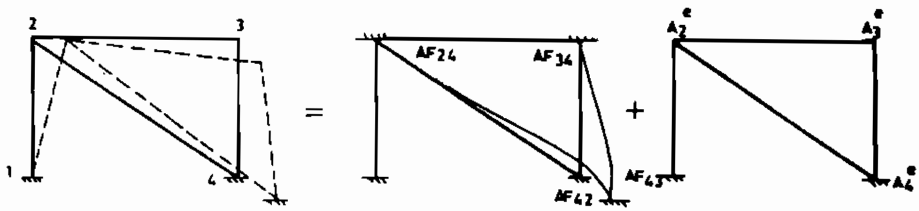


Figure 5.38

Equation 5.94. After performing the stiffness analysis, the members end actions are determined from Equations 5.95 and 5.96.

5.6.3 Equivalent Joint Actions due to Temperature Changes

Temperature changes could cause detrimental effects in the structures, especially in hot environment where air conditioning is used inside the buildings. In order to determine the equivalent joint actions due to these temperature changes, one proceeds as in the previous sections by keeping the structure kinematically determinate, as shown in Figure 5.39. The fixed end actions for any member subjected to temperature changes are then determined as illustrated next.

Considering, in general, member ij in the plane frame, the fixed end actions are determined as shown in Figure 5.40 from the deformations at the released support.

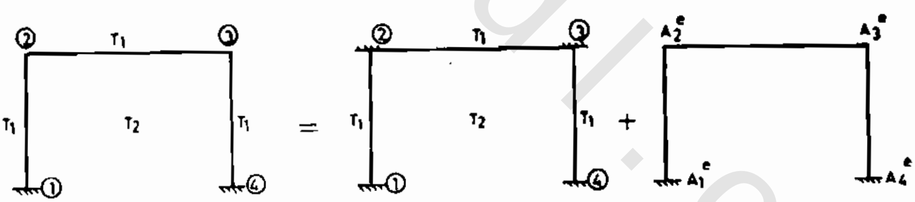


Figure 5.39

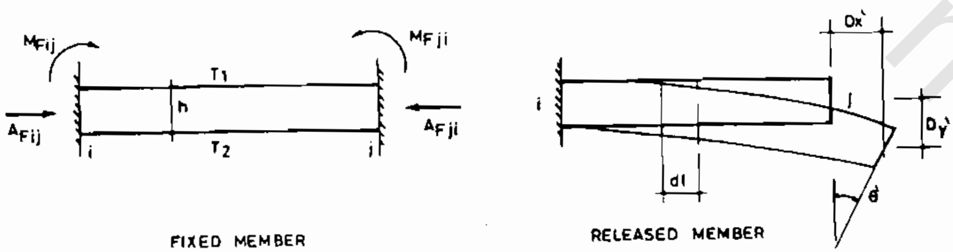


Figure 5.40

These deformations are calculated by considering an infinitesimal element $d\ell$ of member ij as shown in Figure 5.41, and integrating along the member length.

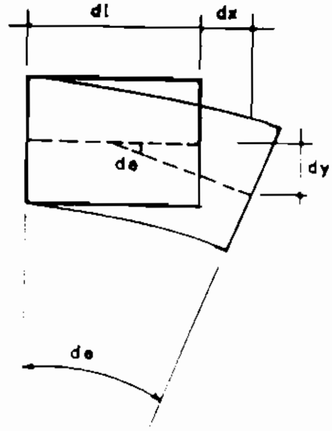


Figure 5.41

The deformations of the element $d\ell$ due to temperature change T_1 on top surface and T_2 on bottom surface are

$$dx = \alpha\left(\frac{T_1 + T_2}{2}\right)d\ell$$

$$dy = d\theta\left(\frac{d\ell}{2}\right)$$

$$d\theta = -\alpha\left(\frac{T_1 - T_2}{h}\right)d\ell \qquad (5.98)$$

The deformation at joint j can be calculated as follows:

$$D'_x = \int_0^L dx = \alpha\left(\frac{T_1 + T_2}{2}\right)L$$

$$D'_y = \int_0^L d\theta\,\frac{d\ell}{2} = -\alpha\left(\frac{T_1 - T_2}{2h}\right)L^2$$

$$\theta'_z = \int_0^L d\theta = -\alpha\left(\frac{T_1 - T_2}{h}\right)L \qquad (5.99)$$

The fixed end actions for plane frame members are determined from Equations 5.1 and 5.2 by substituting for D'_x, D'_y, and θ'_z of Eq. 5.99 as follows:

$$\underline{A}'_{Fij} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} \begin{bmatrix} -D'_x \\ -D'_y \\ -\theta'_z \end{bmatrix} = \begin{bmatrix} -\alpha EA\left(\frac{T_1+T_2}{2}\right) \\ 0 \\ -\alpha EI\left(\frac{T_1-T_2}{h}\right) \end{bmatrix}$$

(5.100)

$$\underline{A}'_{Fji} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} -D'_x \\ -D'_y \\ -\theta'_z \end{bmatrix} + \underline{0} = \begin{bmatrix} -\alpha EA\left(\frac{T_1+T_2}{2}\right) \\ 0 \\ \alpha EI\left(\frac{T_1+T_2}{h}\right) \end{bmatrix} \qquad (5.101)$$

For beams or plane frames, where axial deformations are neglected, the fixed end actions due temperature change become

$$\underline{A}'_{Fij} = \begin{bmatrix} 0 \\ -\alpha EI\left(\frac{T_1-T_2}{h}\right) \end{bmatrix}, \quad \underline{A}'_{Fji} \begin{bmatrix} 0 \\ \alpha EI\left(\frac{T_1-T_2}{h}\right) \end{bmatrix} \qquad (5.102)$$

For truss members, where the axial deformations are only considered, the fixed end actions for member ij are

$$A'_{Fij} = -\alpha T (EA) \quad ; \quad A'_{Fji} = -\alpha T (EA) \qquad (5.103)$$

In case of grids, the fixed end actions due temperature change are

$$\underline{A}'_{Fji} = \underline{A}'_{Fij} = \begin{bmatrix} 0 \\ 0 \\ -\alpha EI\left(\frac{T_1-T_2}{h}\right) EI_y \end{bmatrix} \qquad (5.104)$$

In case of space frames, if member ij is subjected to the temperature change shown in Figure 5.42, the fixed end actions about the principal axes are obtained from

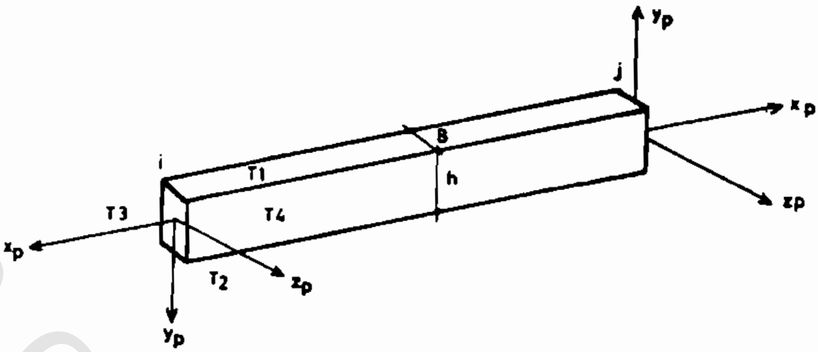


Figure 5.42

$$\underline{\mathbf{A}}'_{Fijp} = \begin{bmatrix} -\alpha EA\left(T_1 + T_2 + T_3 + T_4\right)/4 \\ 0 \\ 0 \\ 0 \\ \alpha EI_y\left(\frac{T_3 - T_4}{B}\right) \\ -\alpha EI_z\left(\frac{T_1 - T_2}{h}\right) \end{bmatrix} \qquad (5.105)$$

$$\underline{\mathbf{A}}'_{Fjip} = \begin{bmatrix} -\alpha EA\left(T_1 + T_2 + T_3 + T_4\right)/4 \\ 0 \\ 0 \\ 0 \\ \alpha EI_y\left(\frac{T_3 - T_4}{B}\right) \\ \alpha EI_z\left(\frac{T_1 - T_2}{h}\right) \end{bmatrix} \qquad (5.106)$$

To determine the fixed end actions in the local coordinates, one uses the following transformation:

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{R}}^T_{ijp}\,\underline{\mathbf{A}}'_{fijp} \qquad (5.107)$$

After determining the fixed end actions in the local coordinates, one can determine the equivalent joint actions in the global coordinates as was shown in section 5.6.1.

5.6.4 Equivalent Joint Actions due to Initial Strains

On many occasions during construction, the construction engineer finds out that there are members which are little shorter or little longer than the required length. Longer members can be shortened to meet the specifications. However, short members could cause waste of materials if they are replaced by other fitted members. Sometimes one can manage the connection of short members in the structure. However, it is necessary to check if the stresses developed due to the initial strains caused by these members are within the acceptable limits or not.

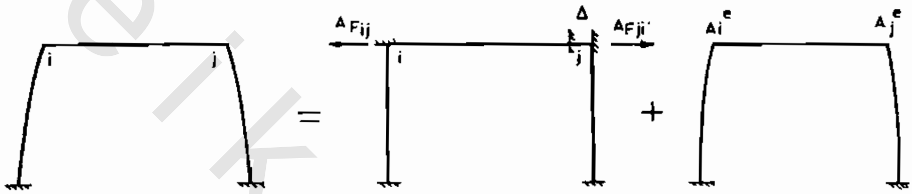


Figure 5.43

The equivalent joint actions due to initial strains can be determined from the fixed end actions. If member ij in the structure shown in Figure 5.43 is short Δ in length at j, then the fixed end actions can be calculated from the actions-deformations relationship as follows:

$$\underline{A}'_{Fij} = \underline{S}'^{j}_{ii}\,\underline{D}'_{ij} + \underline{S}'_{ij}\,\underline{D}'_{ji} = \left[\left(\frac{EA}{L}\Delta\right) \quad 0 \quad 0\right]^T \qquad (5.108)$$

$$\underline{A}'_{Fji} = \underline{S}'_{ji}\,\underline{D}'_{ij} + S'^{i}_{jj}\,\underline{D}'_{ji} = \left[\left(\frac{EA}{L}\Delta\right) \quad 0 \quad 0\right]^T \qquad (5.109)$$

The equivalent joint actions are determined as usual using Equation 5.95.

5.6.5 Numerical Applications

In this section various numerical examples are given, most of which were solved previously by other methods.

Example 5.10

Determine the member forces in the truss shown in Figure 5.44 due to the applied loads and a vertical settlement at joint 5 of 0.01 m downward. EA = 10^6 kN, for all members.

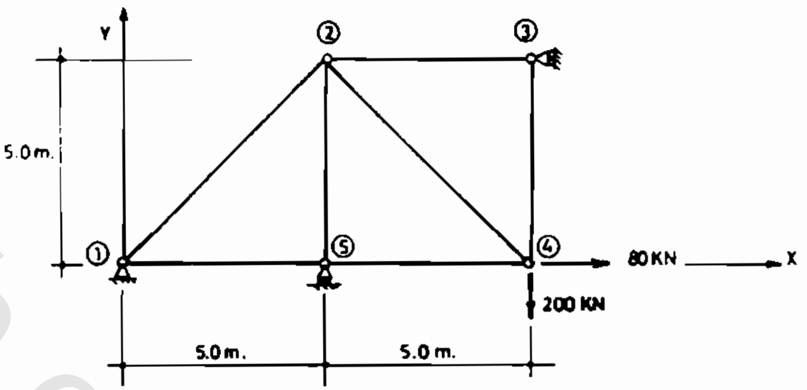


Figure 5.44

Solution

The boundary conditions are $\underline{\mathbf{D}}_1 = \underline{\mathbf{0}}$, $D_{3x} = 0$, $D_{5y} = -0.01\,\text{m}$. However, one may replace D_{5y} by equivalent joint actions.

The action-displacement relation for this structure using $\underline{\mathbf{D}}_1 = \underline{\mathbf{0}}$.

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \\ \underline{\mathbf{A}}_4 \\ \underline{\mathbf{A}}_5 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{\mathbf{S}}_{24} & \underline{\mathbf{S}}_{25} \\ \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} & \underline{\mathbf{0}} \\ \underline{\mathbf{S}}_{42} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44} & \underline{\mathbf{S}}_{45} \\ \underline{\mathbf{S}}_{52} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{54} & \underline{\mathbf{S}}_{55} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \\ \underline{\mathbf{D}}_4 \\ \underline{\mathbf{D}}_5 \end{bmatrix}$$

in which the rows and columns corresponding to D_{3x} and D_{5y}, numbers 3 and 8, will be deleted.

To determine $\underline{\mathbf{S}}_{22}$ one has

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3} + \underline{\mathbf{S}}_{22}^{4} + \underline{\mathbf{S}}_{22}^{5}$$

$$\underline{\mathbf{R}}_{21} = [0.707 \quad 0.707] \quad ; \quad \underline{\mathbf{S}}'^{1}_{22} = \left[\frac{10^6}{7.07}\right]$$

$$\underline{\mathbf{S}}_{22}^{1} = \underline{\mathbf{R}}_{21}^{T}\, \underline{\mathbf{S}}'^{1}_{22}\, \underline{\mathbf{R}}_{21} = 10^5 \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{23} = [-1 \quad 0] \quad ; \quad \underline{\mathbf{S}}'^{3}_{22} = \left[\frac{10^6}{5}\right]$$

$$\underline{S}_{22}^{3} = \underline{R}_{23}^{T}\ \underline{S}'^{3}_{22}\ \underline{R}_{23} = 10^5 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{R}_{24} = [-0.707 \quad 0.707] \quad ; \quad \underline{S}'^{4}_{22} = \left[\frac{10^6}{7.07}\right]$$

$$\underline{S}_{22}^{4} = \underline{R}_{24}^{T}\ \underline{S}'^{4}_{22}\ \underline{R}_{24} = 10^5 \begin{bmatrix} 0.707 & -0.707 \\ -0.707 & 0.707 \end{bmatrix}$$

$$\underline{R}_{25} = [0 \quad 1] \quad ; \quad \underline{S}'^{5}_{22} = \left[\frac{10^6}{5}\right]$$

$$\underline{S}_{22}^{5} = \underline{R}_{25}^{T}\ \underline{S}'^{5}_{22}\ \underline{R}_{25} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Therefore $\underline{S}_{22}$ is given by

$$\underline{S}_{22} = \underline{S}_{22}^{1} + \underline{S}_{22}^{3} + \underline{S}_{22}^{4} + \underline{S}_{22}^{5} = 10^5 \begin{bmatrix} 3.414 & 0 \\ 0 & 3.414 \end{bmatrix}$$

To determine $\underline{S}_{33}$ one has

$$\underline{S}_{33} = \underline{S}_{33}^{2} + \underline{S}_{33}^{4}$$

$$\underline{S}_{33}^{2} = \underline{R}_{32}^{T}\ \underline{S}'^{2}_{33}\ \underline{R}_{32} = 10^5 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{S}_{33}^{4} = \underline{R}_{34}^{T}\ \underline{S}'^{4}_{33}\ \underline{R}_{34} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Therefore, $\underline{S}_{33}$ is given by

$$\underline{S}_{33}^{2} = \underline{S}_{33}^{2} + \underline{S}_{33}^{4} = 10^5 \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

To determine $\underline{S}_{44}$ one has

$$\underline{S}_{44} = \underline{S}_{44}^{2} + \underline{S}_{44}^{3} + \underline{S}_{44}^{5}$$

$$\underline{S}_{44}^{2} = \underline{R}_{42}^{T}\ \underline{S}'^{2}_{44}\ \underline{R}_{42} = 10^5 \begin{bmatrix} 0.707 & -0.707 \\ -0.707 & 0.707 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{44}^{4} = \underline{\mathbf{R}}_{43}^{T}\, \underline{\mathbf{S}}'^{3}_{44}\, \underline{\mathbf{R}}_{43} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{44}^{5} = \underline{\mathbf{R}}_{45}^{T}\, \underline{\mathbf{S}}'^{5}_{44}\, \underline{\mathbf{R}}_{45} = 10^5 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, $\underline{\mathbf{S}}_{44}$ is given by

$$\underline{\mathbf{S}}_{44} = \underline{\mathbf{S}}_{44}^{2} + \underline{\mathbf{S}}_{44}^{3} + \underline{\mathbf{S}}_{44}^{4} = 10^5 \begin{bmatrix} 2.707 & -0.707 \\ -0.707 & 2.707 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}_{55}$ one has

$$\underline{\mathbf{S}}_{55} = \underline{\mathbf{S}}_{55}^{1} + \underline{\mathbf{S}}_{55}^{2} + \underline{\mathbf{S}}_{55}^{4}$$

$$\underline{\mathbf{S}}_{55}^{1} = \underline{\mathbf{R}}_{51}^{T}\, \underline{\mathbf{S}}'^{1}_{55}\, \underline{\mathbf{R}}_{51} = 10^5 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{55}^{2} = \underline{\mathbf{R}}_{52}^{T}\, \underline{\mathbf{S}}'^{2}_{55}\, \underline{\mathbf{R}}_{52} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{55}^{4} = \underline{\mathbf{R}}_{54}^{T}\, \underline{\mathbf{S}}'^{4}_{55}\, \underline{\mathbf{R}}_{54} = 10^5 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, $\underline{\mathbf{S}}_{55}$ is given by

$$\underline{\mathbf{S}}_{55} = \underline{\mathbf{S}}_{55}^{1} + \underline{\mathbf{S}}_{55}^{2} + \underline{\mathbf{S}}_{55}^{4} = 10^5 \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}_{21}, \underline{\mathbf{S}}_{23}, \underline{\mathbf{S}}_{24}, \underline{\mathbf{S}}_{25}, \underline{\mathbf{S}}_{34}$ and $\underline{\mathbf{S}}_{45}$ one has

$$\underline{\mathbf{S}}_{21} = \underline{\mathbf{R}}_{21}^{T}\, \underline{\mathbf{S}}'_{21}\, \underline{\mathbf{R}}_{12} = 10^5 \begin{bmatrix} -0.707 & -0.707 \\ -0.707 & -0.707 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{23} = \underline{\mathbf{R}}_{23}^{T}\, \underline{\mathbf{S}}'^{2}_{23}\, \underline{\mathbf{R}}_{32} = 10^5 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{24} = \underline{\mathbf{R}}_{24}^{T}\, \underline{\mathbf{S}}'_{24}\, \underline{\mathbf{R}}_{42} = 10^5 \begin{bmatrix} -0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{25} = \underline{\mathbf{R}}_{25}^{T}\, \underline{\mathbf{S}}'_{25}\, \underline{\mathbf{R}}_{52} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\underline{S}_{34} = \underline{R}^T_{34}\, \underline{S}'_{34}\, \underline{R}_{43} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\underline{S}_{45} = \underline{R}^T_{45}\, \underline{S}'_{45}\, \underline{R}_{54} = 10^5 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

To determine the equivalent joint loading due to settlement at joint 5 of the truss, only member 2-5 is affected by the settlement. The fixed end actions in this member are determined as follows:

$$\underline{A}'_{F52} = \underline{S}'^{2}_{55}\, \underline{R}_{52}\, \underline{D}_5 + \underline{S}'_{52}\, \underline{R}_{25}\, \underline{D}_2$$
$$= \left(\frac{10^6}{5}\right) \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.01 \end{bmatrix} = 2000\ \text{kN}$$

$$\underline{A}'_{F25} = \underline{S}'_{25}\, \underline{R}_{52}\, \underline{D}_5 + \underline{S}'^{5}_{22}\, \underline{R}_{25}\, \underline{D}_2 = 2000\ \text{kN}$$

Therefore, the equivalent joint loading due to the settlement are

$$\underline{A}^e_{25} = -\underline{R}^T_{25}\, \underline{A}'_{F25} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} 2000 = \begin{bmatrix} 0 \\ -2000 \end{bmatrix}$$

$$\underline{A}^e_{52} = -\underline{R}^T_{52}\, \underline{A}'_{F52} = -\begin{bmatrix} 0 \\ -1 \end{bmatrix} 2000 = \begin{bmatrix} 0 \\ 2000 \end{bmatrix}$$

The force-displacement relationship is thus obtained as

$$\begin{bmatrix} 0 \\ -2000 \\ R_{3x} \\ 0 \\ 80 \\ -200 \\ 0 \\ R_{5y} + 2000 \end{bmatrix} = 10^5 \begin{bmatrix} 3.414 & 0 & -2 & 0 & -0.707 & 0.707 & 0 & 0 \\ 0 & 3.414 & 0 & 0 & 0.707 & -0.707 & 0 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 2.707 & -0.707 & -2 & 0 \\ 0.707 & -0.707 & 0 & -2 & -0.707 & 2.707 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 4 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ 0 \\ D_{3y} \\ D_{4x} \\ D_{4y} \\ D_{5x} \\ 0 \end{bmatrix}$$

The final relation after deleting the third and eighth rows and columns becomes

$$\begin{bmatrix} 0 \\ -2000 \\ 0 \\ 80 \\ -200 \\ 0 \end{bmatrix} = 10^5 \begin{bmatrix} 3.414 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 3.414 & 0 & 0.707 & -0.707 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ -0.707 & 0.707 & 0 & 2.707 & -0.707 & -2 \\ 0.707 & -0.707 & -2 & -0.707 & 2.707 & 0 \\ 0 & 0 & 0 & -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ D_{3y} \\ D_{4x} \\ D_{4y} \\ D_{5x} \end{bmatrix}$$

The solution for this equation is

$$\left[D_{2x}\ D_{2y}\ D_{3y}\ D_{4x}\ D_{4y}\ D_{5x}\right] = 10^{-5}\,[307.09\ \ -892.91\ \ -1602.98\ \ -120\ \ -1602.88\ \ -60]\ \text{m.}$$

The member end actions are determined as follows:

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'^{2}_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 = \frac{10^6}{7.07}[0.707\ \ 0.707]\,\underline{\mathbf{D}}_2 = -585.81\ \text{kN}$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{1}_{22}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{0}} = -585.81\ \text{kN}$$

$$\underline{\mathbf{A}}'_{15} = \underline{\mathbf{S}}'^{5}_{11}\,\underline{\mathbf{R}}_{15}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{15}\,\underline{\mathbf{R}}_{51}\,\underline{\mathbf{D}}_5 = \left(\frac{10^6}{5}\right)[1\quad 0]\begin{bmatrix} D_{5x} \\ 0 \end{bmatrix} = -120\ \text{kN}$$

$$\underline{\mathbf{A}}'_{52} = \underline{\mathbf{S}}'^{2}_{55}\,\underline{\mathbf{R}}_{52}\,\underline{\mathbf{D}}_5 + \underline{\mathbf{S}}'_{52}\,\underline{\mathbf{R}}_{25}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F52}$$
$$= \left(\frac{10^6}{5}\right)[0\ \ -1]\begin{bmatrix} D_{5x} \\ 0 \end{bmatrix} + \left(\frac{10^6}{5}\right)[0\quad 1]\,\underline{\mathbf{D}}_2 + 2000 = 214.2\ \text{kN}$$

$$\underline{\mathbf{A}}'_{54} = \underline{\mathbf{S}}'^{4}_{55}\,\underline{\mathbf{R}}_{54}\,\underline{\mathbf{D}}_5 + \underline{\mathbf{S}}'_{54}\,\underline{\mathbf{R}}_{45}\,\underline{\mathbf{D}}_4$$
$$= \left(\frac{10^6}{5}\right)[-1\ \ 0]\begin{bmatrix} D_{5x} \\ 0 \end{bmatrix} + \left(\frac{10^6}{5}\right)[1\ \ 0]\,\underline{\mathbf{D}}_4 = -120\ \text{kN}$$

$$\underline{\mathbf{A}}'_{24} = \underline{\mathbf{S}}'^{4}_{22}\,\underline{\mathbf{R}}_{24}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{24}\,\underline{\mathbf{R}}_{42}\,\underline{\mathbf{D}}_4 = 282.91\ \text{kN}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3 = -614.18\ \text{kN}$$

$$\underline{\mathbf{A}}'_{34} = \underline{\mathbf{S}}'^{4}_{33}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{24}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 = 0\ \text{kN}$$

The member forces are shown in Figure 5.45.

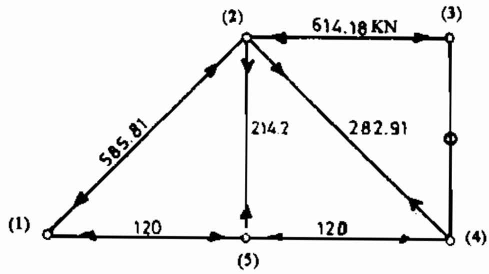


Figure 5.45

Example 5.11

Determine the member forces in the truss shown in Figure 5.46 due to the applied load and a rise in temperature for member 3-4 and 4-5 of 20°c (EA = 2 × 10^6 kN for all members, $\alpha = 10^{-5}/°c$).

Solution

The boundary conditions are $\underline{D}_5 = \underline{0}$, $D_{1x} = 0$, and $D_{3y} = 0$. The force-displacement relation which needs to be determine is

$$\begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \\ \underline{A}_3 \\ \underline{A}_4 \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} & \underline{0} & \underline{S}_{14} \\ \underline{S}_{21} & \underline{S}_{22} & \underline{S}_{23} & \underline{S}_{24} \\ \underline{0} & \underline{S}_{32} & \underline{S}_{33} & \underline{S}_{34} \\ \underline{S}_{41} & \underline{S}_{42} & \underline{S}_{43} & \underline{S}_{44} \end{bmatrix} \begin{bmatrix} \underline{D}_1 \\ \underline{D}_2 \\ \underline{D}_3 \\ \underline{D}_4 \end{bmatrix}$$

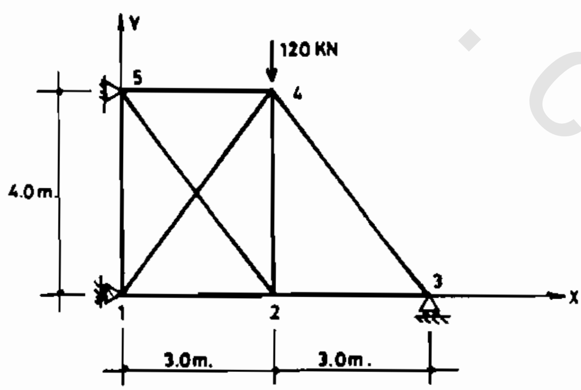


Figure 5.46

The matrix $\underline{S}_{11}$ is obtained as follows:

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}_{11}^{2} + \underline{\mathbf{S}}_{11}^{4} + \underline{\mathbf{S}}_{11}^{5}$$

$$\underline{\mathbf{S}}_{11}^{2} = \underline{\mathbf{R}}_{12}^{T}\ \underline{\mathbf{S}}_{11}^{\prime 2}\ \underline{\mathbf{R}}_{12}$$
$$= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \left[\frac{EA}{3}\right] [-1 \quad 0] = 10^5 \begin{bmatrix} 6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{11}^{4} = \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix} \left[\frac{EA}{5}\right] [-0.6 \quad -0.8] = 10^5 \begin{bmatrix} 1.444 & 1.92 \\ 1.92 & 2.56 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{11}^{5} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \left[\frac{EA}{4}\right] [0 \quad -1] = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}_{11}^{2} + \underline{\mathbf{S}}_{11}^{4} + \underline{\mathbf{S}}_{11}^{5} = 10^5 \begin{bmatrix} 8.107 & 1.92 \\ 1.92 & 7.56 \end{bmatrix}$$

The matrix $\underline{\mathbf{S}}_{22}$ is obtained from

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3} + \underline{\mathbf{S}}_{22}^{4} + \underline{\mathbf{S}}_{22}^{5}$$

$$\underline{\mathbf{S}}_{22}^{1} = 10^5 \begin{bmatrix} 6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22}^{3} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \left[\frac{EA}{3}\right] [-1 \quad 0] = 10^5 \begin{bmatrix} 6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22}^{4} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \left[\frac{EA}{4}\right] [0 \quad -1] = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22}^{5} = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix} \left[\frac{EA}{5}\right] [0.6 \quad -0.8] = 10^5 \begin{bmatrix} 1.44 & -1.92 \\ -1.92 & 2.56 \end{bmatrix}$$

Therefore, one has

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3} + \underline{\mathbf{S}}_{22}^{4} + \underline{\mathbf{S}}_{22}^{5} = 10^5 \begin{bmatrix} 14.774 & -1.92 \\ -1.92 & 7.56 \end{bmatrix}$$

The matrix $\underline{\mathbf{S}}_{33}$ is obtained from

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}_{33}^{2} + \underline{\mathbf{S}}_{33}^{4}$$

$$\underline{S}_{33}^2 = 10^5 \begin{bmatrix} 6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{S}_{33}^4 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix} \begin{bmatrix} \frac{EA}{5} \end{bmatrix} [0.6 \quad -0.8] = 10^5 \begin{bmatrix} 1.44 & -1.92 \\ -1.92 & 2.56 \end{bmatrix}$$

$$\underline{S}_{33} = \underline{S}_{33}^2 + \underline{S}_{33}^4 = 10^5 \begin{bmatrix} 8.1967 & -1.92 \\ -1.92 & 2.56 \end{bmatrix}$$

The calculations for $\underline{S}_{44}$ are

$$\underline{S}_{44} = \underline{S}_{44}^1 + \underline{S}_{44}^2 + \underline{S}_{44}^3 + \underline{S}_{44}^5$$

$$\underline{S}_{44}^1 = 10^5 \begin{bmatrix} 1.44, & 1.92 \\ 1.92 & 2.56 \end{bmatrix} \quad ; \quad \underline{S}_{44}^2 = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \quad ; \quad \underline{S}_{44}^3 = 10^5 \begin{bmatrix} 1.44 & 1.92 \\ -1.92 & 2.56 \end{bmatrix}$$

$$\underline{S}_{44}^5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{EA}{3} \end{bmatrix} [1 \quad 0] = 10^5 \begin{bmatrix} 6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{S}_{44} = \underline{S}_{44}^1 + \underline{S}_{44}^2 + \underline{S}_{44}^3 + \underline{S}_{44}^5 = 10^5 \begin{bmatrix} 9.547 & 0 \\ 0 & 10.12 \end{bmatrix}$$

Calculations of $\underline{S}_{12}, \underline{S}_{23}, \underline{S}_{24}, \underline{S}_{34}$ and $\underline{S}_{14}$, are as follows:

$$\underline{S}_{12} = \underline{R}_{12}^T \, \underline{S}'_{12} \, \underline{R}_{21} = 10^5 \begin{bmatrix} -6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{S}_{23} = \underline{R}_{23}^T \, \underline{S}'_{23} \, \underline{R}_{32} = 10^5 \begin{bmatrix} -6.667 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{S}_{24} = \underline{R}_{24}^T \, \underline{S}'_{24} \, \underline{R}_{42} = 10^5 \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\underline{S}_{34} = \underline{R}_{34}^T \, \underline{S}'_{34} \, \underline{R}_{43} = 10^5 \begin{bmatrix} -1.44 & 1.92 \\ 1.92 & -2.56 \end{bmatrix}$$

$$\underline{S}_{14} = \underline{R}_{14}^T \, \underline{S}'_{14} \, \underline{R}_{41} = 10^5 \begin{bmatrix} -1.44 & -1.92 \\ -1.92 & 2.56 \end{bmatrix}$$

The equivalent joint actions due to temperature change are obtained for members 3-4 and 4-5 as follows:

$$\underline{\mathbf{A}}'_{F54} = \underline{\mathbf{A}}'_{F45} = -EA\alpha T = -(2\times10^{6})(10^{-5})(20) = -400 \text{ kN}$$

$$\underline{\mathbf{A}}'_{F43} = \underline{\mathbf{A}}'_{F34} = -400 \text{ kN}$$

$$\underline{\mathbf{A}}_4^e = -\underline{\mathbf{R}}_{45}^T\,\underline{\mathbf{A}}'_{F45} - \underline{\mathbf{R}}_{43}^T\,\underline{\mathbf{A}}'_{F43}$$
$$= -\begin{bmatrix}1\\0\end{bmatrix}(-400) - \begin{bmatrix}-0.6\\0.8\end{bmatrix}(-400) = \begin{bmatrix}160\\320\end{bmatrix} \text{ kN}$$

$$\underline{\mathbf{A}}_3^e = -\underline{\mathbf{R}}_{34}^T\,\underline{\mathbf{A}}'_{F34} = -\begin{bmatrix}0.6\\-0.8\end{bmatrix}(-400) = \begin{bmatrix}240\\320\end{bmatrix} \text{ kN}$$

Therefore, the total loads at joint 4 are

$$\underline{\mathbf{A}}_4 = \begin{bmatrix}0\\-120\end{bmatrix} + \begin{bmatrix}160\\320\end{bmatrix} = \begin{bmatrix}160\\200\end{bmatrix} \text{ kN}$$

The final force-displacement relationship becomes

$$\begin{bmatrix}R_{1x}\\0\\0\\0\\240\\R_{cy}-320\\160\\200\end{bmatrix} = 10^5\begin{bmatrix}8.107 & 1.92 & -6.67 & 0 & 0 & 0 & -1.44 & -1.92\\ 1.92 & 7.56 & 0 & 0 & 0 & 0 & -1.92 & -2.56\\ -6.67 & 0 & 14.774 & -1.92 & -6.67 & 0 & 0 & 0\\ 0 & 0 & -1.92 & 7.56 & 0 & 0 & 0 & -5\\ 0 & 0 & -6.67 & 0 & 8.107 & -1.92 & -1.44 & 1.92\\ 0 & 0 & 0 & 0 & -1.92 & 7.56 & 1.92 & -2.56\\ -1.44 & -1.92 & 0 & 0 & -1.44 & 1.92 & 9.547 & 0\\ -1.92 & -2.56 & 0 & -5 & 1.92 & -2.56 & 0 & 10.12\end{bmatrix}\begin{bmatrix}0\\D_{1y}\\D_{2x}\\D_{2y}\\D_{3x}\\0\\D_{4x}\\D_{4y}\end{bmatrix}$$

By deleting the first and sixth rows and columns where the boundary conditions are zero, one obtains

$$\begin{bmatrix}0\\0\\0\\240\\160\\200\end{bmatrix} = 10^5\begin{bmatrix}7.56 & 0 & 0 & 0 & -1.92 & -2.56\\ 0 & 14.774 & -1.92 & -6.667 & 0 & 0\\ 0 & -1.92 & 7.56 & 0 & 0 & -5\\ 0 & -6.667 & 0 & 8.107 & -1.44 & 1.92\\ -1.92 & 0 & 0 & -1.44 & 9.547 & 0\\ -2.56 & 0 & -5 & 1.92 & 0 & 10.12\end{bmatrix}\begin{bmatrix}D_{1y}\\D_{2x}\\D_{2y}\\D_{3x}\\D_{4x}\\D_{4y}\end{bmatrix}$$

The solution is

$$\left[D_{1y}\ D_{2x}\ D_{2y}\ D_{3x}\ D_{4x}\ D_{4y}\right] = 10^{-5}\,[15.807 \quad 25.175 \quad 23.715 \quad 48.959 \quad 27.323 \quad 26.190]\ \text{m.}$$

The member end actions are calculated as follows:

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_{1} + \underline{\mathbf{S}}'_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_{2}$$

$$= \left[\frac{EA}{3}\right][-1\ \ 0]\begin{bmatrix}0\\15.807\end{bmatrix}\times 10^{-5} + \left[\frac{EA}{3}\right][1\ \ 0]\begin{bmatrix}25.175\\23.175\end{bmatrix}\times 10^{-5} = 168\ \text{kN}$$

$$\underline{\mathbf{A}}'_{14} = \underline{\mathbf{S}}'^{4}_{11}\,\underline{\mathbf{R}}_{14}\,\underline{\mathbf{D}}_{1} + \underline{\mathbf{S}}'_{14}\,\underline{\mathbf{R}}_{41}\,\underline{\mathbf{D}}_{4} = 98.8\ \text{kN}$$

$$\underline{\mathbf{A}}'_{15} = \underline{\mathbf{S}}'^{5}_{11}\,\underline{\mathbf{R}}_{15}\,\underline{\mathbf{D}}_{1} + \underline{\mathbf{S}}'_{15}\,\underline{\mathbf{R}}_{51}\,\underline{\mathbf{D}}_{5} = -79\ \text{kN}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_{2} + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_{3} = 158\ \text{kN}$$

$$\underline{\mathbf{A}}'_{24} = \underline{\mathbf{S}}'^{4}_{22}\,\underline{\mathbf{R}}_{24}\,\underline{\mathbf{D}}_{2} + \underline{\mathbf{S}}'_{24}\,\underline{\mathbf{R}}_{42}\,\underline{\mathbf{D}}_{4} = 12.5\ \text{kN}$$

$$\underline{\mathbf{A}}'_{25} = \underline{\mathbf{S}}'^{5}_{22}\,\underline{\mathbf{R}}_{25}\,\underline{\mathbf{D}}_{2} + \underline{\mathbf{S}}'_{25}\,\underline{\mathbf{R}}_{52}\,\underline{\mathbf{D}}_{5} = -15.4\ \text{kN}$$

$$\underline{\mathbf{A}}'_{34} = \underline{\mathbf{S}}'^{4}_{33}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{R}}_{3} + \underline{\mathbf{S}}'_{34}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_{4} + \underline{\mathbf{A}}'_{F34}$$

$$= \left[\frac{EA}{5}\right][0.6\ \ -0.8]\ \underline{\mathbf{D}}_{3} + \left[\frac{EA}{5}\right][-0.6\ \ 0.8]\ \underline{\mathbf{D}}_{4} + (-400) = -264.3\ \text{kN}$$

$$\underline{\mathbf{A}}'_{45} = \underline{\mathbf{S}}'^{5}_{44}\,\underline{\mathbf{R}}_{45}\,\underline{\mathbf{D}}_{4} + \underline{\mathbf{S}}'_{45}\,\underline{\mathbf{R}}_{54}\,\underline{\mathbf{D}}_{5} + \underline{\mathbf{A}}'_{F45}$$

$$= \left[\frac{EA}{3}\right][1\ \ 0]\ \underline{\mathbf{D}}_{4} + 0 + (-400) = -218\ \text{kN}$$

The member forces are displayed in Figure 5.47.

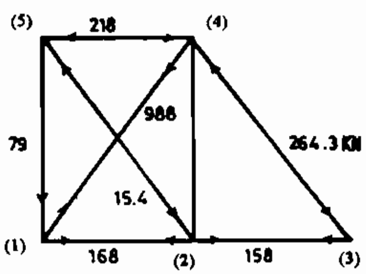


Figure 5.47

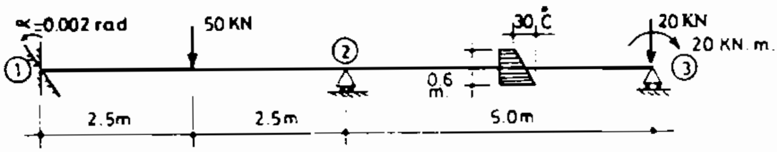


Figure 5.48

Example 5.12

Neglecting the axial and shear deformations, determine the bending moment and shear force diagrams for the beam shown in Figure 5.48 due to the applied loads, a rotation at support (1) of 0.002 rad, and a rise in temperature for member (2)–(3). ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}$/°c).

Solution

The rotation of support 1 is replaced by equivalent joint actions. The boundary conditions are $\underline{D}_1 = \underline{0}$, $D_{2v} = 0$ and $D_{3v} = 0$. The actions-deformations relation which needs to be determined is

$$\begin{bmatrix} \underline{A}_2 \\ \underline{A}_3 \end{bmatrix} = \begin{bmatrix} \underline{S}_{22} & \underline{S}_{23} \\ \underline{S}_{32} & \underline{S}_{33} \end{bmatrix} \begin{bmatrix} \underline{D}_2 \\ \underline{D}_3 \end{bmatrix}$$

To determine $\underline{S}_{22}$ one has $\quad \underline{S}_{22} = \underline{S}^1_{22} + \underline{S}^3_{22}$

$$\underline{R}_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \underline{S}'^1_{22} = 10^4 \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 8 \end{bmatrix}$$

$$\underline{S}^1_{22} = \underline{R}^T_{21}\, \underline{S}'^1_{22}\, \underline{R}_{21} = 10^4 \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 8 \end{bmatrix}$$

$$\underline{R}_{23} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \underline{S}'^3_{22} = \underline{S}'^1_{22}$$

$$\underline{S}^3_{22} = \underline{R}^T_{23}\, \underline{S}'^3_{22}\, \underline{R}_{23} = 10^4 \begin{bmatrix} 0.96 & 2.4 \\ 2.4 & 8 \end{bmatrix}$$

Therefore, $\underline{S}_{22} = \underline{S}^1_{22} + \underline{S}^3_{22} = 10^4 \begin{bmatrix} 1.92 & 0 \\ 0 & 16 \end{bmatrix}$

To determine $\underline{S}_{33}$ and $\underline{S}_{23}$ one has

$$\underline{S}_{33} = \underline{S}^2_{33} = \underline{S}'^2_{33} = 10^4 \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 8 \end{bmatrix}$$

$$\underline{S}_{23} = \underline{R}^T_{23}\, \underline{S}'_{23}\, \underline{R}_{32} = 10^4 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} -0.96 & 2.4 \\ -2.4 & 4 \end{bmatrix}$$

The fixed end actions due to loading in member (1)–(2) are

$$\underline{A}'_{F12} = \begin{bmatrix} -25 \\ 31.25 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix} \quad ; \quad \underline{A}'_{F21} = \begin{bmatrix} 25 \\ -32.15 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix}$$

The fixed end actions due to rotation of support (1) are

$$\underline{A}'_{F12} = \begin{bmatrix} -\dfrac{6EI\theta}{L^2} \\ \dfrac{4EI\theta}{L} \end{bmatrix} = \begin{bmatrix} -48 \\ 160 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix} \quad ; \quad \underline{A}'_{F21} = \begin{bmatrix} -\dfrac{6EI\theta}{L^2} \\ \dfrac{2EI\theta}{L} \end{bmatrix} = \begin{bmatrix} -48 \\ 80 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix}$$

The fixed end actions due to temperature change in member (2)–(3) are

$$\underline{A}'_{F23} = \begin{bmatrix} 0 \\ -\alpha\ EI\left(\dfrac{T_1 - T_2}{h}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix} \quad ; \quad \underline{A}'_{F32} = \begin{bmatrix} 0 \\ -50 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix}$$

The total equivalent joint actions are obtained as follows:

$$\underline{A}^e_2 = -\underline{R}^T_{21}\, \underline{A}'_{F21} - \underline{R}^T_{23}\, \underline{A}'_{F23}$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 25 \\ -31.25 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -48 \\ 80 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 50 \end{bmatrix} = \begin{bmatrix} 23 \\ -98.75 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix}$$

$$\underline{A}^e_3 = -\underline{R}^T_{32}\, \underline{A}'_{F32} = -\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -50 \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix}$$

Therefore, the total actions at joint 3 are

$$\underline{A}_3 = \begin{bmatrix} 0 \\ 50 \end{bmatrix} + \begin{bmatrix} -20 \\ -20 \end{bmatrix} = \begin{bmatrix} -20 \\ 30 \end{bmatrix} \begin{matrix} kN \\ kN.m. \end{matrix}$$

Substitute into the actions-deformations relationship, one has

$$\begin{bmatrix} 23+R_{2y} \\ -98.75 \\ -20+R_{3y} \\ 30 \end{bmatrix} = 10^4 \begin{bmatrix} 1.92 & 0 & -0.96 & 2.4 \\ 0 & 16 & -2.4 & 4 \\ -0.96 & -2.4 & 0.96 & -2.4 \\ 2.4 & 4 & -2.4 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ \theta_2 \\ 0 \\ \theta_3 \end{bmatrix}$$

By deleting the first and third rows and columns, one finally obtains

$$\begin{bmatrix} -98.75 \\ 30 \end{bmatrix} = 10^4 \begin{bmatrix} 16 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix}$$

The solution is $\theta_2 = -8.125 \times 10^{-4}$ rad ; $\theta_3 = 7.8125 \times 10^{-4}$ rad

The member end actions are obtained as follows:

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'^{2}_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F12}$$

$$= \underline{\mathbf{0}} + \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -8.125 \end{bmatrix} + \begin{bmatrix} -25 \\ +31.25 \end{bmatrix} + \begin{bmatrix} -48 \\ 160 \end{bmatrix} = \begin{bmatrix} -53.5 \\ 158.75 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{1}_{22}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{0}} + \underline{\mathbf{A}}'_{F21}$$

$$= \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -8.125 \end{bmatrix} + \begin{bmatrix} -25 \\ -31.25 \end{bmatrix} + \begin{bmatrix} -48 \\ 80 \end{bmatrix} = \begin{bmatrix} -3.5 \\ -16.25 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{A}}'_{F23}$$

$$= \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 8 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -8.125 \end{bmatrix} + \begin{bmatrix} 0.96 & -2.4 \\ -2.4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 7.8125 \end{bmatrix} + \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \\ 16.25 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'^{2}_{33}\,\mathbf{R}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{A}}'_{F32} = \begin{bmatrix} 0.75 \\ -20 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

The bending moment and shear force diagrams can be plotted as shown in Figure 5.49.

