

TABLES OF THE LEGENDRE POLYNOMIALS OF COMPLEX ARGUMENT

By

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§ 1. Introduction :

In this paper we produce the double entry tables of the function $P_n(x + iy)$. The argument x , appearing in a vertical column on each page covers the range $x = 0, (0.1); 0.9$, and the argument y arranged horizontally, takes the values $y = 0.1, (0.1); 1$.

Seven decimal places are given throughout, and the error does not exceed one unit in the last figure.

§ 2. The calculation of this table is based on the formula

$$P_n(x+iy) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r y^{2r} (1-x^2)^{-r}}{(2r)!} P_n^{2r}(x) + i \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^r y^{2r+1} (1-x^2)^{-r-1/2}}{(2r+1)!} P_n^{2r+1}(x).$$

This formula is deduced from the theorem

$$P_n(x+iy) = \sum_{m=0}^{\infty} \frac{y^m}{m!} (1-x^2)^{-\frac{m}{2}} P_n^m(x)$$

which is given by the authors in (1). To give an example, choose $x = 0.2, y = 0.2$ and $n = 3$, then

$$P_3(0.2 + 0.2i) = -0.34 - 0.2599999i$$

where

$$P_3^0(0.2) = -0.28, \quad P_3^1(0.2) = -1.17575508,$$

$$P_3^2(0.2) = 2.88, \quad P_3^3(0.2) = 14.10906092$$

$$\text{and } \sqrt{6} = 2.4494897.$$

This value in the table, which is calculated by the computer I.B.M. 1620 is read as

$P_3(0.2 + 0.2 i) = - 0.3399999 - 0.2599999 i$,
and the true value is

$$P_3(0.2 + 0.2 i) = - 0.34 - 0.26 i$$

The error done by the computer does not exceed one unit in the last figure.

Interpolation in the x or y direction is performed by quadratic interpolation with a Lagrangian three-point interpolation formula [2], namely

$$f(x) = \frac{(x - a_1)(x - a_2)}{(a_0 - a_1)(a_0 - a_2)} f(a_0) + \frac{(x - a_0)(x - a_2)}{(a_1 - a_0)(a_1 - a_2)} f(a_1) \\ + \frac{(x - a_0)(x - a_1)}{(a_2 - a_0)(a_2 - a_1)} f(a_2)$$

To interpret the use of this formula, write the Lagrangian formula in the form.

$$f(x) = A_0 f(a_0) + A_1 f(a_1) + A_2 f(a_2),$$

So if we take.

$$a_0 = 0.1, a_1 = 0.2 \text{ and } a_2 = 0.3, \text{ then}$$

$$A_0 = - 0.08, A_1 = 0.96, A_2 = 0.12$$

$$f(a_0) = - 0.545 + 0.06 i,$$

$$f(a_1) = - 0.5 + 0.12 i, \text{ and}$$

$$f(a_2) = - 0.425 + 0.18 i.$$

Hence

$$P_2(0.22 + 0.2 i) = - 0.4874 + 0.1320 i$$

which is identical with the accurate value.

§ 3. The following tables give the values of $P_n(x + iy) = u + iv$,
for $n = 1, 2, \dots, 10$.

One thousand values are calculated for $P_n(x + iy)$.

$P_1(x + iy)$

| $y \ x$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 0.0 | 0.000000 +0.100000i | 0.000000 +0.200000i | 0.000000 +0.300000i | 0.000000 +0.400000i | 0.000000 +0.500000i |
| 0.1 | 0.100000 +0.100000i | 0.100000 +0.200000i | 0.100000 +0.300000i | 0.100000 +0.400000i | 0.100000 +0.500000i |
| 0.2 | 0.200000 +0.100000i | 0.200000 +0.200000i | 0.200000 +0.300000i | 0.200000 +0.400000i | 0.200000 +0.500000i |
| 0.3 | 0.300000 +0.100000i | 0.300000 +0.200000i | 0.300000 +0.300000i | 0.300000 +0.400000i | 0.300000 +0.500000i |
| 0.4 | 0.400000 +0.100000i | 0.400000 +0.200000i | 0.400000 +0.300000i | 0.400000 +0.400000i | 0.400000 +0.500000i |
| 0.5 | 0.500000 +0.100000i | 0.500000 +0.200000i | 0.500000 +0.300000i | 0.500000 +0.400000i | 0.500000 +0.500000i |
| 0.6 | 0.600000 +0.100000i | 0.600000 +0.200000i | 0.600000 +0.300000i | 0.600000 +0.400000i | 0.600000 +0.500000i |
| 0.7 | 0.700000 +0.100000i | 0.700000 +0.200000i | 0.700000 +0.300000i | 0.700000 +0.400000i | 0.700000 +0.500000i |
| 0.8 | 0.800000 +0.100000i | 0.800000 +0.200000i | 0.800000 +0.300000i | 0.800000 +0.400000i | 0.800000 +0.500000i |
| 0.9 | 0.900000 +0.100000i | 0.900000 +0.200000i | 0.900000 +0.300000i | 0.900000 +0.400000i | 0.900000 +0.500000i |

$P_1(x + iy)$

| x | y | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|-----|-----|------------------------|------------------------|------------------------|------------------------|------------------------|
| 0.0 | | 0.000000 +0.600000i | 0.000000 +0.700000i | 0.000000 +0.800000i | 0.000000 +0.900000i | 0.000000 +1.000000i |
| 0.1 | | 0.100000 +0.600000i | 0.100000 +0.700000i | 0.100000 +0.800000i | 0.100000 +0.900000i | 0.100000 +1.000000i |
| 0.2 | | 0.200000 +0.600000i | 0.200000 +0.700000i | 0.200000 +0.800000i | 0.200000 +0.900000i | 0.200000 +1.000000i |
| 0.3 | | 0.300000 +0.600000i | 0.300000 +0.700000i | 0.300000 +0.800000i | 0.300000 +0.900000i | 0.300000 +1.000000i |
| 0.4 | | 0.400000 +0.600000i | 0.400000 +0.700000i | 0.400000 +0.800000i | 0.400000 +0.900000i | 0.400000 +1.000000i |
| 0.5 | | 0.500000 +0.600000i | 0.500000 +0.700000i | 0.500000 +0.800000i | 0.500000 +0.900000i | 0.500000 +1.000000i |
| 0.6 | | 0.600000 +0.600000i | 0.600000 +0.700000i | 0.600000 +0.800000i | 0.600000 +0.900000i | 0.600000 +1.000000i |
| 0.7 | | 0.700000 +0.600000i | 0.700000 +0.700000i | 0.700000 +0.800000i | 0.700000 +0.900000i | 0.700000 +1.000000i |
| 0.8 | | 0.800000 +0.600000i | 0.800000 +0.700000i | 0.800000 +0.800000i | 0.800000 +0.900000i | 0.800000 +1.000000i |
| 0.9 | | 0.900000 +0.600000i | 0.900000 +0.700000i | 0.900000 +0.800000i | 0.900000 +0.900000i | 0.900000 +1.000000i |

$P_2 (x + iy)$

| xy | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.0 | -0.5150000 | -0.5600000 | -0.6350000 | -0.7400000 | -0.8750000 |
| 0.1 | -0.5000000 +0.0300000i | -0.5450000 +0.0600000i | -0.6200000 +0.0900000i | -0.7250000 +0.1200000i | -0.8600000 +0.1500000i |
| 0.2 | -0.4550000 +0.0600000i | -0.5000000 +0.1200000i | -0.5750000 +0.1800000i | -0.6800000 +0.2400000i | -0.8150000 +0.3000000i |
| 0.3 | -0.3800000 +0.0900000i | -0.4250000 +0.1800000i | -0.5000000 +0.2700000i | -0.6050000 +0.3600000i | -0.7400000 +0.4500000i |
| 0.4 | -0.2750000 +0.1200000i | -0.3200000 +0.2400000i | -0.3950000 +0.3600000i | -0.5000000 +0.4800000i | -0.6350000 +0.6000000i |
| 0.5 | -0.1400000 +0.1500000i | -0.1850000 +0.3000000i | -0.2600000 +0.4500000i | -0.3650000 +0.6000000i | -0.5000000 +0.7500000i |
| 0.6 | 0.0250000 +0.1800000i | -0.0200000 +0.3600000i | -0.0950000 +0.5400000i | -0.2000000 +0.7200000i | -0.3350000 +0.9000000i |
| 0.7 | 0.2200000 +0.2100000i | 0.1750000 +0.4200000i | 0.1000000 +0.6300000i | -0.0050000 +0.8400000i | -0.1400000 +1.0500000i |
| 0.8 | 0.4450000 +0.2400000i | 0.4000000 +0.4860000i | 0.3250000 +0.7200000i | 0.2200000 +0.9600000i | 0.0850000 +1.2000000i |
| 0.9 | 0.7000000 +0.2700000i | 0.6550000 +0.5400000i | 0.5800000 +0.8100000i | 0.4750000 +1.0800000i | 0.3400000 +1.3500000i |

$P_2(x+iy)$

| xy | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.0 | -1.0400000 | -1.2350000 | -1.4600000 | -1.7150000 | -2.0000000 |
| 0.1 | -1.0250000 +0.1800000i | -1.2200000 +0.2100000i | -1.4450000 +0.2400000i | -1.7000000 +0.2700000i | -1.9850000 +0.3000000i |
| 0.2 | -0.9800000 +0.3600000i | -1.1750000 +0.4200000i | -1.4000000 +0.4800000i | -1.6550000 +0.5400000i | -1.9400000 +0.6000000i |
| 0.3 | -0.9050000 +0.5400000i | -1.1000000 +0.6300000i | -1.3250000 +0.7200000i | -1.5800000 +0.8100000i | -1.8650000 +0.9000000i |
| 0.4 | -0.8000000 +0.7200000i | -0.9950000 +0.8400000i | -1.2200000 +0.9600000i | -1.4750000 +1.0800000i | -1.7600000 +1.2000000i |
| 0.5 | -0.6650000 +0.9000000i | -0.8600000 +1.0500000i | -1.0850000 +1.2000000i | -1.3400000 +1.3500000i | -1.6250000 +1.5000000i |
| 0.6 | -0.5000000 +1.0800000i | -0.6950000 +1.2600000i | -0.9200000 +1.4400000i | -1.1750000 +1.6200000i | -1.4600000 +1.8000000i |
| 0.7 | -0.3050000 +1.2600000i | -0.5000000 +1.4700000i | -0.7250000 +1.6800000i | -0.9800000 +1.8900000i | -1.2650000 +2.1000000i |
| 0.8 | -0.0800000 +1.4400000i | -0.2750000 +1.6800000i | -0.5000000 +1.9200000i | -0.7550000 +2.1600000i | -1.0400000 +2.4000000i |
| 0.9 | 0.1750000 +1.6200000i | -0.0200000 +1.8900000i | -0.2450000 +2.1600000i | -0.5000000 +2.4300000i | -0.7850000 +2.7000000i |