Example 5.13

Determine the bending moment, shear force, and axial force diagrams for the plane structure shown in Figure 5.50 due to: (a) The applied loads only (b) Member 1-3 is 1 cm short (EA = 0.5×10^5 kN, EI = 10^5 kN.m^2).

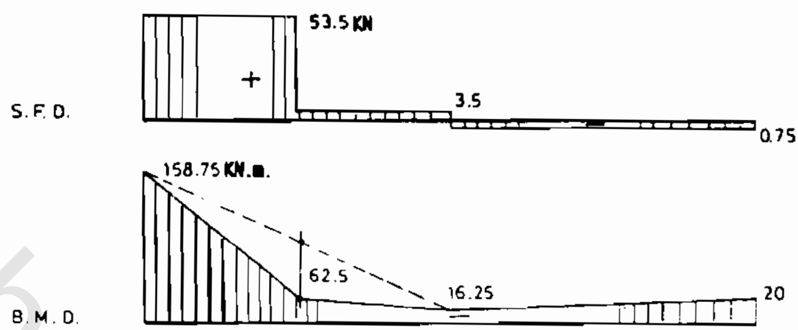


Figure 5.49

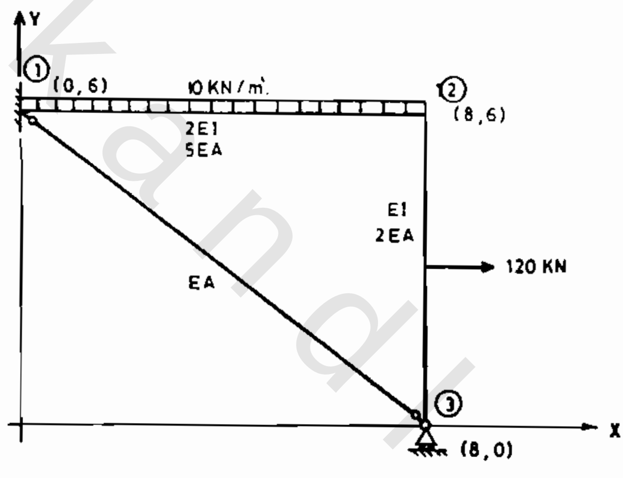


Figure 5.50

Solution

The boundary conditions are $\underline{\mathbf{D}}_1 = \underline{0}$ and $D_{3y} = 0$. The actions-deformations relationship which needs to be determined is

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} \\ \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \end{bmatrix}$$

The determination of $\underline{\mathbf{S}}_{22}$ is obtained as follows:

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3}$$

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{1}_{22} = 10^4 \begin{bmatrix} 3.125 & 0 & 0 \\ 0 & 0.4687 & -1.875 \\ 0 & -1.875 & 10 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{1}_{22} = \underline{\mathbf{R}}^{T}_{21}\, \underline{\mathbf{S}}'^{1}_{22}\, \underline{\mathbf{R}}_{21} = \underline{\mathbf{S}}'^{1}_{22}$$

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{3}_{22} = 10^4 \begin{bmatrix} 1.667 & 0 & 0 \\ 0 & 0.5556 & -1.667 \\ 0 & -1.667 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{3}_{22} = \underline{\mathbf{R}}^{T}_{23}\, \underline{\mathbf{S}}'^{3}_{22}\, \underline{\mathbf{R}}_{23} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 \\ 0 & 1.667 & 0 \\ 1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^{1}_{22} + \underline{\mathbf{S}}^{3}_{22} = 10^4 \begin{bmatrix} 3.6806 & 0 & 1.667 \\ 0 & 2.1357 & -1.875 \\ 1.667 & -1.875 & 16.667 \end{bmatrix}$$

The determination of $\underline{\mathbf{S}}_{33}$ and $\underline{\mathbf{S}}_{32}$ are obtained as follows:

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}^{1}_{33} + \underline{\mathbf{S}}^{2}_{33}$$

$$\underline{\mathbf{R}}_{31} = [0.8 \quad -0.6] \quad ; \quad \underline{\mathbf{S}}'^{1}_{33} = \left[\frac{EA}{10}\right] = [5000]$$

$$\underline{\mathbf{S}}^{1}_{33} = \underline{\mathbf{R}}^{T}_{31}\, \underline{\mathbf{S}}'^{1}_{33}\, \underline{\mathbf{R}}_{31} = 10^4 \begin{bmatrix} 0.32 & -0.24 \\ -0.24 & 0.18 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{2}_{33} = \underline{\mathbf{R}}^{T}_{32}\, \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32} = 10^4 \begin{bmatrix} 0.5556 & 0 & -1.667 \\ 0 & 1.667 & 0 \\ -1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{32} = \underline{\mathbf{R}}^{T}_{32}\, \underline{\mathbf{S}}'_{32}\, \underline{\mathbf{R}}_{23} = 10^4 \begin{bmatrix} -0.5556 & 0 & -1.667 \\ 0 & -1.667 & 0 \\ 1.667 & 0 & 3.333 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}^{1}_{33} + \underline{\mathbf{S}}^{2}_{33} = 10^4 \begin{bmatrix} 0.8756 & -0.24 & -1.667 \\ -0.24 & 1.847 & 0 \\ -1.667 & 0 & 6.667 \end{bmatrix}$$

The actions-deformations relation is thus obtained as

$$\begin{bmatrix} A_{2x} \\ A_{2y} \\ M_{2z} \\ A_{3x} \\ A_{3y} \\ M_{3z} \end{bmatrix} = 10^4 \begin{bmatrix} 3.6806 & 0 & 1.667 & -0.5556 & 0 & 1.667 \\ 0 & 2.1357 & -1.875 & 0 & -1.667 & 0 \\ 1.667 & -1.875 & 1.667 & -1.667 & 0 & 3.333 \\ -0.5556 & 0 & -1.667 & 0.8756 & -0.24 & -1.667 \\ 0 & -1.667 & 0 & -0.24 & 1.847 & 0 \\ 1.667 & 0 & 3.333 & -1.667 & 0 & 6.667 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ 0 \\ \theta_3 \end{bmatrix}$$

Part (a)

The equivalent joint actions due to loading in members 1-2 and 2-3 are obtained as follows:

$$\underline{\mathbf{A}}'_{F12} = \begin{bmatrix} 0 \\ -40 \\ 53.333 \end{bmatrix} \begin{matrix} \\ kN \\ kN.m. \end{matrix} \quad ; \quad \underline{\mathbf{A}}'_{F21} = \begin{bmatrix} 0 \\ 40 \\ -53.333 \end{bmatrix} \begin{matrix} \\ kN \\ kN.m. \end{matrix}$$

$$\underline{\mathbf{A}}'_{F23} = \begin{bmatrix} 0 \\ +60 \\ -90 \end{bmatrix} \begin{matrix} \\ kN \\ kN.m. \end{matrix} \quad ; \quad \underline{\mathbf{A}}'_{F32} = \begin{bmatrix} 0 \\ -60 \\ 90 \end{bmatrix} \begin{matrix} \\ kN \\ kN.m. \end{matrix}$$

$$\underline{\mathbf{A}}_2^e = -\underline{\mathbf{R}}_{21}^T \underline{\mathbf{A}}'_{F21} - \underline{\mathbf{R}}_{23}^T \underline{\mathbf{A}}'_{F23} = \begin{bmatrix} 60 \\ -40 \\ 143.333 \end{bmatrix} \begin{matrix} kN \\ kN \\ kN.m \end{matrix}$$

$$\underline{\mathbf{A}}_3^e = -\underline{\mathbf{R}}_{32}^T \underline{\mathbf{A}}'_{F32} = \begin{bmatrix} 60 \\ 0 \\ -90 \end{bmatrix} \begin{matrix} kN \\ kN \\ kN.m \end{matrix}$$

The final actions-deformations relationship after deleting the fifth rows and columns becomes

$$\begin{bmatrix} 60 \\ -40 \\ 143.333 \\ 60 \\ -90 \end{bmatrix} = 10^4 \begin{bmatrix} 3.6806 & 0 & 1.667 & -0.555 & 1.667 \\ 0 & 2.1357 & -1.875 & 0 & 0 \\ 1.667 & -1.875 & 16.667 & -1.667 & 3.333 \\ -0.5556 & 0 & -1.667 & 0.8756 & -1.667 \\ 1.667 & 0 & 3.333 & -1.667 & 6.667 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ \theta_3 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} D_{2x} & D_{2y} & \theta_2 & D_{3x} & \theta_3 \end{bmatrix} = 10^{-5}\,[25.96 \;\; -3.439 \;\; 17.419 \;\; 121.235 \;\; 1.614]$$

The member end actions are obtained as follows:

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F12}$$

$$= \begin{bmatrix} 81.22 \\ -34.24 \\ 93.45 \end{bmatrix} + \begin{bmatrix} 0 \\ -40 \\ -53.33 \end{bmatrix} = \begin{bmatrix} 81.22 \\ -74.24 \\ 146.8 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{A}}'_{F23}$$

$$= \begin{bmatrix} -5.76 \\ 21.22 \\ -37.12 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \\ -90 \end{bmatrix} = \begin{bmatrix} -5.76 \\ 81.22 \\ -127.12 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F32}$$

$$= \begin{bmatrix} -5.76 \\ 21.23 \\ -90 \end{bmatrix} + \begin{bmatrix} 0 \\ -60 \\ 90 \end{bmatrix} = \begin{bmatrix} -5.73 \\ -38.77 \\ 0 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{31} = \underline{\mathbf{S}}'^{1}_{33}\,\mathbf{R}_{31}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{31}\,\underline{\mathbf{R}}_{13}\,\underline{\mathbf{D}}_1 = 48.5 \text{ kN}$$

The axial force, shear force, and bending moment diagrams can now be plotted as shown in Figure 5.51.

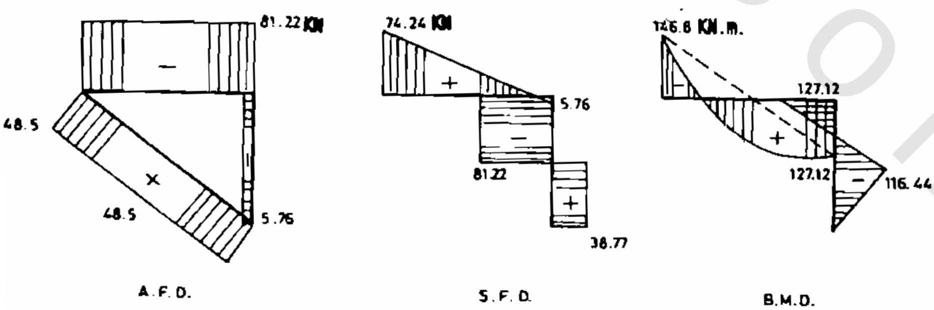


Figure 5.51

Part (b)

The equivalent joint actions due to initial strain due to shortening are obtained as follows:

$$\underline{\mathbf{A}}'_{F13} = \underline{\mathbf{A}}'_{F31} = \left[\frac{EA}{10}(0.01)\right] = [50] \text{ kN}$$

$$\underline{\mathbf{A}}_3^e = -\underline{\mathbf{R}}_{31}^T \underline{\mathbf{A}}'_{F31} = -\begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} [50] = \begin{bmatrix} -40 \\ 30 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \end{matrix}$$

The total loading at joint (3) is

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} -40 \\ 30 \\ 0 \end{bmatrix} + \begin{bmatrix} 60 \\ 0 \\ -90 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ -90 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The left hand side of the actions-deformations relationship in part (a) becomes

$$\left[\underline{A}_2^T \quad \underline{A}_3^T\right] = [60 \ \ -40 \ \ 143.333 \ \ 20 \ \ -90]$$

The solution for the unknown deformations is

$$\left[D_{2x} \ \ D_{2y} \ \ \theta_2 \ \ D_{3x} \ \ \theta_3\right] = 10^{-4}[23.31 \ \ -8.71 \ \ 11.41 \ \ 22.33 \ \ -19.45]$$

The member end actions are obtained from

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'_{12} \underline{\mathbf{R}}_{21} \underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F12}$$

$$= \begin{bmatrix} 72.84 \\ -25.48 \\ 73.38 \end{bmatrix} + \begin{bmatrix} 0 \\ -40 \\ 53.33 \end{bmatrix} = \begin{bmatrix} 72.84 \\ -65.48 \\ 126.71 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{I}_{22} \underline{\mathbf{R}}_{21} \underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F21}$$

$$= \begin{bmatrix} 72.84 \\ -25.48 \\ 130.43 \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \\ -53.33 \end{bmatrix} = \begin{bmatrix} 72.84 \\ 14.52 \\ 77.10 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_{2} + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_{3} + \underline{\mathbf{A}}'_{F23}$$

$$= \begin{bmatrix} -14.52 \\ -31.96 \\ 114.9 \end{bmatrix} + \begin{bmatrix} 0 \\ 44.82 \\ -102.05 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \\ -90 \end{bmatrix} = \begin{bmatrix} -14.52 \\ 72.86 \\ -77.15 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_{3} + \underline{\mathbf{S}}'_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_{2} + \underline{\mathbf{A}}'_{F32}$$

$$= \begin{bmatrix} 0 \\ 44.82 \\ -166.88 \end{bmatrix} + \begin{bmatrix} -14.52 \\ -31.96 \\ 76.87 \end{bmatrix} + \begin{bmatrix} 0 \\ -60 \\ 90 \end{bmatrix} = \begin{bmatrix} -14.52 \\ -47.14 \\ 0 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{31} = \underline{\mathbf{S}}'^{1}_{33}\,\underline{\mathbf{R}}_{31}\,\underline{\mathbf{D}}_{3} + \underline{\mathbf{S}}'_{31}\,\underline{\mathbf{R}}_{13}\,\underline{\mathbf{D}}_{1} + \underline{\mathbf{A}}'_{F31}$$

$$= \begin{bmatrix} 8.832 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 58.932 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The axial force, shear force, and bending moment diagrams can be plotted as shown in Figure 5.52.

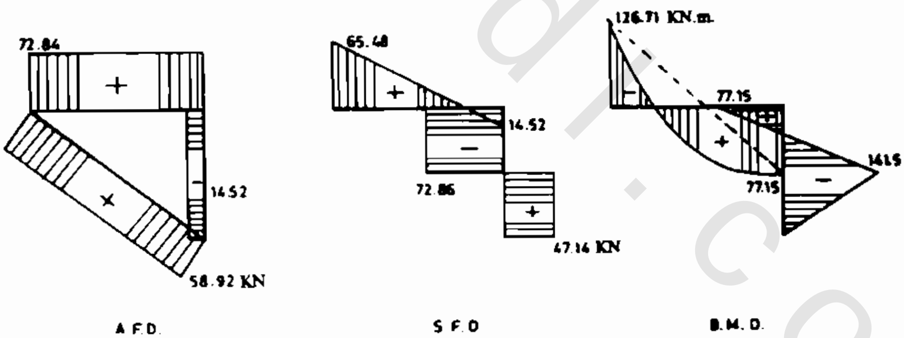


Figure 5.52

Example 5.14

Determine the axial force, shear force, and bending moment diagrams for the plane frame shown in Figure 5.53 due to the applied loads and a rise in temperature of member (2)–(3). ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}/°c$, $EA = 10^5$ kN).

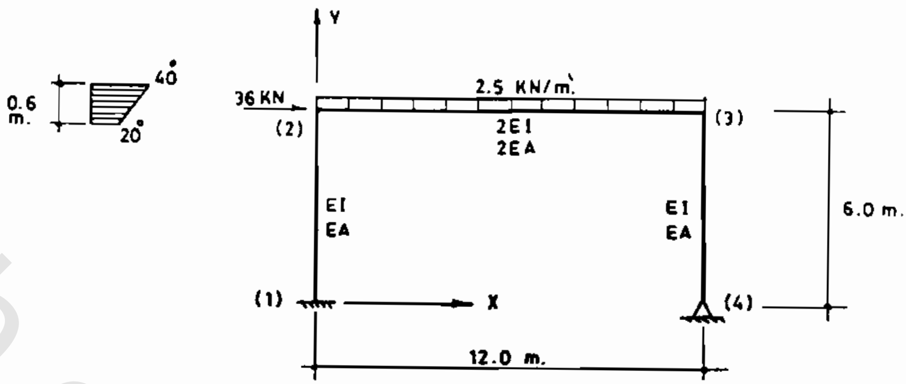


Figure 5.53

Solution

The boundary conditions are $\underline{\mathbf{D}}_1 = \underline{0}$ and $D_{4x} = D_{4y} = 0$. The actions-deformations relationship which needs to be determined is

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \\ \underline{\mathbf{A}}_4 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{0} \\ \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} \\ \underline{0} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \\ \underline{\mathbf{D}}_4 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}_{22}$ one has $\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{3}$

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{1}_{22} = 10^4 \begin{bmatrix} 1.667 & 0 & 0 \\ 0 & 0.5556 & -1.667 \\ 0 & -1.667 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22}^{1} = \underline{\mathbf{R}}_{21}^{T}\ \underline{\mathbf{S}}'^{1}_{22}\ \underline{\mathbf{R}}_{21} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 \\ 0 & 1.667 & 0 \\ 1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{3}_{22} = 10^4 \begin{bmatrix} 1.667 & 0 & 0 \\ 0 & 0.1389 & -0.833 \\ 0 & -0.833 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22}^{3} = \underline{\mathbf{R}}_{23}^{T}\ \underline{\mathbf{S}}'^{3}_{22}\ \underline{\mathbf{R}}_{23} = 10^4 \begin{bmatrix} 1.667 & 0 & 0 \\ 0 & 0.1389 & 0.833 \\ 0 & 0.833 & 6.667 \end{bmatrix}$$

$$\underline{S}_{22} = \underline{S}_{22}^{1} + \underline{S}_{22}^{3} = 10^{4}\begin{bmatrix} 2.222 & 0 & 1.067 \\ 0 & 1.805 & 0.833 \\ 1.667 & 0.844 & 13.333 \end{bmatrix}$$

The determination of $\underline{S}_{33}$ one has $\underline{S}_{33} = \underline{S}_{33}^{2} + \underline{S}_{33}^{4}$

$$\underline{S}_{33}^{2} = \underline{R}_{32}^{T}\ \underline{S}'^{2}_{33}\ \underline{R}_{32} = 10^{4}\begin{bmatrix} 1.667 & 0 & 0 \\ 0 & 0.1389 & -0.833 \\ 0 & -0.833 & 6.667 \end{bmatrix}$$

$$\underline{R}_{34} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ ; \quad \underline{S}_{33}^{4} = \underline{R}_{34}^{T}\ \underline{S}'^{4}_{33}\ \underline{R}_{34} = \underline{S}_{22}^{1}$$

$$\underline{S}_{33} = \underline{S}_{33}^{2} + \underline{S}_{33}^{4} = 10^{4}\begin{bmatrix} 2.222 & 0 & 1.667 \\ 0 & 1.805 & -0.8333 \\ 1.667 & -0.833 & 13.333 \end{bmatrix}$$

To determine $\underline{S}_{44}, \underline{S}_{23}, \underline{S}_{34}$, one has

$$\underline{S}_{44} = \underline{S}_{44}^{3} = \underline{R}_{43}^{T}\ \underline{S}'^{3}_{44}\ \underline{R}_{34} = 10^{4}\begin{bmatrix} 0.5556 & 0 & -1.667 \\ 0 & 1.667 & 0 \\ -1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{S}_{23} = \underline{R}_{23}^{T}\ \underline{S}'_{23}\ \underline{R}_{32} = 10^{4}\begin{bmatrix} -1.667 & 0 & 0 \\ 0 & -0.1389 & 0.8333 \\ 0 & -0.8333 & 3.333 \end{bmatrix}$$

$$\underline{S}_{34} = \underline{R}_{34}^{T}\ \underline{S}'_{34}\ \underline{R}_{43} = 10^{4}\begin{bmatrix} -0.5556 & 0 & 1.66 \\ 0 & -1.667 & 0 \\ -1.667 & 0 & 3.33 \end{bmatrix}$$

The equivalent joint actions due to direct loading are obtained from the fixed end actions as follows:

$$\underline{A}'_{F23} = \begin{bmatrix} 0 \\ -15 \\ 30 \end{bmatrix} \begin{matrix} \\ kN \\ kN.m. \end{matrix} \quad ; \quad \underline{A}'_{F32} = \begin{bmatrix} 0 \\ 15 \\ -30 \end{bmatrix} \begin{matrix} \\ kN \\ kN.m. \end{matrix}$$

$$\underline{\mathbf{A}}^{e}_{23} = -\underline{\mathbf{R}}^{T}_{23}\,\underline{\mathbf{A}}'_{F23} = \begin{bmatrix} 0 \\ -15 \\ 30 \end{bmatrix} \quad ; \quad \underline{\mathbf{A}}^{c}_{32} = -\mathrm{R}^{T}_{32}\,\mathrm{A}'_{F32} = \begin{bmatrix} 0 \\ -15 \\ 30 \end{bmatrix} \begin{matrix} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The equivalent joint actions due to temperature change are

$$\underline{\mathbf{A}}'_{F23} = \begin{bmatrix} -\alpha(2EA)\left(\dfrac{T_1+T_2}{2}\right) \\ 0 \\ -\alpha(2EI)\left(\dfrac{T_1-T_2}{h}\right) \end{bmatrix} = \begin{bmatrix} -60 \\ 0 \\ -66.67 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix} \quad ; \quad \underline{\mathbf{A}}'_{F32} = \begin{bmatrix} -60 \\ 0 \\ -66.67 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}^{e}_{23} = -\underline{\mathbf{R}}^{T}_{23}\,\underline{\mathbf{A}}'_{F23} = \begin{bmatrix} -60 \\ 0 \\ 66.67 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix} \quad ; \quad \underline{\mathbf{A}}^{c}_{32} = -\underline{\mathbf{R}}^{T}_{32}\,\underline{\mathbf{A}}'_{F23} = \begin{bmatrix} +60 \\ 0 \\ -66.67 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The total joint loads are obtained from adding the joint loading to the equivalent joint loading as follows:

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 36 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -15 \\ -30 \end{bmatrix} + \begin{bmatrix} -60 \\ 0 \\ 66.7 \end{bmatrix} = \begin{bmatrix} -24 \\ -15 \\ 36.67 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -15 \\ 30 \end{bmatrix} + \begin{bmatrix} 60 \\ 0 \\ -66.67 \end{bmatrix} = \begin{bmatrix} 60 \\ -15 \\ -36.67 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m.} \end{matrix}$$

The final actions-deformations relationship, after deleting the rows and columns corresponding to D_{4x} and D_{4y} is

$$\begin{bmatrix} -24 \\ -15 \\ 36.67 \\ 60 \\ -15 \\ -36.67 \\ 0 \end{bmatrix} = 10^4 \begin{bmatrix} 2.222 & 0 & 1.667 & -1.667 & 0 & 0 & 0 \\ 0 & 1.805 & 0.833 & 0 & -0.1389 & 0.8333 & 0 \\ 1.667 & 0.8333 & 13.333 & 0 & -0.8333 & 3.333 & 0 \\ -1.667 & 0 & 0 & 2.222 & 0 & 1.667 & 1.667 \\ 0 & -0.1389 & -0.8333 & 0 & 1.805 & -0.8333 & 0 \\ 0 & 0.8333 & 3.333 & 1.667 & -0.833 & 13.333 & 3.333 \\ 0 & 0 & 0 & 1.667 & 0 & 3.333 & 6.667 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ D_{3y} \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

The solution for the unknown deformations is

$$\left[D_{2x}\ D_{2y}\ \theta_2\ \ D_{3x}\ \ D_{3y}\ \ \theta_3\ \ \theta_4\right] = 10^{-4}\left[66.36\ \ -3.09\ \ -3.78\ \ 99.1\ \ -14.9\ \ -9.98\ \ -19.78\right.$$

The members end actions are obtained, using $\underline{\mathbf{D}}_4^T = 10^{-4}\left[0 \quad 0 \quad -19.78\right]$

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 = [-5.17\ \ -30.55\ \ 97.97]^T$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{1}_{22}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{21}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 = [-5.17\ \ -30.55\ \ 85.35]^T$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\,\underline{\mathbf{A}}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{A}}'_{F23} = [-5.44\ \ -5.17\ \ -85.35]^T$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\,\underline{\mathbf{R}}_{32}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F32} = [-5.44\ \ 24.83\ \ -32.69]^T$$

$$\underline{\mathbf{A}}'_{34} = \underline{\mathbf{S}}'^{4}_{33}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{34}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 = [-24.83\ \ -5.44\ \ +32.69]^T$$

$$\underline{\mathbf{A}}'_{43} = \underline{\mathbf{S}}'_{43}\,\underline{\mathbf{R}}_{34}\,\underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'^{3}_{44}\,\underline{\mathbf{R}}_{43}\,\underline{\mathbf{D}}_4 = [-24.83\ \ -5.44\ \ 0]^T$$

The axial force, shear force, and bending moment diagrams can now be plotted as shown in Figure 5.54.

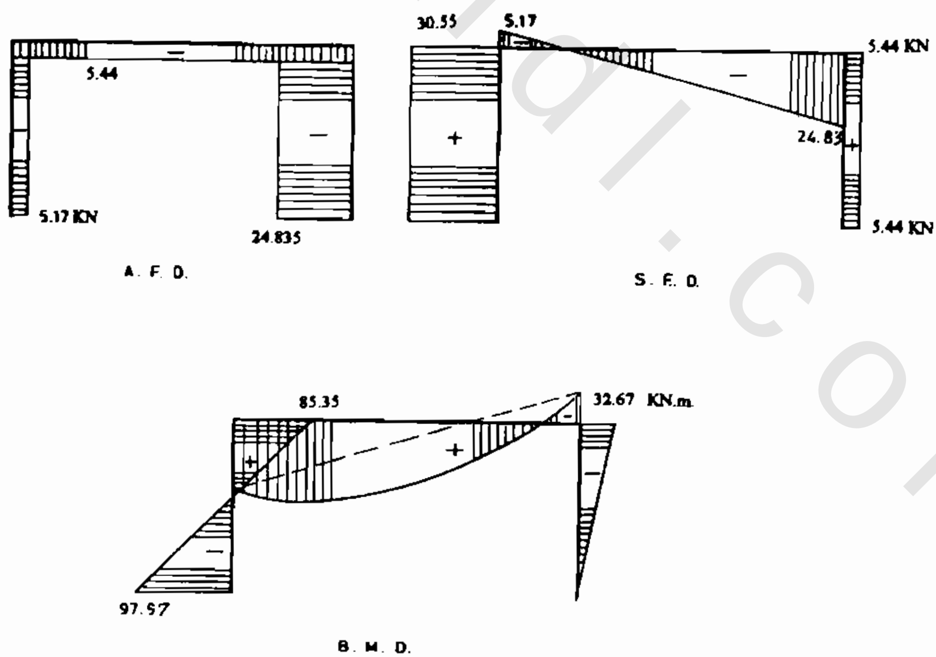


Figure 5.54

Example 5.15

Determine the axial force, shear force, and bending moment diagrams for the frame of example 5.14, neglecting the axial deformations.

Solution

The boundary conditions become $\underline{\mathbf{D}}_1 = \underline{\mathbf{0}}$, $D_{2y} = 0$, $D_{3y} = 0$, $D_{4x} = 0$, $D_{4y} = 0$, and $D_{2x} = D_{3x}$. The members stiffness matrices in case of neglecting axial deformations become of dimensions 2 × 2. The stiffness matrices in global coordinates are determined as follows:

To determine $\underline{\mathbf{S}}_{22}$ one has $\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^1_{22} + \underline{\mathbf{S}}^3_{22}$

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^1_{22} = 10^4 \begin{bmatrix} 0.5556 & -1.667 \\ -1.667 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}^1_{22} = \underline{\mathbf{R}}^T_{21}\ \underline{\mathbf{S}}'^1_{22}\ \underline{\mathbf{R}}_{21} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 \\ 0 & 0 & 0 \\ 1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^3_{22} = 10^4 \begin{bmatrix} 0.1389 & -0.883 \\ -0.883 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}^3_{22} = \underline{\mathbf{R}}^T_{23}\ \underline{\mathbf{S}}'^3_{22}\ \underline{\mathbf{R}}_{23} = 10^4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1389 & 0.833 \\ 0 & 0.833 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^1_{22} + \underline{\mathbf{S}}^3_{22} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 \\ 0 & 0.1389 & 0.8333 \\ 1.667 & 0.833 & 13.333 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}_{33}$ one has

$$\underline{\mathbf{R}}_{34} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^4_{33} = 10^4 \begin{bmatrix} 0.5556 & -1.667 \\ -1.667 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}^4_{33} = \underline{\mathbf{R}}^T_{34}\ \underline{\mathbf{S}}'^4_{33}\ \underline{\mathbf{R}}_{34} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 \\ 0 & 0 & 0 \\ 1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{R}}_{32} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{2}_{33} = 10^4 \begin{bmatrix} 0.1389 & -0.883 \\ -0.883 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{2}_{33} = \underline{\mathbf{R}}^{T}_{32}\, \underline{\mathbf{S}}'^{2}_{33}\, \underline{\mathbf{R}}_{32} = 10^4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1389 & -0.833 \\ 0 & -0.833 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{33} = \underline{\mathbf{S}}^{2}_{33} + \underline{\mathbf{S}}^{4}_{33} = 10^4 \begin{bmatrix} 0.555 & 0 & 1.667 \\ 0 & 0.1389 & -0.833 \\ 1.667 & -0.833 & 13.333 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}_{44}, \underline{\mathbf{S}}_{23}, \underline{\mathbf{S}}_{34}$ one has

$$\underline{\mathbf{S}}_{44} = \underline{\mathbf{S}}^{3}_{44} = \underline{\mathbf{R}}^{T}_{43}\, \underline{\mathbf{S}}'^{3}_{44}\, \underline{\mathbf{R}}_{43} = 10^4 \begin{bmatrix} 0.5556 & 0 & -1.667 \\ 0 & 0 & 0 \\ -1.667 & 0 & 6.667 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{23} = \underline{\mathbf{R}}^{T}_{23}\, \underline{\mathbf{S}}'^{3}_{22}\, \underline{\mathbf{R}}_{32} = 10^4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.1389 & 0.8333 \\ 0 & -0.8333 & 3.333 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{34} = \underline{\mathbf{R}}^{T}_{34}\, \underline{\mathbf{S}}'^{4}_{33}\, \underline{\mathbf{R}}_{43} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 \\ 0 & 0 & 0 \\ -1.667 & 0 & 3.333 \end{bmatrix}$$

The equivalent joint actions have been determined in the previous example. However, the axial force due to temperature change should be neglected. Therefore one has

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 36 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -15 \\ -30 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 66.67 \end{bmatrix} = \begin{bmatrix} 36 \\ -15 \\ 36.667 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m} \end{matrix}$$

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -15 \\ 30 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -66.67 \end{bmatrix} = \begin{bmatrix} 0 \\ -15 \\ -36.667 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN.m} \end{matrix}$$

The actions-deformations relationship is

$$\begin{bmatrix} 36 \\ -15 \\ 36.667 \\ 0 \\ -15 \\ -36.667 \\ 0 \end{bmatrix} = 10^4 \begin{bmatrix} 0.5556 & 0 & 1.667 & 0 & 0 & 0 & 0 \\ 0 & 0.1389 & 0.8333 & 0 & -0.1389 & 0.8333 & 0 \\ 1.667 & 0.8333 & 13.333 & 0 & -0.8333 & 3.333 & 0 \\ 0 & 0 & 0 & 0.5556 & 0 & 1.667 & 1.667 \\ 0 & -0.1389 & -0.8333 & 0 & 0.1389 & -0.8333 & 0 \\ 0 & 0.83333 & 3.333 & 1.667 & -0.8333 & 13.333 & 3.333 \\ 0 & 0 & 0 & 1.667 & 0 & 3.333 & 6.667 \end{bmatrix} \begin{bmatrix} D_{2x} \\ D_{2y} \\ \theta_2 \\ D_{3x} \\ D_{3y} \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

Since $D_{2x} = D_{3x}$, $D_{2y} = 0$ and $D_{3y} = 0$, one obtains the final relation by deleting the rows and columns corresponding to $D_{2y} = 0$ and $D_{3y} = 0$, and adding the first and fourth rows and columns. Therefore, one obtains

$$\begin{bmatrix} 36 \\ 36.667 \\ -36.667 \\ 0 \end{bmatrix} = 10^4 \begin{bmatrix} 1.111 & 1.667 & 1.667 & 1.667 \\ 1.667 & 13.333 & 3.333 & 0 \\ 1.667 & 3.333 & 13.333 & 3.333 \\ 1.667 & 0 & 3.333 & 6.667 \end{bmatrix} \begin{bmatrix} D_{2x} \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

The solution is obtained as

$$\begin{bmatrix} D_{2x} & \theta_2 & \theta_3 & \theta_4 \end{bmatrix} = 10^{-4}\begin{bmatrix} 70.623 & -4.343 & -6.948 & -14.1855 \end{bmatrix}$$

The deformations can thus be expressed as

$$\begin{bmatrix} D_{2x} & D_{2y} & \theta_2 & D_{3x} & D_{3y} & \theta_3 & \theta_4 \end{bmatrix} = 10^{-4}\begin{bmatrix} 70.623 & 0 & -4.343 & 70.623 & 0 & -6.948 & -14.1855 \end{bmatrix}$$

The member end actions which represent the shear force and bending moment are obtained as follows:

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{2}_{11}\ \underline{\mathbf{R}}_{12}\ \underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'_{12}\ \underline{\mathbf{R}}_{21}\ \underline{\mathbf{D}}_2 = \begin{bmatrix} -32 \\ 103.253 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{1}_{22}\ \underline{\mathbf{R}}_{21}\ \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{21}\ \underline{\mathbf{R}}_{12}\ \underline{\mathbf{D}}_1 = \begin{bmatrix} -32 \\ 88.77 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{3}_{22}\ \underline{\mathbf{R}}_{23}\ \underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'_{23}\ \underline{\mathbf{R}}_{32}\ \underline{\mathbf{D}}_3 + \underline{\mathbf{A}}'_{F32} = \begin{bmatrix} -5.59 \\ -88.77 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{2}_{33}\ \underline{\mathbf{R}}_{32}\ \underline{\mathbf{D}}_3 + \underline{\mathbf{S}}'_{32}\ \underline{\mathbf{R}}_{23}\ \underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'_{F32} = \begin{bmatrix} 24.41 \\ -24.13 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{A}'_{34} = \underline{S}'^{4}_{33}\ \underline{R}_{34}\ \underline{D}_3 + \underline{S}'_{34}\ \underline{R}_{43}\ \underline{D}_4 = \begin{bmatrix} -4 \\ 24.13 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{A}'_{43} = \underline{S}'_{43}\ \underline{R}_{34}\ \underline{D}_3 + \underline{S}'^{3}_{44}\ \underline{R}_{43}\ \underline{D}_4 = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

The bending moment and shear force diagrams can now be plotted as shown in Figure 5.55. The axial force diagram can be determined using statics principles.

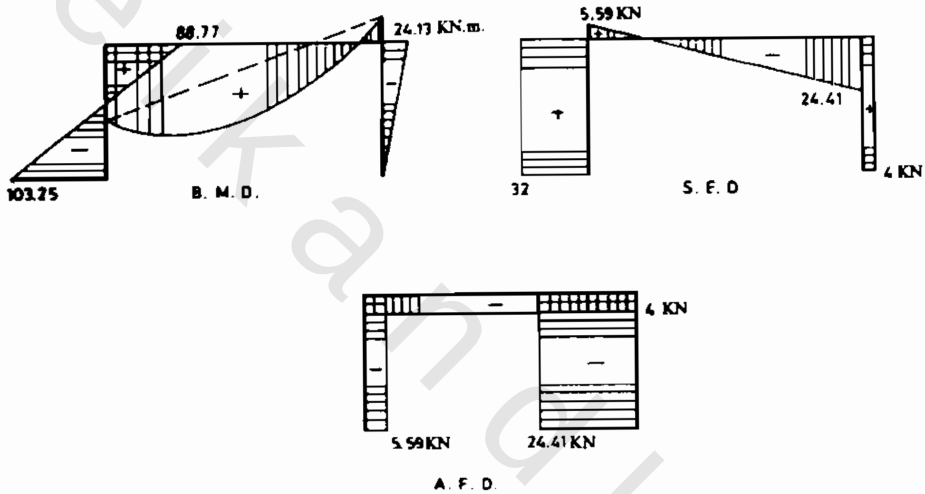


Figure 5.55

5.7 SPECIAL PROBLEMS

In this section, the application of the stiffness matrix method for special problems is shown. The problems considered in this section include: members with moment releases, inclined supports, elastic supports, nonprismatic members, series connected members, and shear deformation effect.

5.7.1 Members with Moment Releases

In the presence of a hinge at a joint, one has to modify the stiffness matrices of the members connected with this joint. For example, if member ij of a plane frame is hinged at j, the stiffness matrices in the local coordinates can be determined by the unit displacement method according to Figure 5.56 and 5.57 as follows:

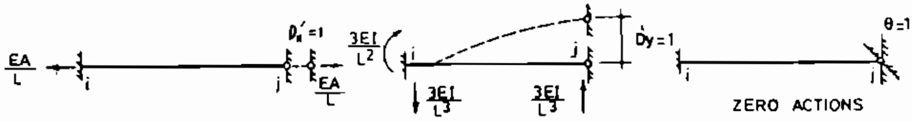


Figure 5.56

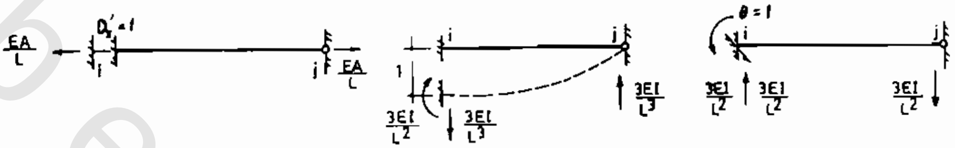


Figure 5.57

$$\underline{S}'^{oj}_{ii} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & \frac{-3EI}{L^2} \\ 0 & \frac{-3EI}{L^2} & \frac{3EA}{L} \end{bmatrix} \; ; \; \underline{S}'^{o}_{ji} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & \frac{-3EI}{L^2} \\ 0 & 0 & 0 \end{bmatrix} \qquad (5.110)$$

$$\underline{S}'^{oj}_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \; ; \; \underline{S}'^{o}_{ji} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & 0 \\ 0 & \frac{-3EI}{L2} & 0 \end{bmatrix} \qquad (5.111)$$

in which the superscript o is used to indicate a moment release exists in member ij, and the elements of matrix $\underline{S}'^{oi}_{jj}$ indicate that the moment release is at joint j.

For other types of structures, one has to state the kind of releases. In grids, for example, the moment releases could be the moment M'_x or M'_y. In space frames, the moment releases could be M'_x, M'_y or M'_z. In any case, the stiffness matrices can be found by the unit displacement method as has been done for the member of the plane frame above. Sometimes one faces a moment release at the support. In this case one has the choice either to include the moment release in the boundary conditions, or to modify the stiffness matrices of the members connected with this joint. When a joint with a moment release is connecting two members, one may attribute the joint to any member, or to both members. Whenever a moment release is taken into consideration, one has to use the fixed end actions and equivalent joint actions considering the presence of the moment release in the member.

5.7.2 Inclined Supports

Sometimes the supports of the structure are built on inclined surfaces, and, therefore, the determination of the reactions becomes easier when dealing with the supports local coordinates. In other words, the analyst may find it is beneficial to describe the reactions in the support local coordinates and not in the global coordinates. For example, the plane frame shown in Figure 5.58 has an inclined roller at joint (1). Therefore, one may state that $D_{1ys} = 0$ in the support local coordinates. One, thus, reduces the dimensions of the stiffness matrix when using the support local coordinates. In order to transform part of the actions-deformations relationship into the support local coordinates, one has to determine the transformation matrix between the support coordinates and the global coordinates. The actions-deformations relationship for the plane frame of Figure 5.58 is described in global coordinates by

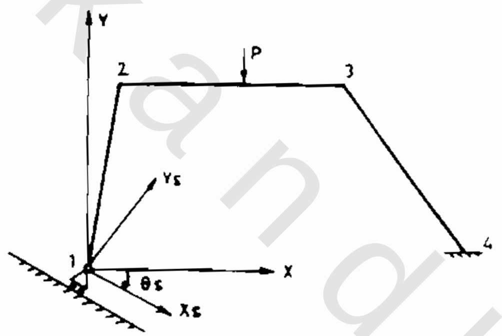


Figure 5.58

$$\begin{bmatrix} \underline{\mathbf{A}}_1 \\ \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_3 \\ \underline{\mathbf{A}}_4 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_3 \\ \underline{\mathbf{D}}_4 \end{bmatrix} \qquad (5.112)$$

The boundary conditions are $D_{1ys} = 0$ in the support local coordinates and $\underline{\mathbf{D}}_4 = \underline{\mathbf{0}}$ in global coordinates. Therefore, one has to transform $\underline{\mathbf{A}}_1$ and $\underline{\mathbf{D}}_1$ into the supports local coordinates. The actions at joint (1) expressed in the support local coordinates which are denoted by $\underline{\mathbf{A}}_{1s}$ are related to the global coordinates by

$$\underline{\mathbf{A}}_{1s} = \begin{bmatrix} \cos(x_s x) & \cos(x_s y) & \cos(x_s z) \\ \cos(y_s x) & \cos(y_s y) & \cos(y_s z) \\ 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{A}}_1 = \underline{\mathbf{R}}_{1s}\, \underline{\mathbf{A}}_1 \qquad (5.113)$$

Similarly, the relationship between the deformations at joint (1) in the support local coordinates and the global coordinates is

$$\underline{\mathbf{D}}_{1s} = \underline{\mathbf{R}}_{1s}\,\underline{\mathbf{D}}_{1} \tag{5.114}$$

In order to replace $\underline{\mathbf{A}}_1$ and $\underline{\mathbf{D}}_1$ in Equations 5.112 by $\underline{\mathbf{A}}_{1s}$ and $\underline{\mathbf{D}}_{1s}$, respectively, one uses Equations 5.113 and 5.114 and therefore, Equation 5.112 becomes

$$\begin{bmatrix} \underline{\mathbf{A}}_{1s} \\ \underline{\mathbf{A}}_{2} \\ \underline{\mathbf{A}}_{3} \\ \underline{\mathbf{A}}_{4} \end{bmatrix} = \begin{bmatrix} (\underline{\mathbf{R}}_{1s}\,\underline{\mathbf{S}}_{11}) & (\underline{\mathbf{R}}_{1s}\,\underline{\mathbf{S}}_{12}) & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_{1} \\ \underline{\mathbf{D}}_{2} \\ \underline{\mathbf{D}}_{3} \\ \underline{\mathbf{D}}_{4} \end{bmatrix}$$

or

$$\begin{bmatrix} \underline{\mathbf{A}}_{1s} \\ \underline{\mathbf{A}}_{2} \\ \underline{\mathbf{A}}_{3} \\ \underline{\mathbf{A}}_{4} \end{bmatrix} = \begin{bmatrix} (\underline{\mathbf{R}}_{1s}\,\underline{\mathbf{S}}_{11}\,\underline{\mathbf{R}}_{1s}^{T}) & (\underline{\mathbf{R}}_{1s}\,\underline{\mathbf{S}}_{12}) & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ (\underline{\mathbf{S}}_{21}\,\underline{\mathbf{R}}_{1s}^{T}) & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{23} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{32} & \underline{\mathbf{S}}_{33} & \underline{\mathbf{S}}_{34} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{43} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_{1s} \\ \underline{\mathbf{D}}_{2} \\ \underline{\mathbf{D}}_{3} \\ \underline{\mathbf{D}}_{4} \end{bmatrix} \tag{5.115}$$

in which $D_{1sy} = 0$. Notice that one may also consider member (1)–(2) having a moment release at joint (1). In this case, $\underline{\mathbf{S}}_{11}$ is calculated as was illustrated in section 5.7.1.

After the determination of the free deformations, the members' end actions are calculated as usual. However, for members with inclined supports, the deformations in the support local coordinates or in the global coordinates can be used as follows:

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{S}}'^{j}_{ii}\,\underline{\mathbf{R}}_{ij}\,\underline{\mathbf{D}}_{i} + \underline{\mathbf{S}}'_{ij}\,\underline{\mathbf{R}}_{ji}\,\underline{\mathbf{D}}_{j} \tag{5.116}$$

$$\underline{\mathbf{A}}'_{ij} = \underline{\mathbf{S}}'^{j}_{ii}\,\underline{\mathbf{R}}_{ij}\,\underline{\mathbf{R}}_{is}^{T}\,\underline{\mathbf{D}}_{i} + \underline{\mathbf{S}}'_{ij}\,\underline{\mathbf{R}}_{ji}\,\underline{\mathbf{D}}_{j} \tag{5.117}$$

in which Equation 5.114 has been used.

Therefore, from Equation 5.115 it is understood that in order to modify certain actions and deformations at support (i) into the support local coordinates, one has to premultiply the corresponding rows by the transformation matrix $\underline{\mathbf{R}}_{is}$ and post multiply the corresponding columns by $\underline{\mathbf{R}}_{is}^{T}$.

5.7.3 Elastic Supports

The deformations of some supports in the structure depend on the magnitude of their reactions. If the relationship between the reactions and the corresponding deformations is linear, the support is called a linear elastic support. Linear elastic supports exist in many structures, especially those which are supported on sandy soil, or springs. When the reactions-deformations relationship is nonlinear, one may idealize such a relation by a bilinear relation, such as shown in Figure 5.59.

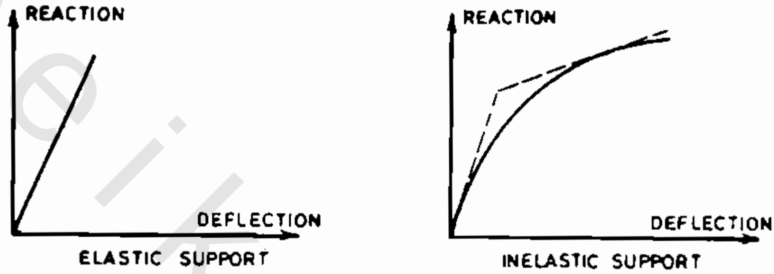


Figure 5.59

In the presence of an elastic support, the support is considered as a member connected with another joint. The joint stiffness matrix is found as usual by summing up all direct stiffness matrices at the support. For example, if the support at joint (1) in the plane frame shown in Figure 5.60 is idealized by the three springs in the horizontal, vertical, and rotational directions, the stiffness matrix $\underline{S}_{11}$ is obtained as follows:

$$\underline{S}_{11} = \underline{S}_{11}^{2} + S_{11}^{3} + \underline{S}_{11}^{4} + \underline{S}_{11}^{5} + \underline{S}_{11}^{6} \qquad (5.118)$$

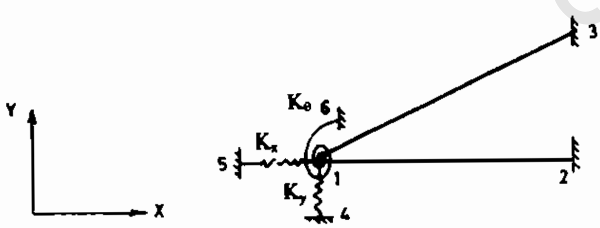


Figure 5.60

According to the global coordinates shown in Figure 5.60, the stiffness matrices, $\underline{S}_{11}^{4}$, $\underline{S}_{11}^{5}$ and $\underline{S}_{11}^{6}$ are, respectively, given by:

$$\underline{S}_{11}^{4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.119)$$

$$\underline{S}_{11}^{5} = \begin{bmatrix} K_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.120)$$

$$\underline{S}_{11}^{6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_z \end{bmatrix} \quad (5.121)$$

The analysis is continued as usual, but the deformations at joint (1) are considered as unknowns.

Example 5.16

Determine the bending moment and shear force diagrams for the plane frame shown in Figure 5.61, in which joint (1) is an elastic support and joint (2) is a hinge. (EA = 30000 kN, EI = 30000 kN.m^2, $K_x = K_y = 0.36$ kN/cm, $K_\theta = 50$ kN.m/rad).

Solution

The boundary conditions are $\underline{D}_3 = \underline{0}$. The moment release at joint (2) can be attributed to member (2)–(1) or (2)–(3) or to both members. The actions-deformations relationship, considering the boundary conditions, is

$$\begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{S}_{21} & \underline{S}_{22} \end{bmatrix} \begin{bmatrix} \underline{D}_1 \\ \underline{D}_2 \end{bmatrix}$$

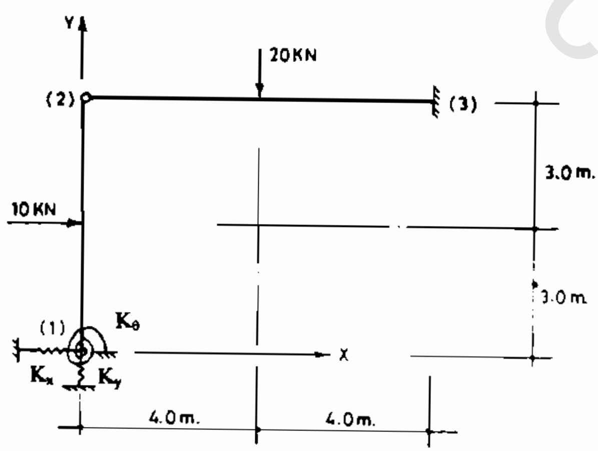


Figure 5.61

A solution considering the hinge is common between members (2)–(1) and (2)–(3) is given. The matrix $\underline{\mathbf{S}}_{11}$ in the global coordinates is obtained as follows:

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}_{11}^{o2} + \underline{\mathbf{S}}_{11s}$$

$$\underline{\mathbf{S}}_{11s} = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 50 \end{bmatrix} \quad ; \quad \underline{\mathbf{R}}_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{o2}_{11} = 10^3 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.4167 & -2.5 \\ 0 & -2.5 & 15 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{11}^{o2} = \underline{\mathbf{R}}_{12}^{T}\ \underline{\mathbf{S}}'^{o2}_{11}\ \underline{\mathbf{R}}_{12}$$

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}_{11}^{o2} + \underline{\mathbf{S}}_{11s} = 10^3 \begin{bmatrix} 0.4527 & 0 & -2.5 \\ 0 & 5.036 & 0 \\ -2.5 & 0 & 15.05 \end{bmatrix}$$

The matrices $\underline{\mathbf{S}}_{22}$ and $\underline{\mathbf{S}}_{12}$ are obtained as follows:

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{o1} + \underline{\mathbf{S}}_{22}^{o3}$$

$$\underline{\mathbf{S}}_{22}^{o1} = \underline{\mathbf{R}}_{21}^{T}\ \underline{\mathbf{S}}'^{o1}_{22}\ \underline{\mathbf{R}}_{21} \quad ; \quad \underline{\mathbf{S}}_{22}^{o3} = \underline{\mathbf{R}}_{23}^{T}\ \underline{\mathbf{S}}'^{o3}_{22}\ \underline{\mathbf{R}}_{23}$$

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{R}}_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{S}}'^{o1}_{22} = 10^3 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.4167 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{o3}_{22} = 10^3 \begin{bmatrix} 3.75 & 0 & 0 \\ 0 & 0.1758 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{o1} + \underline{\mathbf{S}}_{22}^{o3} = 10^3 \begin{bmatrix} 4.1667 & 0 & 0 \\ 0 & 5.1758 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{12} = \underline{\mathbf{R}}_{12}^{T}\ \underline{\mathbf{S}}'^{o}_{12}\ \underline{\mathbf{R}}_{21} \quad ; \quad \underline{\mathbf{S}}'^{o}_{12} = 10^3 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.4167 & 0 \\ 0 & -2.5 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{12} = 10^3 \begin{bmatrix} -0.4167 & 0 & 0 \\ 0 & -5 & 0 \\ 2.5 & 0 & 0 \end{bmatrix}$$

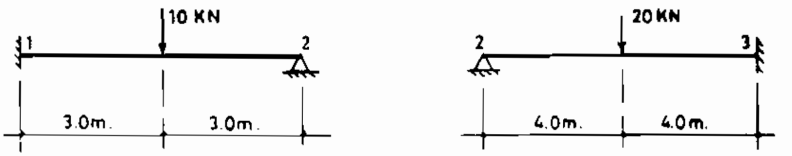


Figure 5.62

The equivalent joint actions are obtained as follows:

The fixed end actions are determined for members (1)-(2) and (2)-(3) as shown in Figure 5.62. They are given by

$$\underline{\mathbf{A}}'^{o}_{F12} = \begin{bmatrix} 0 \\ -6.875 \\ 11.25 \end{bmatrix} \quad ; \quad \underline{\mathbf{A}}'^{o}_{F21} = \begin{bmatrix} 0 \\ 3.125 \\ 0 \end{bmatrix}$$

$$\underline{\mathbf{A}}'^{o}_{F23} = \begin{bmatrix} 0 \\ -6.25 \\ 0 \end{bmatrix} \quad ; \quad \underline{\mathbf{A}}'^{o}_{F32} = \begin{bmatrix} 0 \\ 13.75 \\ -30 \end{bmatrix}$$

The equivalent joint actions at joints (1) and (2) are obtained from

$$\underline{\mathbf{A}}^{e}_{1} = -\underline{\mathbf{R}}^{T}_{12}\,\underline{\mathbf{A}}'^{o}_{F12} = [6.875 \quad 0 \quad -11.25]^T$$

$$\underline{\mathbf{A}}^{e}_{2} = -\underline{\mathbf{R}}^{T}_{21}\,\underline{\mathbf{A}}'^{o}_{F21} - \underline{\mathbf{R}}^{T}_{23}\,\underline{\mathbf{A}}'^{o}_{F23} = [3.125 \quad -6.25 \quad 0]^T$$

The final actions-deformations relationship is obtained as

$$\begin{bmatrix} 6.875 \\ 0 \\ -11.2 \\ 3.125 \\ -6.25 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} 0.4527 & 0 & -2.5 & -0.4167 & 0 & 0 \\ 0 & 5.036 & 0 & 0 & -5 & 0 \\ -2.5 & 0 & 15.05 & 2.5 & 0 & 0 \\ -0.4167 & 0 & 2.5 & 4.1667 & 0 & 0 \\ 0 & -5 & 0 & 0 & 5.1758 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_{1x} \\ D_{1y} \\ \theta_1 \\ D_{2x} \\ D_{2y} \\ \theta_2 \end{bmatrix}$$

The solution is

$$\left[D_{1x} \quad D_{1y} \quad \theta_1 \quad D_{2x} \quad D_{2y}\right] = 10^{-3}\,[133.84 \quad -29.33 \quad 21.25 \quad 1.38 \quad -29.54]$$

The value of θ_2 is undefined since the hinge at joint (2) was attributed to both members, (2)-(1) and (2)-(3).

The member end actions are calculated as follows:

$$\underline{\mathbf{A}}'_{12} = \underline{\mathbf{S}}'^{o2}_{11}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{S}}'^{o}_{12}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'^{o}_{F12} = [-1.05 \quad -4.875 \quad -0.75]^T$$

$$\underline{\mathbf{A}}'_{21} = \underline{\mathbf{S}}'^{o1}_{22}\,\underline{\mathbf{R}}_{21}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{S}}'^{o}_{21}\,\underline{\mathbf{R}}_{12}\,\underline{\mathbf{D}}_1 + \underline{\mathbf{A}}'^{o}_{F21} = [-1.05 \quad 5.12 \quad 0]^T$$

$$\underline{\mathbf{A}}'_{23} = \underline{\mathbf{S}}'^{o3}_{22}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'^{o}_{F23} = [-5.25 \quad -1.056 \quad 0]^T$$

$$\underline{\mathbf{A}}'_{32} = \underline{\mathbf{S}}'^{o}_{32}\,\underline{\mathbf{R}}_{23}\,\underline{\mathbf{D}}_2 + \underline{\mathbf{A}}'^{o}_{F32} = [-5.25 \quad 18.944 \quad -71.48]^T$$

The bending moment and shear force diagrams can be plotted as shown in Figure 5.63.

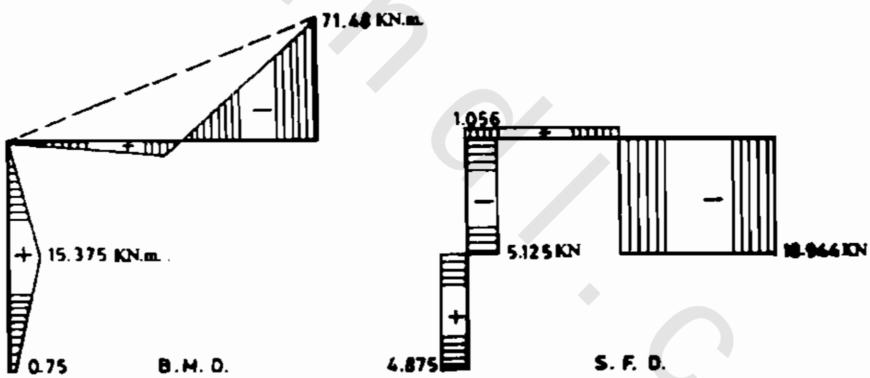


Figure 5.63

A solution considering the hinge at joint (2) belongs only to member (2)-(1) is given below:

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}^{o2}_{11} + \underline{\mathbf{S}}_{11s} = 10^3 \begin{bmatrix} 0.4527 & 0 & -2.5 \\ 0 & 5.036 & 0 \\ -2.5 & 0 & 15.05 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^{o1}_{22} + \underline{\mathbf{S}}^{3}_{22}$$

$$\underline{S}_{22}^{o1} = 10^3 \begin{bmatrix} 0.4167 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{S}'^{3}_{22} = 10^3 \begin{bmatrix} 3.75 & 0 & 0 \\ 0 & 0.703 & -2.815 \\ 0 & -0.8125 & 15 \end{bmatrix}$$

$$\underline{S}_{22}^{3} = \underline{R}_{23}^{T}\, \underline{S}'^{3}_{22}\, \underline{R}_{23} = 10^3 \begin{bmatrix} 3.75 & 0 & 0 \\ 0 & 0.703 & 2.815 \\ 0 & 2.8125 & 15 \end{bmatrix}$$

$$\underline{S}_{22} = \underline{S}_{22}^{o1} + \underline{S}_{22}^{3} = 10^3 \begin{bmatrix} 4.1667 & 0 & 0 \\ 0 & 5.703 & 2.815 \\ 0 & 2.8125 & 15 \end{bmatrix}$$

$$\underline{S}_{12} = \underline{R}_{12}^{T}\, \underline{S}'_{12}\, \underline{R}_{21} = 10^3 \begin{bmatrix} -0.4167 & 0 & 0 \\ 0 & -5 & 0 \\ 2.5 & 0 & 0 \end{bmatrix}$$

Figure 5.64

The fixed end actions in this case are obtained according to Figure 5.64 as follows:

$\underline{A}'^{o}_{F12} = [0 \quad -6.875 \quad 11.25]^T$

$\underline{A}'^{o}_{F21} = [0 \quad 3.125 \quad 0]^T$

$\underline{A}'_{F23} = [0 \quad -10 \quad 20]^T$

$\underline{A}'_{F32} = [0 \quad 10 \quad -20]^T$

The equivalent joint actions are, therefore, given by

$\underline{A}_1^c = -\underline{R}_{12}^T\, \underline{A}'^{o}_{F12} = [6.875 \quad 0 \quad -11.25]^T$

$$\underline{A}_2^c = -\underline{R}_{21}^T\, \underline{A}'^{o}_{F21} - \underline{R}_{23}^T\, \underline{A}'_{F23} = [3.125 \quad -10 \quad -20]^T$$

The free deformations are obtained from the solution of the final actions-deformations relation which is given by

$$\begin{bmatrix} 6.875 \\ 0 \\ -11.25 \\ 3.125 \\ -10 \\ -20 \end{bmatrix} = 10^3 \begin{bmatrix} 0.4527 & 0 & -2.5 & -0.4167 & 0 & 0 \\ 0 & 5.036 & 0 & 0 & -5 & 0 \\ -2.5 & 0 & 15.05 & 2.5 & 0 & 0 \\ -0.4167 & 0 & 2.5 & 4.1667 & 0 & 0 \\ 0 & -5 & 0 & 0 & 5.703 & 2.8125 \\ 0 & 0 & 0 & 0 & 2.8125 & 15 \end{bmatrix} \begin{bmatrix} D_{1x} \\ D_{1y} \\ \theta_1 \\ D_{2x} \\ D_{2y} \\ \theta_2 \end{bmatrix}$$

The solution is given by

$$\begin{bmatrix} D_{1x} & D_{1y} & \theta_1 & D_{2x} & D_{2y} & \theta_2 \end{bmatrix} = 10^{-3}\,[133.84 \quad -29.33 \quad 21.25 \quad 1.38 \quad -29.56 \quad 4.21]$$

It is obvious that this method has provided a value for θ_2 which represents a relative angle of rotation at joint (2).

The member end actions are obtained as follows:

$$\underline{A}'_{12} = \underline{S}'^{o2}_{11}\, \underline{R}_{12}\, \underline{D}_1 + \underline{S}'^{o}_{12}\, \underline{R}_{21}\, \underline{D}_2 + \underline{A}'^{o}_{F12} = [-1.05 \quad -4.875 \quad -0.75]^T$$

$$\underline{A}'_{21} = \underline{S}'^{o1}_{22}\, \underline{R}_{21}\, \underline{D}_2 + \underline{S}'^{o}_{21}\, \underline{R}_{12}\, \underline{D}_1 + \underline{A}'^{o}_{F21} = [-1.05 \quad 5.125 \quad 0]^T$$

$$\underline{A}'_{23} = \underline{S}^{3}_{22}\, \underline{R}_{23}\, \underline{D}_2 + \underline{A}'_{F23} = [-5.25 \quad -1.056 \quad 0]^T$$

$$\underline{A}'_{32} = \underline{S}'_{32}\, \underline{R}_{23}\, \underline{D}_2 + \underline{A}'_{F32} = [-5.25 \quad 18.944 \quad -71.48]^T$$

The shear force and bending moment diagrams are the same as obtained previously.

Example 5.17

Determine the shear force and bending moment diagrams for the frame shown in Figure 5.65 (EI = 10^5 kN.m^2, EA = 10^6 kN).

Solution

The degree of freedom for the frame is six. However, if the modified stiffness matrix for members with hinges are used, the number of the unknowns can be reduced to only four. The actions-deformations relationship which needs to be determined and considering the boundary condition $\underline{D}_3 = \underline{0}$ is

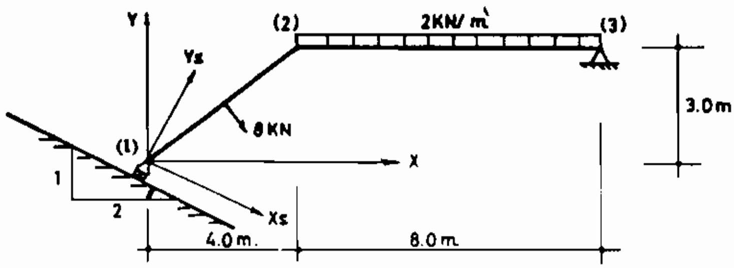


Figure 5.65

$$\begin{bmatrix} \underline{\mathbf{A}}_1 \\ \underline{\mathbf{A}}_2 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \end{bmatrix}$$

where the stiffness matrices are calculated as follows:

$$\underline{\mathbf{R}}_{12} = \begin{bmatrix} -0.8 & -0.6 & 0 \\ 0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{o2}_{11} = 10^4 \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0.24 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{11} = \underline{\mathbf{S}}^{o2}_{11} = \underline{\mathbf{R}}^T_{12}\, \underline{\mathbf{S}}'^{o2}_{11}\, \underline{\mathbf{R}}_{12} = 10^4 \begin{bmatrix} 12.8864 & 9.4848 & 0 \\ 9.4848 & 7.3536 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{12} = \underline{\mathbf{R}}^T_{12}\, \underline{\mathbf{S}}'^{o}_{12}\, \underline{\mathbf{R}}_{21} = 10^4 \begin{bmatrix} -12.8864 & -9.4848 & -0.72 \\ -9.4848 & -7.3536 & 0.96 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^{o1}_{22} + \underline{\mathbf{S}}^{o3}_{22}$$

$$\underline{\mathbf{S}}^{o1}_{22} = \underline{\mathbf{R}}^T_{21}\, \underline{\mathbf{S}}'^{o1}_{22}\, \underline{\mathbf{R}}_{21} \quad ; \quad \underline{\mathbf{S}}'^{o1}_{22} = 10^4 \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0.24 & -1.2 \\ 0 & -1.2 & 6 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{o1}_{22} = 10^4 \begin{bmatrix} 12.8864 & 9.4848 & -0.72 \\ 9.4848 & 7.3536 & -0.96 \\ 0.72 & -0.96 & 6 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{o3}_{22} = \underline{\mathbf{R}}^T_{23}\, \underline{\mathbf{S}}'^{o3}_{22}\, \underline{\mathbf{R}}_{23}$$

$$\underline{\mathbf{R}}_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{o3}_{22} = 10^4 \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 0.05859 & -0.46875 \\ 0 & -0.46875 & 3.75 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{o3}_{22} = 10^4 \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 0.05859 & 0.46875 \\ 0 & 0.46875 & 3.75 \end{bmatrix}$$

Therefore, $\underline{\mathbf{S}}_{22}$ is given by

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}^{o1}_{22} + \underline{\mathbf{S}}^{o3}_{22} = 10^4 \begin{bmatrix} 25.3864 & 9.4848 & 0.72 \\ 9.4848 & 7.41219 & -0.49125 \\ 0.72 & -0.49125 & 9.75 \end{bmatrix}$$

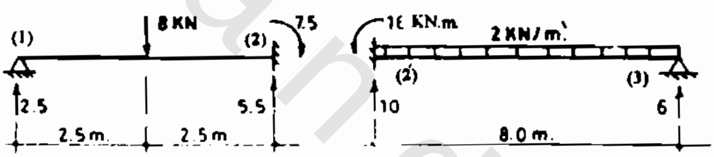


Figure 5.66

According to Figure 5.66, the fixed end actions are obtained as follows:

$$\underline{\mathbf{A}}'^{o}_{F12} = \begin{bmatrix} 0 \\ -2.5 \\ 0 \end{bmatrix} \quad , \quad \underline{\mathbf{A}}'^{o}_{F21} = \begin{bmatrix} 0 \\ 5.5 \\ -7.5 \end{bmatrix}$$

$$\underline{\mathbf{A}}'^{o}_{F23} = \begin{bmatrix} 0 \\ -10 \\ 16 \end{bmatrix} \quad ; \quad \underline{\mathbf{A}}'^{o}_{F32} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

The equivalent joint actions are obtained from

$$\underline{\mathbf{A}}^{e}_{1} = -\underline{\mathbf{R}}^{T}_{12}\, \underline{\mathbf{A}}'^{o}_{F12} = [1.5 \quad -2 \quad 0]^T$$

$$\underline{\mathbf{A}}^{e}_{2} = -\underline{\mathbf{R}}^{T}_{21}\, \underline{\mathbf{A}}'^{o}_{F21} - \underline{\mathbf{R}}^{T}_{23}\, \underline{\mathbf{A}}'^{o}_{F23} = [3.3 \quad -14.4 \quad -8.5]^T$$

$$\underline{A}_3^c = -\underline{R}_{32}^T \underline{A}'^o_{F32} = [0 \quad -6 \quad 0]^T$$

The actions-deformations relationship is, therefore,

$$\begin{bmatrix} 1.5+R_{1x} \\ -2+R_{1y} \\ 0 \\ 3.3 \\ 14.4 \\ -8.5 \end{bmatrix} = 10^4 \begin{bmatrix} 12.8864 & 9.4848 & 0 & -12.8864 & -9.4848 & -0.72 \\ 9.4848 & 7.3536 & 0 & -9.4848 & -7.3536 & 0.96 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -12.8864 & -9.4848 & 0 & 25.3864 & 9.4848 & 0.72 \\ -9.4848 & -7.3536 & 0 & 9.4848 & 7.41219 & -0.49125 \\ -0.72 & 0.96 & 0 & 0.72 & -0.49125 & 9.75 \end{bmatrix} \begin{bmatrix} D_{1x} \\ D_{1y} \\ \theta_1 \\ D_{2x} \\ D_{2y} \\ \theta_2 \end{bmatrix}$$

In order to replace $\underline{A}_1$ by $\underline{A}_{1s}$ and $\underline{D}_1$ by $\underline{D}_{1s}$, the following transformation matrix is used:

$$\underline{R}_{1s} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{A}_{1s} = \underline{R}_{1s}\,\underline{A}_1 = \begin{bmatrix} 2.236 & (R_{1ys} & -1.118) & 0 \end{bmatrix}^T$$

$$\underline{D}_{1s} = \underline{R}_{1s}\,\underline{D}_1 = \begin{bmatrix} D_{1xs} & 0 & \theta_1 \end{bmatrix}^T$$

The matrices $\underline{S}_{11}, \underline{S}_{12}, \underline{S}_{21}$ are transformed into the local coordinates of support (1) as follows:

$$\hat{\underline{S}}_{11} = \underline{R}_{1s}\,\underline{S}_{11}\,\underline{R}_{1s}^T$$

$$= 10^4 \begin{bmatrix} 4.19169 & 7.9035 & 0 \\ 7.9035 & 16.0469 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\underline{S}}_{12} = \underline{R}_{1s}\,\underline{S}_{12}$$

$$= 10^4 \begin{bmatrix} -7.2839 & -5.1946 & -1.0732 \\ -14.246 & -10.8186 & 0.5366 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\underline{S}}_{21} = \underline{S}_{21}\,\underline{R}_{1s}^T$$

The actions-deformations relationship becomes

$$\begin{bmatrix} 2.236 \\ (R_{1ys} - 1.118) \\ 0 \\ 3.3 \\ -14.4 \\ -8.5 \end{bmatrix} = 10^{4} \begin{bmatrix} 4.19169 & 7.9035 & 0 & -7.2839 & -5.1946 & -1.0732 \\ 7.9035 & 16.0469 & 0 & -14.246 & -10.8186 & 0.5366 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -7.2839 & -14.246 & 0 & 25.3864 & 9.4848 & 0.72 \\ -5.1946 & -10.8186 & 0 & 9.4848 & 7.41219 & -0.49125 \\ -1.0732 & 0.5366 & 0 & 0.72 & -0.49125 & 9.75 \end{bmatrix} \begin{bmatrix} D_{1xs} \\ 0 \\ \theta_1 \\ D_{2x} \\ D_{2y} \\ \theta_2 \end{bmatrix}$$

By deleting the second and third rows and columns, the solution is obtained as

$$\begin{bmatrix} D_{1xs} & D_{2x} & D_{2y} & \theta_2 \end{bmatrix} = 10^{-4} [-2566 \quad 0.8992 \quad -21.389 \quad -4.841]$$

The member end actions are obtained as follows:

$$\underline{A}'_{12} = \underline{S}'^{o2}_{11} \underline{R}_{12} \underline{D}_1 + \underline{S}'^{o}_{12} \underline{R}_{21} \underline{D}_2 + \underline{A}'^{o}_{F12}$$

$$= \underline{S}'^{o2}_{11} \underline{R}_{12} \underline{R}^{T}_{1s} \underline{D}_{1s} + \underline{S}'^{o}_{12} \underline{R}_{21} \underline{D}_2 + \underline{A}'^{o}_{F12}$$

$$= \begin{bmatrix} 229.5 \\ -5.508 \\ 0 \end{bmatrix} + \begin{bmatrix} -243.388 \\ 1.5713 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -12.888 \\ -6.436 \\ 0 \end{bmatrix} \begin{matrix} kN \\ kN \\ \end{matrix}$$

$$\underline{A}'_{21} = \underline{S}'^{o1}_{22} \underline{R}_{21} \underline{D}_2 + \underline{S}'^{o}_{21} \underline{R}_{12} \underline{D}_1 + \underline{A}'^{o}_{F21}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & .24 & -1.2 \\ 0 & -1.2 & 6 \end{bmatrix} \begin{bmatrix} -12.1194 \\ -17.6579 \\ -4.841 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0.24 & 0 \\ 0 & -1.2 & 0 \end{bmatrix} \begin{bmatrix} 11.475 \\ -22.95 \\ \theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5.5 \\ -7.5 \end{bmatrix} = \begin{bmatrix} -12.888 \\ 1.5633 \\ 12.184 \end{bmatrix} \begin{matrix} kN \\ kN \\ kN.m \end{matrix}$$

$$\underline{A}'_{32} = \underline{S}'^{o2}_{33} \underline{R}_{32} \underline{D}_3 + \underline{S}'^{o}_{32} \underline{R}_{23} \underline{D}_2 + \underline{A}'^{o}_{F32}$$

$$= \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 0.05859 & -0.46875 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8992 \\ 21.398 \\ -4.841 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -11.24 \\ 9.523 \\ 0 \end{bmatrix} \begin{matrix} kN \\ kN \\ kN.m. \end{matrix}$$

$$\underline{A}'_{23} = \underline{S}'^{o3}_{22} \underline{R}_{23} \underline{D}_2 + \underline{A}'^{o}_{F23}$$

$$= \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 0.05859 & -0.46875 \\ 0 & -0.46875 & 3.75 \end{bmatrix} \begin{bmatrix} -0.8992 \\ 21.398 \\ -4.841 \end{bmatrix} + \begin{bmatrix} 0 \\ -10 \\ 16 \end{bmatrix} = \begin{bmatrix} -11.24 \\ -6.477 \\ -12.184 \end{bmatrix} \begin{matrix} kN \\ kN \\ kN.m. \end{matrix}$$

The axial force, shear force and bending moment diagrams can be plotted as shown in Figure 5.67.

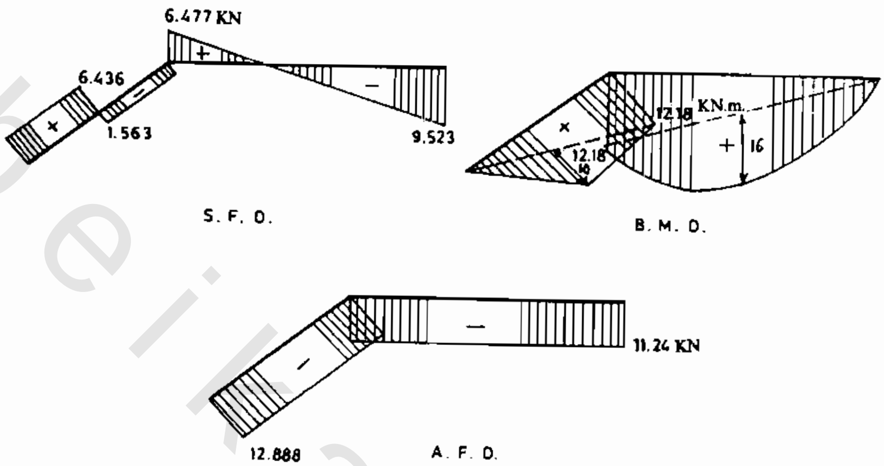


Figure 5.67

Example 5.18

Determine the shear force and bending moment diagrams for the beam shown in Figure 5.68 (EI = 10^5 kN.m^2, $\alpha = 10^{-5}$/°c, K = 10 kN/cm).

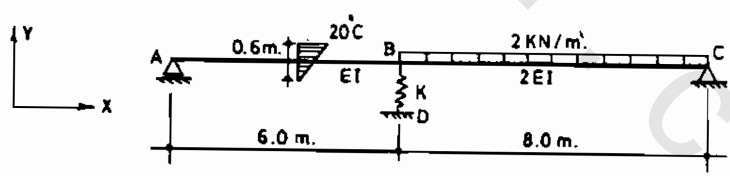


Figure 5.68

Solution

The degree of freedom is four, which represents $\theta_A, \theta_B, D_{By}$ and θ_C. However, if the modified stiffness matrices are used for member AB and BC because of the hinges at A and C, the unknowns are reduced to two, which are θ_B and D_{By}.

The actions-deformations relationship which needs to be determined is

$$\underline{\mathbf{A}}_B = \underline{\mathbf{S}}_{BB}\,\underline{\mathbf{D}}_B$$

where $\underline{\mathbf{D}}_B = \left[D_{By} \quad \theta_B\right]^T$

$$\underline{\mathbf{S}}_{BB} = \underline{\mathbf{S}}_{BB}^{oA} + \underline{\mathbf{S}}_{BB}^{oC} + \underline{\mathbf{S}}_{BB}^{D}$$

$$\underline{\mathbf{S}}_{BB}^{D} = 10^4 \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{BB}^{oA} = \underline{\mathbf{R}}_{BA}^{T}\, \underline{\mathbf{S}}_{BB}^{\prime oA}\, \underline{\mathbf{R}}_{BA} = \begin{bmatrix} \dfrac{3EI}{L^3} & \dfrac{-3EI}{L^2} \\ \dfrac{-3EI}{L^2} & \dfrac{3EI}{L} \end{bmatrix} = 10^4 \begin{bmatrix} 0.13889 & -0.8333 \\ -0.8333 & 5 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{BB}^{oc} = \underline{\mathbf{R}}_{BC}^{T}\, \underline{\mathbf{S}}_{BB}^{\prime oc}\, \underline{\mathbf{R}}_{BC} = \begin{bmatrix} \dfrac{6EI}{8^3} & \dfrac{6EI}{8^2} \\ \dfrac{6EI}{8^2} & \dfrac{6EI}{8} \end{bmatrix} = 10^4 \begin{bmatrix} 0.11718 & 0.9375 \\ 0.9375 & 7.5 \end{bmatrix}$$

Therefore, $\underline{\mathbf{S}}_{BB}$ is given by

$$\underline{\mathbf{S}}_{BB} = \underline{\mathbf{S}}_{BB}^{oA} + \underline{\mathbf{S}}_{BB}^{oc} = 10^4 \begin{bmatrix} 0.35607 & 0.1042 \\ 0.1042 & 12.5 \end{bmatrix}$$

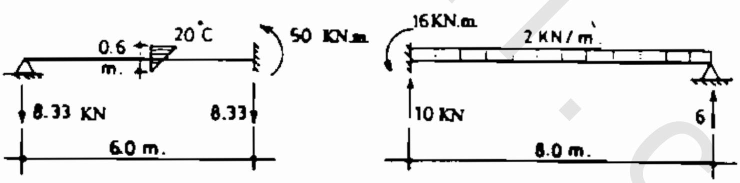


Figure 5.69

From Figures 5.69, the fixed end actions are obtained from

$$\underline{\mathbf{A}}_{FBC}^{\prime o} = [-10 \quad 6]^T \quad ; \quad \underline{\mathbf{A}}_{FBC}^{\prime o} = [6 \quad 0]^T$$

$$\underline{\mathbf{A}}_{FAB}^{\prime o} = [-8.333 \quad 0]^T \quad ; \quad \underline{\mathbf{A}}_{FBA}^{\prime o} = [-8.333 \quad 50]^T$$

The equivalent joint actions are obtained from

$$\underline{\mathbf{A}}_B^e = -R_{BC}^T A'^o_{FBC} - R_{BA}^T A'^o_{FBA}$$

$$= -\begin{bmatrix} 10 \\ 16 \end{bmatrix} - \begin{bmatrix} -8.333 \\ 50 \end{bmatrix} = \begin{bmatrix} -1.667 \\ -66 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

The actions-deformations relationship is, thus, given by

$$\begin{bmatrix} -1.667 \\ -66 \end{bmatrix} = 10^4 \begin{bmatrix} 0.35607 & 0.1042 \\ 0.1042 & 12.5 \end{bmatrix} \begin{bmatrix} D_{yB} \\ \theta_B \end{bmatrix}$$

The solution is $\left[D_{yB} \quad \theta_B\right] = 10^{-4}\,[-3.1442 \quad -5.25379]$

The members end actions are obtained as follows:

$$\underline{\mathbf{A}}'_{BA} = \underline{\mathbf{S}}'^{oA}_{BB}\, \underline{\mathbf{R}}_{BA}\, \underline{\mathbf{D}}_B + \underline{\mathbf{A}}'^o_{FBA}$$

$$= \begin{bmatrix} 0.13889 & -0.8333 \\ -0.8333 & 5 \end{bmatrix} \begin{bmatrix} -3.1442 \\ -5.25379 \end{bmatrix} + \begin{bmatrix} -8.333 \\ 50 \end{bmatrix} = \begin{bmatrix} -4.392 \\ 26.351 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{\mathbf{A}}'_{BC} = \underline{\mathbf{S}}'^{oC}_{BB}\, \underline{\mathbf{R}}_{BC}\, \underline{\mathbf{D}}_B + \underline{\mathbf{A}}'^o_{FBC}$$

$$= \begin{bmatrix} 0.11718 & -0.9375 \\ -0.9375 & 7.5 \end{bmatrix} \begin{bmatrix} 3.1442 \\ -5.25379 \end{bmatrix} + \begin{bmatrix} -10 \\ 16 \end{bmatrix} = \begin{bmatrix} -4.706 \\ -26.351 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

The shear force and bending moment diagrams are shown in Figure 5.70, which are the same results as obtained previously in Chapters 3 and 4.

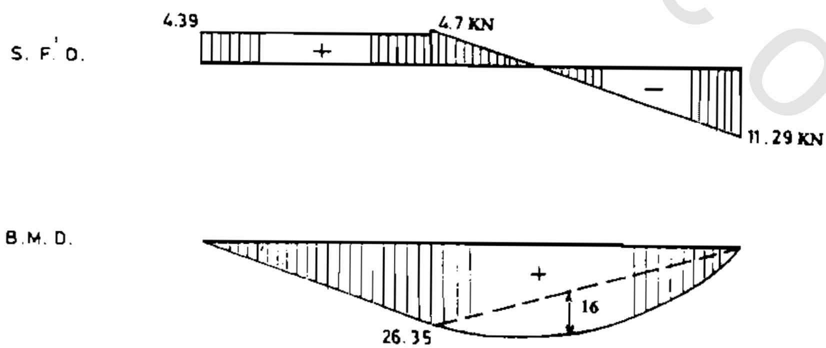


Figure 5.70

5.7.4 Nonprismatic Members

The stiffness coefficients for nonprismatic members can, in general, be determined using the integration procedure as explained in slope deflection equation method. However, for general nonprismatic members, the integration procedure is very lengthy and difficult. One way to simplify the analysis is using the column analogy method as explained in Chapter 3. The analysis becomes much easier if the nonprismatic member is considered to consist of several segments and each segment is prismatic which has a constant cross section area and constant moment of inertia. The nonprismatic member in this case is called a stepped member, as shown in Figure 4.71. One can determine the stiffness coefficients for a stepped member by either the slope deflection equation method, column analogy method, flexibility method, or stiffness matrix method.

In using the flexibility method, the flexibility matrix is determined at one end of the nonprismatic member due to unit actions applied at this end. The inverse of the flexibility matrix provides the direct stiffness matrix at the same end. The rest of the stiffness matrices are determined from the equilibrium equations.

In using the stiffness method, one may consider the joint at each step of the nonprismatic member as a free joint, and the solution is obtained as was previously given in Section 5.4. However, one shall have a structural stiffness matrix of large dimension if the nonprosmatic members consists of many segments. Another approach is to analyze each nonprismatic member as an independent structure by the stiffness matrix method in order to determine the stiffness matrices at the ends of the member.

Consider, for example, the nonprismatic member ij shown in Figure 5.71. The end joint j is kept fixed, and it is required to determine $\underline{S}_{ii}^{j}$. Treating the member as an independent structure the actions-deformations relationship in the global coordinates is given by

$$\begin{bmatrix} \underline{A}_i \\ \underline{A}_1 \\ \underline{A}_2 \\ \underline{A}_n \\ \underline{A}_j \end{bmatrix} = \begin{bmatrix} \underline{S}_{ii}^{1} & \underline{S}_{i1} & \underline{0} & \underline{0} & \underline{0} \\ \underline{S}_{1i} & \underline{S}_{11} & \underline{S}_{12} & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_{21} & \underline{S}_{22} & \underline{S}_{2n} & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_{n2} & \underline{S}_{nn} & \underline{S}_{nj} \\ \underline{0} & \underline{0} & \underline{0} & \underline{S}_{jn} & \underline{S}_{jj}^{n} \end{bmatrix} \begin{bmatrix} \underline{D}_i \\ \underline{D}_1 \\ \underline{D}_2 \\ \underline{D}_n \\ \underline{0} \end{bmatrix} \qquad (5.122)$$

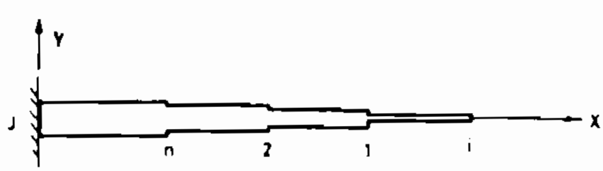


Figure 5.71

By applying the boundary conditions for $\underline{\mathbf{D}}_j$ and zero actions at joints 1, 2, and n, one has the following relationships:

$$\begin{bmatrix} \underline{\mathbf{A}}_i \\ \underline{\mathbf{A}}_1 \\ \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_n \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{A}}_i \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{ii}^1 & \underline{\mathbf{S}}_{i1} & \underline{0} & \underline{0} \\ \underline{\mathbf{S}}_{1i} & \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{0} \\ \underline{0} & \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{2n} \\ \underline{0} & \underline{0} & \underline{\mathbf{S}}_{n2} & \underline{\mathbf{S}}_{nn} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_i \\ \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_n \end{bmatrix}$$

$$\underline{\mathbf{A}}_i = \underline{\mathbf{S}}_{ii}^1 \, \underline{\mathbf{D}}_i + \left[\underline{\mathbf{S}}_{i1} \quad \underline{0} \quad \underline{0}\right] \left[\underline{\mathbf{D}}_1 \quad \underline{\mathbf{D}}_2 \quad \underline{\mathbf{D}}_n\right]^T \tag{5.123}$$

$$\begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{1i} \\ \underline{0} \\ \underline{0} \end{bmatrix} \underline{\mathbf{D}}_i + \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{0} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{2n} \\ \underline{0} & \underline{\mathbf{S}}_{n2} & \underline{\mathbf{S}}_{nn} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_n \end{bmatrix} \tag{5.124}$$

Therefore, one has

$$\begin{bmatrix} \underline{\mathbf{D}}_1 \\ \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_n \end{bmatrix} = - \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{0} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{2n} \\ \underline{0} & \underline{\mathbf{S}}_{n2} & \underline{\mathbf{S}}_{nn} \end{bmatrix}^{-1} \begin{bmatrix} \underline{\mathbf{S}}_{1i} \\ \underline{0} \\ \underline{0} \end{bmatrix} \underline{\mathbf{D}}_i \tag{5.125}$$

$$\underline{\mathbf{A}}_i = \left\{ \underline{\mathbf{S}}_{ii}^1 - \left[\underline{\mathbf{S}}_{i1} \quad \underline{0} \quad \underline{0}\right] \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{0} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{2n} \\ \underline{0} & \underline{\mathbf{S}}_{n2} & \underline{\mathbf{S}}_{nn} \end{bmatrix}^{-1} \begin{bmatrix} \underline{\mathbf{S}}_{1i} \\ \underline{0} \\ \underline{0} \end{bmatrix} \right\} \underline{\mathbf{D}}_i \tag{5.126}$$

It is obvious that the net matrix between brackets in equation 5.126 represents $\underline{\mathbf{S}}_{ii}^j$ expressed in the global coordinates, which is the same as $\underline{\mathbf{S}}_{ii}^{'j}$ in the local coordinates. In order to determine the rest of the stiffness matrices, the equilibrium equations are used.

Figure 5.72

From Figure 5.72, the relationship between $\underline{\mathbf{A}}'_{ij}$ and $\underline{\mathbf{A}}'_{ji}$ is determined as

$$\underline{\mathbf{A}}'_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L & -1 \end{bmatrix} \underline{\mathbf{A}}'_{ij} = \underline{\mathbf{E}}_{ji}\ \underline{\mathbf{A}}'_{ij} = \underline{\mathbf{E}}_{ji}\ \underline{\mathbf{S}}'^{j}_{ii}\ \underline{\mathbf{D}}'_{ij} \tag{5.127}$$

This indicates that the matrix $[\underline{\mathbf{E}}_{ji}\ \underline{\mathbf{S}}'^{j}_{ii}]$ is the same as the matrix $\underline{\mathbf{S}}'_{ji}$. In order to determine the matrices $\underline{\mathbf{S}}'^{i}_{jj}$ and $\underline{\mathbf{S}}'_{ij}$, one uses the following equalities:

$$\begin{aligned} \underline{\mathbf{A}}'_{ji} &= \underline{\mathbf{S}}'_{ji}\ \underline{\mathbf{D}}'_{ij} + \underline{\mathbf{S}}'^{i}_{jj}\ \underline{\mathbf{D}}'_{ji} \\ &= \underline{\mathbf{E}}_{ji}\ (\underline{\mathbf{S}}'^{j}_{ii}\ \underline{\mathbf{D}}'_{ij} + \underline{\mathbf{S}}'_{ij}\ \underline{\mathbf{D}}'_{ji}) \end{aligned} \tag{5.128}$$

Hence, one has

$$\underline{\mathbf{S}}'^{i}_{jj} = \underline{\mathbf{E}}_{ji}\ \underline{\mathbf{S}}'_{ij} \tag{5.129}$$

$$\underline{\mathbf{S}}'_{ij} = \underline{\mathbf{S}}'^{T}_{ji} \tag{5.130}$$

Therefore, one obtains the stiffness matrix for the nonprismatic member ij in the local coordinates. The stiffness analysis is then continued as usual.