$P_3(x + iy)$

| $x y$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.0 | 0.0000000 -0.1524999i | 0.0000000 -0.3199999i | 0.0000000 -0.5174999i | 0.0000000 -0.7599999i | 0.0000000 -1.0624999i |
| 0.1 | -0.1549999 -0.1449999i | -0.1774999 -0.3049999i | -0.2149999 -0.4949999i | -0.2674999 -0.7299999i | -0.3349999 -1.0249999i |
| 0.2 | -0.2949999 -0.1224999i | -0.3399999 -0.2599999i | -0.4149999 -0.4274999i | -0.5199999 -0.6399999i | -0.6549999 -0.9124999i |
| 0.3 | -0.4049999 -0.0849999i | -0.4724999 -0.1849999i | -0.5849999 -0.3149999i | -0.7424999 -0.4899999i | -0.9449999 -0.7249999i |
| 0.4 | -0.4699999 -0.0325000i | -0.5599999 -0.0800000i | -0.7099999 -0.1575000i | -0.9199999 -0.2499999i | -1.1899999 -0.4624999i |
| 0.5 | -0.4749999 -0.0350000i | -0.5874999 +0.0549999i | -0.7749999 +0.0450000i | -1.0374999 -0.0100000i | -1.3749999 -0.1250000 |
| 0.6 | -0.4049999 +0.1174999i | -0.5399999 +0.2199999i | -0.7649999 +0.2924999i | -1.0799999 +0.3199999i | -1.4849999 +0.2874999i |
| 0.7 | -0.2450000 +0.2149999i | -0.4025000 +0.4149999i | -0.6649999 +0.5849999i | -1.0324999 +0.7099999i | -1.5049999 +0.7749999i |
| 0.8 | 0.0199999 +0.3274999i | -0.1600000 +0.6399999i | -0.4599999 +0.9225000i | -0.8799999 +1.1600000i | -1.4199999 +1.3374999i |
| 0.9 | 0.4050000 +0.4549999i | 0.2024999 +0.8949999i | -0.1350000 +1.3050000i | -0.6074999 +1.6699999i | -1.2149999 +1.9749999i |

$P_3(x + iy)$

| x y | 0.6 | 0.7 | 0.8 | 0.9 | 1.— |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.0 | 0.0000000 -1.4399999i | 0.0000000 -1.9074999i | 0.0000000 -2.4799999i | 0.0000000 -3.1724999i | 0.0000000 -3.9999999i |
| 0.1 | -0.4174999 -1.3949999i | -0.5149999 -1.8549999i | -0.6274999 -2.4199999i | -0.7549999 -3.1049999i | -0.8974999 -3.9249999i |
| 0.2 | -0.8199999 -1.2599999i | -1.0149999 -1.6974999i | -1.2399999 -2.2399999i | -1.4949999 -2.9024999i | -1.7799999 -3.6999999i |
| 0.3 | -1.1924999 -1.0349999i | -1.4849999 -1.4349999i | -1.8224999 -1.9399999i | -2.2049999 -2.5649999i | -2.6324999 -3.3249999i |
| 0.4 | -1.5199999 -0.7199999i | -1.9099999 -1.0674999i | -2.3599999 -1.5199999i | -2.8699999 -2.0924999i | -3.4399999 -2.7999999i |
| 0.5 | -1.7874999 -0.3150000i | -2.2749999 -0.5949999i | -2.8374999 -0.9799999i | -3.4749999 -1.4849999i | -4.1874999 -2.1249999i |
| 0.6 | -1.9799999 +0.1799999i | -2.5649999 -0.0175000i | -3.2399999 -0.3200000i | -4.0049999 -0.7424999i | -4.8599999 -1.2999999i |
| 0.7 | -2.0824999 +0.7650000i | -2.7649999 +0.6650000i | -3.5524999 +0.4599999i | -4.4449999 +0.1349999i | -5.4424999 -0.3250000i |
| 0.8 | -2.0799999 +1.4400000i | -2.8599999 +1.4524999i | -3.7599999 +1.3599999i | -4.7799999 +1.1475000i | -5.9199999 +0.8000000i |
| 0.9 | -1.9574999 +2.2049999i | -2.8349999 +2.3449999i | -3.8474999 +2.3799999i | -4.9949999 +2.3799999i | -6.2774999 +2.0749999i |

$P_i(x+iy)$

| xy | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.0 | 0.4129374 | 0.5319999 | 0.7479374 | 1.0869999 | 1.5859374 |
| 0.1 | 0.3732500 -0.0749999i | 0.4844374 -0.1604999i | 0.6872500 -0.2669999i | 1.0079374 -0.4049999i | 1.4832499 -0.5849999i |
| 0.2 | 0.2594375 -0.1394999i | 0.3470000 -0.2999999i | 0.5104375 -0.5024999i | 0.7759999 -0.7679999i | 1.1804374 -1.1174999i |
| 0.3 | 0.0872500 -0.1829999i | 0.1354375 -0.3974999i | 0.2332500 -0.6749999i | 0.4069375 -1.0469999i | 0.6932500 -1.5449999i |
| 0.4 | -0.1170624 -0.1949999i | -0.1239999 -0.4319999i | -0.1180624 -0.7529999i | -0.0729999 -1.1999999i | 0.0479375 -1.8149999i |
| 0.5 | -0.3167499 -0.1649999i | -0.3945624 -0.3825000i | -0.5067499 -0.7049999i | -0.6270624 -1.1849999i | -0.7187499 -1.8749999i |
| 0.6 | -0.4645624 -0.0825000i | -0.6289999 -0.2280000i | -0.8855624 -0.4995000i | -1.2079999 -0.9599999i | -1.5595624 -1.6724999i |
| 0.7 | -0.5027499 +0.0629999i | -0.7695624 +0.0524999i | -1.1967499 -0.1050000i | -1.7580624 -0.4829999i | -2.4167499 -1.1549999i |
| 0.8 | -0.3630625 +0.2819999i | -0.7479999 +0.4799999i | -1.3720624 +0.5100000i | -2.2089999 +0.2880000i | -3.2220624 -0.2700000i |
| 0.9 | 0.0332500 +0.5849999i | -0.4855625 +1.0754999i | -1.3327500 +1.3770000i | -2.4020624 +1.3950000i | -3.8967499 +1.0350000i |