If the nonprismatic member is subjected to direct loading, one has to determine the fixed end actions $\underline{\mathbf{A}}'_{Fij}$ and $\underline{\mathbf{A}}'_{Fji}$. They represent the reactions of member ij which is fixed at i and j. Therefore, in Equation 5.122, when $\underline{\mathbf{D}}_i$ and $\underline{\mathbf{D}}_j$ are zeros, and $\underline{\mathbf{A}}_1$, $\underline{\mathbf{A}}_2$ and $\underline{\mathbf{A}}_n$ represent the equivalent joint actions at joints (1), (2), and (n), determined for the prismatic segments (i)-(1), (1)-(2), (2)-(n), and (n)-(j) as in section 5.6, the fixed end actions in the global coordinates are obtained from:

$$\begin{bmatrix} \underline{\mathbf{A}}_i \\ \underline{\mathbf{A}}_j \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{A}}_{Fij} \\ \underline{\mathbf{A}}_{Fji} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{i1} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}_{jn} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{S}}_{11} & \underline{\mathbf{S}}_{12} & \underline{\mathbf{0}} \\ \underline{\mathbf{S}}_{21} & \underline{\mathbf{S}}_{22} & \underline{\mathbf{S}}_{2n} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{n2} & \underline{\mathbf{S}}_{nn} \end{bmatrix}^{-1} \begin{bmatrix} \underline{\mathbf{A}}^{e}_{1} \\ \underline{\mathbf{A}}^{e}_{2} \\ \underline{\mathbf{A}}^{e}_{n} \end{bmatrix} \tag{5.131}$$

The fixed end actions in the local coordinates, $\underline{\mathbf{A}}'_{Fij}$ and $\underline{\mathbf{A}}'_{Fji}$ are then determined using the transformation matrices.

It is obvious that if either of the joints (I) or (j) is a hinge, one may include the moment release in the stiffness matrix of the nonprismatic member by considering the release in the actions-deformations relationship of Equation 5.122. The same principle can be applied in determining the fixed end actions for a nonprismatic member with a moment release.

In the previous analysis, one may separate the axial force and axial displacement from the stiffness matrices. This will reduce the order of the stiffness matrices to dimension of (2 × 2) for plane frames, instead of (3 × 3). The stiffness coefficients corresponding to axial deformations are obtained from:

$$\underline{S}_{ii}^{j}(1,1) = \frac{1}{\sum_{k=1}^{n}\left(\frac{L}{EA}\right)_K} \qquad (5.132)$$

where k is the number of segments in the member.

Example 5.19

Determine the axial force, shear force, and bending moment diagrams for the frame shown in Figure 5.73. ($EA = 10^3$ Kip, $EI = 10^5$ K.ft^2).

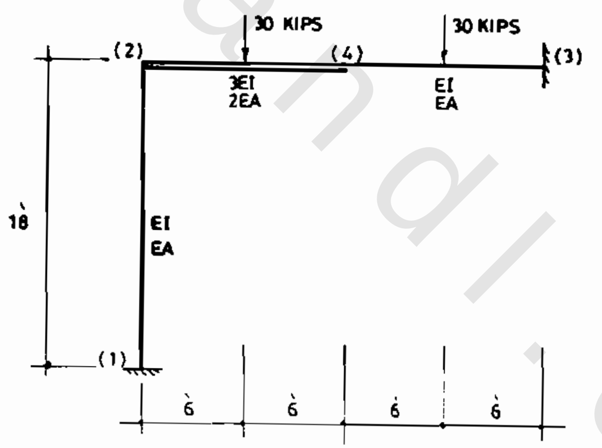


Figure 5.73

Solution

The boundary conditions are $\underline{D}_1 = \underline{D}_3 = \underline{0}$. The actions-deformations relationship which needs to be determined is

$$\underline{A}_2 = \underline{S}_{22}\,\underline{D}_2 \quad ; \quad \text{where,} \quad \underline{S}_{22} = \underline{S}_{22}^{1} + \underline{S}_{22}^{3}$$

Therefore, $\underline{S}_{22}$ is obtained as follows:

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \underline{\mathbf{S}}'^{1}_{22} = 10^2 \begin{bmatrix} 0.5555 & 0 & 0 \\ 0 & 2.0576 & -18.5185 \\ 0 & -18.5185 & 222.22 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{1}_{22} = \underline{\mathbf{R}}^{T}_{21} \underline{\mathbf{S}}'^{1}_{22} \, \underline{\mathbf{R}}_{21} = 10^2 \begin{bmatrix} 2.0576 & 0 & 18.5185 \\ 0 & 0.5555 & 0 \\ 18.5185 & 0 & 222.22 \end{bmatrix}$$

To determine $\underline{\mathbf{S}}^{3}_{22}$ for the nonprismatic member (2)-(3), first the element $S^{3}_{22}(1,1)$ is determined from

$$S^{3}_{22}(1,1) = \frac{1}{\left(\frac{12}{2} + \frac{12}{1}\right)10^{-3}} = \left(\frac{1}{18}\right)10^{3}$$

Since the matrix $\underline{\mathbf{S}}^{3}_{22}$ is required, member (2)-(3) is considered fixed at joint (3). The actions-deformations relationship for member (2)-(4) is

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_4 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}^{4}_{22} & \underline{\mathbf{S}}_{24} \\ \underline{\mathbf{S}}_{42} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_4 \end{bmatrix}$$

where, $\underline{\mathbf{S}}_{44} = \underline{\mathbf{S}}^{2}_{44} + \underline{\mathbf{S}}^{3}_{44}$

Therefore, one has

$$\underline{\mathbf{A}}_2 = \left[\underline{\mathbf{S}}^{4}_{22} - \underline{\mathbf{S}}_{24} \ \underline{\mathbf{S}}^{-1}_{44} \ \underline{\mathbf{S}}_{42}\right] \underline{\mathbf{D}}_2$$

which means that

$$\underline{\mathbf{S}}^{3}_{22} = \underline{\mathbf{S}}^{4}_{22} - \underline{\mathbf{S}}_{24} \ \underline{\mathbf{S}}^{-1}_{44} \ \underline{\mathbf{S}}_{42}$$

The calculations of these matrices follow

$$\underline{\mathbf{R}}_{24} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{S}}'^{4}_{22} = 10^3 \begin{bmatrix} 2.0833 & -12.5 \\ -12.5 & 100 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{4}_{22} = \underline{\mathbf{R}}^{T}_{24} \ \underline{\mathbf{S}}'^{4}_{22} \ \underline{\mathbf{R}}_{24} = 10^3 \begin{bmatrix} 2.0833 & 12.5 \\ 12.5 & 100 \end{bmatrix}$$

$$\underline{\mathbf{S}}^{T}_{42} = \underline{\mathbf{S}}_{24} = \underline{\mathbf{R}}^{T}_{24} \ \underline{\mathbf{S}}'_{24} \ \underline{\mathbf{R}}_{42} = 10^{-3} \begin{bmatrix} -2.0833 & 12.5 \\ -12.5 & 100 \end{bmatrix}$$

$$\underline{S}_{44}^{2} = 10^{3}\begin{bmatrix} 2.0833 & -12.5 \\ -12.5 & 100 \end{bmatrix} \quad ; \quad \underline{S}_{44}^{3} = 10^{3}\begin{bmatrix} 0.6944 & 4.1667 \\ 4.1667 & 33.3333 \end{bmatrix}$$

$$\underline{S}_{44} = \underline{S}_{44}^{2} + \underline{S}_{44}^{3} = 10^{3}\begin{bmatrix} 2.777 & -8.333 \\ -8.333 & 133.333 \end{bmatrix} \quad ; \quad \underline{S}_{44}^{-1} = 10^{-3}\begin{bmatrix} 0.443 & 0.0277 \\ 0.0277 & 0.00923 \end{bmatrix}$$

$$\underline{S}_{22}^{3} = \underline{S}_{44}^{4} - \underline{S}_{24}\ \underline{S}_{44}^{-1}\ \underline{S}_{42} = 10^{3}\begin{bmatrix} 0.16 & 2.4036 \\ 2.4036 & 42.3067 \end{bmatrix}$$

$$\underline{S}_{22} = \underline{S}_{22}^{1} + \underline{S}_{22}^{3} = 10^{3}\begin{bmatrix} 0.20576 & 0 & 1.85185 \\ 0 & 0.0555 & 0 \\ 1.85185 & 0 & 22.222 \end{bmatrix} + 10^{3}\begin{bmatrix} 0.0555 & 0 & 0 \\ 0 & 0.1602 & 2.4036 \\ 0 & 2.4036 & 42.306 \end{bmatrix}$$

$$= 10^{3}\begin{bmatrix} 0.26137 & 0 & 1.85185 \\ 0 & 0.2157 & 2.4036 \\ 1.85185 & 2.4036 & 64.5289 \end{bmatrix}$$

The fixed end actions for member (2)–(3) are determined according to Figure 5.74 as follows:

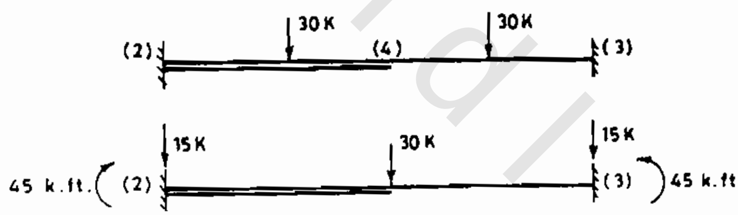


Figure 5.74

$$\begin{bmatrix} \underline{A}_{2F} \\ \underline{A}_{4}^{e} \\ \underline{A}_{3F} \end{bmatrix} = \begin{bmatrix} \underline{S}_{22}^{4} & \underline{S}_{24} & \underline{0} \\ \underline{S}_{42} & \underline{S}_{44} & \underline{S}_{43} \\ \underline{0} & \underline{S}_{34} & \underline{S}_{33}^{4} \end{bmatrix} \begin{bmatrix} \underline{0} \\ \underline{D}_{4} \\ \underline{0} \end{bmatrix}$$

where $\underline{A}_{4}^{e} = [-30 \quad 0]^{T}$

Therefore, $\underline{D}_{4}$ is obtained from

$$\underline{D}_{4} = \underline{S}_{44}^{-1}\ \underline{A}_{4}^{e} = 10^{-3}[-13.2923 \quad -0.8307]^{T}$$

The fixed end actions are

$$\underline{A}_{2F} = \underline{S}_{24}\ \underline{D}_4 = [17.3078 \quad 124.616]^T$$

$$\underline{A}_{3F} = \underline{S}_{34}\ \underline{D}_4 = [12.692 \quad -69.2296]^T$$

where $\underline{A}_{2F}$ and $\underline{A}_{3F}$ represent the fixed end actions for Figure 5.74 (b). The equivalent joint actions are obtained from

$$\underline{A}_2^c = -\underline{A}_{2F} - \underline{A}_{F24} = \begin{bmatrix} -17.3078 \\ -124.616 \end{bmatrix} - \begin{bmatrix} +15 \\ +45 \end{bmatrix} = \begin{bmatrix} -32.3078 \\ -169.6158 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

$$\underline{A}_3^c = -\underline{A}_{3F} - \underline{A}_{F34} = \begin{bmatrix} -12.692 \\ -69.2296 \end{bmatrix} - \begin{bmatrix} +15 \\ -45 \end{bmatrix} = \begin{bmatrix} -27.692 \\ 114.2296 \end{bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m.} \end{matrix}$$

where $\underline{A}_{F24}$ and $\underline{A}_{F34}$ are calculated from Figure 5.74 (a). Thus, the deformations at joint (2) is obtained from

$$\underline{A}_2^c = \underline{S}_{22}\ \underline{D}_2$$

The solution is

$$\underline{D}_2^T = 10^{-3}\,[-54.75 \quad -235.822 \quad 7.727]$$

For member (3)-(2), the local coordinates are the same as the global coordinates at joint (3). Thus, the member end actions are obtained from

$$\underline{A}_{32} = \underline{A}_{32} = \underline{S}_{33}^2\ \underline{D}_3 + \underline{S}_{32}\ \underline{D}_2 + \underline{A}_{F32}$$

where $\underline{A}_{F32}$ is the fixed end actions using Figure 5.74 (a). This means that $\underline{A}_{F32} = -\underline{A}_3^c$.

$$\underline{A}_{32}^T = [3.042 \quad 46.898 \quad -335.272]$$

Similarly, for member (1)-(2) one determines the end actions from

$$\underline{A}_{12} = \underline{S}_{11}^2\ \underline{D}_1 + \underline{S}_{12}\ \underline{D}_2 + \underline{A}_{F12}$$
$$= [-3.0422 \quad 13.101 \quad -15.548]^T$$

From the end actions at joints (1) and (3), the internal actions diagrams can be plotted as shown in Figure 5.75.

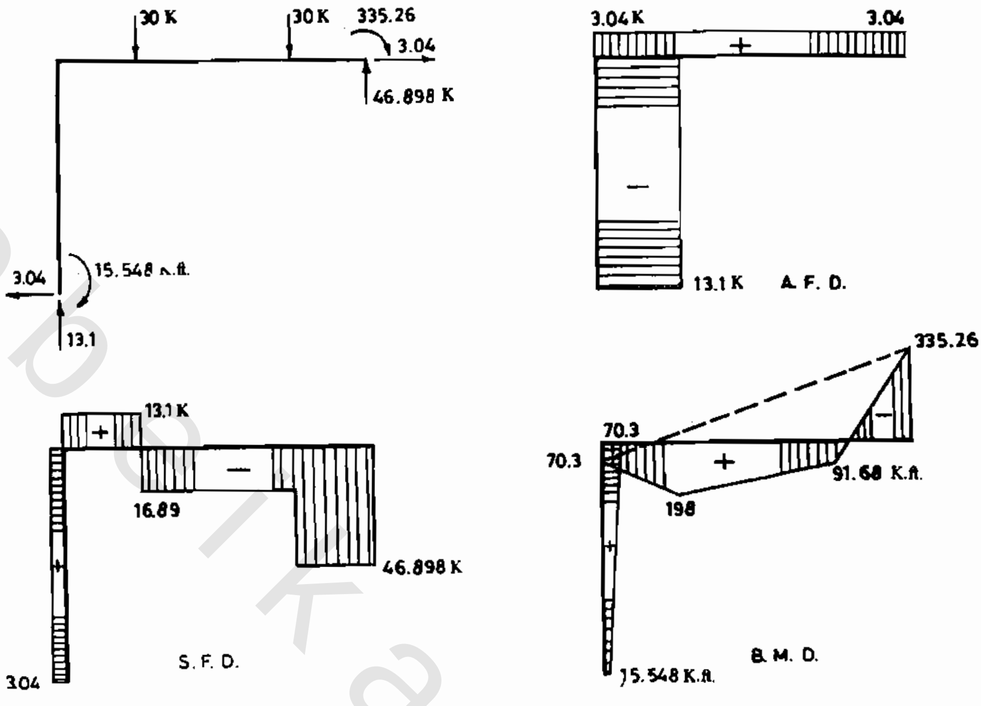


Figure 5.75

Example 5.20

Determine the bending moment diagram for the frame shown in Figure 5.76. (EA = 10^3 Kips, EI = 10^3 K.ft^2).

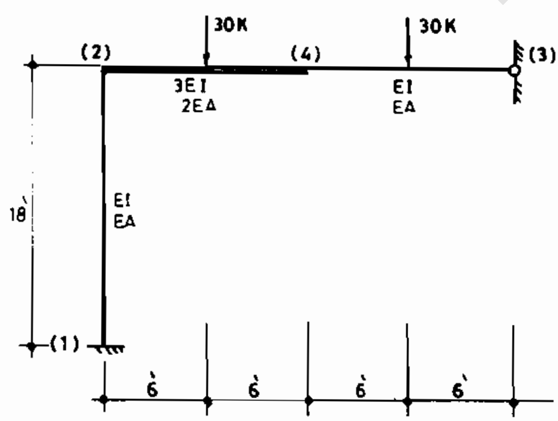


Figure 5.76

Solution

The degree of freedom is four which represents three deformations at joint (2) and one rotation at joint (3). However, one may disregard the rotation at joint (3) by modifying the stiffness matrix of member (2)–(3). In this case, the actions-deformations relationship is

$$\underline{\mathbf{A}}_2 = \underline{\mathbf{S}}_{22}\, \underline{\mathbf{D}}_2 \quad ; \qquad \text{where}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{o3}$$

$$\underline{\mathbf{S}}_{22}^{1} = 10^2 \begin{bmatrix} 2.0576 & 0 & 18.5185 \\ 0 & 0.5555 & 0 \\ 18.5185 & 0 & 222.22 \end{bmatrix}$$

$$\mathbf{S}_{22}^{o3}(1,1) = \frac{10^3}{\left(\dfrac{12}{2} + \dfrac{12}{1}\right)} = \left(\frac{1}{18}\right) 10^3$$

To determine the rest of the element of matrix $\underline{\mathbf{S}}_{22}^{o3}$, one has

$$\begin{bmatrix} \underline{\mathbf{A}}_2 \\ \underline{\mathbf{A}}_4 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22}^{4} & \underline{\mathbf{S}}_{24} \\ \underline{\mathbf{S}}_{42} & \underline{\mathbf{S}}_{44} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_2 \\ \underline{\mathbf{D}}_4 \end{bmatrix}$$

where $\underline{\mathbf{S}}_{44} = \underline{\mathbf{S}}_{44}^{2} + \underline{\mathbf{S}}_{44}^{o3}$

For $\underline{\mathbf{A}}_4 = \underline{\mathbf{0}}$ one has

$$\underline{\mathbf{A}}_2 = \left[\underline{\mathbf{S}}_{22}^{4} - \underline{\mathbf{S}}_{24}\, \underline{\mathbf{S}}_{44}^{-1}\, \underline{\mathbf{S}}_{42}\right] \underline{\mathbf{D}}_2$$

where, $\underline{\mathbf{S}}_{22}^{4} = 10^3 \begin{bmatrix} 2.0833 & 12.5 \\ 12.5 & 100 \end{bmatrix}$

$$\underline{\mathbf{S}}_{42}^{T} = \underline{\mathbf{S}}_{24} = \underline{\mathbf{R}}_{24}^{T}\, \underline{\mathbf{S}}_{24}'\, \underline{\mathbf{R}}_{42} = 10^3 \begin{bmatrix} -2.0833 & 12.5 \\ -12.5 & 50 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{44}^{2} = 10^3 \begin{bmatrix} 2.0833 & -12.5 \\ -12.5 & 100 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{44}^{o3} = \underline{\mathbf{R}}_{43}^{T}\,\underline{\mathbf{S}}_{44}^{\prime o3}\,\underline{\mathbf{R}}_{43}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times 10^3 \begin{bmatrix} -0.1736 & -2.083 \\ -2.083 & 25 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} 0.1736 & 2.083 \\ 2.083 & 25 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{44} = \underline{\mathbf{S}}_{44}^{2} + \underline{\mathbf{S}}_{44}^{o3} = 10^3 \begin{bmatrix} 2.2569 & -10.4167 \\ -10.4167 & 125 \end{bmatrix} \;;\; \underline{\mathbf{S}}_{44}^{-1} = 10^{-3} \begin{bmatrix} 0.72 & 0.06 \\ 0.06 & 0.013 \end{bmatrix}$$

Therefore,

$$\underline{\mathbf{S}}_{22}^{o3} = 10^3 \begin{bmatrix} 2.0833 & 12.5 \\ 12.5 & 100 \end{bmatrix} - 10^3 \begin{bmatrix} -2.0833 & 12.5 \\ -12.5 & 50 \end{bmatrix} \begin{bmatrix} 0.72 & 0.06 \\ 0.06 & 0.013 \end{bmatrix} \begin{bmatrix} -2.0833 & -12.5 \\ 12.5 & 50 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 0.05213 & 1.25 \\ 1.25 & 30 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{32}^{\prime o} = 10^3 \begin{bmatrix} 0.05213 & 1.25 \\ -2.5011 & -60 \end{bmatrix}$$

$$\underline{\mathbf{S}}_{22} = \underline{\mathbf{S}}_{22}^{1} + \underline{\mathbf{S}}_{22}^{o3} = 10^3 \begin{bmatrix} 0.26125 & 0 & 1.85185 \\ 0 & 0.10763 & 1.25 \\ 1.85185 & 1.25 & 52.222 \end{bmatrix}$$

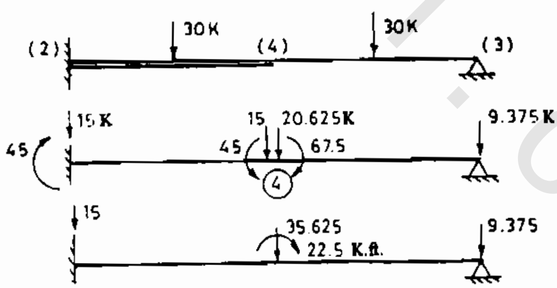


Figure 5.77

From Figure 5.77, the fixed end actions at joints 2 and 3 are determined from

$$\begin{bmatrix} \underline{\mathbf{A}}_{2F} \\ \underline{\mathbf{A}}_{4}^{e} \\ \underline{\mathbf{A}}_{3F}^{\prime} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}}_{22}^{4} & \underline{\mathbf{S}}_{24} & \underline{\mathbf{0}} \\ \underline{\mathbf{S}}_{42} & \underline{\mathbf{S}}_{44} & \underline{\mathbf{S}}_{43}^{o} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}}_{34}^{o} & \underline{\mathbf{S}}_{33}^{o4} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{D}}_{4} \\ \underline{\mathbf{0}} \end{bmatrix}$$

where $\underline{\mathbf{A}}_4^c = [-35.625 \quad -22.5]^T$, and $\underline{\mathbf{A}}_{2F}$ and $\underline{\mathbf{A}}_{3F}$ are the fixed end actions using Figure 5.77 c. They can be determined as follows:

$$\underline{\mathbf{D}}_4 = \underline{\mathbf{S}}_{44}^{-1} \; \underline{\mathbf{A}}_4^c = 10^{-3}[-27 \quad -2.43]^T$$

$$\underline{\mathbf{A}}_{2F} = \underline{\mathbf{S}}_{24} \; \underline{\mathbf{D}}_4 = [25.875 \quad 216]^T$$

$$\underline{\mathbf{A}}_{3F} = \underline{\mathbf{S}}_{34}^o \; \underline{\mathbf{D}}_4 = \begin{bmatrix} -0.1736 & -2.0833 \\ 0 & 0 \end{bmatrix} \underline{\mathbf{D}}_4 = \begin{bmatrix} 9.75 \\ 0 \end{bmatrix} \begin{matrix} \text{Kips} \\ \text{K.ft.} \end{matrix}$$

$$\underline{\mathbf{A}}_2^c = -\underline{\mathbf{A}}_{2F} - \underline{\mathbf{A}}_{F24} = \begin{bmatrix} -25.875 \\ -216 \end{bmatrix} - \begin{bmatrix} 15 \\ 45 \end{bmatrix} = \begin{bmatrix} -40.875 \\ -261 \end{bmatrix} \begin{matrix} \text{Kips} \\ \text{K.ft.} \end{matrix}$$

$$\underline{\mathbf{A}}_3^c = -\underline{\mathbf{A}}_{3F} - \underline{\mathbf{A}}_{F34}^o = \begin{bmatrix} -9.75 \\ 0 \end{bmatrix} - \begin{bmatrix} 9.375 \\ 0 \end{bmatrix} = \begin{bmatrix} -19.125 \\ 0 \end{bmatrix} \begin{matrix} \text{Kips} \\ \text{K.ft.} \end{matrix}$$

where $\underline{\mathbf{A}}_{F24}$ and $\underline{\mathbf{A}}_{F34}^o$ are the fixed end actions using Figure 5.77. Therefore, the final actions-deformations relationship is

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 0 \\ -40.875 \\ -261 \end{bmatrix} = \underline{\mathbf{S}}_{22} \; \underline{\mathbf{D}}_2$$

The solution for $\underline{\mathbf{D}}_2$ is

$$\underline{\mathbf{D}}_2^T = 10^{-3}[-61.636 \quad -480.7608 \quad 8.6954]$$

The member end actions are determined from:

$$\underline{\mathbf{A}}_{12} = \underline{\mathbf{S}}_{11}^2 \; \underline{\mathbf{D}}_1 + \underline{\mathbf{S}}_{12} \; \underline{\mathbf{D}}_2 + \underline{\mathbf{A}}_{F12}$$

$$= \begin{bmatrix} -0.2057 & 0 & -1.85185 \\ 0 & -0.0555 & 0 \\ 1.85185 & 0 & 11.111 \end{bmatrix} \begin{bmatrix} -61.636 \\ -480.7608 \\ 8.6954 \end{bmatrix} = \begin{bmatrix} -3.424 \\ 26.682 \\ -17.526 \end{bmatrix} \begin{matrix} \text{Kip} \\ \text{Kip} \\ \text{Kip.ft.} \end{matrix}$$

since $\underline{\mathbf{D}}_1 = \underline{\mathbf{0}}$, and $\underline{\mathbf{A}}_{F12} = \underline{\mathbf{0}}$

$$\underline{A}_{32} = \underline{S}^{o2}_{33}\ \underline{D}_3 + \underline{S}^{o}_{32}\ \underline{D}_2 + \underline{A}^{o}_{F32}$$

$$= \begin{bmatrix} -0.0555 & 0 & 0 \\ 0 & -0.05213 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -61.636 \\ -480.7608 \\ 8.6954 \end{bmatrix} + \begin{bmatrix} 0 \\ 19.125 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.424 \\ 33.318 \\ 0 \end{bmatrix} \begin{matrix} \text{Kip} \\ \text{Kip} \\ \text{Kip.ft.} \end{matrix}$$

where $\underline{A}^{o}_{F32} = -\underline{A}^{e}_{3}$.

The axial force, shear force, and bending moment diagrams are shown in Figure 5.78.

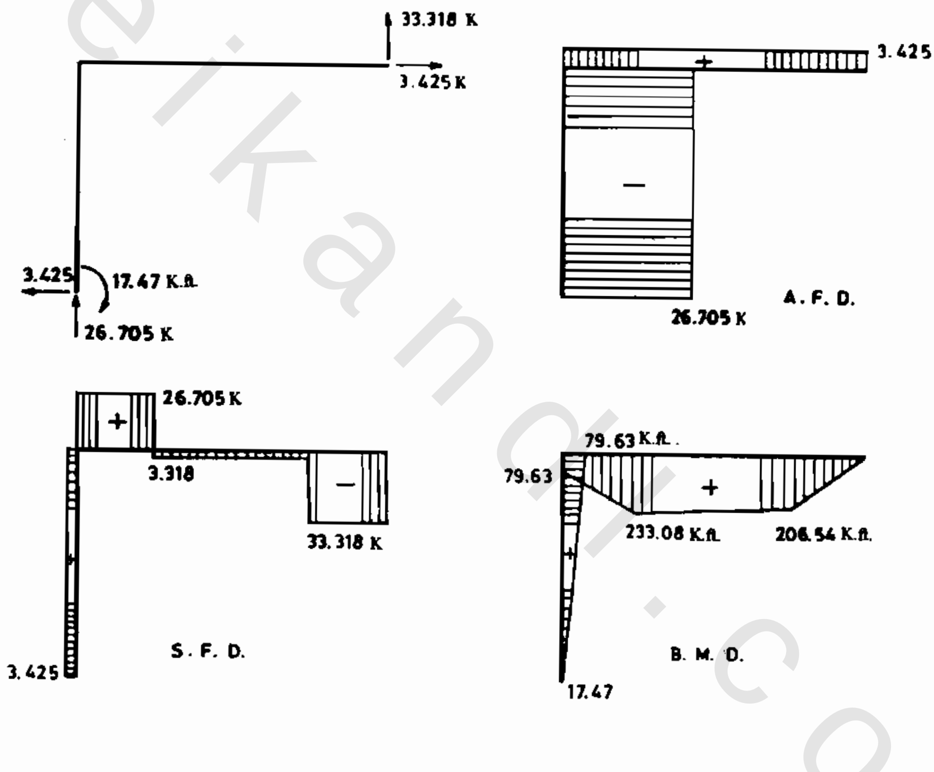


Figure 5.78

5.7.5 Series Connected Members

When a structure consists of a set of members connected in series such that each joint is connecting two members, the structure has a high degree of freedom. Examples of these kinds of structures are given in Figure 5.79.

The analysis of these structures by the stiffness method is time consuming due to the large dimension of the stiffness matrix. Since these structures, at most, have three degree of static indeterminacy, and because of the lengthy integration if the

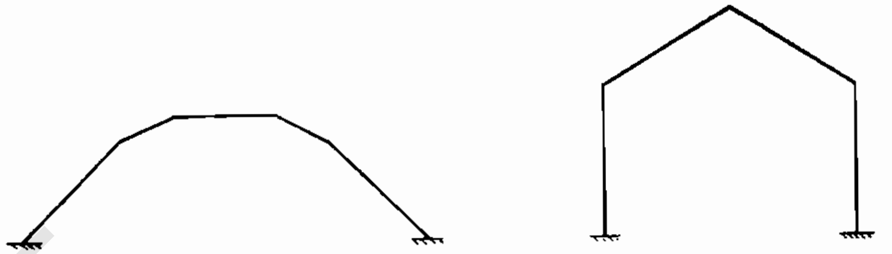


Figure 5.79

flexibility method is used, one may combine the systematic analysis of the stiffness method, and the small number of unknowns in the flexibility method. The resulting method is called the mixed method.

Consider members i-j and j-k in the structure shown in Figure 5.80. The actions-deformations relationships for member i-j are

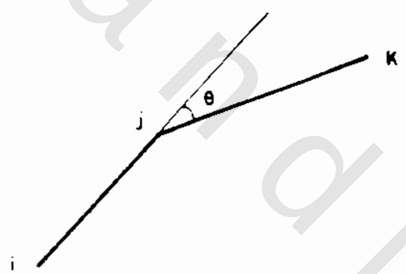


Figure 5.80

$$\underline{A}'_{ij} = \underline{S}'^{j}_{ii}\ \underline{D}'_{ij} + \underline{S}'_{ij}\ \underline{D}'_{ji} \tag{5.133}$$

$$\underline{A}'_{ji} = \underline{S}'_{ji}\ \underline{D}'_{ij} + \underline{S}'^{i}_{jj}\ \underline{D}'_{ji} \tag{5.134}$$

From Equation 5.134 one obtains

$$\begin{aligned}\underline{D}'_{ji} &= \underline{S}'^{i^{-1}}_{jj}\ \left[\underline{A}'_{ji} - \underline{S}'^{i}_{ji}\ \underline{D}'_{ij}\right] \\ &= \underline{F}'^{i}_{jj}\ \underline{A}'_{ji} - \underline{F}'^{i}_{jj}\ \underline{S}'_{ji}\ \underline{D}'_{ij}\end{aligned} \tag{5.135}$$

where $\underline{F}'^{i}_{jj}$ is the flexibility matrix of member i-j in the local coordinates.

The relation between $\underline{\mathbf{A}}'_{ji}$ and $\underline{\mathbf{A}}'_{ij}$ is obtained from the equilibrium principle, using Figure 5.81, as follows:

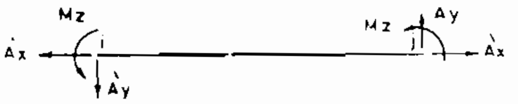


Figure 5.81

$$\underline{\mathbf{A}}'_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L & -1 \end{bmatrix} \underline{\mathbf{A}}'_{ij}$$

$$= \underline{\mathbf{E}}_{ji}\, \underline{\mathbf{A}}'_{ij} \qquad (5.136)$$

From Equations 5.135 and 5.136 one can obtain a relation between $(\underline{\mathbf{D}}'_{ji}, \underline{\mathbf{A}}'_{ji})$ and $(\underline{\mathbf{D}}'_{ij}, \underline{\mathbf{A}}'_{ij})$ as follows:

$$\begin{bmatrix} \underline{\mathbf{D}}'_{ji} \\ \hline \underline{\mathbf{A}}'_{ji} \end{bmatrix} = \left[\begin{array}{c|c} -\underline{\mathbf{F}}'^{i}_{ij}\, \underline{\mathbf{S}}'_{ji} & \underline{\mathbf{F}}'^{i}_{ij}\, \underline{\mathbf{E}}_{ji} \\ \hline \underline{\mathbf{0}} & \underline{\mathbf{E}}_{ji} \end{array}\right] \begin{bmatrix} \underline{\mathbf{D}}'_{ij} \\ \hline \underline{\mathbf{A}}'_{ij} \end{bmatrix} \qquad (5.137)$$

From the definitions of $\underline{\mathbf{F}}'^{i}_{ij}$ in Chapter 2 and $\underline{\mathbf{S}}'_{ji}$ in this chapter, one has

$$\begin{bmatrix} \underline{\mathbf{D}}'_{ji} \\ \hline \underline{\mathbf{A}}'_{ji} \end{bmatrix} = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & \frac{L}{EA} & 0 & 0 \\ 0 & -1 & L & 0 & \frac{-L^3}{6EI} & \frac{-L^2}{2EI} \\ 0 & 0 & 1 & 0 & \frac{-L^2}{2EI} & \frac{-L}{EI} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -L & -1 \end{array}\right] \begin{bmatrix} \underline{\mathbf{D}}'_{ij} \\ \hline \underline{\mathbf{A}}'_{ij} \end{bmatrix} \qquad (5.138)$$

The above relation is expressed as

$$\begin{bmatrix} \underline{\mathbf{D}}'_{ji} \\ \hline \underline{\mathbf{A}}'_{ji} \end{bmatrix} = \underline{\mathbf{G}}_{i,j} \begin{bmatrix} \underline{\mathbf{D}}'_{ij} \\ \hline \underline{\mathbf{A}}'_{ij} \end{bmatrix} \qquad (5.139)$$

in which $\underline{G}_{ij}$ is a matrix of dimension 6×6 as given in Equation 5.138, for member i-j. Similarly, for member j-k one has

$$\begin{bmatrix} \underline{D}'_{kj} \\ \hdashline \underline{A}'_{kj} \end{bmatrix} = \underline{G}_{j,k} \begin{bmatrix} \underline{D}'_{jk} \\ \hdashline \underline{A}'_{jk} \end{bmatrix} \tag{5.140}$$

In order to relate $\left[\underline{D}'^{T}_{jk} \;\vdots\; \underline{A}'^{T}_{jk}\right]$ with $\left[\underline{D}'^{T}_{ij} \;\vdots\; \underline{A}'^{T}_{ij}\right]$, one has to transform $\left[\underline{D}'^{T}_{ji} \;\vdots\; \underline{A}'^{T}_{ji}\right]$ into the coordinates of member jk at joint j. This can be done by applying the compatibility and equilibrium conditions at joint j in the global coordinates. The compatibility conditions at joint j in the global coordinates are

$$\underline{D}_{jk} = \underline{D}_{ji} \tag{5.141}$$

Thus, one has

$$\underline{D}'_{jk} = \underline{R}_{jk}\, \underline{R}^{T}_{ji}\, \underline{D}'_{ji} \tag{5.142}$$

The equilibrium conditions at joint j in the global coordinates give

$$\underline{A}_{j} = \underline{A}_{jk} + \underline{A}_{ji}$$

Therefore, $\underline{A}_{jk}$ and $\underline{A}'_{jk}$ are obtained from

$$\underline{A}_{jk} = \underline{A}_{j} - \underline{A}_{ji} \tag{5.143}$$

Substituting from local coordinates equations, one obtains

$$\underline{A}'_{jk} = \underline{R}_{jk}\, \underline{A}_{j} - \underline{R}_{jk}\, \underline{R}^{T}_{ji}\, \underline{A}'_{ji} \tag{5.144}$$

From Equations 5.142 and 5.144, one obtains

$$\begin{bmatrix} \underline{D}'_{jk} \\ \hdashline \underline{A}'_{jk} \end{bmatrix} = \begin{bmatrix} \underline{R}_{jk}\ \ \underline{A}^{T}_{ji} & \underline{0} \\ \hdashline \underline{0} & -\underline{R}_{jk}\ \ \underline{R}^{T}_{ji} \end{bmatrix} \begin{bmatrix} \underline{D}'_{ji} \\ \hdashline \underline{A}'_{ji} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \hdashline \underline{R}_{jk}\ \ \underline{A}_{j} \end{bmatrix} \tag{5.145}$$

Substituting Equation 5.139 into Equation 5.145 one has

$$\begin{bmatrix} \underline{D}'_{jk} \\ \hdashline \underline{A}'_{jk} \end{bmatrix} = \underline{U}_{j}\ \ \underline{G}_{i,j} \begin{bmatrix} \underline{D}'_{ij} \\ \hdashline \underline{A}'_{ij} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \hdashline \underline{R}_{jk}\ \ \underline{A}_{j} \end{bmatrix} \tag{5.146}$$

where $\underline{U}_j$ is given by

$$\underline{U}_j = \left[\begin{array}{cc|cc} \underline{R}_{jk} & \underline{R}_{ji}^T & & \underline{0} \\ \hline & \underline{0} & -\underline{R}_{jk} & \underline{R}_{ji}^T \end{array}\right] \tag{5.147}$$

By repeating Equations 5.146 for all members connecting joints 1, 2, 3, ..i, j, j, ..n, one has at the last joint n the following relationship:

$$\begin{aligned}
\left[\frac{\underline{D}'_{n,n-1}}{\underline{A}'_{n,n-1}}\right] &= \underline{G}_{n,(n-1)} \left(\underline{U}_{n-1}\right) \underline{G}_{(n-1),(n-2)} \left(\underline{U}_{n-2}\right) \underline{G}_{(n-2),(n-3)} \cdots \underline{G}_{k,j} \left(\underline{U}_j\right) \underline{G}_{j,i} \cdots \\
&\quad \underline{G}_{3,2} \left(\underline{U}_2\right) \underline{G}_{2,1} \left[\frac{\underline{D}'_{12}}{\underline{A}'_{12}}\right] \\
&+ \underline{G}_{n,(n-1)} \left(\underline{U}_{n-1}\right) \underline{G}_{(n-1),(n-2)} \left(\underline{U}_{n-2}\right) \underline{G}_{(n-2),(n-3)} \cdots \underline{G}_{4,3} \left(\underline{U}_3\right) \underline{G}_{3,2} \left[\frac{\underline{0}}{\underline{R}_{23}\,\underline{A}_2}\right] \\
&+ \underline{G}_{(n),(n-1)} \left(\underline{U}_{n-1}\right) \underline{G}_{(n-1),(n-2)} \left(\underline{U}_{n-2}\right) \cdots \underline{G}_{5,4} \left(\underline{U}_4\right) \underline{G}_{4,3} \left[\frac{\underline{0}}{\underline{R}_{34}\,\underline{A}_3}\right] \\
&+ \cdots\cdots + \underline{G}_{(n),(n-1)} \left(\underline{U}_{n-1}\right) \underline{G}_{(n-1),(n-2)} \left[\frac{\underline{0}}{\underline{R}_{(n-2),\,(n-1)}\,\underline{A}_{(n-2)}}\right] \\
&+ \underline{G}_{(n),(n-1)} \left[\frac{\underline{0}}{\underline{R}_{(n-1),n}\,\underline{A}_{(n-1)}}\right]
\end{aligned} \tag{5.148}$$

It is obvious the simplicity of programming Equation 5.148. Since both $\underline{U}$ and $\underline{G}$ matrices are of dimension 6×6, the result of the multiplications is of dimension 6×6. The final results of Equation 5.148 can be expressed as

$$\left[\frac{\underline{D}'_{n,(n-1)}}{\underline{A}'_{n,(n-1)}}\right] = \left[\begin{array}{c|c} \underline{G}_{I,I} & \underline{G}_{I,II} \\ \hline \underline{G}_{II,I} & \underline{G}_{II,II} \end{array}\right] \left[\frac{\underline{D}'_{12}}{\underline{A}'_{12}}\right] + \left[\frac{\underline{C}_1}{\underline{C}_2}\right] \tag{5.149}$$

The solution of Equation 5.149 is obtained by specifying the boundary conditions. For all types of boundary conditions one always has six linear simultaneous equations in six unknowns, which can easily be solved. For example, if the structure is fixed at both ends one has $\underline{D}'_{n,(n-1)} = \underline{D}'_{1,2} = \underline{0}$. Thus, Equation 5.149 can be written as

$$\underline{0} = \underline{G}_{I,II}\,\underline{A}'_{12} + \underline{C}_1 \tag{5.150}$$

$$\underline{A}'_{n,(n-1)} = \underline{G}_{II,II}\,\underline{A}'_{12} + \underline{C}_2 \tag{5.151}$$

Equation 5.150 is solved for $A'_{1,2}$ and $A'_{n,(n-1)}$ is obtained from Equation 5.151. By the determination of the reactions, one can determine the end actions and free deformations at every joint by using Equation 5.146. The obtained results are in the members' local coordinates, and they can easily be transformed into the global coordinates.

The above approach is called the transfer matrix approach and can easily be used to solve polygonal frames, arches, or circular structures. Whatever the number of members in the series, the final equation 5.149 always provides six linear simultaneous equations in six unknowns. In fact, using this approach to solve arches or circular structures is the first application to the popular finite element method, and can be realised from Figure 5.82.

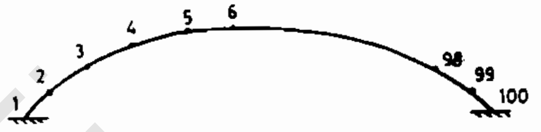


Figure 5.82

Example 5.21

Determine the bending moment diagram for the plane frame shown in Figure 5.83 (EI = 10^5 kN.m^2, EA = 5 × 10^5 kN).