$P_4(x+iy)$

| xy | 0.6 | 0.7 | 0.8 | 0.9 | 1.— |
|-----|---------------------------|---------------------------|---------------------------|----------------------------|----------------------------|
| 0.0 | 2.2919999 | 3.2629374 | 4.5669999 | 6.2829374 | 8.4999999 |
| 0.1 | 2.1604374 -0.8174999i | 3.0972499 -1.1129999i | 4.3619374 -1.4819999i | 6.0332499 -1.9319999i | 8.2004374 -2.4824999i |
| 0.2 | 1.7709999 -1.5719999i | 2.6054374 -2.1524999i | 3.7519999 -2.8799999i | 5.2894374 -3.7754999i | 7.3069999 -4.8599999i |
| 0.3 | 1.1394374 -2.2004999i | 1.8032500 -3.0449999i | 2.7529375 -4.1099999i | 4.0672499 -5.4269999i | 5.8354374 -7.0274999i |
| 0.4 | 0.2920000 -2.6399999i | 0.7169375 -3.7169999i | 1.3910000 -5.0879999i | 2.3929374 -6.7949999i | 3.8119999 -8.8799999i |
| 0.5 | -0.7345624 -2.8274999i | -0.6167500 -4.0949999i | -0.5970625 -5.7299999i | 0.3032500 -7.7849999i | 1.2784274 -10.3124999i |
| 0.6 | -1.8929999 -2.6999999i | -2.1505624 -4.1054999i | -2.2639999 -5.9519999i | -2.1545624 -8.3024999i | 1.7329999 -11.2199999i |
| 0.7 | -3.1255624 -2.1944999i | -3.8267499 -3.6749999i | -4.4520524 -5.6699999i | -4.9227499 -8.2529999i | -5.1495624 -11.4974999i |
| 0.8 | -4.3639999 -1.2479999i | -5.5770624 -2.7299999i | -6.7929999 -4.7999999i | -7.9330624 -7.5419999i | -8.9079999 -11.0399999i |
| 0.9 | -5.5295624 +0.2025000i | -7.3227499 -1.1969999i | -9.2080624 -3.2579999i | -11.1067499 -6.0749999i | -12.9295624 -9.7424999i |

$P_5(x + iy)$

| x y | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.0 | 0.0000000 +0.1963287i | 0.0000000 +0.4475199i | 0.0000000 +0.8178862i | 0.0000000 +1.3906399i | 0.0000000 +2.2773437i |
| 0.1 | 0.2046849 +0.1696849i | 0.2869787 +0.3895074i | 0.4388849 +0.7199549i | 0.6870387 +1.2368149i | 1.0614849 +2.0496249i |
| 0.2 | 0.3545074 +0.0944787i | 0.5049199 +0.2249200i | 0.7271974 +0.4837362i | 1.0488199 +0.7942399i | 1.9547074 +1.3909937i |
| 0.3 | 0.4040549 -0.0151149i | 0.5942362 -0.0746974i | 0.9364549 +0.0134590i | 1.5945862 +0.1194150i | 2.5208549 +0.3696250i |
| 0.4 | -0.3268150 +0.1354712i | -0.5162399 +0.2916790i | 0.7299749 -0.4191371i | 1.5244899 -0.6925599i | 2.6200149 -0.8936562i |
| 0.5 | 0.1246249 -0.2335149i | 0.2375937 -0.5302920i | 0.5736249 -0.9417449i | 1.1783437 -1.5095849i | 2.1407249 -2.2813749i |
| 0.6 | -0.1628774 -0.2667212i | -0.1852359 -0.6486799i | -0.0746774 -1.2516637i | 0.2585400 -2.1625599i | 1.0085225 -3.4399062i |
| 0.7 | -0.4488049 -0.1831150i | -0.6665487 -0.5428925i | -0.9192049 -1.2465449i | -1.9413987 -2.4428649i | -0.8016049 -4.2503749i |
| 0.8 | -0.5895699 +0.0707287i | -1.1219199 -0.6900800i | -1.8231699 -0.7445237i | -2.6843199 -2.1037599i | -3.2607699 -4.3586562i |
| 0.9 | -0.3754349 +0.5896049i | -1.3357912 +0.2515074i | -2.7946350 +0.4670550i | -4.5393412 -0.8637849i | -6.2722349 -3.4103749i |

$P_5(x + iy)$

| $x \ y$ | 0.6 | 0.7 | 0.8 | 0.9 | 1. |
|---------|---------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| 0.0 | 0.0000000 +3.6273599i | 0.0000000 +5.6373012i | 0.0000000 +8.5604799i | 0.0000000 +12.7163587i | 0.0000000 +18.4999999i |
| 0.1 | 1.6057787 +3.3021224i | 2.3718849 +5.1861949i | 3.4212287 +7.9504299i | 4.8246849 +11.9095649i | 6.6625787 +17.4539374i |
| 0.2 | 2.9913199 -2.3547599i | 4.4021074 +3.8659512i | 6.4899199 -6.1580799i | 9.2165074 +9.5317087i | 12.8025199 -14.3629999i |
| 0.3 | 3.9458362 +0.8703224i | 5.9984549 +1.7757950i | 8.8629862 +3.2268300i | 12.7520549 +5.7103649i | 17.9066362 +9.3689374i |
| 0.4 | 4.2774399 -1.0094399i | 6.7276149 -0.9188987i | 10.2162399 -0.4443199i | 15.0268149 +0.6581587i | 21.4806399 +02.7079999i |
| 0.5 | 3.8225937 -3.0860774i | 6.4246249 -3.9866049i | 10.2538437 -4.8007699i | 15.6646249 -5.3272349i | 23.0585937 -5.2890624i |
| 0.6 | 2.4555600 -5.1044399i | 4.9028224 -7.1296487i | 8.7177599 -9.4323199i | 14.3271224 -11.8630912i | 22.2123599 -14.1969999i |
| 0.7 | 0.0978512 -6.7526774i | 2.0207949 -9.9842049i | 5.3972012 -13.9231699i | 10.7231949 -18.4816349i | 18.5610512 -23.4960624i |
| 0.8 | -3.2723200 -7.6622399i | -2.3031699 -12.1202987i | 0.1380800 -17.7819199i | 4.6184299 -24.6300412i | 11.7804799 -32.5719999i |
| 0.9 | -7.6105912 -7.4078774i | -8.0866350 -13.0418049i | -7.1475412 -20.4415699i | -4.1554349 -29.6704349i | 1.6126087 -40.7160624i |

$P_6(x + iy)$

| x y | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|----------------------------|
| 0.0 | -0.3801081 | -0.6074239 | -1.0731186 | -1.9256359 | -3.4091796 |
| 0.1 | -0.3046250 +0.1311344i | -0.4939045 +0.3103852i | -0.8875429 +0.5962634i | -1.6209903 +1.0680704i | -2.9202589 +1.8362924i |
| 0.2 | -0.1035329 +0.2158852i | -0.1865000 +0.5176079i | -0.3769634 +1.0117957i | -0.7713919 +1.8466559i | -1.5412244 +3.2327662i |
| 0.3 | 0.1522929 +0.2182634i | 0.2205259 +0.5392957i | 0.3253749 +1.0970504i | 0.4353401 +2.0878514i | 0.4699389 +3.8015774i |
| 0.4 | 0.3620528 +0.1230705i | 0.5873919 +0.3346560i | 1.0147223 +0.7648515i | 1.7034999 +1.6269119i | 2.7005813 +3.2588324i |
| 0.5 | 0.4210089 -0.0537075i | 0.7547869 -0.0747337i | 1.4408109 +0.0215775i | 2.6554811 +0.4238325i | 4.6093749 +1.4765625i |
| 0.6 | 0.2568690 -0.2545042i | 0.5802519 -0.5841359i | 1.3442385 -1.0016527i | 2.8681599 -1.3950719i | 5.5626975 -1.4653012i |
| 0.7 | -0.1234369 -0.3697784i | -0.0150415 -0.9848632i | 0.5032449 -2.0177954i | 1.9196726 -3.4955864i | 4.8814089 -5.2269524i |
| 0.8 | -0.5765841 -0.2276190i | -0.9891199 -0.9434879i | -1.2091146 -2.5526969i | -0.5534119 -5.2940159i | -1.8980243 -9.1567349 |
| 0.9 | -0.7552849 +0.4166504i | -2.1116395 +0.0189472i | -3.7574030 -1.9139084i | -4.7885253 -5.9156054i | -3.9757189 -12.2391674i |

$P_6(x + iy)$

| $x \ y$ | 0.6 | 0.7 | 0.8 | 0.9 | 1.— |
|---------|----------------------------|----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0.0 | -5.9000959 | -9.9536511 | -16.3612039 | -26.2177731 | -40.9999999 |
| 0.1 | -5.1383065 +3.0529957i | -8.8018129 +4.9222214i | -14.6683533 + 7.7103009i | -23.7939649 +11.7566504i | -37.6111105 +17.4833662i |
| 0.2 | -2.9692519 +5.4638639i | -5.4963959 +8.9376367i | - 9.7788799 +14.1765119i | -16.7557979 +21.8484472i | -27.7270759 +32.7877199i |
| 0.3 | 0.2633239 +6.6528472i | -0.4824950 +11.2122044i | - 2.2548228 +18.2373029i | - 5.7978470 +28.7046914i | -12.1906000 +43.8379987i |
| 0.4 | 4.0033399 +6.1649279i | 5.5153893 +11.0574134i | 6.9344119 +18.8979839i | 7.9395878 +30.9393944i | 7.6111359 +48.7670399i |
| 0.5 | 7.5108649 +3.7321987i | 11.5193409 +8.0030474i | 16.6880381 +15.4132649i | 22.8964689 +27.4508324i | 29.7724609 +46.0195312i |
| 0.6 | 9.8934999 -0.6637679i | 16.3344060 +1.8699502i | 25.3102719 + 7.3704959i | 37.1294040 +17.5091017i | 51.9055959 +34.4559599i |
| 0.7 | 10.1581564 - 6.7295497i | 8.5953749 -7.1572094i | 31.0571096 - 5.2271729i | 48.3584229 +00.8526735i | 71.1874324 +13.4565637i |
| 0.8 | 7.2657279 -13.7975039i | 16.8239528 -18.4571729i | 31.9434999 -21.8776319i | 54.0241508 -22.2186509i | 84.4167039 -16.9747199i |
| 0.9 | 0.1991584 -20.7633982i | 9.4963269 -30.8993264i | 25.8609116 -41.4966509i | 51.3553749 -50.7501854i | 88.0815544 -56.1063037i |