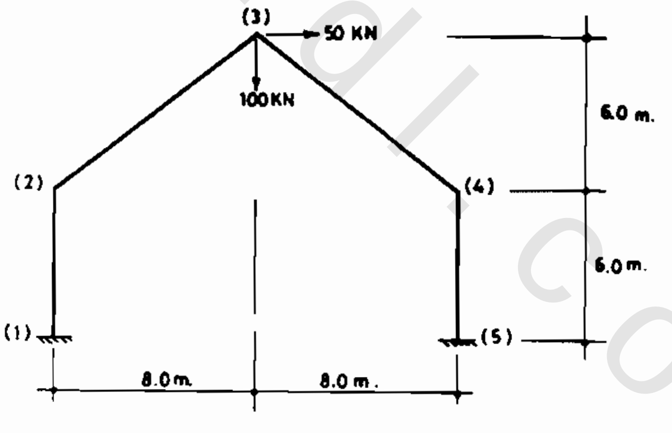


Figure 5.83

Solution

If the stiffness method is used to solve this frame one has 9 degrees of freedom, while the mixed method shall provide six equations in six unknowns. By the mixed method one has

$$\begin{bmatrix} \underline{\mathbf{D}}'_{23} \\ \underline{\mathbf{A}}'_{23} \end{bmatrix} = \underline{\mathbf{U}}_2 \; \underline{\mathbf{G}}_{2,1} \begin{bmatrix} \underline{\mathbf{D}}'_{12} \\ \underline{\mathbf{A}}'_{12} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{R}}_{23} \; \underline{\mathbf{A}}_2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{\mathbf{D}}'_{34} \\ \underline{\mathbf{A}}'_{34} \end{bmatrix} = \underline{\mathbf{U}}_3 \; \underline{\mathbf{G}}_{3,2} \begin{bmatrix} \underline{\mathbf{D}}'_{23} \\ \underline{\mathbf{A}}'_{23} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{R}}_{34} \; \underline{\mathbf{A}}_3 \end{bmatrix}$$

$$\begin{bmatrix} \underline{\mathbf{D}}'_{45} \\ \underline{\mathbf{A}}'_{45} \end{bmatrix} = \underline{\mathbf{U}}_4 \; \underline{\mathbf{G}}_{4,3} \begin{bmatrix} \underline{\mathbf{D}}'_{34} \\ \underline{\mathbf{A}}'_{12} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{R}}_{45} \; \underline{\mathbf{A}}_4 \end{bmatrix}$$

$$\begin{bmatrix} \underline{\mathbf{D}}'_{54} \\ \underline{\mathbf{A}}'_{54} \end{bmatrix} = \underline{\mathbf{G}}_{5,4} \begin{bmatrix} \underline{\mathbf{D}}'_{45} \\ \underline{\mathbf{A}}'_{45} \end{bmatrix}$$

The successive substitution of the above equations, gives

$$\begin{bmatrix} \underline{\mathbf{D}}'_{54} \\ \underline{\mathbf{A}}'_{54} \end{bmatrix} = \underline{\mathbf{G}}_{5,4} \, \underline{\mathbf{U}}_4 \, \underline{\mathbf{G}}_{4,3} \, \underline{\mathbf{U}}_3 \, \underline{\mathbf{G}}_{3,2} \, \underline{\mathbf{U}}_2 \, \underline{\mathbf{G}}_{2,1} \begin{bmatrix} \underline{\mathbf{D}}'_{12} \\ \underline{\mathbf{A}}'_{12} \end{bmatrix} + \underline{\mathbf{G}}_{5,4} \, \underline{\mathbf{U}}_4 \, \underline{\mathbf{G}}_{4,3} \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{R}}_{34} \, \underline{\mathbf{A}}_3 \end{bmatrix}$$

which can be evaluated as follows:

$$\underline{\mathbf{R}}_{21} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{R}}_{23} = \begin{bmatrix} -0.8 & -0.6 & 0 \\ 0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{U}}_2 = \left[\begin{array}{ccc|ccc} -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.6 & -0.8 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\underline{\mathbf{G}}_{2,1} = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1.2\times10^{-5} & 0 & 0 \\ 0 & -1 & 6 & 0 & -3.6\times10^{-4} & -18\times10^{-5} \\ 0 & 0 & 1 & 0 & -18\times10^{-5} & -6\times10^{-5} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -6 & -1 \end{array}\right]$$

$$\underline{\mathbf{R}}_{32} = \begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{R}}_{34} = \begin{bmatrix} -0.8 & 0.6 & 0 \\ -0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{U}}_3 = \left[\begin{array}{ccc|ccc} -0.28 & 0.96 & 0 & 0 & 0 & 0 \\ -0.96 & -0.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.28 & -0.96 & 0 \\ 0 & 0 & 0 & 0.96 & 0.28 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\underline{\mathbf{G}}_{3,2} = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 2\times10^{-5} & 0 & 0 \\ 0 & -1 & 10 & 0 & -16.67\times10^{-4} & -5\times10^{-4} \\ 0 & 0 & 1 & 0 & -5\times10^{-4} & -10\times10^{-4} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -10 & -1 \end{array}\right]$$

$$\underline{\mathbf{R}}_{43} = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\mathbf{R}}_{45} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} s$$

$$\underline{\mathbf{U}}_4 = \left[\begin{array}{ccc|ccc} -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.6 & -0.8 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\underline{\mathbf{G}}_{4,3} = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 2\times10^{-5} & 0 & 0 \\ 0 & -1 & 10 & 0 & -1.67\times10^{-4} & -5\times10^{-4} \\ 0 & 0 & 1 & 0 & -5\times10^{-4} & -10^{-4} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -10 & -1 \end{array}\right]$$

$$\underline{\mathbf{G}}_{5,4} = \underline{\mathbf{G}}_{2,1}$$

Substituting, one obtains

$$\underline{\mathbf{G}}_{5,4}\,\underline{\mathbf{U}}_4\;\underline{\mathbf{G}}_{4,3}\,\underline{\mathbf{U}}_3\;\underline{\mathbf{G}}_{3,2}\,\underline{\mathbf{U}}_2\,\underline{\mathbf{G}}_{2,1} = \left[\begin{array}{ccc|ccc} 1 & 0 & -16 & 0.0085 & 0.0173 & 0.00256 \\ 0 & 1 & 0 & -0.0173 & -0.0182 & -0.00216 \\ 0 & 0 & 1 & -0.00256 & -0.00216 & -3.2\times10^{-4} \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -16 & 0 & -1 \end{array}\right]$$

$$\underline{\mathbf{R}}_{34}\,\underline{\mathbf{A}}_3 = \begin{bmatrix} -0.8 & 0.6 & 0 \\ -0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 50 \\ -100 \\ 0 \end{bmatrix} = \begin{bmatrix} -100 \\ 50 \\ 0 \end{bmatrix}$$

$$\underline{\mathbf{G}}_{5,4}\,\underline{\mathbf{U}}_4\,\underline{\mathbf{G}}_{4,3} \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{R}}_{34}\,\underline{\mathbf{A}}_3 \end{bmatrix} = \begin{bmatrix} 0.06458 & -0.2728 & -0.0456 & -100 & -5 & -200 \end{bmatrix}^T$$

The final substitution provide the following matrix equation:

$$\begin{bmatrix} \underline{\mathbf{D}}'_{54} \\ \hline \underline{\mathbf{A}}'_{54} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & -16 & 0.0085 & 0.0173 & 0.00256 \\ 0 & 1 & 0 & -0.0173 & -0.0182 & -0.00216 \\ 0 & 0 & 1 & -0.00256 & -0.00216 & -3.2\times10^{-4} \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -16 & 0 & -1 \end{array}\right] \begin{bmatrix} \underline{\mathbf{D}}'_{12} \\ \hline \underline{\mathbf{A}}'_{12} \end{bmatrix} + \begin{bmatrix} 0.06458 \\ -0.2728 \\ -0.0456 \\ -100 \\ -50 \\ -200 \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} \underline{\mathbf{D}}_{54} \\ \underline{\mathbf{A}}_{54} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{G}}_{I,I} & \underline{\mathbf{G}}_{I,II} \\ \underline{\mathbf{G}}_{II,I} & \underline{\mathbf{G}}_{II,II} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_{12} \\ \underline{\mathbf{A}}_{12} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{C}}_1 \\ \underline{\mathbf{C}}_2 \end{bmatrix}$$

The boundary conditions are $\underline{\mathbf{D}}'_{54} = \underline{\mathbf{D}}'_{12} = \underline{\mathbf{0}}$. Thus, one has

$$\underline{\mathbf{0}} = \underline{\mathbf{G}}_{I,II}\,\underline{\mathbf{A}}'_{12} + \underline{\mathbf{C}}_1 \quad ; \quad \text{which gives}$$

$$\underline{\mathbf{A}}'_{12} = -\underline{\mathbf{G}}_{I,II}^{-1}\,\underline{\mathbf{C}}_1 = [-25.21 \;\; 10.28 \;\; -10.97]^T$$

$$\underline{\mathbf{A}}'_{54} = -\underline{\mathbf{G}}_{II,II}\,\underline{\mathbf{A}}'_{1,2} + \underline{\mathbf{C}}_2 = [-74.79 \;\; -60.28 \;\; 214.33]^T$$

The reactions at joint 1 and 5 are shown in Figure 5.84.

The end actions at joints 2, 3, and 4 can be obtained by substituting into the previous relations. The deformations and end actions at joint 2 in the local coordinates of member 2-3 are given by

$$\begin{bmatrix} \underline{\mathbf{D}}'_{23} \\ \hline \underline{\mathbf{A}}'_{23} \end{bmatrix} = \underline{\mathbf{U}}_2\,\underline{\mathbf{G}}_{2,1} \begin{bmatrix} \underline{\mathbf{0}} \\ \hline \underline{\mathbf{A}}'_{12} \end{bmatrix}$$

$$\underline{\mathbf{D}}'^{T}_{23} = [-0.0012 \;\; 0.00128 \;\; -0.0012]$$

$$\underline{\mathbf{A}}'^{T}_{23} = [-23.35 \;\; -14 \;\; 5.71]$$

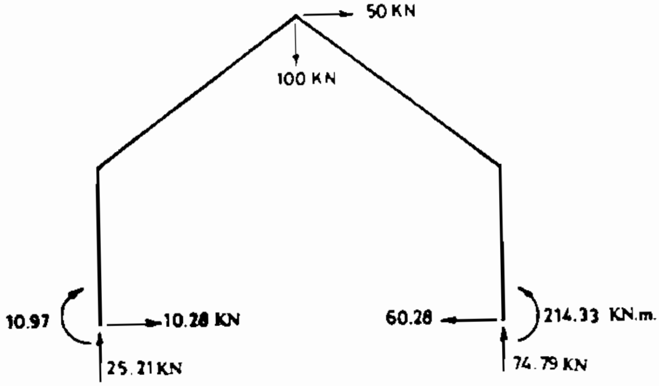


Figure 5.84

The deformations and end actions at joint 3 in the local coordinates of member 3-4 are

$$\begin{bmatrix} \underline{D}'_{34} \\ \underline{A}'_{34} \end{bmatrix} = \underline{U}_3 \; \underline{G}_{32} = \begin{bmatrix} \underline{D}'_{23} \\ \underline{A}'_{23} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{R}_{34} \; \underline{A}_3 \end{bmatrix}$$

$$\underline{D}'^{T}_{34} = [-0.0149 \quad 0.0036 \quad 7.29 \times 10^{-4}]$$

$$\underline{A}'^{T}_{34} = [-93.1 \quad 23.66 \quad -89.29]$$

The deformations and end actions at joint 4 in the local coordinates of member 4-5 are

$$\begin{bmatrix} \underline{D}'_{45} \\ \underline{A}'_{45} \end{bmatrix} = \underline{U}_4 \; \underline{G}_{4,3} = \begin{bmatrix} \underline{D}'_{34} \\ \underline{A}'_{34} \end{bmatrix}$$

$$\underline{D}'^{T}_{45} = [-0.00086 \quad -0.00158 \quad -0.0022]$$

$$\underline{A}'^{T}_{45} = [-74.79 \quad -60.28 \quad 147.31]$$

The axial force, shear force, and bending moment diagrams can be drawn as shown in Figure 5.85.

Example 5.22

Determine the reactions of the plane frame of Example 5.21 if the supports at joints 1 and 5 are hinged as shown in Figure 5.86.

Solution

The final equation from the previous example is

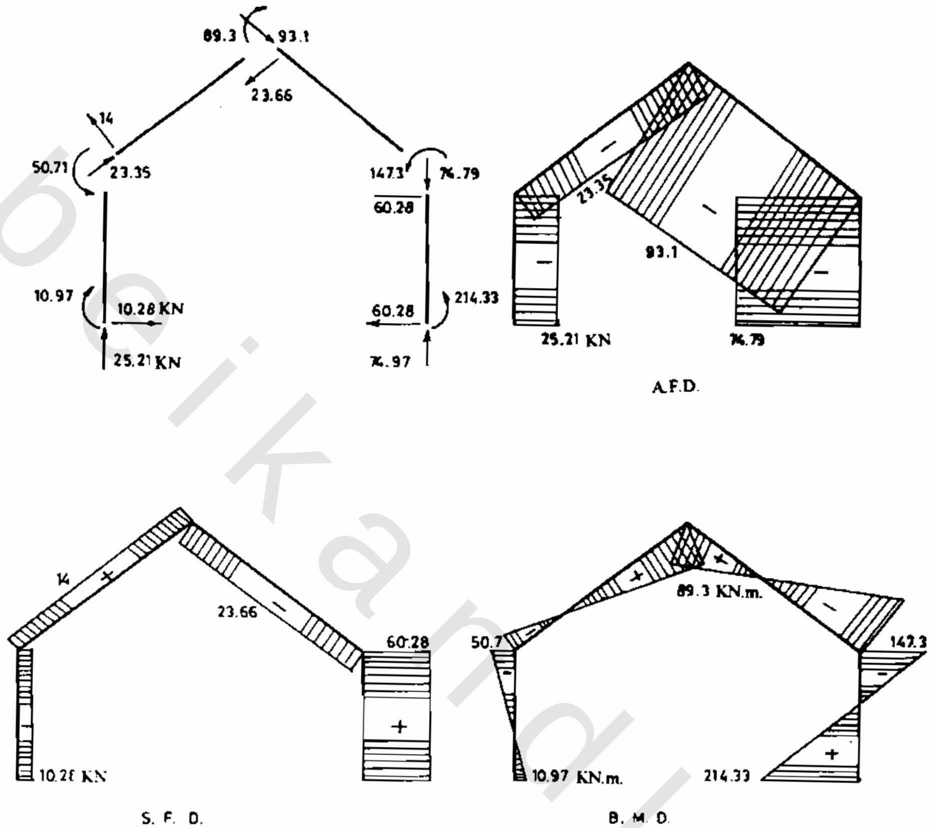


Figure 5.85

$$\begin{bmatrix} \mathbf{D'}_{54} \\ \hline \mathbf{A'}_{54} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & -16 & 0.0085 & 0.0173 & 0.00256 \\ 0 & 1 & 0 & -0.0173 & -0.0182 & -0.00216 \\ 0 & 0 & 1 & -0.00256 & -0.00216 & -3.2\times10^{-4} \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -16 & 0 & -1 \end{array}\right] \begin{bmatrix} \mathbf{D'}_{12} \\ \hline \mathbf{A'}_{12} \end{bmatrix} + \begin{bmatrix} 0.06458 \\ -0.2728 \\ -0.0456 \\ -100 \\ -50 \\ -200 \end{bmatrix}$$

The boundary conditions are $D'_{1x} = D'_{1y} = M_{1z} = 0$ and $D'_{5x} = D'_{5y} = M_{5z} = 0$. Therefore, collecting the unknowns at joint 1 one obtains

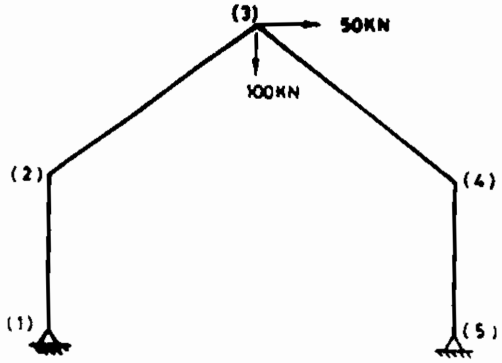


Figure 5.86

$$\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}-16 & 0.0085 & 0.0173\\0 & -0.0173 & -0.018\\0 & -16 & 0\end{bmatrix}\begin{bmatrix}\theta_1\\A'_{x12}\\A'_{y12}\end{bmatrix}+\begin{bmatrix}0.06458\\-0.2728\\-200\end{bmatrix}$$

The solution is obtained as

$$[\theta_1 \quad A'_{x12} \quad A'_{y12}]^T=[-7823\times10^{-3} \quad 12.5 \quad 3.107]$$

Similarly, collecting the unknowns at joint 5 one obtains

$$\begin{bmatrix}\theta_5\\A'_{x54}\\A'_{y54}\end{bmatrix}=\begin{bmatrix}1 & -0.00256 & -0.00216\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\A'_{x12}\\A'_{y12}\end{bmatrix}+\begin{bmatrix}0.0456\\-100\\-50\end{bmatrix}=\begin{bmatrix}-92.11\times10^{-3}\\-112.5\\-53.107\end{bmatrix}\begin{matrix}\text{rad.}\\\text{kN}\\\text{kN}\end{matrix}$$

The obtained reactions are shown in Figure 5.87.

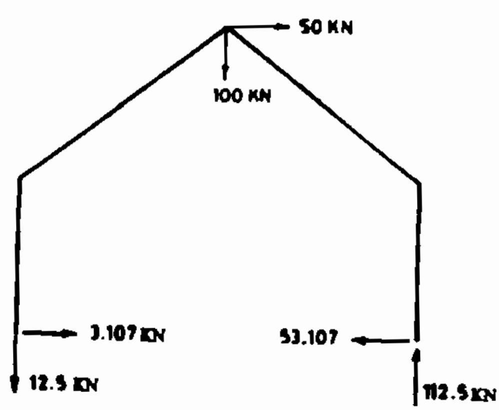


Figure 5.87

5.7.6 Shear Deformation Effect

In all previous sections, the shear deformations have been neglected. This is true in most one-dimensional structures where the member length is very large as compared with its depth. In some situations, however, the member either sustains heavy shear force or its depth is so deep that shear deformations have a significant effect on the analysis. The shear deformation effect should be considered when the span/depth ratio is less than 6, as can be realized from Figure 5.88, where ρ represents the ratio between the total deformation which includes shear deformation and the deformation due to bending only.

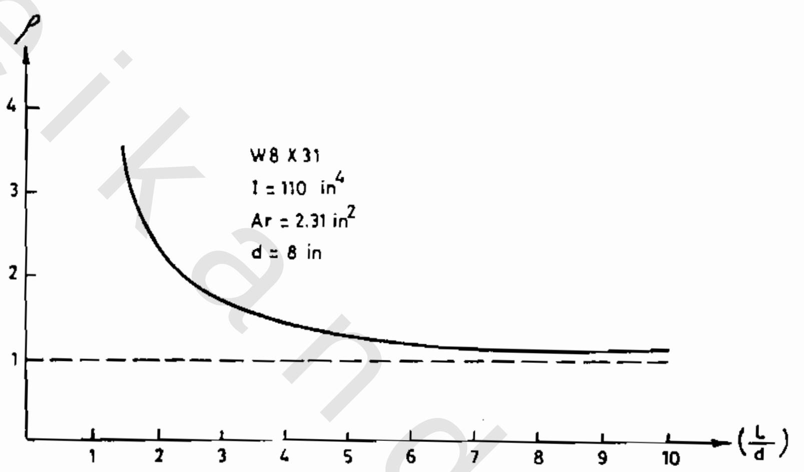


Figure 5.88

In order to include the shear effect in the stiffness matrix, one has to calculate first the flexibility matrix considering shear deformations. For member ij, which is fixed at i and free at j, as shown in Figure 5.89, the flexibility matrix has been obtained in Chapter 2 as follows:

$$\underline{F}'_{jj} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{L}{G A_{ry}} + \frac{L^3}{3EI_z}\right) & 0 & 0 & 0 & \frac{L^2}{2EI_z} \\ 0 & 0 & \left(\frac{L}{G A_{rz}} + \frac{L^3}{3EI_y}\right) & 0 & \frac{-L^2}{2EI_y} & 0 \\ 0 & 0 & 0 & \frac{L}{GJ_x} & 0 & 0 \\ 0 & 0 & -\frac{L^2}{2EI_y} & 0 & \frac{L}{EI_y} & 0 \\ 0 & \frac{L^2}{2EI_z} & 0 & 0 & 0 & \frac{L}{EI_z} \end{bmatrix} \qquad (5.152)$$

Figure 5.89

For a rectangular cross section of width B and depth D one has

$$\frac{1}{A_{ry}} = \int_0^{D/2} \frac{Q_z^2\, dy}{I_z^2\, B} \qquad (5.153)$$

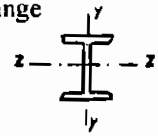
$$\frac{1}{A_{rz}} = \int_0^{B/2} \frac{Q_y^2\, dy}{I_y^2\, B} \qquad (5.154)$$

where Q_z and Q_y are the first moment of area of a section about the neutral axis. Values of A_r for some typical cross sections are given in Table 5.1.

The stiffness matrix $\underline{S}'^{i}_{jj}$ is obtained from the inverse of $\underline{F}'^{i}_{jj}$. The stiffness matrices $\underline{S}'^{j}_{ii}$, $\underline{S}'_{ij}$ and $\underline{S}'_{ji}$ are then obtained from the equilibrium principles as described in section 5.7.4. For example, in plane frame one has

$$\underline{A}'_{ij} = \underline{E}_{ij}\, \underline{A}'_{ji} = \underline{E}_{ij}\, \underline{S}'^{i}_{jj}\, \underline{D}'_{ji} = \underline{S}'_{ji}\, \underline{D}'_{ji} \qquad (5.155)$$

Table 5.1 Reduced Area for Shear

Section	A_r
Rectangular	$\frac{5}{6}A$
Solid circular	$\frac{9}{10}A$
Wide flange 	$A_{ry} = \frac{A_f}{1.2A}$ $A_{rz} = \frac{A_w}{A}$
A = total area; A_f = area of flange; A_w = area of web	

$$\underline{S}'_{ij} = \underline{E}_{ij}\,\underline{S}'^{i}_{jj} \tag{5.156}$$

where $\underline{E}_{ij}$ is given by

$$\underline{E}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L & -1 \end{bmatrix} \tag{5.157}$$

Also, one has

$$\underline{A}'_{ij} = \underline{S}'^{j}_{ii}\,\underline{D}'_{ij} + \underline{S}'_{ij}\,\underline{D}'_{ji} = \underline{E}_{ij}\,(\underline{S}'_{ji}\,\underline{D}'_{ij} + \underline{S}'^{i}_{jj}\,\underline{D}'_{ji}) \tag{6.158}$$

$$\underline{S}'^{j}_{ii} = \underline{E}_{ij}\,\underline{S}'_{ji} = \underline{E}_{ij}\,\underline{S}'^{T}_{ij} \tag{6.159}$$

Example 5.23

Determine the stiffness matrices for a member made of a rectangular section of width 50 cm and depth 150 cm in a plane frame considering shear deformations. Compare the results with those of neglecting shear deformations (I_z = 0.140625 m^4, E = 2×10^8 kN/m^2, EA = 0.75 E, L = 6 m).

Solution

When neglecting shear deformation, one has

$$\underline{S}'^{j}_{ii} = \underline{S}'^{i}_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} = 10^6 \begin{bmatrix} 25 & 0 & 0 \\ 0 & 1.5625 & -4.6875 \\ 0 & -4.6875 & 18.75 \end{bmatrix}$$

$$\underline{\mathbf{S}}'_{ij} = \underline{\mathbf{S}}'_{ji} = 10^6 \begin{bmatrix} 25 & 0 & 0 \\ 0 & 1.5625 & -4.6875 \\ 0 & -4.6875 & 18.75 \end{bmatrix}$$

When considering shear deformations, one has

$$\underline{\mathbf{F}}'^{i}_{jj} = \begin{bmatrix} \dfrac{L}{EA} & 0 & 0 \\ 0 & \left(\dfrac{L}{GA_r} + \dfrac{L^3}{3EI_z}\right) & \dfrac{L^2}{2EI_z} \\ 0 & \dfrac{L^2}{2EI_z} & \dfrac{L}{EI_z} \end{bmatrix} = 10^{-6} \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 2.667 & 0.64 \\ 0 & 0.64 & 0.21333 \end{bmatrix}$$

$$\underline{\mathbf{S}}'^{i}_{jj} = [\underline{\mathbf{F}}'^{i}_{jj}]^{-1} = 10^6 \begin{bmatrix} 25.0003 & 0 & 0 \\ 0 & 1.339266 & -4.01795 \\ 0 & -4.01795 & 16.74169 \end{bmatrix}$$

$$\underline{\mathbf{S}}'^{i}_{ij} = \underline{\mathbf{E}}_{ij}\,\underline{\mathbf{S}}'^{i}_{jj} = 10^6 \begin{bmatrix} 25.0003 & 0 & 0 \\ 0 & 1.339266 & -4.01795 \\ 0 & -4.01765 & 7.366 \end{bmatrix}$$

$$\underline{\mathbf{S}}'_{ji} = \underline{\mathbf{S}}'^{T}_{ij} = 10^6 \begin{bmatrix} 25.0003 & 0 & 0 \\ 0 & 1.339266 & -4.01766 \\ 0 & -4.01795 & 7.366 \end{bmatrix}$$

$$\underline{\mathbf{S}}'^{j}_{ii} = \underline{\mathbf{E}}_{ij}\,\underline{\mathbf{S}}'_{ji} = 10^6 \begin{bmatrix} 25.0003 & 0 & 0 \\ 0 & 1.339266 & -4.01765 \\ 0 & -4.01765 & 16.7399 \end{bmatrix}$$

Comparing the stiffness matrices, one finds that the elements which include shear deformations have different values than the case of neglecting shear deformations.

Exercises

Use the stiffness matrix method, approach II, to solve the following problems:

1. Determine the axial force, shear force and bending moment diagrams in Figure 1 due to:

   (a) the applied loading only ;
   (b) member AC is short by 1 cm.

   (EA = 0.5×10^5 kN, EI = 10^5 kN.m^2)

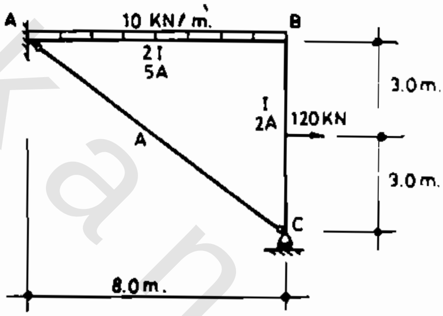


Figure 1

2. Determine the member forces and the deformations for the loaded truss shown in Figure 2 due to a rise in temperature for member ED and DC of 20°C (EA = 2×10^6 kN for all members, $\alpha = 10^{-5}$/°c).

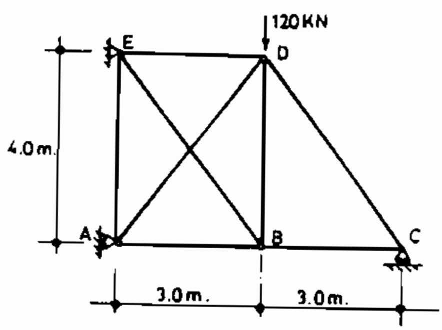


Figure 2

3 Determine the shear force and bending moment diagrams for the beam shown in Figure 3 due to the applied loading, a rise in temperature for member 2-3, and a rotation at support 1 as shown. ($EI = 10^5$ kN.m^2, $\alpha = 10^{-5}$/°c).

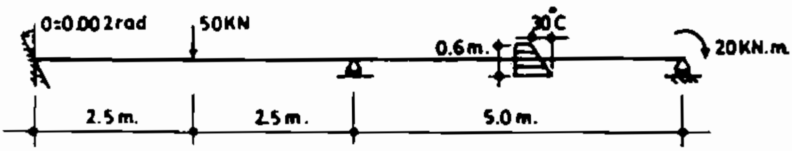


Figure 3

4. Determine the member forces in the truss shown in Figure 4 due to the loading and a vertical settlement at joint 5 of 1 cm downward ($EA = 10^6$ kN for all members).

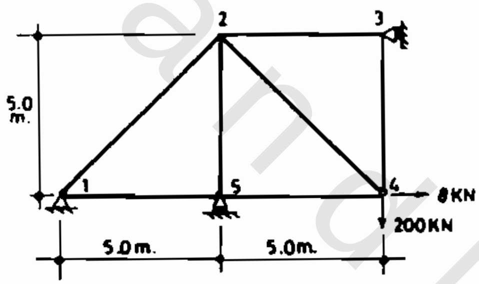


Figure 4

5. Determine the bending moment diagram for the beam shown in Figure 5.

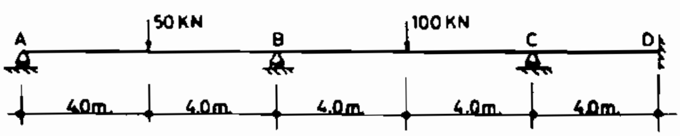


Figure 5

6. Determine the bending moment diagram for the beam shown in Figure 6 due to the applied loading.

7. Analyze the space truss shown in Figure 7 due to the applied loads, using the given global coordinates. (EA = constant).

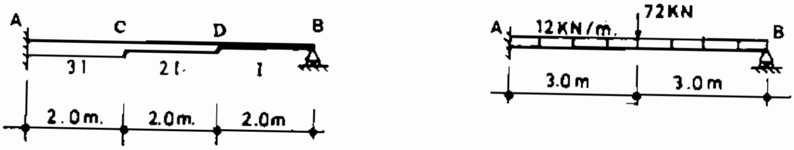


Figure 6

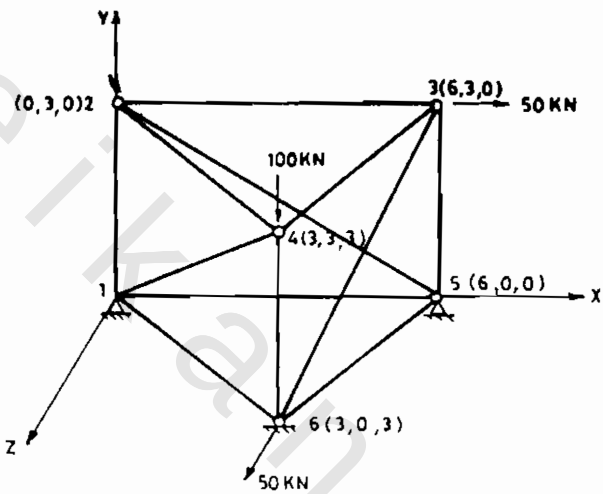


Figure 7

8. Determine the reactions and bending moment diagram for the frame shown in Figure 8. ($A = 6$ in^2, $I = 144$ in^4, $E = 30000$ Ksi).

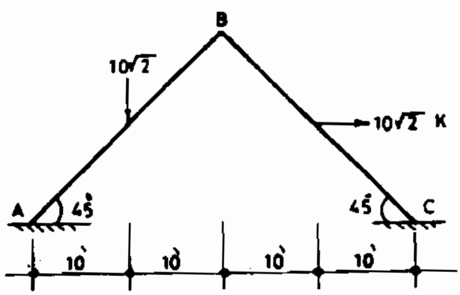


Figure 8

9. Determine the member forces in the truss shown in Figure 9 due to the applied loads and a downward settlement at A of 0.5 inch. (EA = 1000 Kips).

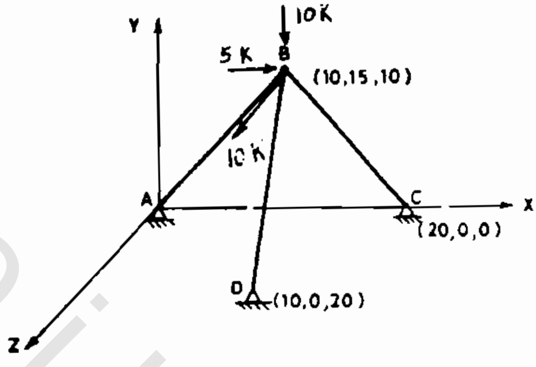


Figure 9

10. Explain the possible approaches which can be used in analyzing a structure subjected to settlement at the supports. Make your discussion applicable to any simple structure.

11. Determine the member forces in the truss shown in Figure 10. (EA = constant for all members).

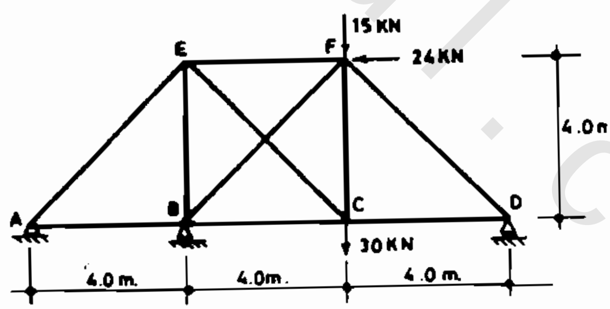


Figure 10

12. Analyze the truss shown in Figure 11 using stiffness matrix approach II, and approach I. (EA = constant for all members).

13. The structure shown in Figure 12 can be analyzed either as a series of connected members or by using the stiffness method approach II. Determine the bending moment diagram using any of the methods for the structure due to a vertical settlement of 2 inches downward at C. (EA = 10^3 Kips, EI = 10^3 K.ft^2).

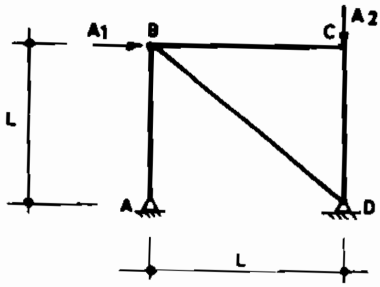


Figure 11

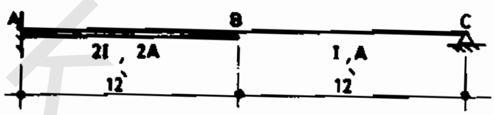


Figure 12

14. Determine the axial force and bending moment diagrams for the frame shown in Figure 13 due to the loading and rise in temperature ($K_1 = K_2 = 80$ Kips/ft., $K_3 = 100$ K.ft/rad, $EA = 10^4$ Kips, $EI = 10^5$ K.ft^2, $\alpha = 10^{-6}$/°F).

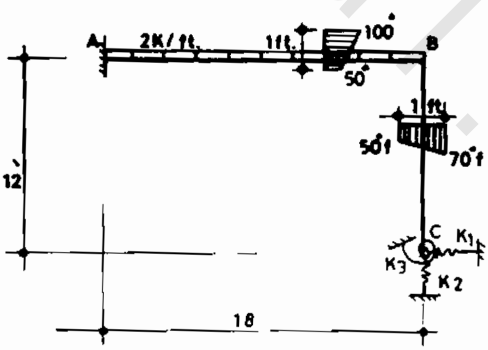


Figure 13

15. For the frame shown in Figure 14, the fixed support at C is allowed to slide on the plane x'–x'. Determine the reactions and draw the axial force, shear force and bending moment diagrams for the frame. ($EA = 10^5$ kN, $EI = 10^6$ kN.m^2).

16. Determine the axial force, shear force and bending moment diagrams for the frame shown in Figure 15. ($EA = 3000$ Kips, $EI = 12000$ K.ft^2).

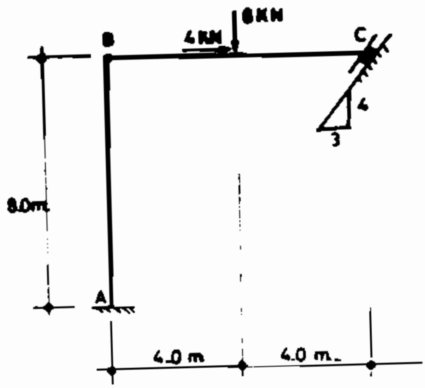


Figure 14

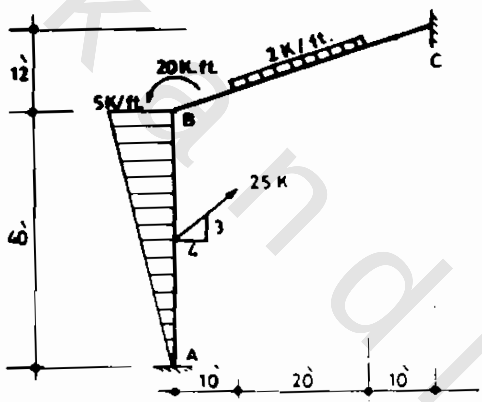


Figure 15

17. Analyze the series connected structure shown in Figure 16 and draw the internal actions diagrams. ($EA = 5\times10^5$ kN, $EI = 10^5$ kN.m^2).

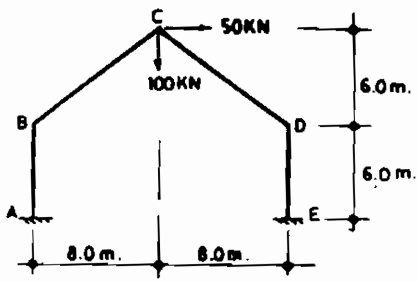


Figure 16

18. Analyze the series connected structure shown in Figure 17 and draw the bending moment diagram.

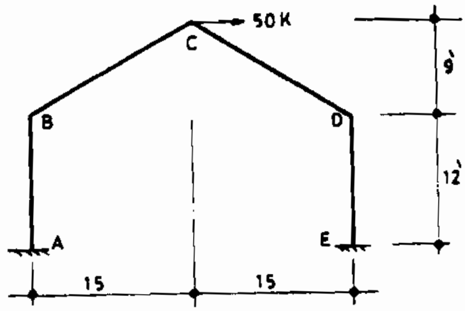


Figure 17

19. Analyze the structure of Problem 18 considering that support E is a hinge.

20. Determine the reactions for the frame shown in Figure 18. The fixed support B is allowed to slide on an inclined plane as shown. (EA = 10^5 kN, EI = 10^5 kN.m^2).

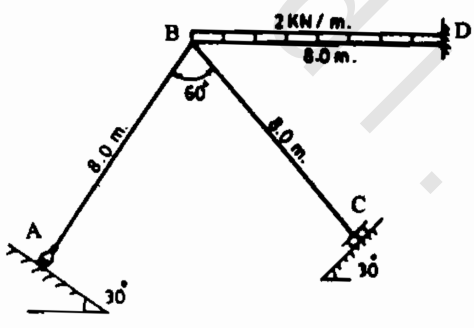


Figure 18

21. Determine the reactions for the frame shown in Figure 19.

22. Determine the axial force, shear force and bending moment diagrams for the frame shown in Figure 20 considering the moment releases at supports A and C. (EA = 3000 Kips, EI = 4.32 × 10^6 K.ft^2).

23. Determine the bending moment diagram for the frame shown in Figure 21. (EA = 30000 Kips, EI = 30000 K.ft^2, K_x = K_y = 3 Kip/in., K_θ = 50 K.ft/rad).

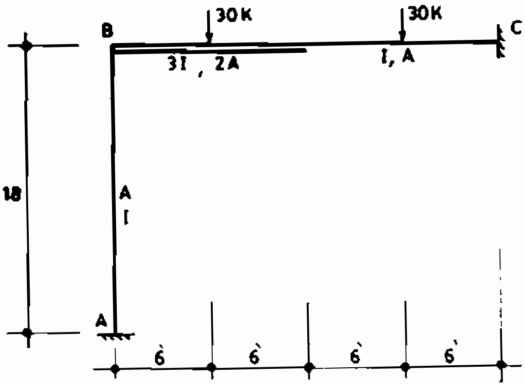


Figure 19

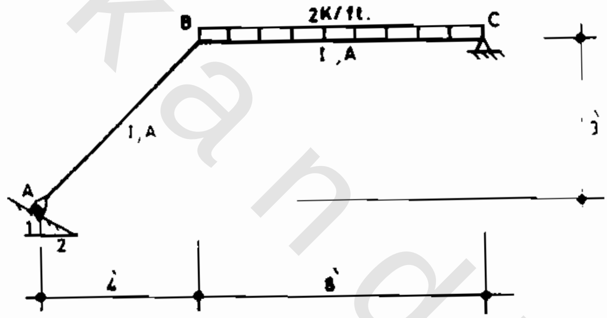


Figure 20

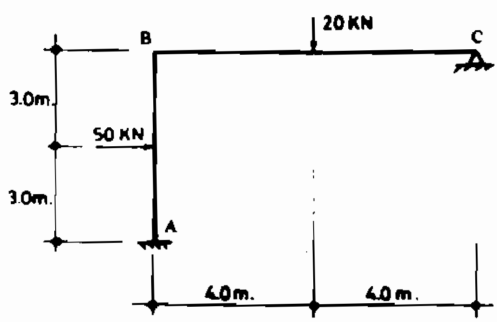


Figure 21

24. Determine the shear force and bending moment diagrams for the frame shown in Figure 22 neglecting axial deformations. ($EI = 10^5$ kN.m^2).

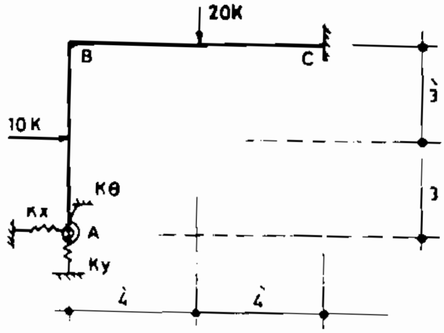


Figure 22

25. Determine the internal forces in the grid shown in Figure 23. (J_x = 110 in^4, I_y = 400 in^4, G = 10,000 Ksi, E = 30000 Ksi).

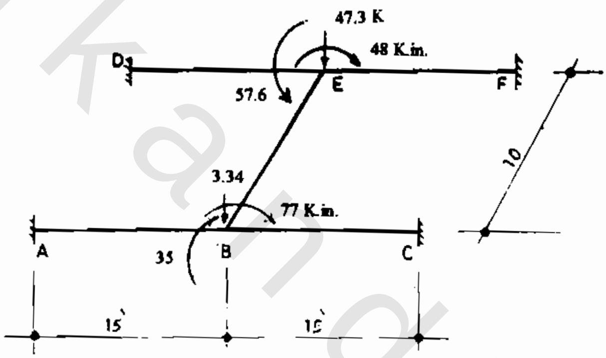


Figure 23

26. Analyze the space frame shown in Figure 24 considering that the angle β = 0°. (J_x = 50 in^4, I_y = 100 in^4, I_z = 1000 in^4, A = 10 in^2, G = 10000 Ksi, E = 30000 Ksi).

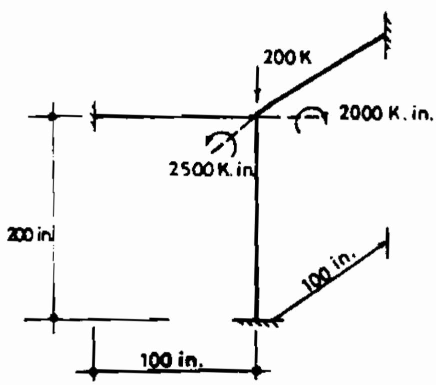


Figure 24

27. Analyze the floor system shown in Figure 25. ($I_x = I_y = 1\ m^4$, $G = 0.4\times10^5$ $kN.m^2$. $E = 10^5\ kN.m^2$).

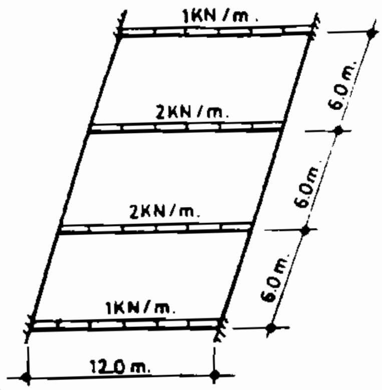


Figure 25

28. Analyze the frame shown in Figure 26 and determine its bending moment diagram.

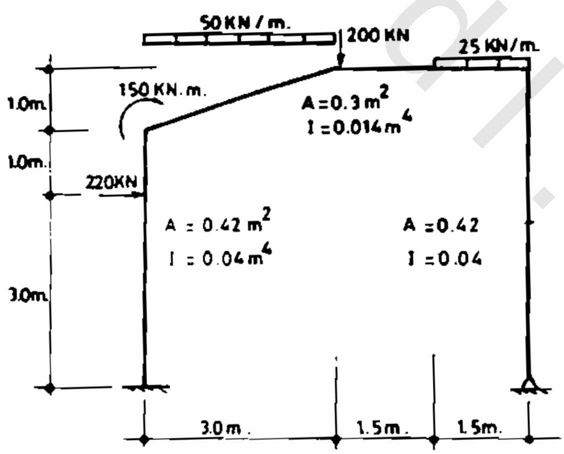


Figure 26

29. Compute all members internal actions for the structure shown in Figure 27 and draw the bending moment diagram. ($E = 30000$ Ksi, $A = 5\ in^2$, $I = 50\ in^4$).

30. Determine the axial force, shear force and bending moment diagrams for the frame shown in Figure 28, neglecting axial deformations.

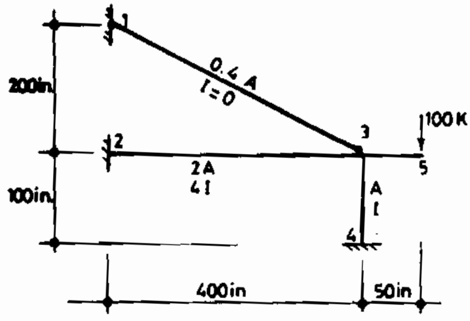


Figure 27

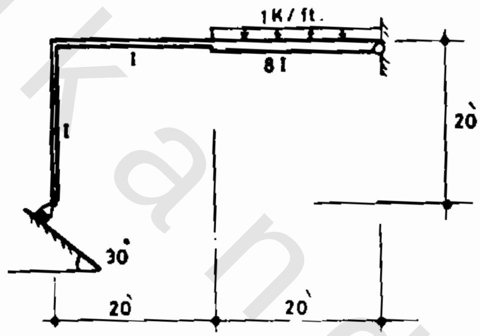


Figure 28

31. Determine the axial force, shear force and bending moment diagrams for the frame shown in Figure 29. (EA = 30000 Kips, EI = 100000 K.in^2).

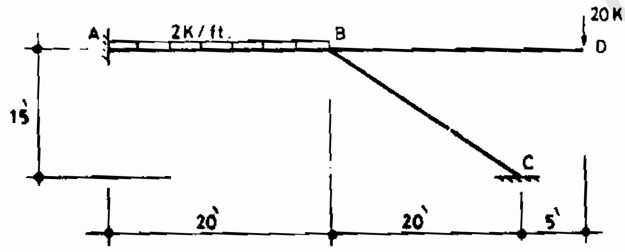


Figure 29

32. Determine the internal actions for the grid shown in Figure 30. (GJ_x = 600 Ksi, EI_y = 3000 Ksi).

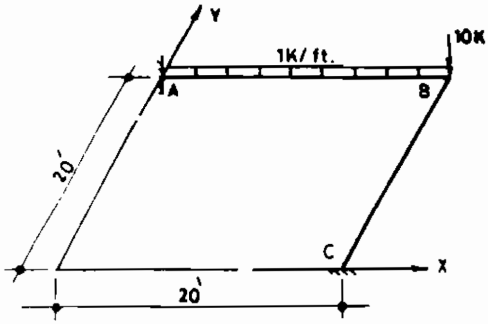


Figure 30

33. Determine the degrees of static indeterminacy and degrees of freedom for the structures shown in Figure 31

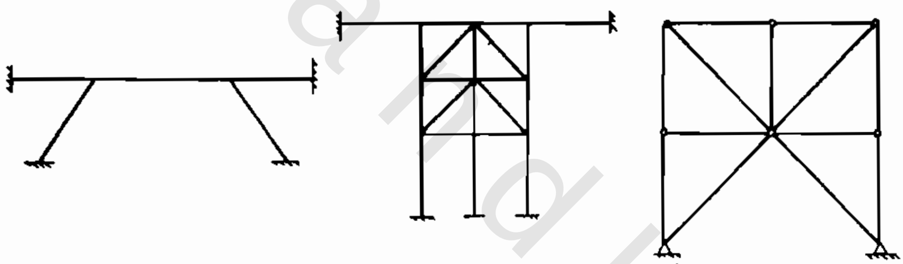


Figure 31

34. Determine the reactions in the grid shown in Figure 32. (GJ_x = 600 Ksi, EI_y = 1200 Ksi).

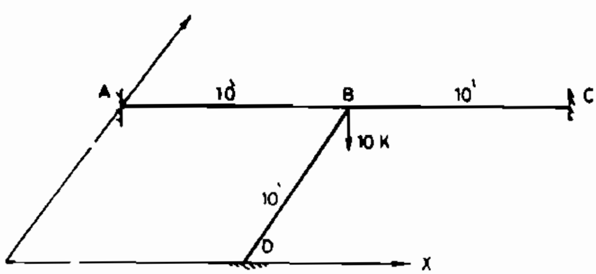


Figure 32

35. Analyze the space truss shown in Figure 33. (EA = constant).

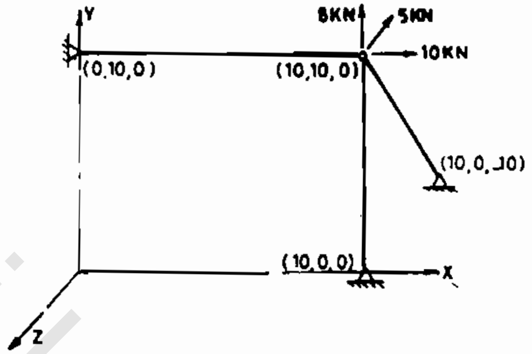


Figure 33

36. Determine the axial force and bending moment diagrams for the frame shown in Figure 34. (EA = 2000 K, EI = 10000 K.ft^2).