$P_7(x+iy)$

| xy | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|----------------------------|
| 0.0 | 0.0000000 -0.2388733i | 0.0000000 -0.6092031i | 0.0000000 -1.2989257i | 0.0000000 -2.6224495i | 0.0000000 -5.1176757i |
| 0.1 | -0.2563710 -0.1776639i | -0.4529928 -0.4596708i | -0.8740776 -1.0900864i | -1.6683466 -2.0659354i | -3.1573068 -4.1274647i |
| 0.2 | -0.3824116 -0.0200233i | -0.6943143 -0.0598056i | -1.2783937 -0.1025579i | -2.7283071 -0.5679104i | -5.2497727 -1.4219011i |
| 0.3 | -0.3020185 +0.1628420i | -0.5867967 -0.3977106i | -1.2514563 +0.7009613i | -2.6320740 +1.3816399i | -5.4289459 +2.2373722i |
| 0.4 | -0.0340295 +0.2747803i | -0.1237296 +0.7167871i | -0.4346619 +1.5428783i | -1.2604032 +3.0376131i | -3.2655897 +5.6945207i |
| 0.5 | 0.2940897 +0.2284718i | 0.5121229 +0.6654902i | 0.8509527 +1.6320409i | 1.1910360 +3.6601747i | 1.0742187 +7.5604062i |
| 0.6 | 0.4730998 -0.0972679i | 1.0031655 +0.1200392i | 2.1189387 +0.7656602i | 4.0175231 +2.4297471i | 6.6942205 +6.4810883i |
| 0.7 | 0.2928950 -0.3466803i | 0.9175798 -0.8205726i | 2.5063088 -1.2742876i | 5.9819233 -1.0247470i | 11.8865204 +1.3808772i |
| 0.8 | -0.3090215 -0.5127385i | -0.1574655 -1.6919295i | 1.2399636 -3.8289733i | 5.4113311 -6.4732783i | 14.1175500 -8.1058497i |
| 0.9 | -1.0179814 +0.0406900i | -2.3915283 -1.4825178i | -2.8185088 -5.6927044i | 0.2816357 -12.7013294i | 10.0960123 -21.2254547i |

$P_7(x+iy)$

| $x y$ | 0.6 | 0.7 | 0.8 | 0.9 | 1.— |
|-------|----------------------------|-----------------------------|-----------------------------|-----------------------------|------------------------------|
| 0.0 | 0.0000000 -9.6835583i | 0.0000000 -17.7713190i | 0.0000000 -31.6456287i | 0.0000000 -54.7208712i | 0.0000000 -91.9999999i |
| 0.1 | -5.7325482 -7.9889473i | -10.0665546 -14.9735400i | -17.1120277 -27.1757084i | -28.2047250 -47.7947335i | -45.1648110 -81.5628123i |
| 0.2 | -9.7551591 -3.2975255i | -17.4851098 -7.1392935i | -30.2571903 -14.5414143i | -50.6415646 -28.0610179i | -82.1636967 -51.6259879i |
| 0.3 | -10.6486089 +3.2540137i | -19.9862173 +4.0974461i | -35.9485253 +3.9849063i | -62.1374005 +1.4069938i | -103.5538773 -6.2461758i |
| 0.4 | -7.5615871 +9.9063167i | -16.0440179 +16.1717128i | -31.6453615 +24.7961887i | -58.6951355 +35.6898680i | -103.3253503 +48.0407679i |
| 0.5 | -0.4608700 +14.4800719i | -5.2142522 +25.8236345i | -16.1926014 +43.2209199i | -38.0479202 +68.3260726i | -77.5834960 +102.5576171i |
| 0.6 | 9.6590471i +14.6597927i | 11.5682692 +29.4295142i | 9.7800575 +53.9015167i | -0.1728391 +91.7376526i | -25.1911416 +146.9507479i |
| 0.7 | 20.6203720 +8.3586832i | 31.7462498 +23.4275051i | 43.5141556 +51.2808123i | 52.2494570 +97.7775268i | 51.6434284 +169.8382466i |
| 0.8 | 28.9740031 -5.8354176i | 50.9640576 +4.8378805i | 79.844703 +30.1966495i | 113.4426864 +78.3984059i | 146.8460287 +159.4731519i |
| 0.9 | 29.9926010 -28.1330083i | 62.9729581 -28.1221020i | 111.0032978 -13.4154164i | 1742239525 +26.4433324i | 250.0372118 +104.7018326i |

$P_9(x - iy)$

| $x - y$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---------|---------------------------|---------------------------|---------------------------|----------------------------|----------------------------|
| 0.0 | 0.3773834 | 0.7599471 | 1.6696245 | 3.6517686 | 7.7808532 |
| 0.1 | 0.2517892 -0.1961242i | 0.5196069 -0.5276477i | 1.1752591 -1.2009154i | 2.6521905 -2.5844345i | 5.8327297 -5.3406307i |
| 0.2 | -0.0490586 -0.2691165i | -0.0716932 -0.7394519i | -0.0697406 -1.7388767i | 0.6777856 -3.8750207i | 0.7129417 -8.2835464i |
| 0.3 | -0.3336748 -0.1560100i | -0.6721715 -0.4682179i | -1.4279437 -1.2245782i | -2.9107893 -3.0373693i | -5.5627027 -7.1573827i |
| 0.4 | -0.3938397 +0.0920180i | -0.8793163 +0.1964927i | -2.0830231 +0.2447998i | -4.7434072 -0.0749280i | -10.1508516 -1.6421251i |
| 0.5 | -0.1355122 +0.3163282i | -0.4298821 +0.2813352i | -1.3733707 +1.5943752i | -3.9514508 +3.9540026i | -10.1196289 +6.8085937i |
| 0.6 | 0.3088396 +0.3032209i | 0.5778510 +1.0237751i | 0.8117885 +2.8627794i | 0.1877631 +6.9672959i | -3.4123154 +14.8492515i |
| 0.7 | 0.5574346 +0.0765439i | 1.5251996 +0.1288461i | 3.6447277 +1.5366172i | 6.9440586 +6.2023502i | 10.0352527 +17.5295977i |
| 0.8 | 0.1371173 +0.6278826i | 1.2637552 +1.7713919i | 5.0712120 +2.8110206i | 13.4561910 +1.0191551i | 27.1147879 +9.0885716i |
| 0.9 | -1.0664737 -0.4699012i | -1.6320752 -3.4151508i | 1.7336401 -9.5171800i | 14.1938419 -16.0520180i | 40.4146387 -15.6436717i |