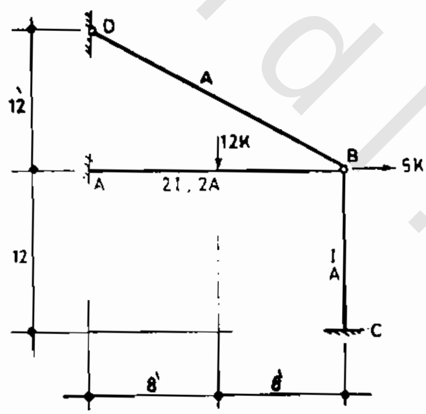


Figure 34

37. Analyze the frame shown in Figure 35 and draw the bending moment diagram. (EA = 2000 Kips, EI = 10000 K.ft^2).

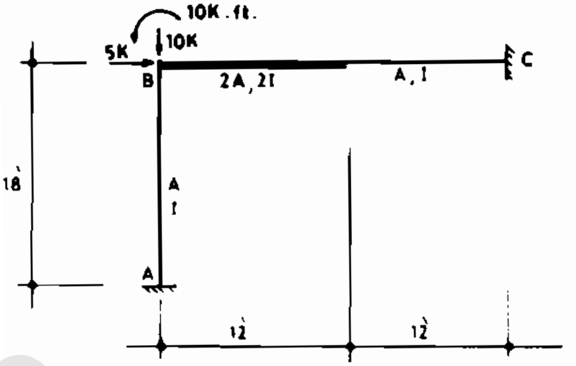


Figure 35

CHAPTER 6

ADDITIONAL TOPICS

6.1 INTRODUCTION

This chapter presents some topics of interest to the structural engineers. First, an introduction to the finite element method is given as an outcome of the stiffness matrix method approach II of Chapter 5. Second, the influence lines of statically indeterminate structures are presented. Finally, some examples are given utilizing the computer softwares to verify some of the examples presented in this book.

6.2 AN INTRODUCTION TO THE FINITE ELEMENT METHOD

The finite element method is a tool to solve one-dimensional, two-dimensional, and three-dimensional structures with approximation instead of solving complicated partial differential equations. The structure is discretized into a set of elements joined together at some points called nodes or nodal points. These nodes are similar to the joints in the one-dimensional structures which were investigated in the previous chapters. The nodes could be the common corners between the elements, or chosen between the boundaries of the elements, as shown in Figure 6.1. The similarity in the concept between one-dimensional skeleton structures and two- or three-dimensional structures, in terms of discretization is also shown in Figure 6.1.

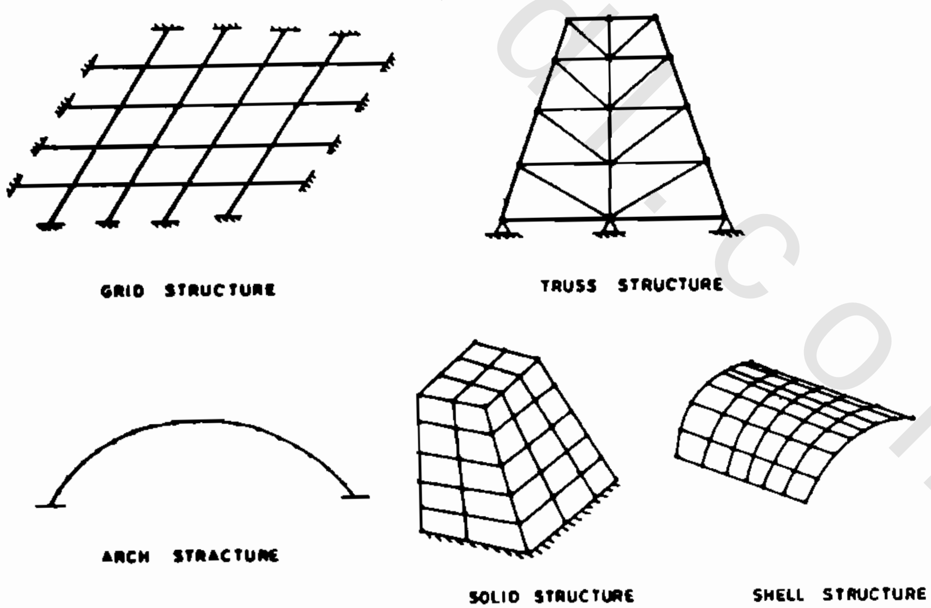


Figure 6.1

It is obvious that in one-dimensional structures the element is a one-dimensional member, as was analyzed in Chapters 3, 4, and 5. In two-dimensional structures, the element is a two-dimensional plate or shell element. The three-dimensional element could be a cube, prism, or a tetrahedron, either with straight sides or curved sides, as shown in Figure 6.2

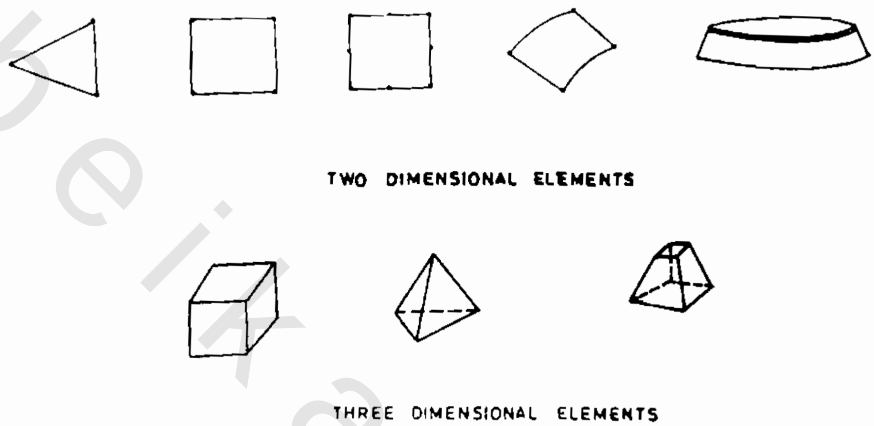


Figure 6.2

The solution of the finite element method is almost the same as the direct stiffness matrix method presented in Chapter 5. Once the elements' stiffness matrices are found, these matrices are augmented according to the compatibility and equilibrium conditions at every node. The free nodal displacements can be determined after specifying the boundary conditions at the boundary nodes. However, what interests the analyst in two-dimensional and three-dimensional structures is the stresses and strains not forces and displacements. Therefore, it is necessary to relate the strains at a point within the element with the nodal displacements.

6.2.1 The Relationship Between Strains and Nodal Displacements

The displacements $\underline{\mathbf{d}}$ at any point of coordinates (x,y) in a plane element can be related to the nodal displacement $\underline{\mathbf{D}}_c$ by

$$\underline{\mathbf{d}} = \underline{\mathbf{N}}\,\underline{\mathbf{D}}_c \tag{6.1}$$

where the elements of matrix $\underline{\mathbf{N}}$ are function of the location of the point (x,y).

An easy approach to determine matrix $\underline{\mathbf{N}}$ is to express $\underline{\mathbf{d}}$ in terms of some displacement functions $\underline{\mathbf{P}}$ and weighting parameters $\underline{\alpha}$, as follows:

$$\underline{\mathbf{d}} = \underline{\mathbf{P}}\,\underline{\alpha} \tag{6.2}$$

The elements of $\underline{\mathbf{P}}$ differ from one finite element to another. The dimension of the weighting parameters $\underline{\alpha}$ is preferably equal the number of the nodal displacements of the finite element.

Equation 6.2 can, be applied at every node in the element to obtain

$$\begin{bmatrix} D_{c1} \\ \vdots \\ D_{ci} \\ \vdots \\ D_{cn} \end{bmatrix}_k = \begin{bmatrix} P_1^T \\ \vdots \\ P_i^T \\ \vdots \\ P_n^T \end{bmatrix}_k \underline{\alpha}_k = \underline{\mathbf{C}}_k \, \underline{\alpha}_k \tag{6.3}$$

which indicates that the nodal displacements $\underline{\mathbf{D}}_c$ for the finite element number k, which have nodes, 1, ..., i, ..., n, are related to the parameters $\underline{\alpha}_k$ by the matrix $\underline{\mathbf{C}}_k$, whose elements depend on the locations of the nodes 1, ..., i, ..., n, in the global coordinates. For proper choice of global coordinates and the elements, one can relate $\underline{\mathbf{d}}$ to $\underline{\mathbf{D}}_c$ for any element using Equations 6 2 to 6.3 as follows:

$$\underline{\alpha}_k = \underline{\mathbf{C}}_k^{-1} \, \underline{\mathbf{D}}_{ck} \tag{6.4}$$

$$\underline{\mathbf{d}} = \underline{\mathbf{P}}\,\underline{\mathbf{C}}^{-1} \, \underline{\mathbf{D}}_c \tag{6.5}$$

Thus $\underline{\mathbf{N}}$ for any element is obtained from

$$\underline{\mathbf{N}} = \underline{\mathbf{P}}\,\underline{\mathbf{C}}^{-1} \tag{6.6}$$

The inverse of matrix $\underline{\mathbf{C}}$ exists when choosing appropriate displacement functions for each specific element For the present introduction we shall deal only with plate elements.

Example 6.1

Show how to determine $\underline{\mathbf{N}}$ for the triangular membrane element shown in Figure 6.3.

Solution

Any point of coordinates (x,y) within the element is subjected to the displacement u in the x-direction and displacement v in the y-direction of the global coordinates. Therefore, the nodal displacements at i, j, and k are given by

$$\underline{\mathbf{D}}_c^T = \begin{bmatrix} u_i & v_i & u_j & v_j & u_k & v_k \end{bmatrix}$$

The weighting parameters $\underline{\alpha}$ are chosen to be 6 × 1 as follows:

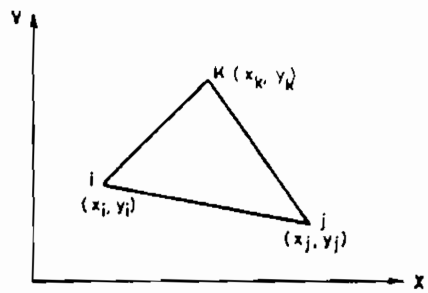


Figure 6.3

$$\underline{\alpha}_e^T = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6]$$

The displacement functions which relate $\underline{d}$ to $\underline{\alpha}$ are chosen to be in the form

$$\underline{P} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}$$

Substituting into Equation 6.3 one has

$$\begin{bmatrix} u_i \\ v_i \\ \hline u_j \\ v_j \\ \hline u_k \\ v_k \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ \hline 1 & x_j & y_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ \hline 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$

Thus, the matrix $\underline{N}$ is determined from the following equation:

$$\underline{N} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix}^{-1}$$

The relationships between strains and displacements can be obtained using Figure 6.4 which can be generalized into y-z and z-x planes as follows:

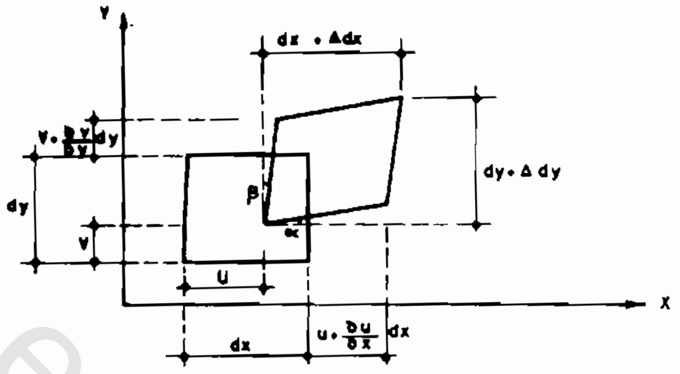


Figure 6.4

$$\varepsilon_x = \frac{\Delta dx}{dx} = \frac{u + \frac{\partial u}{\partial x}dx - u}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{6.7}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

where w is the displacement in z-direction.

Since the displacements $\underline{d}$ are related to the nodal displacements $\underline{\mathbf{D}}_e$, it is possible to relate the strains $\underline{\varepsilon}$ to the nodal displacements $\underline{\mathbf{D}}_e$. Such a relationship is expressed in general by

$$\underline{\varepsilon} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \underline{\mathbf{N}}\,\underline{\mathbf{D}}_c = \underline{\mathbf{B}}\,\underline{\mathbf{D}}_c \qquad (6.8)$$

Example 6.2

Show how to determine matrix $\underline{B}$ for the triangular membrane element considered in Example 6.1.

Solution

The strains of interest in the membrane elements are $\varepsilon_x, \varepsilon_y$, and γ_{xy}, since w = 0. One thus has

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

But $\underline{\mathbf{d}}$ is given by

$$\underline{\mathbf{d}} = \begin{bmatrix} u \\ v \end{bmatrix} = \underline{\mathbf{P}}\,\underline{\mathbf{C}}^{-1}\,\underline{\mathbf{D}}_c = \underline{\mathbf{N}}\,\underline{\mathbf{D}}_c$$

The strains $\underline{\varepsilon}$ are thus obtained from

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \underline{\mathbf{C}}^{-1}\,\underline{\mathbf{D}}_c$$

The matrix $\underline{\mathbf{B}}$, is therefore, given by the following equation:

$$\underline{\mathbf{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix}^{-1}$$

6.2.2 Stresses and Strains Relationships

For a general infinitesimal element dv = dx dy dz, the stresses corresponding to the strains of Equation 6.7 are given as

$$\underline{\sigma}^T = \left[\sigma_x \;\; \sigma_y \;\; \sigma_z \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{zx}\right] \qquad (6.9)$$

where σ_x, σ_y and σ_x indicate normal stresses and τ_{xy}, τ_{yz} and τ_{zx} indicate shear stresses as shown in Figure 6.5. For the linear elastic materials of our interest, the stress-strain relationships are

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x - \nu\sigma_z\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_x - \nu\sigma_y\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \qquad (6.10)$$

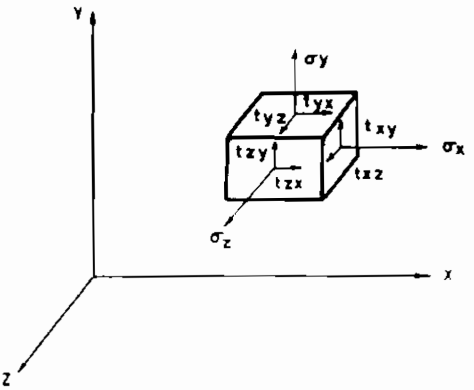


Figure 6.5

where E is the modulus of elasticity; G is the shearing modulus; and ν is the Poisson's ratio. The shearing modulus is related to the modulus of elasticity by

$$G = \frac{E}{2(1+\nu)} \qquad (6.11)$$

Expressing the stresses $\underline{\sigma}$ in terms of strains $\underline{\varepsilon}$ one has, in a general expression, the following equation:

$$\underline{\sigma} = \underline{\mathbf{K}}\,(\underline{\varepsilon} - \underline{\varepsilon}_0) + \sigma_0 \qquad (6.12)$$

in which $\underline{\varepsilon}_o$ represents initial strains, $\underline{\sigma}_o$ represents initial stresses, and $\underline{\mathbf{K}}$ is given by

$$\underline{\mathbf{K}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \qquad (6.13)$$

6.2.3 Plane-Stress and Plane-Strain for Isotropic Materials

If the element under consideration is in x-y plane and subjected to plane-stresses or plane-strains, the stresses and strains in this case are

$$\underline{\sigma}^T = [\sigma_x \;\; \sigma_y \;\; \tau_{xy}] \qquad (6.14)$$

$$\underline{\varepsilon}^T = [\varepsilon_x \;\; \varepsilon_y \;\; \tau_{xy}] \qquad (6.15)$$

In case of plane stresses, the element is subjected to $\sigma_z = 0$. In this case, the stress-strain relationship is obtained from Equation 6.10 as follows:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{bmatrix} \qquad (6.16)$$

If the element is also subjected to a change in temperature T, the initial strains become

$$\underline{\varepsilon}_o^T = \left[\alpha T \quad \alpha T \quad o\right] \tag{6.17}$$

where α is the coefficient of thermal expansion.

In case of plane strains, the element strain ε_z is zero but $\sigma_z \neq 0$. In this case, the stress-strain relationship is given by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{6.18}$$

If the plain strains element is subjected to a change in temperature T, the initial strains become

$$\underline{\varepsilon}_0^T = \left[(1+\nu)\alpha T \quad (1+\nu)\alpha T \quad 0\right] \tag{6.19}$$

6.2.4 Element Stiffness Matrix

In order to determine the stiffness matrix of any element, the principle of virtual work is applied. This principle, as presented in Chapter 2, states that the work done due to a virtual displacement is equal to the strain energy due to this virtual displacement. This can be expressed as

$$\delta \underline{\mathbf{D}}_e^T \underline{\mathbf{A}}_e = \int_{vol.} \delta \underline{\varepsilon}^T \underline{\sigma} \, d\,(vol) \tag{6.20}$$

where $\underline{\mathbf{A}}_e$ is the actions applied at the nodes of the element. Substituting Equations 6.8 and 6.12 into Equation 6.20, one has

$$\delta \underline{\mathbf{D}}_e^T \underline{\mathbf{A}}_e = \int_{vol.} \delta \underline{\mathbf{D}}_e^T \underline{\mathbf{B}}^T \left[\underline{\mathbf{K}}(\underline{\varepsilon} - \underline{\varepsilon}_0) + \underline{\sigma}_0\right] d\,(vol) \tag{6.21}$$

Equation 6.21 can be written as

$$\underline{\mathbf{A}}_e = \left[\int_{vol.} \underline{\mathbf{B}}^T \underline{\mathbf{K}}\, \underline{\mathbf{B}} \, d\,(vol)\right] \underline{\mathbf{D}}_e - \int_{vol.} \underline{\mathbf{B}}^T \underline{\mathbf{K}}\, \underline{\varepsilon}_0 \, d\,(vol) + \int_{vol.} \underline{\mathbf{B}}^T \underline{\sigma}_0 \, d\,(vol) \tag{6.22}$$

It is obvious that the equivalent nodal actions due to initial strains and initial stresses are, respectively, given by

$$\underline{\mathbf{A}}_{c\varepsilon_0} = + \int_{vol.} \underline{\mathbf{B}}^T \, \underline{\mathbf{K}} \, \underline{\varepsilon}_0 \, d\,(vol) \tag{6.23}$$

$$\underline{\mathbf{A}}_{c\sigma_0} = - \int_{vol.} \underline{\mathbf{B}}^T \, \underline{\sigma}_0 \, d\,(vol) \tag{6.24}$$

The relationship between nodal actions and nodal displacements is, in general,

$$\underline{\mathbf{A}}_c = \left[\int_{vol.} \underline{\mathbf{B}}^T \, \underline{\mathbf{K}} \, \underline{\mathbf{B}} \, d\,(vol) \right] \underline{\mathbf{D}}_e \tag{6.25}$$

From which it is obvious that the element stiffness matrix can be obtained from

$$\underline{\mathbf{S}} = \int_{vol.} \underline{\mathbf{B}}^T \, \underline{\mathbf{K}} \, \underline{\mathbf{B}} \, d\,(vol) \tag{6.26}$$

6.2.5 Equivalent Nodal Actions

Equations 6.23 and 6.24 gave, respectively, the equivalent nodal actions due to initial strains and initial stresses. In this section, the nodal actions due to distributed forces will be found. In continuum structures, the forces could be distributed within the body of the structure or on its surface. We shall indicate the body forces by $\underline{\mathbf{g}}$ kN/m^3 and the surface forces by $\underline{\mathbf{q}}$ kN/m^2. The work done by these forces, due to virtual nodal displacements $\delta \, \underline{\mathbf{D}}_e$ in an element, is given by

$$W = \int_{vol.} \delta \, \underline{\mathbf{d}}^T \, \underline{\mathbf{g}} \, d\,(vol) + \int_{surface} \delta \, \underline{\mathbf{d}}^T \, \underline{\mathbf{q}} \, d\,(area) \tag{6.27}$$

Substituting Equation 6.5 into Equation 6.27 one has

$$W = \int_{vol.} \delta \, \underline{\mathbf{D}}_e^T \, \mathbf{N}^T \underline{\mathbf{g}} \, d\,(vol) + \int_{surface} \delta \, \underline{\mathbf{D}}_e^T \, \underline{\mathbf{N}}^T \, \underline{\mathbf{q}} \, d\,(area) \tag{6.28}$$

Equating Equation 6.28 with the strain energy due to virtual displacements, one realizes that the equivalent nodal forces due to body and surface forces are, respectively, given by

$$\underline{\mathbf{A}}_{c_{body}} = \int_{vol.} \underline{\mathbf{N}}^T \underline{\mathbf{g}} \, d\,(vol) \tag{6.29}$$

$$\underline{\mathbf{A}}_{c_{surface}} = \int_{surface} \underline{\mathbf{N}}^T \underline{\mathbf{q}} \, d\,(area) \tag{6.30}$$

6.2.6 Example 6.3

Analyze the plate shown in Figure 6.6, considering plane stresses, $E = 10^7$ N.cm^2 and $\nu = 0$.

Solution

For clear illustration, the plate is divided into only two triangular elements as shown in Figure 6.7. Element 1 has nodes number 1, 3 and 4. Element 2 has nodes number 1, 2 and 4. The two elements are joined at nodes 1 and 4.

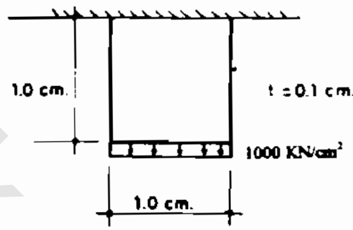


Figure 6.6

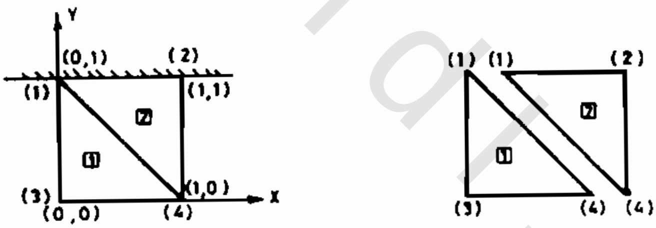


Figure 6.7

According to the coordinates of the nodes with respect to global x-y axes shown in Figure 6.7, the matrices $\underline{C}_1$ and $\underline{C}_2$ are determined as

$$\underline{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \}\text{node 3} \\ \\ \}\text{node4} \\ \\ \}\text{node 1} \end{matrix}$$

$$\underline{\mathbf{C}}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \}\text{node 4} \\ \\ \}\text{node 2} \\ \\ \}\text{node 1} \\ \end{matrix}$$

The matrices $\underline{\mathbf{B}}_1$ and $\underline{\mathbf{B}}_2$ are, respectively, obtained as

$$\underline{\mathbf{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \underline{\mathbf{C}}^{-1}$$

$$\underline{\mathbf{B}}_1 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad ; \quad \underline{\mathbf{B}}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

The stress-strain relationship for the plane stresses problem with $\nu = 0.0$ is thus

$$\underline{\mathbf{K}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

The elements' stiffness matrices are determined from

$$\underline{\mathbf{S}}_1 = \int \underline{\mathbf{B}}_1^T \, \underline{\mathbf{K}} \, \underline{\mathbf{B}}_1 \, d\,(\text{vol})$$

$$= \frac{E}{40} \begin{bmatrix} \overbrace{\begin{matrix} 3 & 1 \end{matrix}}^{\text{node 3}} & \overbrace{\begin{matrix} -2 & -1 \end{matrix}}^{\text{node 4}} & \overbrace{\begin{matrix} -1 & 0 \end{matrix}}^{\text{node 1}} \\ \begin{matrix} 1 & 3 \end{matrix} & \begin{matrix} 0 & -1 \end{matrix} & \begin{matrix} -1 & 0 \end{matrix} \\ \begin{matrix} -2 & 0 \end{matrix} & \begin{matrix} 2 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} -1 & -1 \end{matrix} & \begin{matrix} 0 & 1 \end{matrix} & \begin{matrix} 1 & 0 \end{matrix} \\ \begin{matrix} -1 & -1 \end{matrix} & \begin{matrix} 0 & 1 \end{matrix} & \begin{matrix} 1 & 0 \end{matrix} \\ \begin{matrix} 0 & -2 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 2 \end{matrix} \end{bmatrix} \begin{matrix} \}\text{node 3} \\ \\ \}\text{node 4} \\ \\ \}\text{node 1} \\ \end{matrix}$$

$$\underline{\mathbf{S}}_2 = \int \underline{\mathbf{B}}_2^T \, \underline{\mathbf{K}} \, \underline{\mathbf{B}}_2 \, d\,(\text{vol})$$

$$= \frac{\mathbf{E}}{40} \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}}^{\text{node 4}} & \overbrace{\begin{matrix} -1 & -1 \\ 0 & -2 \end{matrix}}^{\text{node 2}} & \overbrace{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}^{\text{node 1}} \\ \begin{matrix} -1 & 0 \\ -1 & -2 \end{matrix} & \begin{matrix} 3 & 1 \\ 1 & 3 \end{matrix} & \begin{matrix} -2 & -1 \\ 0 & -1 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & \begin{matrix} -2 & 0 \\ -1 & -1 \end{matrix} & \begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \begin{matrix} \}\ \text{node 4} \\ \\ \}\ \text{node 2} \\ \\ \}\ \text{node 1} \end{matrix}$$

The structure stiffness matrix is obtained by superposition to have

$$\underline{\mathbf{S}} = \frac{\mathbf{E}}{40} \begin{bmatrix} \overbrace{\begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix}}^{\text{node 1}} & \overbrace{\begin{matrix} -2 & 0 \\ -1 & -1 \end{matrix}}^{\text{node 2}} & \overbrace{\begin{matrix} -1 & -1 \\ 0 & -2 \end{matrix}}^{\text{node 3}} & \overbrace{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}^{\text{node 4}} \\ \begin{matrix} -2 & -1 \\ 0 & -1 \end{matrix} & \begin{matrix} 3 & 1 \\ 1 & 3 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -1 & 0 \\ -1 & -2 \end{matrix} \\ \begin{matrix} -1 & 0 \\ -1 & -2 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 3 & 1 \\ 1 & 3 \end{matrix} & \begin{matrix} -2 & -1 \\ 0 & -1 \end{matrix} \\ \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} -1 & -1 \\ 0 & -2 \end{matrix} & \begin{matrix} -2 & 0 \\ -1 & -1 \end{matrix} & \begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix} \end{bmatrix} \begin{matrix} \}\ \text{node 1} \\ \\ \}\ \text{node 2} \\ \\ \}\ \text{node 3} \\ \\ \text{node 4} \end{matrix}$$

The equivalent nodal forces are determined from

$$\underline{\mathbf{A}}_{el}^T = t \begin{bmatrix} \underbrace{0 \quad -500}_{\text{node 3}} & \underbrace{0 \quad -500}_{\text{node 4}} & \underbrace{0 \quad 0}_{\text{node 1}} \end{bmatrix}$$

The boundary conditions are $\underline{\mathbf{D}}_{c1} = \underline{\mathbf{D}}_{c2} = \underline{0}$. Therefore, the final stiffness relationship is

$$\begin{bmatrix} 0 \\ -500 \\ 0 \\ -500 \end{bmatrix} = \frac{\mathbf{E}}{40} \begin{bmatrix} 3 & 1 & -2 & -1 \\ 1 & 3 & 0 & -1 \\ -2 & 0 & 3 & 0 \\ -1 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

The solution is obtained as

$$\left[\underline{D}_{e3}^T \quad \underline{D}_{e4}^T\right] = 10^{-4}\,[0 \quad -1 \quad 0 \quad -1]\ \text{cm}$$

Now, the stresses at any point within any element can be determined as follows:

$$\underline{\sigma} = \underline{K}\,\underline{\varepsilon} = \underline{K}\,\underline{B}\,\underline{D}_e$$

$$\underline{\sigma}_1 = \underline{K}\,\underline{B}_1\,[\underline{D}_e]_1 = [0 \quad 1000 \quad 0]$$

$$\underline{\sigma}_2 = \underline{K}\,\underline{B}_2\,[\underline{D}_e]_2 = [0 \quad 1000 \quad 0]$$

where $[\underline{D}_e]_1^T = 10^{-4}\,[0 \quad -1 \quad 0 \quad -1 \quad 0 \quad 0]$

$[\underline{D}_e]_2^T = 10^{-4}\,[0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0]$

The results indicate that the elements are subjected to tensile stress $\sigma_y = 1000$ kN/cm^2.

6.2.7 Displacement Functions of Some Membrane Elements

It was pointed out, in the previous sections, that the displacement functions for a triangular membrane element are given by

$$\underline{P} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \qquad (6.31)$$

where the weighting parameters $\underline{\alpha}$ are chosen to be 6 parameters. Another element used for membrane structures is shown in Figure 6.8. The triangle in this case has six nodes. The weighting parameters $\underline{\alpha}$ are thus chosen to be 12 parameters. The displacement functions in this case are chosen as follows:

$$\underline{P} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \qquad (6.32)$$

One also may use a rectangular element with four nodes as shown in Figure 6.9. The displacement functions in this case could be bilinear as

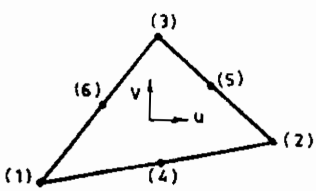


Figure 6.8

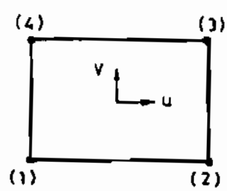


Figure 6.9

$$\underline{P} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \quad (6.33)$$

If the rectangular element has 8 nodes as shown in Figure 6.10, the displacement functions in this case could be chosen as

$$\underline{P} = \begin{bmatrix} 1 & x & y & xy & y^2 & xy^2 & y^3 & xy^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & xy & x^2 & x^2y & x^3 & x^3y \end{bmatrix}$$

(6.34)

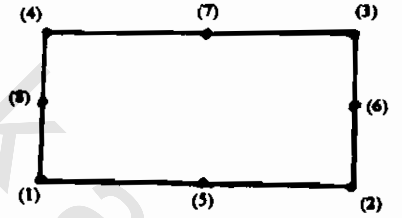


Figure 6.10

6.2.8 Stiffness Matrix of Rectangular Plate Bending Element

The displacements of interest at any node of the rectangular plate element shown in Figure 6.11 are given by

$$\underline{D}_{ei}^{T} = \begin{bmatrix} w_i & \theta_{x_i} & -\theta_{y_i} \end{bmatrix} = \begin{bmatrix} w_i & \dfrac{\partial w_i}{\partial y} & \dfrac{\partial w_i}{\partial x} \end{bmatrix} \quad (6.35)$$

The nodal displacements of the element are thus

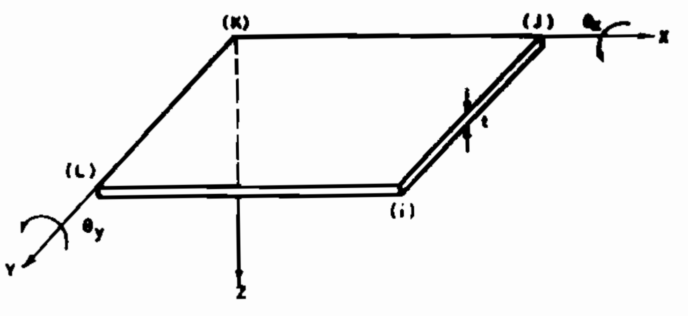


Figure 6.11

$$\underline{\mathbf{D}}_c^T = \left[\underline{\mathbf{D}}_{ci}^T \quad \underline{\mathbf{D}}_{cj}^T \quad \underline{\mathbf{D}}_{ck}^T \quad \underline{\mathbf{D}}_{c\ell}^T\right]_{12\times 1} \tag{6.36}$$

The strains at any point (x,y) within the element consist of the curvatures and are given by

$$\underline{\varepsilon}^T = \left[\frac{-\partial^2 w}{\partial x^2} \quad \frac{-\partial^2 w}{\partial y^2} \quad \frac{-2\partial^2 w}{\partial x\,\partial y}\right] \tag{6.37}$$

The corresponding stresses represent the bending and twisting moments. They are expressed as

$$\underline{\sigma}^T = D\left[-\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \quad -\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad (-1+\nu)\frac{\partial^2 w}{\partial x\,\partial y}\right] \tag{6.38}$$

where

$$D = \frac{E\,t^3}{12(1-\nu^2)} \tag{6.39}$$

Thus, by using Equation 6.37, one can write Equation 6.38 as follows:

$$\underline{\sigma} = D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\underline{\varepsilon} \tag{6.40}$$

The displacement function is taken as follows:

$$\mathbf{P} = \left[1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^3 \quad x^2y \quad xy^2 \quad y^3 \quad x^3y \quad xy^3\right] \tag{6.41}$$

where the deflection $\underline{\mathbf{w}}$ is expressed as

$$\underline{\mathbf{w}} = \underline{\mathbf{P}}\,\underline{\alpha} \tag{6.42}$$

Now, substituting Equation 6.42 into Equation 6.35 and 6.36 one obtains $\underline{\mathbf{C}}$. The displacements at any node i are, for example,

$$\underline{\mathbf{D}}_{ei} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & 1 & 0 & 2x & y & 0 & 3x^2 & 2xy & y^2 & 0 & 3x^2y & y^3 \end{bmatrix}\underline{\alpha} \tag{6.43}$$

in which x and y are coordinates of joint i which are substituted by x_i and y_i.

The relation between $\underline{\varepsilon}$ and $\underline{D}_e$ is obtained from Equation 6.37 and 6.41 as follows:

$$\underline{\varepsilon} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x & 6y \end{bmatrix} \underline{\alpha} \quad (6.44)$$

Matrix $\underline{B}$ is obtained from the inverse of matrix $\underline{C}$ and Equation 6.44, as follows:

$$\underline{B} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x & 6y \end{bmatrix} \underline{C}^{-1} \quad (6.45)$$

The element stiffness matrix can then be determined from Equation 6.26.

6.2.9 Elements for General Solids

The stress-strain relationships for general solids has been given in Equations 6.10 – 6.13. In this section, the displacement functions of some popular solid elements are given.

Tetrahedra with Constant Strain

The element in this case has four nodes as shown in Figure 6.12. The displacements at any point (x, y, z) are

$$\underline{d}^T = [u \quad v \quad w] \quad (6.46)$$

The displacement function in this case is

$$\underline{P} = \begin{bmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & z \end{bmatrix} \quad (6.47)$$

Tetrahedra with Linear Strain

The number of nodes in each element is 10, as shown in Figure 6.13. The displacement function is

$$\underline{P} = \begin{bmatrix} 1 & x & y & z & xy & yz & zx & x^2 & y^2 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\text{similar}} & \ldots\ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & \boxed{\text{similar}} \end{bmatrix} \quad (6.48)$$

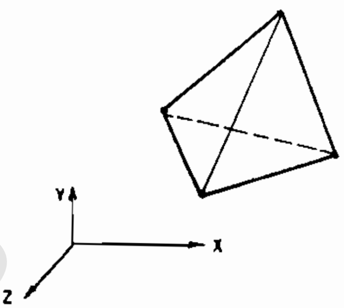


Figure 6.12

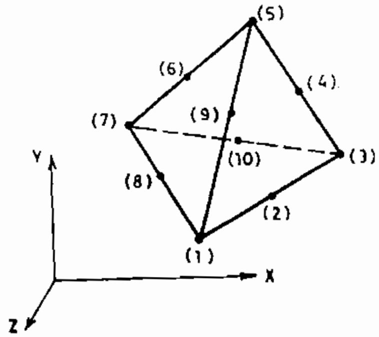


Figure 6.13

which has a dimension of 3 × 30.

Rectangular Solid

The number of nodes in this case is 8 as shown in Figure 6.14. The displacement function is trilinear and given by

$$\underline{\mathbf{P}} = \begin{bmatrix} 1 & x & y & z & xy & yz & zx & xyz & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\text{similar}} & \ldots\ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & \boxed{\text{similar}} \end{bmatrix} \tag{6.49}$$

which has a dimension of 3 × 24.

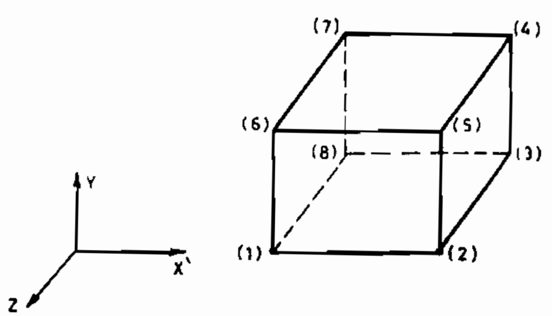


Figure 6.14

6.3 INFLUENCE LINES

6.3.1 Introduction

In the design of a structural element, the designer looks for the greatest stresses applied on the element in order to provide sufficient strength to withstand the maximum stresses. Structures subjected to static or stationary loading are easy to design since the critical section can be found directly from the internal action diagrams. However, for structures subjected to moving loads, as in bridges, it is necessary to investigate the positions of the moving loads which result in maximum stresses. Moreover, the positions of the moving loads which cause maximum normal stresses could be different from those which cause maximum shearing stresses. Instead of solving such problems by rigeorous analysis considering all possible moving loads positions, the problem can greatly be simplified by studying a unit moving load. The variation of a specific internal action at a certain section can be plotted according to the position of the unit load. From these diagrams, which are called influence lines, one can determine the position of the moving loads which cause maximum stresses at a specific section.

6.3.2 Definition

The influence line is a diagram, its ordinate at any point along the structure, gives the magnitude of a specific force function due to a unit load at that point.

For example, the influence line of the reaction R_a of a simple beam is shown in Figure 6.15. The ordinate at any point along this diagram gives the reaction R_a when a unit load is applied at that point. For statically determinate structures, it is very easy to construct the influence lines, as will be illustrated by some numerical examples. However, the construction of influence lines for statically indeterminate structures is more involved.

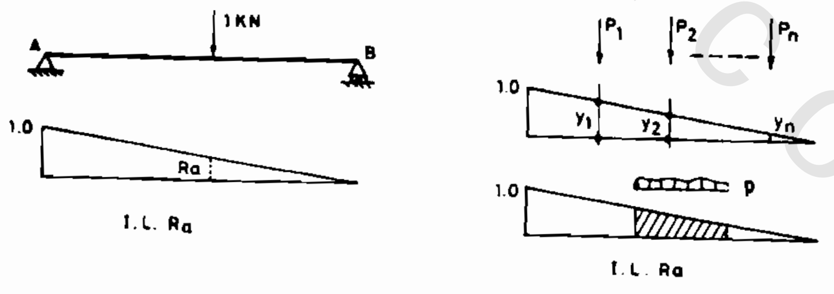


Figure 6.15

The process of determining the maximum internal action caused by the moving loads comes after the construction of the influence lines. If a set of concentrated loads is moving along the beam as shown in Figure 6.15, then for a selected position, the reaction at A is obtained from

$$R_a = P_1\, y_1 + P_2\, y_2 + \ldots + P_n\, y_n \tag{6.50}$$

On the other hand, if the moving load is distributed as shown in Figure 6.15, then the reaction for a certain load position would be

$$R_a = \int (P\, dx)\, y \tag{6.51}$$

For a uniformly distributed moving load, the reaction is the load intensity times the occupied area from the influence line.

6.3.3 Examples for Statically Determinate Structures

In this section, numerical examples on the construction of influence lines for statically determinate structures are given. These examples serve as bases for the next section.

Example 6.4

Determine the influence lines of the reactions at A, B, and C for the beam shown in Figure 6.16. From these influence lines, determine the influence lines of the shearing force and bending moment at the middle of span AB.

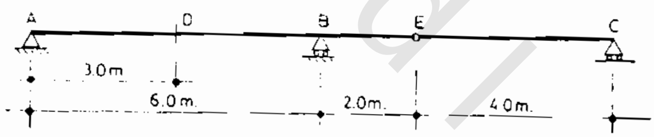


Figure 6.16

Solution

The influence lines of the reactions are determined by finding the reactions when a unit load takes various positions on the beam. If, for example, the unit load is located at A, the value of R_a is unity. If the unit load is located at B, the value of R_a is zero. If the unit load is positioned at E, the value of R_a is $(-\frac{1}{3})$. When the unit load is positioned at C, the value of R_a is zero. The influence lines of R_a, R_b, and R_c are shown in Figure 6.17.

The construction of the influence lines for the reactions eases the determination of the influence lines for the internal actions. The shear force at section D equal to R_a if the unit load is moving between D and C, and equals to $(-R_b)$ if the unit load is moving between A and D. Similarly, the moment at D equals to $(3R_a)$ if the unit load is moving between D and C, and equals to $(3R_b)$ if the load is moving between A and D. The influence lines of A_y and M_z at section D are shown in Figure 6.18.

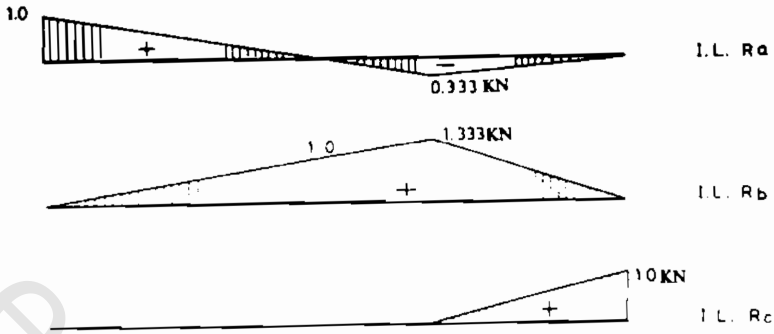


Figure 6.17

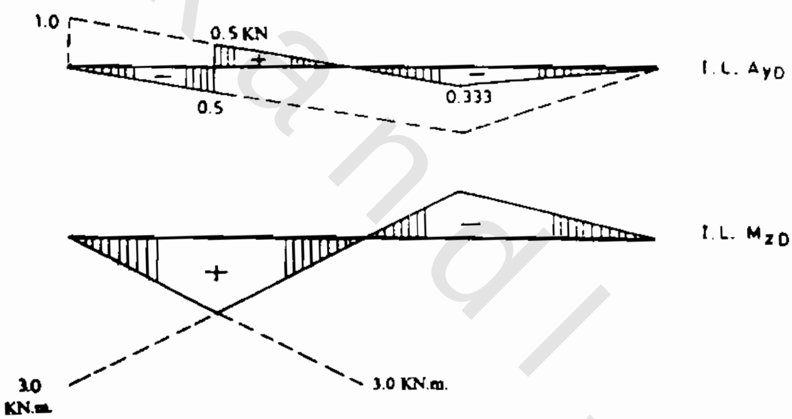


Figure 6.18

Example 6.5

Determine the influence lines for the shear force and bending moment at section s for the simple span bridge shown in Figure 6.19.

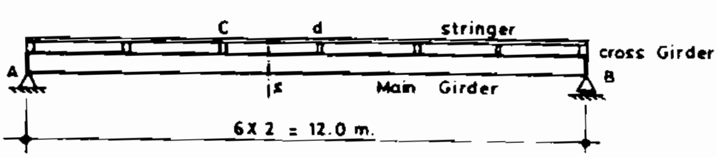


Figure 6.19

Solution

This is a typical structural system for simple span bridges. The moving loads' effects are transmitted to the main girders through the stringers and cross girders. Therefore, the influence of the unit load on the main girder is only through these cross girders. The influence lines for shear force and bending moment are determined from the influence lines of the reactions at A and B.

When the unit load is moving between d and B, $A_{yc} = 2R_a$, and $M_{zc} = 4R_a$. When the unit load is moving between A and C, $A_{yd} = R_b$, and $M_{zd} = 6R_b$. The influence lines of A_{ys}, and M_{zs} are as shown in Figure 6.20.

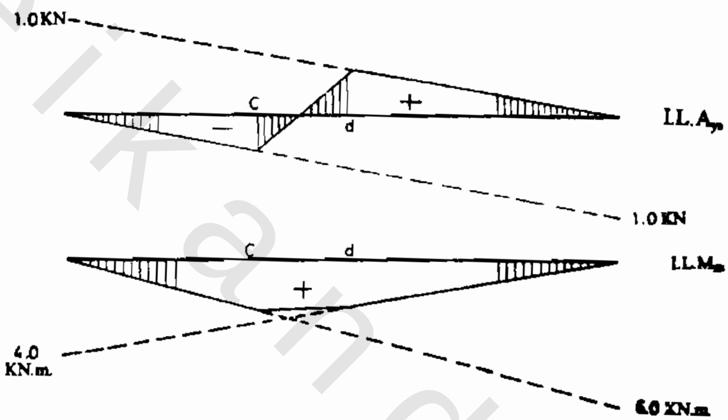


Figure 6.20

Example 6.6

Determine the influence lines for members 1-2, 3-5, 4-6, and 3-6 for the plane truss shown in Figure 6.21.

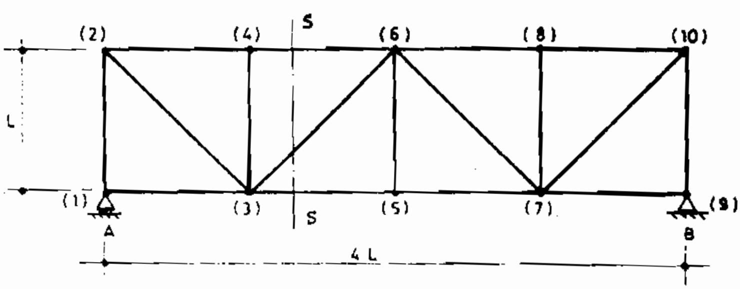


Figure 6.21

Solution

The effect of the moving unit load is transmitted through the floor system to the truss joints. If the unit load is moving along the lower chord of the truss, then its effect is transmitted to joints 1, 3, 5, 7, and 9.

From the equilibrium of joint (1) it is obvious that the influence line of member 1-2 is the same as R_a. By studying the equilibrium of the part on the left hand side of section s-s, it is obvious that when the unit load is moving along the right side of joint s, one has

$$A'_{x3-5} = +2R_a \text{ (tension)}$$

$$A'_{x4-6} = -R_a \text{ (compression)}$$

$$A'_{x3-6} = -1.414\,R_a \text{ (compression)}$$

If the unit load is moving towards the left side of joint 3, the forces in the members are

$$A'_{x3-5} = +2R_b \text{ (tension)}$$

$$A'_{x4-6} = -3R_b \text{ (compression)}$$

$$A'_{x3-6} = +1.414\,R_b \text{ (compression)}$$

The influence lines can then be constructed as shown in Figure 6.22.

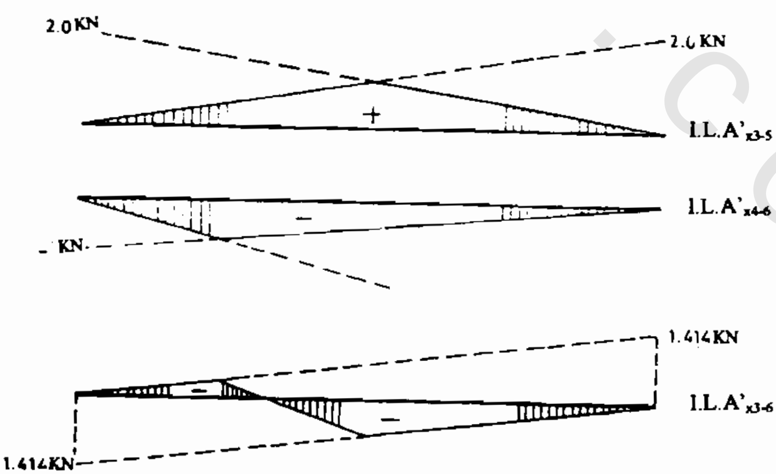


Figure 6.22

6.3.4 Examples for Statically Indeterminate Structures

It was shown in the previous section that the influence line of any internal action can easily be constructed after determining the influence lines of the reactions. A similar process is used for statically indeterminate structures, except that the reactions are usually statically indeterminate. In some other structures, internal members could be statically indeterminate. Therefore, one first has to determine the influence lines of the redundants in the statically indeterminate structures.

Example 6.7

Determine the influence lines of the reactions R_a, R_b and R_c for the beam shown in Figure 6.23.

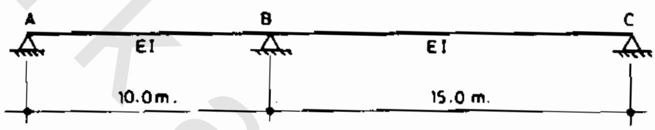


Figure 6.23

Solution

If R_a is considered as a redundant then the equation of consistent deformation in the force method is

$$\Delta_{10} + R_a f_{11} = 0 \qquad ; \qquad R_a = -\frac{\Delta 10}{f_{11}}$$

where Δ_{10} is the deflection at A due to the unit load at position x, and f_{11} is the deflection at A due to the unit load at A. However, from the reciprocal theorem one has Δ_{10} is the same as f_{x1}. Therefore, one only needs to place a unit load at A and calculate the deflections f_{x1} at various locations along the primary structure.

The deflection at various locations along the beam can be calculated by any suitable method. The conjugate beam method is perhaps the fastest method in this example. The calculated deflection is given in Figure 6.24. The influence line of R_a can now be determined by dividing all deflections by f_{11} = 833.333/EI.

In order to determine the influence lines of R_b and R_c one uses the equilibrium conditions. When R_b is written in terms of R_a one has

$$R_b = \frac{25-x}{15} - \frac{25}{15} R_a$$

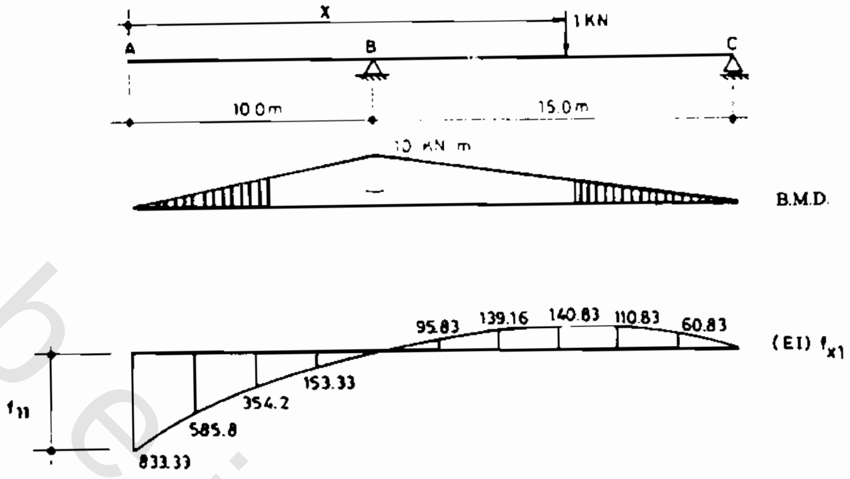


Figure 6.24

Similarly, R_c is obtained as

$$R_c = \frac{x - 10}{15} + \frac{10}{15} R_a$$

The influence lines of the reactions are shown in Figure 6.25.

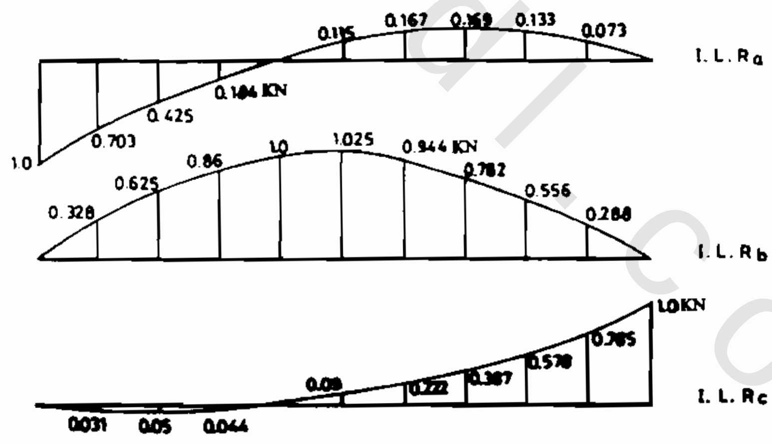


Figure 6.25

Example 6.8

Determine the influence lines of R_b and the forces in members 3-5 and 5-6 in the truss shown in Figure 6.26 (EA = 252000 kN for horizontal members, and EA = 126000 kN for diagonal members).

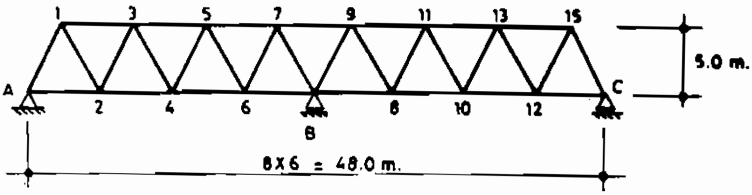


Figure 6.26

Solution

D.S.I. = $m + r - 2j = 31 + 4 - 2(17) = 1$

Select the reaction at B as the redundant. Calculating the deflection at the lower joints due to a unit load at B, the influence line of R_b can be constructed as shown in Figure 6.27.

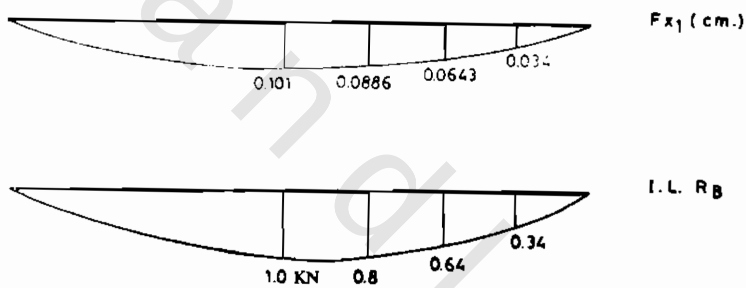


Figure 6.27

In order to determine the influence lines for the forces in members 3-5 and 5-6, one finds the value of these forces in terms of the reactions, as follows:

For member 3-5 one has

$$A'_{x3-5} = -\frac{12}{5} R_a \qquad \text{(if the load moves towards the right side of joint 4)}$$

$$A'_{x3-5} = -\frac{12}{5} R_b - \frac{36}{5} R_c \qquad \text{(if the load moves towards the left of joint 4)}$$

The relationships between R_a, R_c, and R_b are

$$R_a = -\frac{48 - x}{48} - \frac{R_b}{2} \qquad \text{for} \quad 0 \le x \le 48$$

$$R_c = -\frac{x}{48} - 2R_b \qquad \text{for} \quad 0 \le x \le 48$$

The influence lines of the members forces can thus be constructed as shown in Figure 6.28.

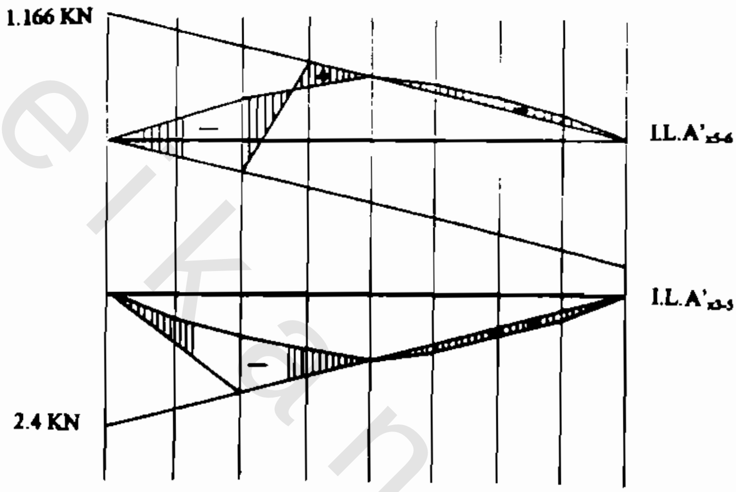


Figure 6.28

6.4 STRUCTURAL ANALYSIS USING COMPUTER

The principles of matrix structural analysis presented in chapters 5 and 6 can easily be programmed for computer use. One has to be familiar with the programming language of communicating with the computer machine, the numerical techniques needed to program matrix operations like addition, subtraction, multiplication, inverse and the solution of linear simultaneous equations. At present, the substantial skills in software techniques have resulted in many professional structural analysis and/or design softwares able to solve complicated structural analysis problems using personal computers. Programs like STAAD-III/ISDS, SAP90, ETAB among many others are characterized by easy use. In these programs, the user introduces the data of geometry, members properties, boundary conditions, applied loading, and the type of analysis or design required. It became possible by these softwares to solve large scale problems for different cases of loading in very short times. Most of the structural analysis programs have also excellent graphic capabilities to enable the user to check out the input geometry and to provide plots of deformations, internal actions, and stresses contours in the structure.

In this section, some examples are solved using program STAAD-III/ISDS to show first the verification of the results which have previously been obtained manually. Secondly, to show the effect of the current practice in assuming the distribution of floors loading on the surrounding beams on the accuracy of the structural analysis of skeleton structures.

Example 6.9

This example is the same as example 5.14 of chapter 5. The geometry of the structure is shown by the computer plot of Figure 6.29. The input file is shown in Figure 6.30. The shear force and bending moment diagrams are given in the computer plots of Figures 6.31 and 6.32. It is obvious that the results are in close agreement with the results of example 5.14.

Example 6.10

This example is the same as example 5.17. The inclined roller support can either be introduced as an inclined link as shown in Figure 6.33, or by rotating the structure such that the new x-axis coincide with the rolling plane as shown in Figure 6.34. The input files of both cases are shown in Figures 6.35 and 6.36, respectively. The shear force and bending moment diagrams of both cases are, respectively, given in Figures 6.37 and 6.38. The results are the same as obtained in example 5.17.

Example 6.11

This example is the same as example 5.19. The geometry of the structure is shown in Figure 6.39. The input file is given in Figure 6.40. The shear force and bending moment diagrams are shown, respectively, in Figures 6.41 and 6.42 which are the same as obtained in example 5.19.

Example 6.12

The space frame with a roof slab shown in Figure 6.43 can be analyzed by distributing the slab weight on the surrounding girders using the attributed slab areas as shown in Figure 6.44. In this case, the problem becomes a one dimensional skeleton space frame. The input file of this problem is as shown in Figure 6.45. The bending moment about z-axis is shown in Figure 6.46. The bending moment along girder 1-2-3 is shown in Figure 6.47.

This structure can also be solved by discretizing the slab into finite elements as shown in Figure 6.48. The input file is given in Figure 6.49. The bending moment about z-axis is shown in Figure 6.50. The bending moment along the girder of nodes 1-2-3-4-5-6-7 is shown in Figure 6.51.

By comparing Figures 6.46 and 6.50 one finds out that in the second case, the girders are subjected to torsional moment which was not discovered in the first case due to neglecting the slab-girders interaction. Moreover, these are difference in the results of the bending moment of the girder and columns.

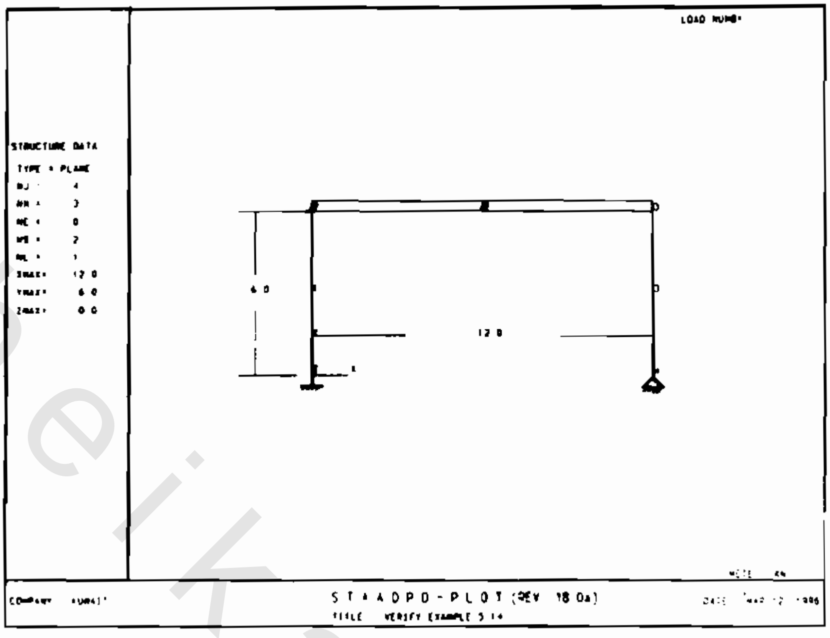


Figure 6.29

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**************************************************
*                                                *
*          S T A A D - III                       *
*          Revision 18.0a                        *
*          Proprietary Program of                *
*          RESEARCH ENGINEERS, Inc.              *
*          Date=    MAR  9, 1996                 *
*          Time=    23:27:40                     *
*                                                *
**************************************************

     1. STAAD PLANE  VERIFY EXAMPLE 5.14
     2. UNITS METER KN
     3. JOINT COORDINATES
     4. 1 0.0 0.0 ; 2 0.0 6.0 ; 3 12.0 6.0 ; 4 12.0 0.0
     5. MEMBER INCIDENCES
     6. 1 1 2 3
     7. MEMBER PROPERTY
     8. 1 3 PRIS  AX 100000.0 IZ 100000.0
     9. 2 PRIS YD 0.6 ZD 1.0 AX 200000.0 IZ 200000.0
    10. CONSTANTS
    11. E 1.0 ALL
    12. ALPHA 0.00001
    13. SUPPORTS
    14. 1 FIXED ; 4 PINNED
    15. LOADING 1
    16. JOINT LOAD
    17. 2 FX 36.0
    18. MEMBER LOAD
    19. 2 UNIFORM GY -2.5
    20. TEMPERATURE LOAD
    21. 2 TEMP 30.0 20.0
    22. PERFORM ANALYSIS

           P R O B L E M   S T A T I S T I C S
           -----------------------------------

     NUMBER OF JOINTS/MEMBER+ELEMENTS/SUPPORTS =      4/     3/     2
     ORIGINAL/FINAL BAND-WIDTH =      1/     1
     TOTAL PRIMARY LOAD CASES  =      1, TOTAL DEGREES OF FREEDOM =      7
     SIZE OF STIFFNESS MATRIX =      42 DOUBLE PREC. WORDS
     TOTAL REQUIRED DISK SPACE =    12.01 MEGA-BYTES


  ++ PROCESSING ELEMENT STIFFNESS MATRIX.             23:27:41
  ++ PROCESSING GLOBAL STIFFNESS MATRIX.              23:27:41
  ++ PROCESSING TRIANGULAR FACTORIZATION.             23:27:41
  ++ CALCULATING JOINT DISPLACEMENTS.                 23:27:41
  ++ CALCULATING MEMBER  FORCES.                      23:27:41

    23. PRINT ANALYSIS RESULTS
```

Figure 6.30

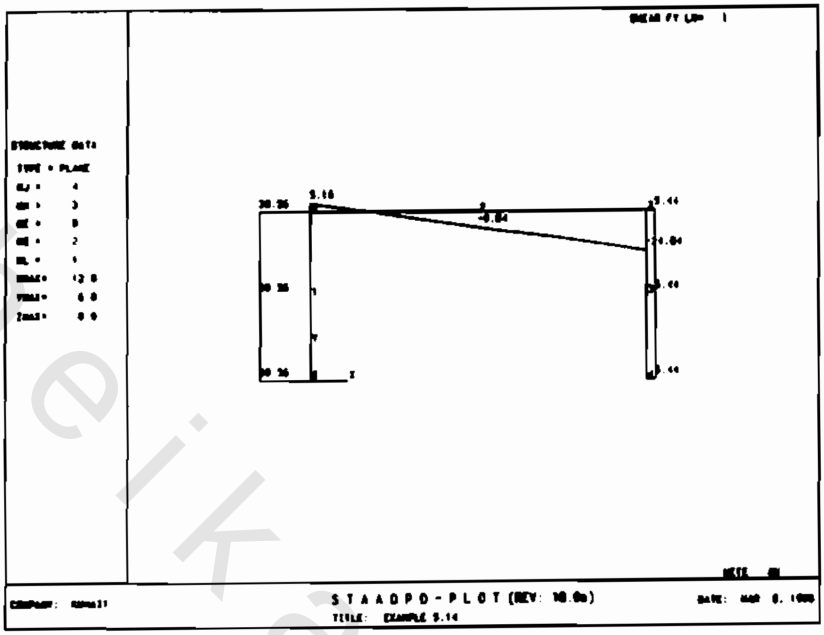

Figure 6.31

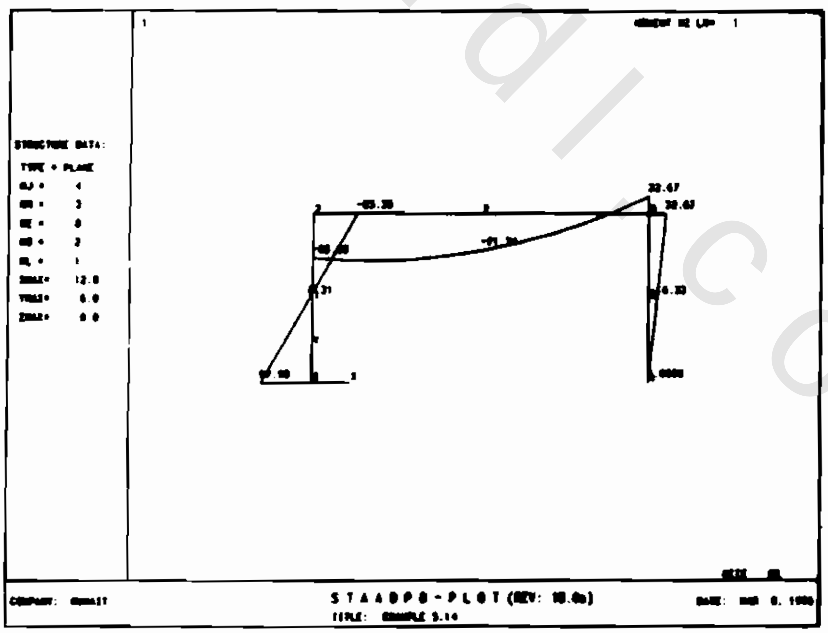

Figure 6.32

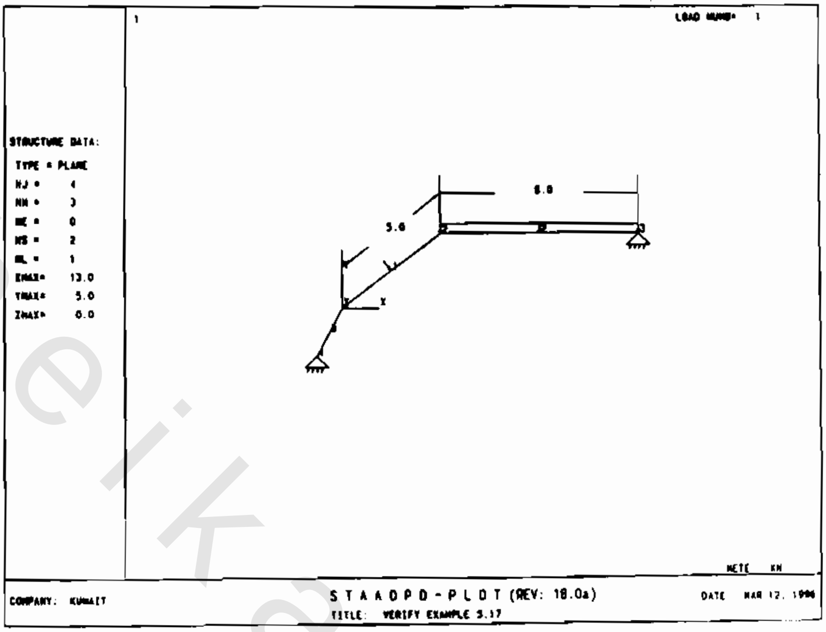

Figure 6.33

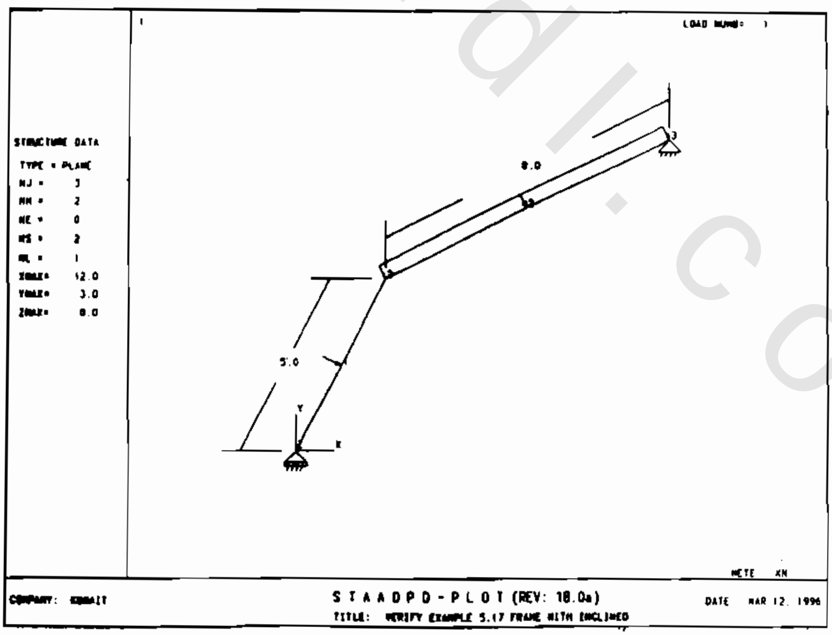

Figure 6.34

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*           S T A A D - III                      *
*           Revision 18.0a                       *
*           Proprietary Program of               *
*           RESEARCH ENGINEERS, Inc.             *
*           Date=     MAR  9, 1996               *
*           Time=     23:26:38                   *
*                                                *
**************************************************

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   3. JOINT COORDINATES
   4. 1 0.0 0.0 ; 2 4.0 3.0 ; 3 12.0 3.0 ; 4 -1.0 -2.0
   5. MEMBER INCIDENCES
   6. 1 1 2 ; 2 2 3 ; 3 1 4
   7. MEMBER RELEASE
   8. 1 3 START MZ
   9. MEMBER PROPERTY
  10. 1 TO 3 PRIS AX 100000.0 IZ 1.0
  11. CONSTANTS
  12. E 2500000000.0 ALL
  13. SUPPORTS
  14. 4 PINNED
  15. 3 PINNED
  16. LOADING 1
  17. MEMBER LOADING
  18. 2 UNIFORM GY -2.0
  19. 1 CONCENTRATED Y -8.0 2.5
  20. PERFORM ANALYSIS

        P R O B L E M   S T A T I S T I C S
        -----------------------------------

   NUMBER OF JOINTS/MEMBER+ELEMENTS/SUPPORTS =     4/     3/     2
   ORIGINAL/FINAL BAND-WIDTH =    3/     1
   TOTAL PRIMARY LOAD CASES  =    1, TOTAL DEGREES OF FREEDOM =      8
   SIZE OF STIFFNESS MATRIX =     48 DOUBLE PREC. WORDS
   TOTAL REQUIRED DISK SPACE =   12.01 MEGA-BYTES

++ PROCESSING ELEMENT STIFFNESS MATRIX.              23:26:38
++ PROCESSING GLOBAL STIFFNESS MATRIX.               23:26:38
++ PROCESSING TRIANGULAR FACTORIZATION.              23:26:38
++ CALCULATING JOINT DISPLACEMENTS.                  23:26:38
++ CALCULATING MEMBER  FORCES.                       23:26:38

  21. PLOT BENDING FILES
  22. PRINT ANALYSIS RESULTS
```
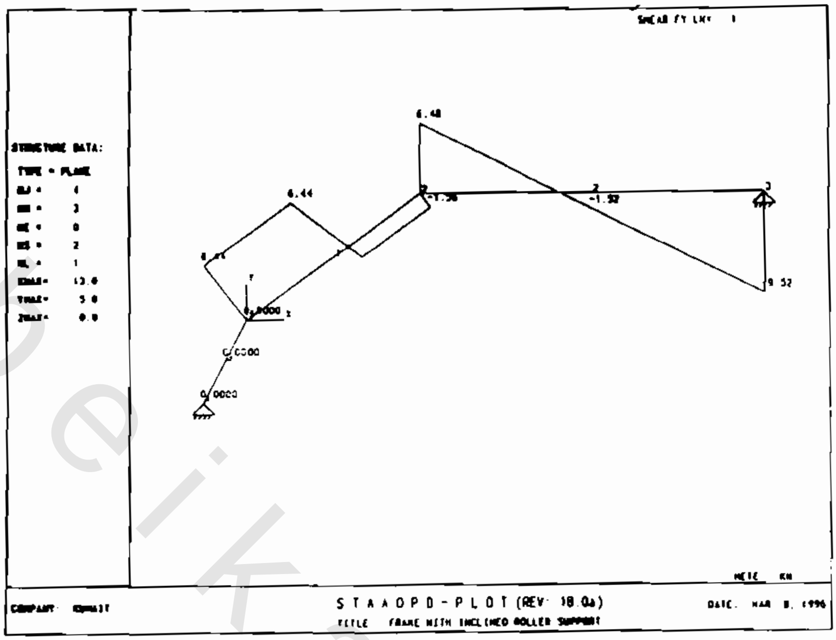

Figure 6.35

```
**************************************************
*                                                *
*             S T A A D - III                    *
*             Revision 18.0a                     *
*             Proprietary Program of             *
*             RESEARCH ENGINEERS, Inc.           *
*             Date=    MAR  9, 1996              *
*             Time=    23:24:53                  *
*                                                *
**************************************************

   1. STAAD PLANE VERIFY EXAMPLE 5.17 FRAME WITH INCLINED ROLLER SUPPORT
   2. UNITS METER KN
   3. JOINT COORDINATES
   4. 1 0.0 0.0 ; 2 4.0 3.0 ; 3 12.0 3.0
   5. MEMBER INCDENCES
   6. 1 1 2 ; 2 2 3
   7. PERFORM ROTATION Z 26.56
   8. MEMBER PROPERTY
   9. 1 2 PRIS AX 100000.0 IZ 1.0
  10. CONSTANTS
  11. E 2500000000.0 ALL
  12. SUPPORTS
  13. 1 FIXED BUT FX MZ
  14. 3 PINNED
  15. LOADING 1
  16. MEMBER LOADING
  17. 2 UNIFORM Y -2.0
  18. 1 CONCENTRATED Y -8.0 2.5
  19. PERFORM ANALYSIS

         P R O B L E M   S T A T I S T I C S
         -----------------------------------

   NUMBER OF JOINTS/MEMBER+ELEMENTS/SUPPORTS =      3/     2/     2
   ORIGINAL/FINAL BAND-WIDTH =     1/     1
   TOTAL PRIMARY LOAD CASES  =     1, TOTAL DEGREES OF FREEDOM =      6
   SIZE OF STIFFNESS MATRIX =       30 DOUBLE PREC. WORDS
   TOTAL REQUIRED DISK SPACE =    12.00 MEGA-BYTES

++ PROCESSING ELEMENT STIFFNESS MATRIX.                23:24:54
++ PROCESSING GLOBAL STIFFNESS MATRIX.                 23:24:54
++ PROCESSING TRIANGULAR FACTORIZATION.                23:24:54
++ CALCULATING JOINT DISPLACEMENTS.                    23:24:54
++ CALCULATING MEMBER  FORCES.                         23:24:54

  20. PLOT BENDING FILES
  21. PRINT ANALYSIS RESULTS
```

Figure 6.36


STRUCTURE DATA:
TYPE = PLANE
6.48
6.44
-1.52
9.52
COMPANY: KUWAIT
STAADPD-PLOT (REV: 18.0a)
TITLE FRAME WITH INCLINED ROLLER SUPPORT

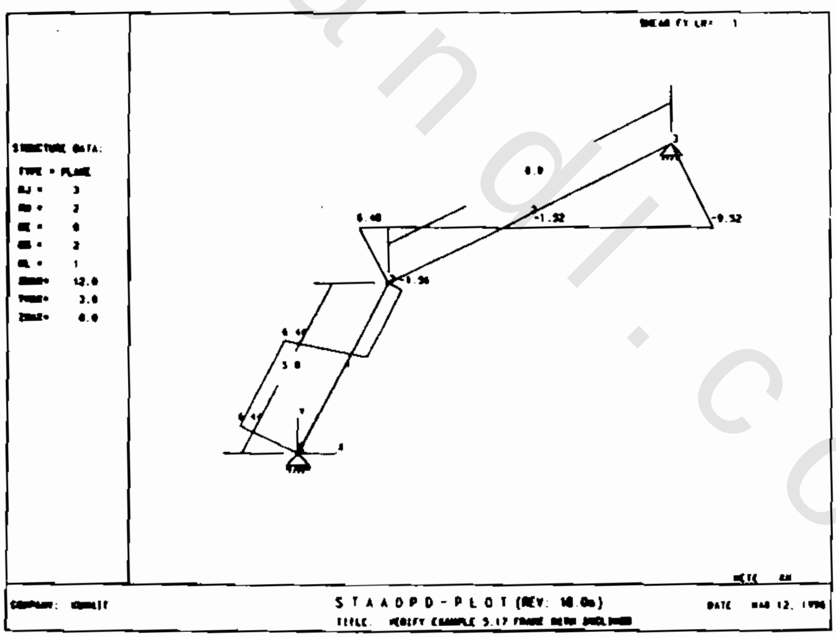
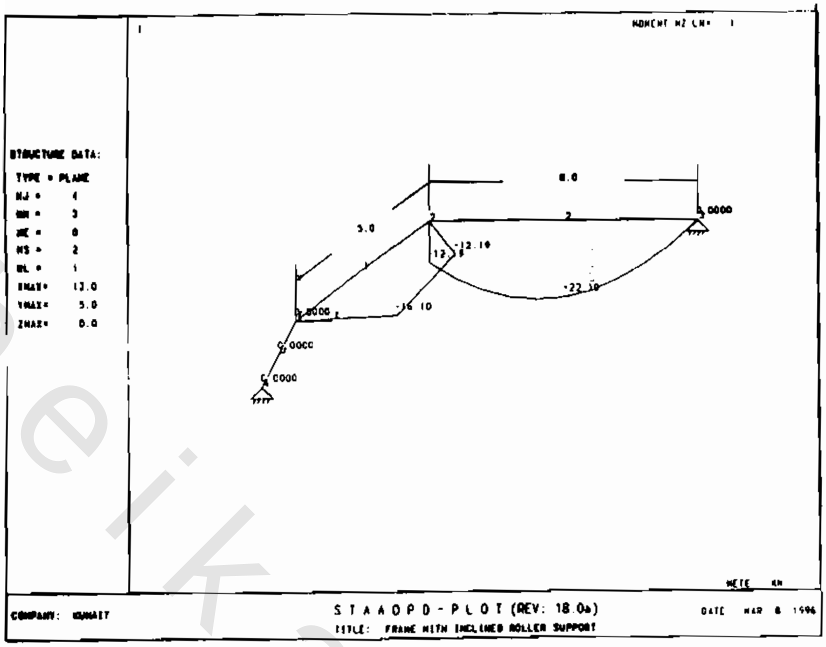

STRUCTURE DATA:
TYPE = PLANE
6.48
-1.52
-9.52
COMPANY: KUWAIT
STAADPD-PLOT (REV: 18.0a)

Figure 6.37


MOMENT MZ LN= 1
STRUCTURE DATA:
TYPE = PLANE
NJ = 4
NM = 3
NE = 0
NS = 2
NL = 1
XMAX= 13.0
YMAX= 5.0
ZMAX= 0.0
8.0
5.0
-12.19
-22.10
-16.10
COMPANY: KUWAIT
S T A A D P O - P L O T (REV: 18.0a)
TITLE: FRAME WITH INCLINED ROLLER SUPPORT
DATE MAR 8 1996

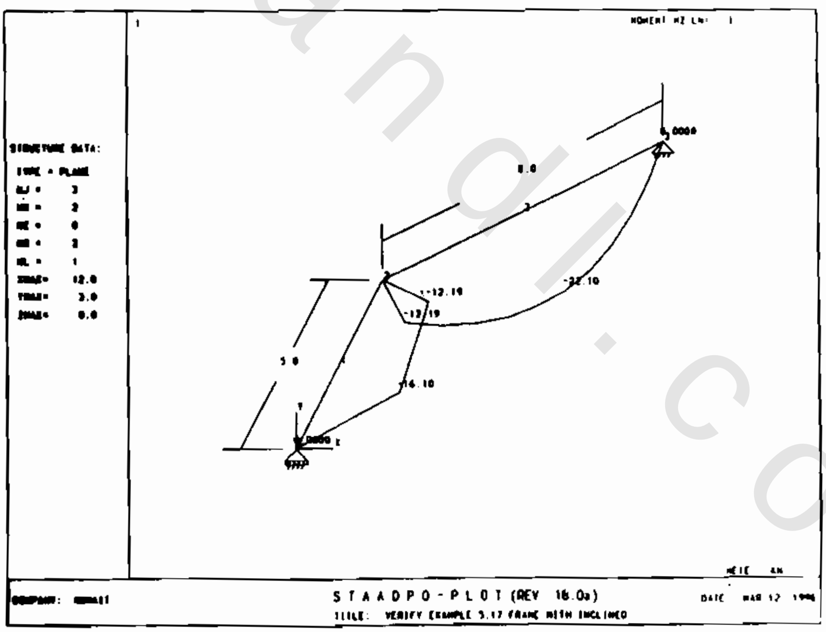

MOMENT MZ LN= 1
STRUCTURE DATA:
TYPE = PLANE
NJ = 3
NM = 2
NE = 0
NS = 2
NL = 1
XMAX= 12.0
YMAX= 3.0
ZMAX= 0.0
8.0
5.0
-12.19
-22.10
-16.10
COMPANY: KUWAIT
S T A A D P O - P L O T (REV 18.0a)
TITLE: VERIFY EXAMPLE 5.17 FRAME WITH INCLINED
DATE MAR 12 1996

Figure 6.38

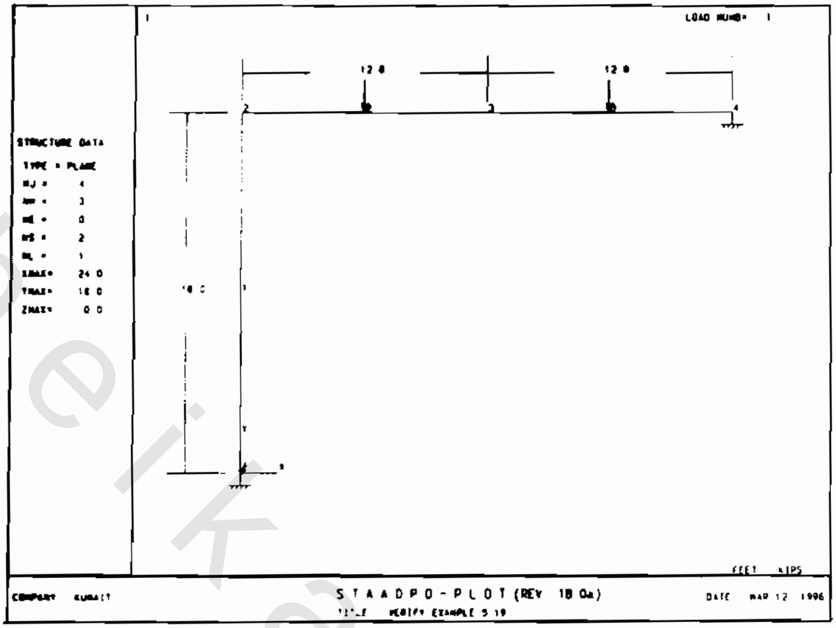

LOAD NUMB= 1
STRUCTURE DATA
TYPE = PLANE
FEET KIPS
DATE MAR 12 1996

Figure 6.39

```
**************************************************
*                                                *
*            S T A A D - III                     *
*            Revision 18.0a                      *
*            Proprietary Program of              *
*            RESEARCH ENGINEERS, Inc.            *
*            Date=     MAR  8, 1996              *
*            Time=     21:13:22                  *
*                                                *
**************************************************

   1. STAAD PLANE EXAMPLE 5.19
   2. UNITS FEET KIPS
   3. JOINT COORDINATES
   4. 1 0.0 0.0 ; 2 0.0 18.0 ; 3 12.0 18.0 ; 4 24.0 18.0
   5. MEMBER INCIDENCES
   6. 1 1 2 3
   7. MEMBER PROPERTY
   8. 1 3 PRIS AX 1.0 IZ 100.0
   9. 2 PRIS AX 2.0 IZ 300.0
  10. CONSTANTS
  11. E 1000.0 ALL
  12. SUPPORTS
  13. 1 4 FIXED
  14. LOAD 1
  15. MEMBER LOAD
  16. 2 CONCEN GY -30.0 6.0
  17. 3 CONCEN GY -30.0 6.0
  18. PERFORM ANALYSIS

       P R O B L E M   S T A T I S T I C S
       -----------------------------------

   NUMBER OF JOINTS/MEMBER+ELEMENTS/SUPPORTS =      4/    3/    2
   ORIGINAL/FINAL BAND-WIDTH =     1/     1
   TOTAL PRIMARY LOAD CASES  =     1, TOTAL DEGREES OF FREEDOM =      6
   SIZE OF STIFFNESS MATRIX  =      36 DOUBLE PREC. WORDS
   TOTAL REQUIRED DISK SPACE =    12.00 MEGA-BYTES


++ PROCESSING ELEMENT STIFFNESS MATRIX.              21:13:22
++ PROCESSING GLOBAL STIFFNESS MATRIX.               21:13:22
++ PROCESSING TRIANGULAR FACTORIZATION.              21:13:22
++ CALCULATING JOINT DISPLACEMENTS.                  21:13:23
++ CALCULATING MEMBER  FORCES.                       21:13:23

  19. PRINT ANALYSIS RESULTS
```