$P_8(x + iy)$

| $x \cdot y$ | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|-------------|-----------------------------|-----------------------------|------------------------------|-------------------------------|-------------------------------|
| 0.0 | 16.0565871 | 32.0348260 | 61.7844966 | 115.2820217 | 208.3749999 |
| 0.1 | 12.4087312 -10.6184157i | 25.4668786 -20.3268355i | 50.3898666 -37.5100703i | 96.1849462 -66.8440552i | 177.3715927 -115.2749935i |
| 0.2 | 2.6496271 -16.9920071i | 7.6227530 -33.4468739i | 19.0221952 -63.2432639i | 43.0237042 -115.0979133i | 90.2485326 -202.1059319i |
| 0.3 | 9.8810164 -15.9705437i | -16.1979622 -33.7377757i | -24.2254350 -67.6389183i | -32.2534738 -129.1815094i | -35.8707013 -236.0352428i |
| 0.4 | -20.3191567 -6.4713600i | -38.0843526 -18.6042257i | -67.0396648 -45.4066367i | -111.1951433 -99.3526104i | -174.2301967 -200.3756159i |
| 0.5 | -23.2941534 +9.7909146i | -48.8613052 +10.3632247i | -94.6140522 +2.7439833i | -171.0045832 -24.1696507i | -291.0809631 -69.5683789i |
| 0.6 | -11.2826512 +27.9394919i | -39.5045397 +46.6553504i | -91.9961983 +68.8601087i | -187.4699614 +87.5927292i | -349.3050332 +87.9462359i |
| 0.7 | 8.7723327 +40.0570462i | -5.3526007 +78.6781117i | -46.9838601 +137.1510761i | -138.7357243 +215.7578734i | -312.9537161 +307.9696387i |
| 0.8 | 43.6683375 +35.9154431i | 55.3754095 +90.2971728i | 46.5211686 +184.2046079i | -9.4044123 +328.4734619i | -152.6077327 +529.3989119i |
| 0.9 | 82.0878851 +4.4351980i | 134.8678396 +62.2326710i | 184.8128919 +180.1760010i | 204.4430432 +383.0324557i | 148.5504986 +694.5971307i |

$P_9(x + iy)$

| $x + iy$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|----------|---------------------------|---------------------------|----------------------------|----------------------------|-----------------------------|
| 0.0 | 0.0000000 +0.2836153i | 0.0000000 +0.8286051i | 0.0000000 +2.1007212i | 0.0000000 +5.0001804i | 0.0000000 +11.0076287i |
| 0.1 | 0.3124912 +0.1684380i | 0.7001518 +0.5052254i | 1.6791190 +1.3381013i | 3.2350018 +3.3510924i | 8.9321618 +8.1687609i |
| 0.2 | 0.3720312 -0.0927542i | 0.8696930 +0.2441225i | 2.1812533 +0.5110108i | 5.3028411 -0.9003161i | 12.7791476 -1.1920910i |
| 0.3 | 0.1088471 -0.2961821i | 0.5173759 -0.8727757i | 0.9971711 -2.1972836i | 3.0028532 -5.1507701i | 8.4332819 -11.2924670i |
| 0.4 | -0.2847004 -0.2491164i | -0.6241721 -0.8208668i | -1.2110129 -2.3098123i | -2.3082659 -6.5678139i | -3.2158454 -15.2894254i |
| 0.5 | -0.4491477 +0.0799716i | -1.1941690 +0.0784253i | -3.1909297 -0.3356928i | -7.4797002 -2.5942047i | -16.9426219 -9.8527232i |
| 0.6 | -0.1277899 +0.4084472i | -0.6253421 +1.271343i | -2.5816047 +3.0712438i | -3.6225126 +5.8783590i | -23.2420557 +7.8154415i |
| 0.7 | 0.4911596 +0.3122454i | 1.1523511 +1.4759476i | 1.6072268 +5.2297333i | -0.8245634 +14.3800926i | -13.2526929 +31.2283159i |
| 0.8 | 0.6004853 -0.4671329i | 2.7188364 -0.6954140i | 8.1530867 +2.0216342i | 16.2937560 +14.3008664i | 19.8408780 +46.5474522i |
| 0.9 | -0.8193737 -1.0453349i | 0.6414431 -5.1045245i | 10.8455982 -10.1356282i | 36.0073799 -5.2710128i | 74.5052316 +30.4422099i |