Figure 6.40

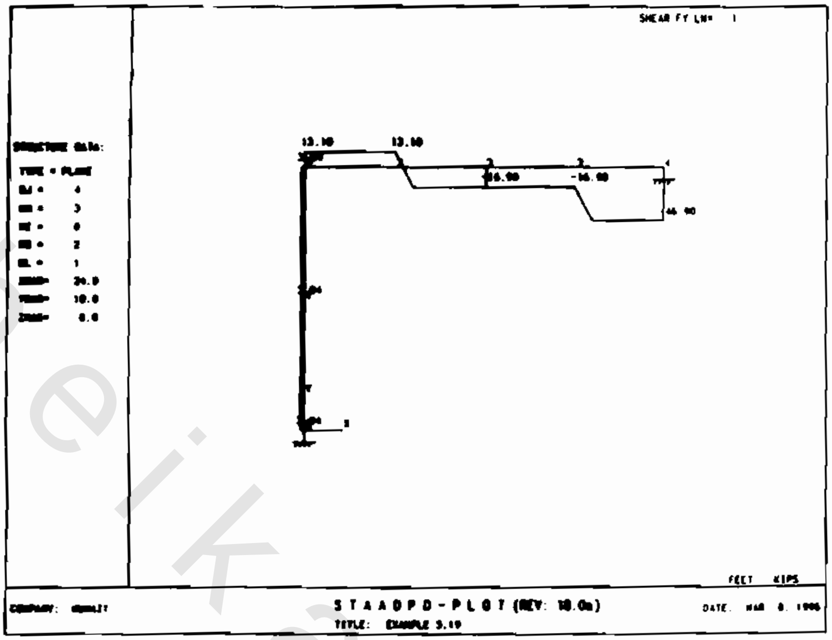


Figure 6.41

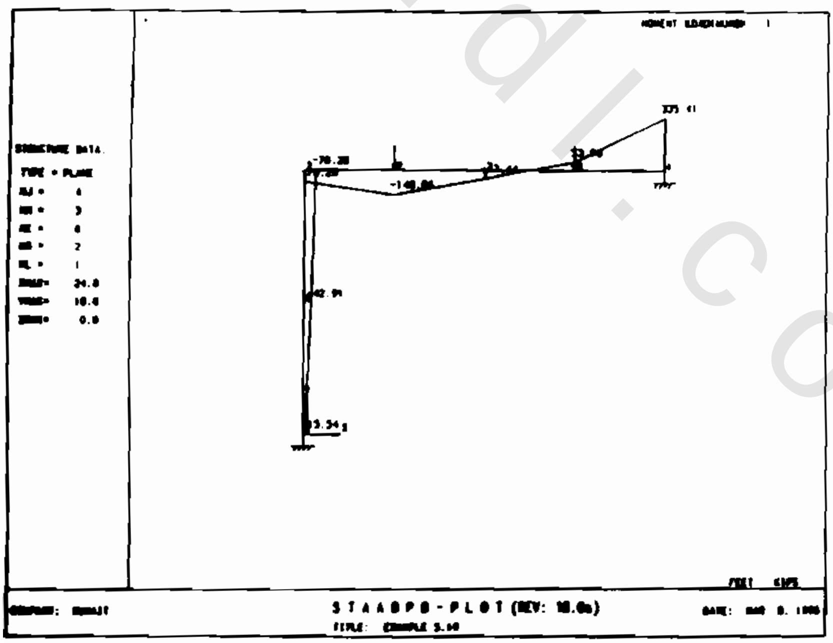


Figure 6.42

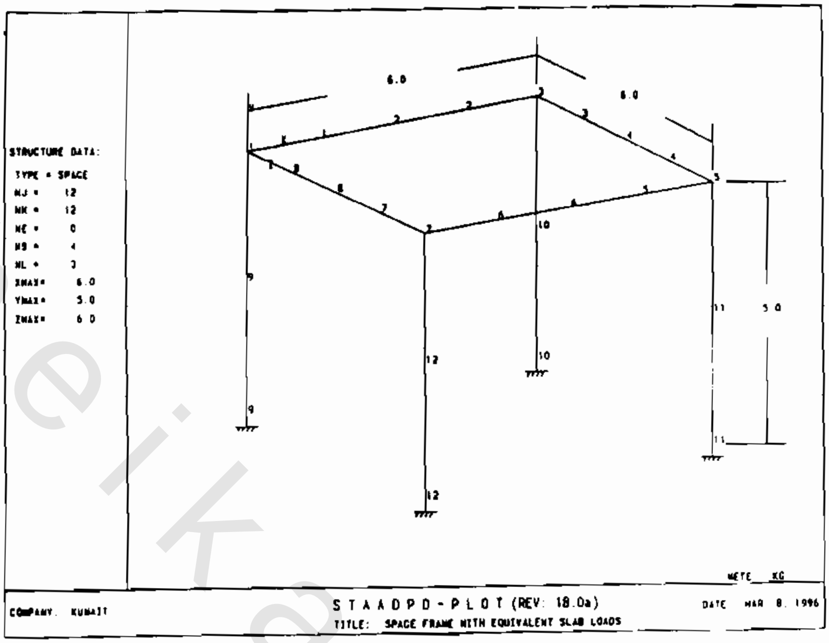


Figure 6.43

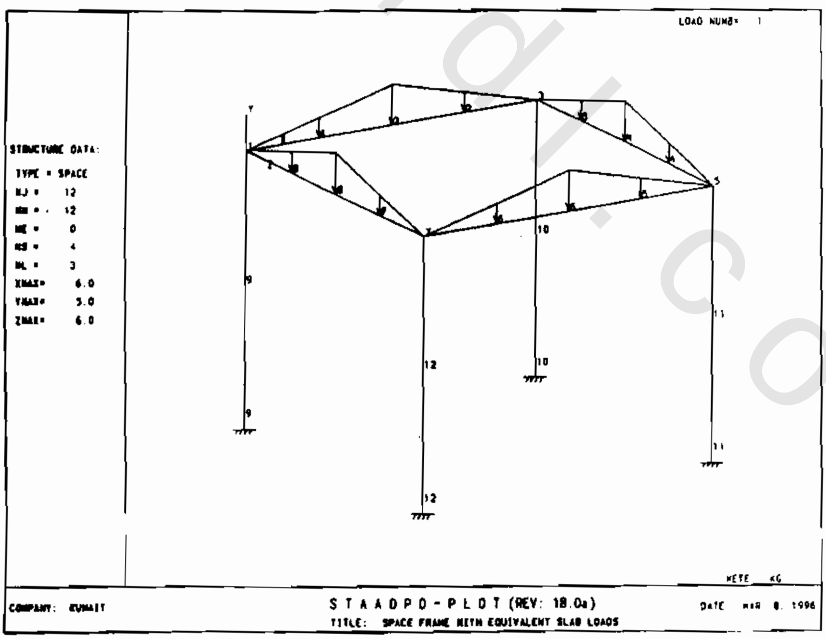


Figure 6.44

```
*****************************************************
*                                                   *
*            S T A A D - III                        *
*            Revision 18.0a                         *
*            Proprietary Program of                 *
*            RESEARCH ENGINEERS, Inc.               *
*            Date=    MAR  4, 1996                  *
*            Time=       1: 3:26                    *
*                                                   *
*****************************************************

  1. STAAD SPACE TESTING SLAB EFFECT
  2. UNIT METER KG
  3. JOINT COORDINATES
  4. 1  0.0 0.0 0.0 3 6.0 0.0 0.0
  5. 4 6.0 0.0 3.0 ; 5 6.0 0.0 6.0
  6. 6 3.0 0.0 6.0 ; 7 0.0 0.0 6.0 ; 8 0.0 0.0 3.0
  7. 9  0.0  -5.0  0.0
  8. 10 6.0  -5.0  0.0
  9. 11 6.0  -5.0  6.0
 10. 12 0.0  -5.0  6.0
 11. MEMBER INCIDENCES
 12. 1 1 2 7; 8 8 1
 13. 9 1 9 ; 10 3 10 ; 11 5 11 ; 12 7 12
 14. MEMBER PROP
 15. 1 TO 8 PRIS YD 0.5 ZD 0.4
 16. 9 TO 12 PRIS YD 0.6 ZD 0.6
 17. CONSTANT
 18. E 2500000000.0 ALL
 19. DEN 2450.0 ALL
 20. SUPPORTS
 21. 9 TO 12 FIXED
 22. LOAD 1
 23. SELFWEIGHT Y -1.0
 24. MEMB LOAD
 25. 1 3 5 7 LIN GY 0.0 -1335.0
 26. 2 4 6 8 LIN GY -1335.0 0.0
 27. LOAD 2
 28. MEMB LOAD
 29. 1 3 5 7 LIN GY 0.0 -1500.0
 30. 2 4 6 8 LIN GY -1500.0 0.0
 31. LOAD COMBINATION  3
 32. 1 1.0  2 1.0
 33. LOAD COMBINATION 4
 34. 1 1.4  2 1.7
 35. PERFORM ANALYSIS

          P R O B L E M   S T A T I S T I C S
          -----------------------------------

   NUMBER OF JOINTS/MEMBER+ELEMENTS/SUPPORTS =     12/    12/     4
   ORIGINAL/FINAL BAND-WIDTH =     8/     4
   TOTAL PRIMARY LOAD CASES  =     2, TOTAL DEGREES OF FREEDOM =     48
   SIZE OF STIFFNESS MATRIX =     864 DOUBLE PREC. WORDS
   TOTAL REQUIRED DISK SPACE =    12.03 MEGA-BYTES

++ PROCESSING ELEMENT STIFFNESS MATRIX.                    1: 3:28
++ PROCESSING GLOBAL STIFFNESS MATRIX.                     1: 3:28
++ PROCESSING TRIANGULAR FACTORIZATION.                    1: 3:29
++ CALCULATING JOINT DISPLACEMENTS.                        1: 3:29
++ CALCULATING MEMBER  FORCES.                             1: 3:29

 36. PRINT ANALYSIS RESULTS
```

Figure 6.45

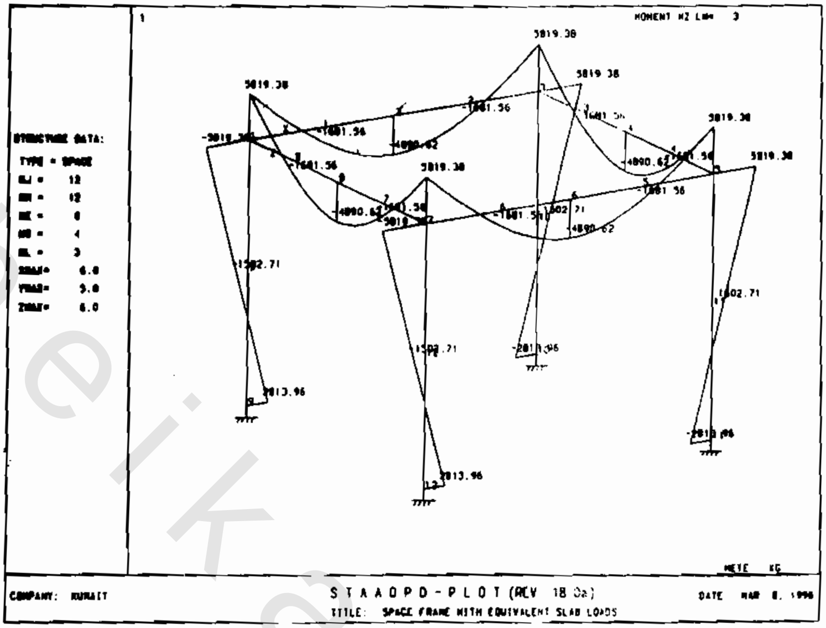


Figure 6.46

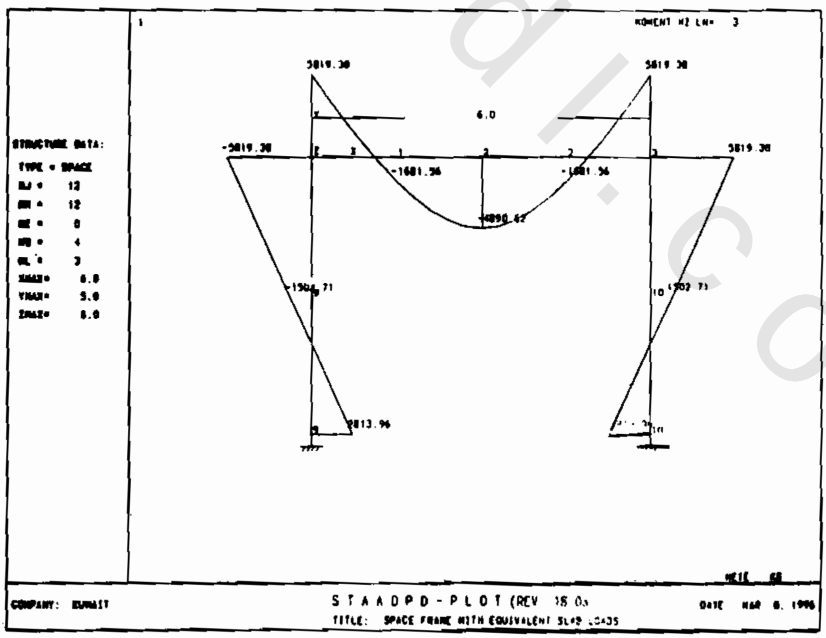


Figure 6.47

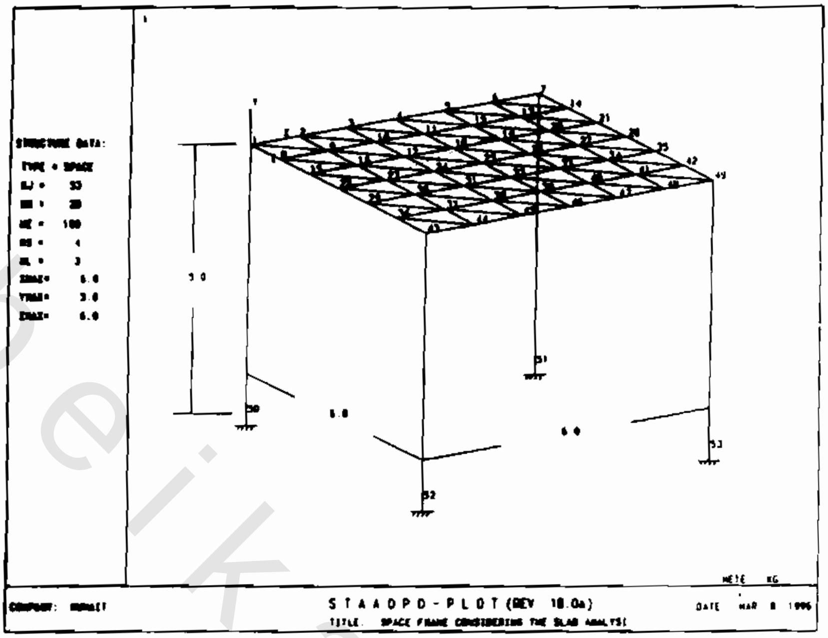


Figure 6.48

```
***************************************************
*                                                 *
*            S T A A D - III                      *
*            Revision 18.0a                       *
*            Proprietary Program of               *
*            RESEARCH ENGINEERS, Inc.             *
*            Date=    MAR  8, 1996                *
*            Time=    17:45:36                    *
*                                                 *
***************************************************

 1. STAAD SPACE  SPACE FRAME CONSIDERING THE SLAB ANALYSIS
 2. UNITS METER KG
 3. JOINT COORDINATE
 4. 1 0.0 0.0 0.0 7 6.0 0.0 0.0 ; 8 0.0 0.0 1.0 14 6.0 0.0 1.0
 5. 15 0.0 0.0 2.0 21 6.0 0.0 2.0 ; 22 0.0 0.0 3.0 28 6.0 0.0 3.0
 6. 29 0.0 0.0 4.0 35 6.0 0.0 4.0 ; 36 0.0 0.0 5.0 42 6.0 0.0 5.0
 7. 43 0.0 0.0 6.0 49 6.0 0.0 6.0
 8. 50 0.0 -5.0 0.0 ; 51 6.0 -5.0 0.0 ; 52 0.0 -5.0 6.0 ; 53 6.0 -5.0 6.
 9. MEMBER INCIDENCES
10. 1 1 2 6 ; 7 7 14 12 1 7 ; 13 1 8 18 1 7 ; 19 43 44 24
11. 25 1 50 ; 26 7 51 ; 27 43 52 ; 28 49 53
12. ELEMENT INCIDENCES
13. 29 1 2 9 TO 34
14. 35 8 1 9 TO 40
15. 41 8 9 16 TO 46
16. 47 15 8 16 TO 52
17. 53 15 16 23 TO 58
18. 59 22 15 23 TO 64
19. 65 22 23 30 TO 70
20. 71 29 22 30 TO 76
21. 77 29 30 37 TO 82
22. 83 36 29 37 TO 88
23. 89 36 37 44 TO 94
24. 95 43 36 44 TO 100
25. CONSTANTS
26. E 2500000000.0 ALL
27. DEN 2450.0 ALL
28. MEMB PROP
29. 1 TO 24 PRIS YD 0.5 ZD 0.4
30. 25 TO 28 PRIS YD 0.6 ZD 0.6
31. ELEM PROP
32. 29 TO 100 THICK 0.10
33. SUPPORTS
34. 50 TO 53 FIXED
35. LOAD 1 DEAD LOAD
36. SELFWEIGHT Y -1.0
37. ELEM LOAD
38. 29 TO 100 PR GY -200.0
39. LOAD 2 LIVE LOAD
40. ELEM LOAD
41. 29 TO 100 PR GY -500.0
42. LOAD COMBINATION 3
43. 1 1.0 2 1.0
44. PERFORM ANALYSIS

          P R O B L E M   S T A T I S T I C S
          -----------------------------------

   NUMBER OF JOINTS/MEMBER+ELEMENTS/SUPPORTS =    53/   100/     4
   ORIGINAL/FINAL BAND-WIDTH =    49/     8
   TOTAL PRIMARY LOAD CASES  =     2, TOTAL DEGREES OF FREEDOM =    294
   SIZE OF STIFFNESS MATRIX =   14112 DOUBLE PREC. WORDS
   TOTAL REQUIRED DISK SPACE =    12.69 MEGA-BYTES

++ PROCESSING ELEMENT STIFFNESS MATRIX.               17:45:38
++ PROCESSING GLOBAL STIFFNESS MATRIX.                17:45:38
++ PROCESSING TRIANGULAR FACTORIZATION.               17:45:39
++ CALCULATING JOINT DISPLACEMENTS.                   17:45:40
++ CALCULATING MEMBER  FORCES.                        17:45:40

  45. PRINT ANALYSIS RESULTS
```

Figure 6.49

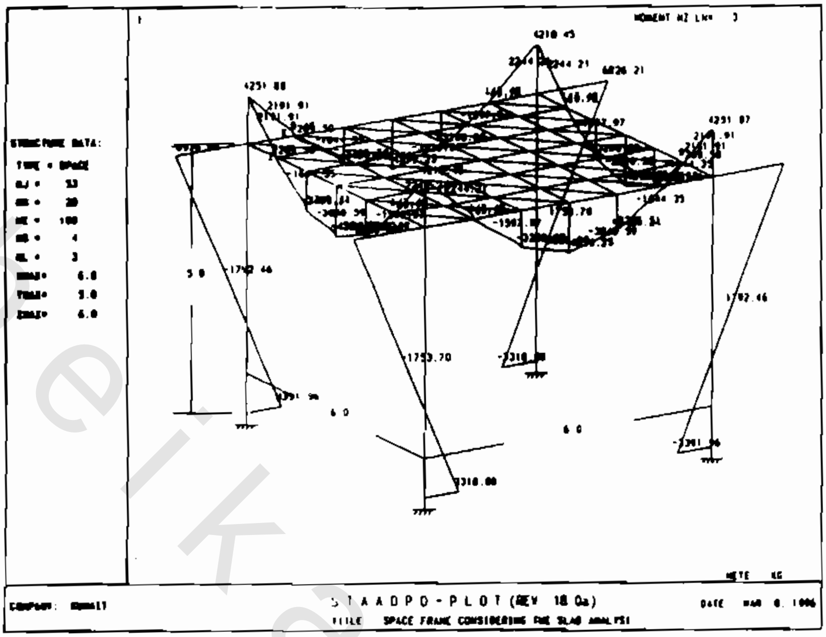


Figure 6.50

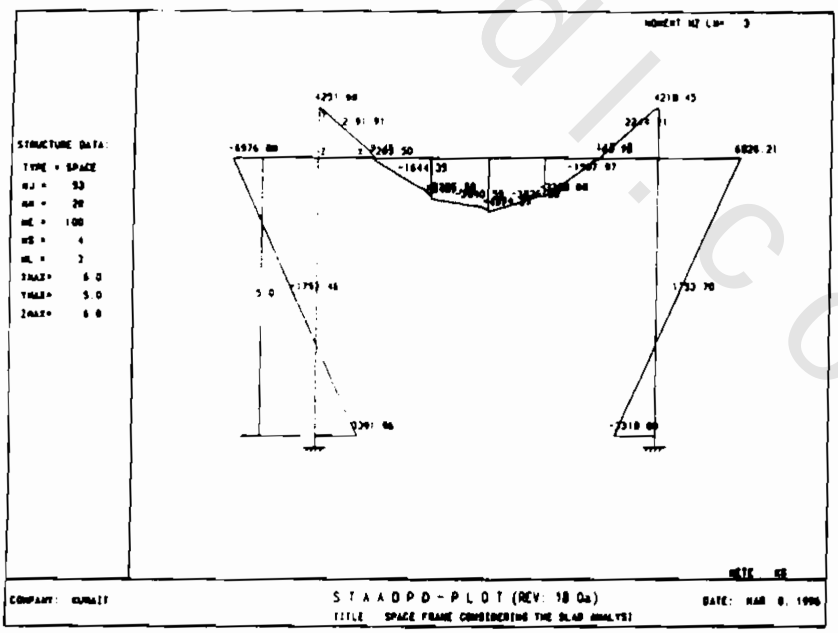


Figure 6.51

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Answers to Exercises

Chapter 2

2. (a) 60.355×10^{-5} m (↓) (b) 306.274×10^{-5} m (↓)

3. (a) $\dfrac{PL^3}{3}$ (b) -2.8336 (↓) (c) $-0.70833\ PL^3$ (↓)

9. 0.00779 m (↓)

10. $\theta_D = 26.67 \times 10^{-5}$ rad , $\Delta_C = 938.67 \times 10^{-5}$ m (↓)

Chapter 3

2. Elastic Centre at D

$I_x = 4096/EI$; $I_y = 2730.67/EI$; $I_{xy} = -2048/EI$
$M_A = -166.4$ kN.m ; $M_B = +35.46$ kN.m ; $M_C = -54.13$ kN.m.

5. $R_B = 2.829$ kN (↓) ; $\Delta_B = 0.2829$ cm (↓)

6. (a) $M_B = 46.42$ kN.m (b) $M_B = 47.695$ kN.m (c) $M_B = -0.0766$ kN.m

7. $F_{BD} = 22.937$ kN ; $M_C = +1.2$ kN.m
$F_{AX} = 1.45$ kN (→) ; $M_B = -8.7$ kN.m
$\Delta_C = 0.0028$ cm

8. $x_1 = R_A = 9.43$ Kips ; $\Delta_{10} = -0.2$ ft ; $f_{11} = 0.03$ ft ; $\Delta_1 = 1$ inch

10. $M_B = 3525.82$ kN.m ; $M_C = -4598.7$ kN.m

11. $x_1 = F_{CF} = -2/59$ kN ; $x_2 = R_c = 23.87$ kN (↑)
$\Delta_{10} = -\dfrac{34.15 \times 5}{EA}$; $\Delta_{20} = -\dfrac{93.23 \times 5}{EA}$
$F_B = 10$ kN ; $F_{BC} = 10$ Kn ; $F_{CB} = 20$ kN

12. $x_1 = R_{AX} = -1.42$ kN (←) ; $x_2 = R_{AY} = -0.284$ kN (↓)
$\Delta_{10} = 0.0045$ ft ; $\Delta_{20} = 0.00225$ ft

13. (a) $\Delta_B = 1.867$ in (↓) ; $\Delta_C = 1.15$ in (↓)
$R_C = 1.133$ kips (↑) ; $R_B = 3.734$ Kips (↑)
(b) $M_B = -6.44$ K.ft ; $M_C = -26.59$ K.ft

14. $x_1 = F_{BG}$; $x_2 = F_{GD}$
$x_3 = R_{CX}$; $x_4 = R_{CY}$
$F_{AB} = 0.32$ kN ; $F_{BC} = 5.09$ kN
$F_{CD} = -0.64$ kN ; $F_{DE} = -0.32$ kN etc...

16. $x_1 = A_{yCD}$; $x_2 = A_{xCD}$; $x_3 = M_{zCD}$
$x_4 = A_{yBE}$; $x_5 = A_{xBE}$; $x_6 = M_{zBE}$

18. Elastic centre at 7.5 ft from AD.

$A = 16/EI$; $I_x = 252/EI$; $M_i = 16.25 + 3.124\,y$
$M_A = -7.85$ K.ft ; $M_B = -29.28$ K.ft

19. $x_1 = F_{BD} = -8.536$ Kips
$\Delta_{10} = 617.95/EA$; $f_{11} = 72.39/EA$
$F_{AD} = -13.97$ Kips ; $F_{AB} = 6.03$ Kips

24. $x_1 = R_C = 50$ kN ($\leftarrow$)
$\Delta_{10} = \dfrac{-1207.1}{EA}$; $f_{11} = \dfrac{24.142}{EA}$; $F_{AB} = -30$ kN

25. $x_1 = R_{BX} = -6.653$ Kips ; $x_2 = R_{By} = 0.54$ Kips
$M_C = -47.03$ K.ft ; $M_A = -13.98$ K.ft

27. $M_B = 570.9$ kN.m. ; $M_C = -1030.46$ kN.m.

29. At elastic centre $x_1 = 4.581$ kN ($\leftarrow\,\rightarrow$) ; $x_2 = 0$; $x_3 = -25.9$ kN.m
$I_x = 268.19/EI$; $I_y = 320/EI$
$M_A = +5.613$ kN.m ; $M_D = -11.035$ kN.m.

30. For $x_1 = F_{AD}$; $x_2 = R_E$

$$\underline{\Delta}_0 = \begin{bmatrix} -1866.67/EI \\ -864/EI \end{bmatrix} \quad ; \quad [f] = \frac{1}{EI}\begin{bmatrix} 146.88 & 28.8 \\ 28.8 & 34.56 \end{bmatrix}$$

$F_{AB} = 1.34$ kN ; $F_{AC} = 10.33$ kN ; $F_{CD} = -1.34$ kN
$F_{DE} = +12.44$ kN ; $F_{BE} = -15.55$ kN ; $F_{BC} = -17.22$ kN
$F_{AD} = +17.22$ kN

31. For $x_1 = F_{DX} = R_{DX} = -16.495$ kN ; $x_2 = R_{Dy} = 46.726$ kN
$x_3 = M_D = -169.695$ kN.m

$$\underline{\Delta}_0 = \begin{bmatrix} -28.6 \\ -48.67 \\ 2.5 \end{bmatrix} , \quad [\underline{f}] = \begin{bmatrix} 1.3533 & 0.8 & -0.08 \\ 0.8 & 1.1067 & -0.06 \\ -0.08 & -0.06 & 0.006 \end{bmatrix}$$

32. $$[f] = \frac{20^3}{EI}\begin{bmatrix} 0.4167 & 0.25 \\ 0.25 & 1.333 \end{bmatrix} \; ; \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3.662 \\ 19.43 \end{bmatrix} \begin{matrix} (\leftarrow) \\ (\uparrow) \end{matrix}$$

$x_1 = R_{Dx}$, $x_2 = R_{Dy}$, $M_A = 25.35$ K.ft , $R_{Ay} = 20.56$ K.ft ($\uparrow$)

34. $x_1 = R_B$, $x_2 = R_C$
$R_B = 0.768$ Kips (↑) , $R_C = 0.44$ Kips (↑)
$R_A = 0.12$ Kips (↓) , $R_D = 0.089$ Kips (↓)

35. $x_1 = R_D = 1.0285$ Kip (↑) , $x_2 = F_{FC} = -6.465$ Kips
$\Delta_{10} = -22.4 \times 10^{-3}$, $\Delta_{20} = 22.8 \times 10^{-3}$

36. (a) $K_{AB} = K_{BA} = 0.447$, $C_{AB} = C_{BA} = 0.5928$

(b) $M_{FAB} = -19.9$ K.ft , $M_{FBA} = -26$ K.ft

37. $A = \frac{24}{EI}$; $I_x = 160/EI$, $I_y = 197.33/EI$
$M_i = 2.22 + 0.23\,y$
$M_B = M_E = -5.78$ kN.m ; $M_A = M_F = 1.42$ kN.m
$M_C = M_D = 3.01$ kN.m

40. $F_{AD} = 0.747$ kN , $F_{CB} = -5.5$ kN , $\Delta_C = 38.65/EA$
$R_{AY} = 6.25$ kN ; $R_{AX} = 0.6$ kN (←)

41. $M_A = 30.125$ K.ft ; $M_B = -60.25$ K.ft
$R_B = 30.295$ Kips , $\theta_A = 0$ from B.M.D.

42. $x_1 = F_{BD} = 1.65$ Kips , $x_2 = F_{AC} = 4.128$ Kips
$M_B = -15.62$ K.ft , $M_C = -15.84$ K.ft
$\Delta_{10} = -0.81778$ ft , $\Delta_{20} = -1.3308$ ft

43. $M_B = -10.97$ kN.m ; $M_C = -5.197$ kN.m
$M_A = -19.133$ kN.m

44. Elastic center at A

$I_x = 369/EI$; $I_y = 1584/EI$, $I_{xy} = 378/EI$
$M_i = -2.496\,y + 2.436\,x$
$M_B = -14.976$ kN.m ; $M_C = 14.256$ kN.m
$M_D = -17.28$ kN.m

Chapter 4

1. $M_{AB} = 1865.53$ kN.m $\theta_B = -0.00603$
$M_{BA} = 397.32$ kN.m $\theta_C = 0.003455$
$M_{CB} = -20$ kN.m $\theta_D = 0.0033569$ $\Delta_D = 0.00676$ m (↑)

2. $M_{AB} = 14.76$ kN.m , $M_{BA} = -22.26$ kN
$M_{BC} = -0.233$ kN.m , $M_{CB} = -24.99$ kN.m
$M_{DC} = -21.24$ kN.m , $M_{DE} = 11.24$ kN.,

4. M_{AB} = -68.65 kN.m M_{BA} = -142.92 kN.m
M_{BC} = 112.92 kN.m M_{CB} = 265.47 kN.m
M_{CD} = -265.47 kN.m

5. M_{AB} = 11.98 kN.m , M_{BA} = -12.56 kN.m
M_{BC} = 12.56 kN.m , M_{CB} = -10.58 kN.m
M_{CE} = -7.7 kN.m , M_{EC} = -25.33 kN.m
M_{CD} = 18.28 kN.m

6. M_{DC} = -27.9 kN.m , M_{DA} = -21.84 kN.m
M_{AD} = -10.77 kN.m , M_{DE} = 49.74 kN.m
M_{ED} = -44.65kN.m , M_{EB} = 25.7 kN.m
M_{BE} = 12.53 kN.m , M_{EF} = 18.95 kN.m
M_{FE} = -22.5 kN.m , M_{FG} = 22.5 kN.m

7. M_{DC} = -18.917 kN.m
M_{DA} = 9.7 kN.m , M_{AD} = 4.77 kN.m
M_{DF} = 6.2 kN.m , M_{DE} = 3.01 kN.m
M_{ED} = -0.415 kN.m , M_{BE} = -8.92 kN.m

11. M_{AB} = -91.67 kN.m , M_{BA} = -126.1 kN.m
M_{CB} = -96.17 kN.m , M_{CD} = 51.44 kN.m
M_{DC} = -2.92 kN.m , M_{CE} = 44.73 kN.m
M_{EC} = -213.98 kN.m , M_{FE} = -218.85 kN.m , M_{GF}= 119.75 kN.m

13. M_{BA} = -56.26 kN.m
M_{BE} = 2.72 kN.m
M_{EB} = 1.36 kN.m
M_{BC} = 54.54 kN.m
M_{CB} = -16.36 kN.m , M_{DC} = 8.18 kN.m

14. (a) S_A = 0.446 EI , C_{AB} = 0.593

(b) K_{AB} = 0.5643 EI , C_{AB} = 0.59276

15. θ_B = -6.25×10^{-4} rad , θ_E = θ_D = 3.125×10^{-4} rad
M_{AB} = -2.09 K.ft , M_{BA} = -7.29 K.ft
M_{BC} = 16.66 K.ft , M_{CB} = -21.35 K.ft
M_{BD} = -4.68 K.ft

16. M_{AB} = 19.09 kN.m ; M_{BA} = 11.04 kN.m
M_{CB} = -5.187 kN.m , M_{CD} = 5.187 kN.m

18. EI = 10000 $K.ft^2$
Δ_B = 0.5 inch (↓)
θ_B = -0.00844 rad
θ_C = 0.022544 rad
M_{AB} = 17 K.ft , M_{BA} = -43.13 K.ft

19. $\theta_B = -6.365/EI$; $\theta_C = 4.52/EI$, $\Delta = 18.17/EI$
$M_{AB} = 9.316$ kN.m ; $M_{BA} = -9.947$ kN.m , $M_{CB} = -11.33$ kN.m
$M_{DC} = 9.07$ kN.m

20. $M_{AB} = 9.86$ K.ft ; $M_{BA} = -46.46$ K.ft
$M_{CB} = -36.71$ K.ft

21. Unknowns θ_B , θ_C , θ_E , θ_D , θ_G , Δ_G , Δ_E
Conditions $M_B = M_E = M_C = M_D = M_G = 0$
$H_{BC} + H_{ED} + H_{HG} = 0$
$H_{AB} + H_{FE} + H_{HG} = 0$

22. $\theta_C = 0$, $\theta_D = 0$, $\Delta = 720/EI$
$M_{AC} = 30$, $M_{FD} = -30$, $M_{CD} = 0$

23. (a) $M_{FAB} = -23.41$ K.ft

(b) $S_A = 0.447$ EI , $C_{AB} = 0.592$

26. $M_{FBA} = -18.1$ kN.m , $M_{FAB} = 18.1$ kN.m
$S_A = 16.824$ kN.m , $C_{AB} = 0.6238$
$M_{AB} = 13.71$ kN.m , $M_{BA} = -24.97$ kN.m

27. $M_{AB} = -10.36$ kN.m , $M_{BA} = -65.73$ kN.m
$M_{BC} = 65.73$ kN.m

28. $A_{BC} = -43.78$ kN , $A_{DC} = 26.21$ kN , $A_{DA} = 0$
$A_{AC} = -8.78$ kN

29. $M_{AB} = 21.48$ kN.m , $M_{BA} = 34.35$ kN.m , $M_{BC} = 15.64$ kN.m
$M_{CB} = 4.91$ kN.m

Chapter 5

1. (a) $\underline{D}_C = \begin{bmatrix} 0.012 \\ 0 \\ 0.00016 \end{bmatrix}$, $\underline{D}_B = \begin{bmatrix} 0.0026 \\ -0.00034 \\ 0.00174 \end{bmatrix}$

$\underline{A}'^T_{AB} = [81.2 \quad -74.24 \quad 146.9]$, $\underline{A}'^T_{BA} = [81.2 \quad 5.76 \quad 127.1]$

$\underline{A}'^T_{BC} = [-5.73 \; 81.2 \; -127.25]$, $\underline{A}'^T_{CB} = [-5.73 \; -38.77 \; 0]$

$\underline{A}'^T_{AC} = 48.48$ kN

(b) $A'^T_{AC} = 58.93$ kN

2. $D_{Ay} = -0.152 \times 10^{-3}$ m , $D_B = 10^{-3}\begin{bmatrix} 0.252 \\ 0.237 \end{bmatrix}$, $D_D = 10^{-3}\begin{bmatrix} 0.273 \\ 0.262 \end{bmatrix}$

$D_{CX} = 10^{-3} \times 0.489$ m

$A'_{AB} = 168$ kN , $A'_{AD} = 98.8$ kN , $A'_{AE} = -79$ kN , $A'_{BC} = 158$ kN

$A'_{BD} = 12.5$ kN , $A'_{BE} = 15.4$ kN

4. $D_2 = 10^{-3}\begin{bmatrix} 3.071 \\ -8.93 \end{bmatrix}$, $D_4 = 10^{-3}\begin{bmatrix} -1.2 \\ -16 \end{bmatrix}$, $D_{3y} = -1.6 \times 10^{-2}$ m

$D_{5x} = -6 \times 10^{-4}$ m , $A'_{12} = -585.8$ kN , $A'_{15} = -120$ kN , $A'_{52} = 214.2$ kN

$A'_{54} = -120$ kN , $A'_{24} = 282.9$ kN , $A'_{23} = -614.18$ kN

6. $\underline{D}_2 = \frac{1}{EI}\begin{bmatrix} -68.38 \\ -48.95 \end{bmatrix}$, $\underline{D}_3 = \frac{1}{EI}\begin{bmatrix} -115.9 \\ -10.59 \end{bmatrix}$, $\theta_4 = 83.63/EI$

$M_{AB} = 164.9$ kN.m

7. $A'_{23} = 49.45$ kN , $A'_{24} = 0$, $A'_{25} = -55.12$ kN

$A'_{12} = 24.4$ kN , $A'_{14} = 9.56$ kN , $A'_{15} = 0$

$A'_{16} = 35.6$ kN , $A'_{35} = 5.52$ kN , A34 = 7.85 kN

$A'_{36} = -9.56$ kN , $A'_{56} = 43.45$ kN , $A'_{46} = -105.6$ kN

8. $\underline{A}'_{12} = \begin{bmatrix} -5.8 \\ -8.45 \\ 79.04 \end{bmatrix}$, $\underline{A}'_{21} = \begin{bmatrix} 4.18 \\ 1.54 \\ 18.71 \end{bmatrix}$, $\underline{A}'_{23} = \begin{bmatrix} -1.54 \\ 4.18 \\ -18.7 \end{bmatrix}$

$\underline{A}'_{32} = \begin{bmatrix} -11.54 \\ -5.8 \\ 41.6 \end{bmatrix}$, $\underline{D}_2 = 10^{-5}\begin{bmatrix} 63.7 \\ -81.68 \\ 3.39 \end{bmatrix}$

9. $\underline{D}_2 = \begin{bmatrix} 2.256 \\ -1.785 \\ 3.176 \end{bmatrix}$, $A'_{21} = 6.86$ kN , $A'_{23} = -3.448$ kN , $A'_{24} = -15$ kN

11. A'_{AB} = -19.51 kN , A'_{BC} = -1.13 kN , A'_{CD} = 12.74 kN

A'_{DF} = -18.02 kN , A'_{FE} = -18.36 kN , A'_{EA} = -6.34 kN

$\underline{D}_C = \begin{bmatrix} -82.56 \\ -263.81 \end{bmatrix}$, $\underline{D}_F = \begin{bmatrix} -86.7 \\ -199.3 \end{bmatrix}$

12. $$\underline{S} = \frac{EA}{L}\begin{bmatrix} 1.353 & -0.353 & -1 & 0 \\ -0.353 & 1353 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

15. $$\underline{D}_B = \begin{bmatrix} -0.13 \\ -0.353 \\ \theta \end{bmatrix}, \quad D'_C = -0.853 \qquad \underline{A}'^{T}_{AB} = [-4.42 \quad 0.76 \quad -6.09]$$

16. $$\underline{D}_B = \begin{bmatrix} 3.75 \\ -1.52 \\ 0.232 \end{bmatrix}, \quad \underline{A}'_{AB} = \begin{bmatrix} -12.38 \\ 91.27 \\ 219.33 \end{bmatrix}, \quad \underline{A}'_{CB} = \begin{bmatrix} 109.53 \\ -14 \\ 453.57 \end{bmatrix}$$

17. $$\underline{A}'_{AB} = \begin{bmatrix} -25.5 \\ 10.28 \\ -10.97 \end{bmatrix}, \quad \underline{A}'_{BC} = \begin{bmatrix} -75.59 \\ -60.28 \\ 214.33 \end{bmatrix}, \quad \underline{A}'_{ED} = \begin{bmatrix} -75.79 \\ -60.28 \\ 214.33 \end{bmatrix}$$

18. $$\underline{A}_A = \begin{bmatrix} -25.52 \\ -22 \\ 188.14 \end{bmatrix}, \quad \underline{A}'_E = \begin{bmatrix} 24.48 \\ 22 \\ 200.27 \end{bmatrix}$$

19. $$\underline{A}_{AB} = \begin{bmatrix} -35.26 \\ -24.77 \\ 307.2 \end{bmatrix}, \quad \underline{A}'_{ED} = \begin{bmatrix} -14.72 \\ 24.77 \\ 0 \end{bmatrix}$$

20. $$\underline{D}'_A = \begin{bmatrix} -81.26 \\ 0 \\ -12.19 \end{bmatrix}, \quad \underline{D}'_B = \begin{bmatrix} 0 \\ -32.5 \\ -12.19 \end{bmatrix}, \quad \underline{D}'_C = \begin{bmatrix} -65 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{A}'_{AB} = \begin{bmatrix} -3.519 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{A}'_{CB} = \begin{bmatrix} -3.519 \\ 0 \\ 1.52 \end{bmatrix}, \quad \underline{A}'_{DB} = \begin{bmatrix} 0 \\ 9.9 \\ -16.76 \end{bmatrix}$$

21. $$\underline{D}'_B = \begin{bmatrix} -54.75 \\ -235.82 \\ 7.727 \end{bmatrix}, \quad \underline{A}'_{AB} = \begin{bmatrix} -3.04 \\ 13.1 \\ -15.55 \end{bmatrix}, \quad \underline{A}'_{CB} = \begin{bmatrix} 3.04 \\ 46.90 \\ -335.26 \end{bmatrix}$$

22. $\underline{D}'_{AS} = 10^{-4}\begin{bmatrix} 3.829 \\ 0 \\ -1.977\times 10^{-3} \end{bmatrix}$, $\underline{D}_B = 10^{-4}\begin{bmatrix} 2.08 \\ -2.4 \\ 0.1265 \end{bmatrix}$, $\underline{D}_C = 10^{-4}\begin{bmatrix} 0 \\ 0 \\ 0.437 \end{bmatrix}$

$\underline{A}'_{AB} = \begin{bmatrix} -12.87 \\ -6.4 \\ 0 \end{bmatrix}$, $\underline{A}'_{BA} = \begin{bmatrix} -12.87 \\ 1.59 \\ 12.12 \end{bmatrix}$, $\underline{A}'_{23} = \begin{bmatrix} -11.24 \\ -6.48 \\ -12.17 \end{bmatrix}$

$\underline{A}'_{32} = \begin{bmatrix} -11.24 \\ 9.52 \\ 0 \end{bmatrix}$

23. $\underline{D}_B = 10^{-3}\begin{bmatrix} 0 \\ 0 \\ -0.378 \end{bmatrix}$, $A'_{AB} = \begin{bmatrix} 0 \\ -18.7 \\ 24.9 \end{bmatrix}$, $A'_{BC} = \begin{bmatrix} 0 \\ -17.8 \\ 62.7 \end{bmatrix}$

$\underline{A}'^{T}_{CB} = [0 \quad 2.2 \quad 0]$

24. $\underline{D}_A = \begin{bmatrix} 0.134 \\ -00293 \\ 0.0213 \end{bmatrix}$, $\underline{D}_B = \begin{bmatrix} 0.0014 \\ -0.0295 \\ 0 \end{bmatrix}$

$\underline{A}'_{AB} = \begin{bmatrix} -1.05 \\ -4.875 \\ -0.75 \end{bmatrix}$, $\underline{A}'_{BA} = \begin{bmatrix} -1.05 \\ 5.125 \\ 0 \end{bmatrix}$, $\underline{A}'_{BC} = \begin{bmatrix} -5.25 \\ -1.056 \\ 0 \end{bmatrix}$

$\underline{A}_{CB} = \begin{bmatrix} -5.25 \\ 18.94 \\ -71.48 \end{bmatrix}$

25. $\underline{D}_B = 10^{-3}\begin{bmatrix} -6.98 \\ -0.134 \\ -96.58 \end{bmatrix}$, $\underline{D}_E = 10^{-3}\begin{bmatrix} -6;54 \\ -0.09 \\ -926.52 \end{bmatrix}$

$\underline{A}_{AB} = \begin{bmatrix} 42.65 \\ -232.5 \\ 2.68 \end{bmatrix}$, $\underline{A}_{CB} = \begin{bmatrix} 47.65 \\ 196.76 \\ 2.086 \end{bmatrix}$

29. $\underline{A}'_{23} = \begin{bmatrix} 29.22 \\ 9.39 \\ -1246.9 \end{bmatrix}$, $\underline{A}'_{43} = \begin{bmatrix} -105.4 \\ 37.18 \\ -1227.65 \end{bmatrix}$, $\underline{A}'_{13} = 8.89\,K$

$\underline{D}_3 = \begin{bmatrix} 0.0389 \\ -0.0703 \\ -0.042 \end{bmatrix}$, $\underline{D}_5 = \begin{bmatrix} 0.0389 \\ -4.95 \\ -0.125 \end{bmatrix}$

31. $\underline{A}'_{AB} = \begin{bmatrix} -28.517 \\ -12.55 \\ 31.97 \end{bmatrix}$, $\underline{A}'_{BA} = \begin{bmatrix} -28.517 \\ 27.44 \\ -180.919 \end{bmatrix}$

$\underline{A}'_{BC} = \begin{bmatrix} -51.28 \\ 20.84 \\ -319.08 \end{bmatrix}$, $\underline{A}'_{CB} = \begin{bmatrix} -51.28 \\ 20.84 \\ -202.1 \end{bmatrix}$, $\underline{D}_B = \begin{bmatrix} -0.019 \\ -0.0965 \\ -0.0146 \end{bmatrix}$

35. $\underline{A}'_{AB} = \begin{bmatrix} -4.03 \\ -2.21 \\ 20.14 \end{bmatrix}$, $\underline{A}'_{CB} = \begin{bmatrix} -4.03 \\ 2.2 \\ -20.14 \end{bmatrix}$, $\underline{A}'_{DA} = \begin{bmatrix} -1.95 \\ 0 \\ -15.03 \end{bmatrix}$

$\underline{D}'_B = \begin{bmatrix} 5.1317 \\ 0.0442 \\ 0 \end{bmatrix}$

36. $\underline{A}'_{AB} = \begin{bmatrix} 3.25 \\ -8.46 \\ 39.37 \end{bmatrix}$, $\underline{A}_{BD} = \begin{bmatrix} 3.25 \\ -8.46 \\ 39.37 \end{bmatrix}$, $\underline{D}_B = \begin{bmatrix} 0.516 \\ -0.1726 \\ -0.00163 \end{bmatrix}$

37. $\underline{A}'_{AB} = \begin{bmatrix} -9.55 \\ -1.35 \\ 10.12 \end{bmatrix}$, $\underline{A}'_{CB} = \begin{bmatrix} -3.65 \\ 0.44 \\ -6.448 \end{bmatrix}$, $\underline{D}_B = \begin{bmatrix} 0.0328 \\ -0.086 \\ 0.0036 \end{bmatrix}$

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