$P_9(x+iy)$

| $x y$ | 0.6.. |0.7 | 0.8 | 0.9 | 1 |
|-------|---------------|---------------|---------------|---------------|----------------|
| 0.0 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| | +26.8050729i | +58.1542424i | -121.4926872i | -244.6202114i | +475.3749999i |
| 0.1 | 19.4736744 | 40.6350526 | 81.4106612 | 156.8740282 | 291.3916762 |
| | +19.1588145i | +43.1431729i | +93.2156371i | +193.3725169i | +385.7613429i |
| 0.2 | 28.9298310 | 62.6462265 | 129.6490419 | 256.9345762 | 488.8839366 |
| | -0.4851581i | +3.7895265i | +17.7784524i | +54.6019902i | +140.0080879i |
| 0.3 | 21.9661370 | 53.1955181 | 120.4364193 | 256.5648425 | 517.5655078 |
| | -23.1409166i | -44.1775528i | -78.4781835i | -129.2844220i | -195.9569171i |
| 0.4 | -1.2966329 | 10.0854257 | 46.0914924 | 137.0587832 | 338.6914374a |
| | -36.7234646i | -78.7873594i | -157.6526759i | -295.8224876i | -523.1992975i |
| 0.5 | -32.6867414 | -55.2144627 | -79.1106851 | -86.5955437 | -36.7241973 |
| | -30.0242406i | -77.7722986i | -178.7994012i | -374.2689706i | -725.5931701i |
| 0.6 | -56.4375821 | -117.1967923 | -217.0110182 | -361.2426278 | -539.6676365 |
| | +2.4468258i | -26.0461735i | -108.8874803i | -301.0057548i | -690.7560880i |
| 0.7 | -52.1282320 | -139.3261641 | -308.0528685 | -596.6718837 | -1041.4200470 |
| | +55.4763532i | +76.1281711i | +64.7634232i | -37.4843356i | -334.8978339i |
| 0.8 | -0.4711283 | -81.0159165 | -279.0278375 | -673.4539407 | -1361.1127667 |
| | +108.9500457i | +205.3673200i | +321.8108184i | +410.6849251i | +369.9676143i |
| 0.9 | 107.8627570 | 91.0136796 | -56.7536388 | -458.4663767 | -1281.7362542 |
| | +125.5798916i | +309.1185595i | +597.4968310i | +975.2045239i | +1368.3422127i |

$P_{10}(x + iy)$

| $x \ y$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---------|---------------------------|---------------------------|---------------------------|-----------------------------|-----------------------------|
| 0.0 | -0.3935319 | -0.9988224 | -2.7000732 | -7.1551289 | -18.3055152 |
| 0.1 | -0.1992402 +0.2678883i | -0.5266049 +0.8369296i | -1.4956545 +2.2904620i | -4.1840698 +5.9648594i | -11.3088690 +14.8631860i |
| 0.2 | 0.2031479 +0.2767388i | 0.4871530 +0.9032237i | 1.1840786 +2.6158181i | 2.6595228 +7.2359778i | 5.3393150 +19.1233874i |
| 0.3 | 0.4186247 -0.0077338i | 1.1176280 +0.0445932i | 3.1059902 +0.4180579i | 8.2459068 +2.0798733i | 20.5469467 +8.0131360i |
| 0.4 | 0.1854155 -0.3262378i | 0.6289379 -1.0378877i | 2.2170770 -2.7763155i | 7.2850870 -6.5948615i | 21.7866809 -13.6531061i |
| 0.5 | -0.3180429 -0.3034654i | -0.7773682 -1.1724818i | -1.5983852 -3.9418069i | -1.9310316 -11.8507430i | 2.3723144 -31.5833740i |
| 0.6 | -0.5012412 +0.1684508i | -1.7160663 +0.2903033i | -5.4333677 -0.5428087i | -14.4662041 -6.1223467i | -31.5620290 -27.0704760i |
| 0.7 | 0.0922245 +0.5774963i | -0.4009127 +2.2849423i | -4.0438266 +6.5229916i | -18.2587013 +12.8878800i | -57.3127092 +12.8629640i |
| 0.8 | 0.8780874 -0.0308555i | 3.2595091 +1.5703812i | 6.6734840 +10.2611884i | 1.7264772 +35.1594143i | -38.4652618 +81.4212423i |
| 0.9 | -0.2426891 -1.5202927i | 4.5054548 -5.4113528i | 22.7635748 -2.5861812i | 52.8041316 +32.7089930i | 62.1106716 +136.9154537i |

$P_{10}(x + iy)$

| $x + y$ | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---------|---------------------------------|---------------------------------|----------------------------------|----------------------------------|-----------------------------------|
| 0.0 | — 45.0087116 | — 106.1764859 | — 240.2749316 | — 522.0543812 | — 1090.7499999 |
| 0.1 | — 29.3099086 + 35.2967377i | — 72.5799508 + 89.5359749i | — 171.5706227 + 175.2142395i | — 387.4273903 + 365.1550162i | — 837.2165665 + 730.6863342i |
| 0.2 | 9.1617516 + 48.0284527i | 11.9050180 + 114.8616878i | 5.1234124 + 260.7412932i | — 34.4555981 + 563.6950037i | — 161.4531505 + 1163.9778917i |
| 0.3 | 47.7942579 — 26.2245631i | 103.6557566 + 75.9328321i | 209.7394895 + 199.2050193i | 396.3464403 + 481.2971186i | 699.6141132 + 1084.1107406i |
| 0.4 | 59.1665497 — 28.5687707i | 146.5280290 — 29.7169737i | 334.9973000 — 3.8909920i | 710.0967581 + 98.9627782i | 1403.2913350 + 426.2203193i |
| 0.5 | 24.1399490 — 74.8077870i | 94.9585922 — 156.6458754i | 281.7725860 — 292.5772575i | 711.6382933 — 481.8812247i | 1605.7119026 — 678.4599456i |
| 0.6 | 51.2136889 — 43.6950049i | 63.0388465 — 227.5541871i | 0.9129678 — 515.9625732i | 271.6462102 — 1039.7049104i | 1011.6584367 — 1891.8680622i |
| 0.7 | — 140.4636907 — 21.6939763i | — 281.7369252 — 154.8536312i | — 465.8652442 — 505.5409757i | — 604.6131855 — 1264.3451736i | — 467.1244335 — 2701.2848789i |
| 0.8 | — 164.2206710 + 132.7430843i | — 446.1205973 + 123.1907018i | — 955.1438039 — 100.7540161i | — 1717.4572407 — 822.9912681i | — 2634.4929132 — 2500.2225036i |
| 0.9 | — 32.5948585 + 333.7134756i | — 376.8753477 + 593.6313467i | — 1171.5755083 + 773.2956490i | — 2635.5766992 + 533.8930215i | — 4925.3146478 — 720.5711168i |

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