INVESTIGATION OF THE STABILITY OF THE SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS

By

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Consider the following system

$$\left.\begin{aligned} \frac{dy}{dt} &= Y(y, x), \\ \frac{dx}{dt} &= Ax + X(y, x). \end{aligned}\right\} \qquad (1)$$

Where :

y is a scalar,

x an n -dimensional vector,

Y a scalar function and X an n -dimensional vector function having the following properties :

$$Y(0,0) = 0, \ \frac{\partial Y(0,0)}{\partial y} = 0, \ \frac{\partial Y(0,0)}{\partial x} = 0$$

$$X(0,0) = 0, \ \frac{\partial X(0,0)}{\partial y} = 0, \ \frac{\partial X(0,0)}{\partial x} = 0$$

A an $n \times n$ real constant matrix, and all the roots of the characteristic equation

$$\det(A - HE) = 0 \qquad (2)$$

have negative real parts.

In [1] it was proved that the following implicit equation

$$Ax + X(y, x) = 0$$

can be solved explicitly with respect to x , the solution being in the form

$$x = u(y).$$

Now if we let

$$x = u(y) + z \qquad (3)$$

we obtain from (1) the following system of differential equations [1] :

$$\left.\begin{aligned} \frac{dy}{dt} &= \Phi(y) + Y^*(y,z) \\ \frac{dz}{dt} &= Az + \Phi(y)\,q(y) + Z(y,z) \end{aligned}\right\} \qquad (4)$$

where

$$\Phi(0) = \Phi'(0) = 0, \; q(0) = 0 \; . \; Y^*(y,0) = 0 \text{ . and } Z(y,0) = 0 \, .$$

In [1], it has been discussed the case where the function $\Phi(y) = o$ for any $y \varepsilon C$, where C is a set of numbers with o as its limit point. It has been proved that under certain conditions, the trivial solution of the system (1) is stable.

For any $y \varepsilon C$, the system (4) has a solution

$y = c$, $z = o$, see [1].

Consequently from (3) the system (1) has the following solution

$$y = c \quad , \quad x = u(c). \qquad (5)$$

Letting

$$y = c + \alpha \quad , \quad x = u(c) + \Psi$$

in (1), we obtain the following system of differential equations :

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= Q(c, \alpha, \Psi) \\ \frac{d\Psi}{dt} &= A\Psi + G(c, \alpha, \Psi) \, , \end{aligned}\right\} \qquad (6)$$

where

$$Q(c,\alpha,\Psi) = Y(c+\alpha, u(c)+\Psi) - Y(c, u(c)),$$

$$G(c,\alpha,\Psi) = X(c+\alpha, u(e)+\Psi) - X(c, u(c)).$$

In the following theorem we are going to discuss the stability of the nonzero solution (5).

Theorem :

i) If there exists a sequence of positive numbers $y_i \xrightarrow[i \to \infty]{} 0$ such that $\Phi(y_i) = 0$, $i = 1, 2, \ldots \infty$ and simultuneously a sequence of negative numbers $\bar{y}_i \xrightarrow[i \to \infty]{} 0$ such that $\Phi(\bar{y}_i) = 0$, $i = 1, 2, \ldots, \infty$ then the nonzero solution $y = c$, $x = u(c)$ is stable.

ii) If a sequence of positive (negative) numbers $y_i \xrightarrow[i \to \infty]{} 0$ exists such that $\Phi(y_i) = 0$, $i = 1, 2, \ldots \infty$ and $\Phi(y) > 0$ for $y < 0$ ($\Phi(y) < 0$ for $y > 0$) then the solution (5) of (1) is stable.

To prove this theorem, we need the following two lemmas which had been proved in [1].

Lemma (1) :

There exists a monotonic function $P(y)$ [$P(y) \geq 0$, $P(y) \xrightarrow[y \to 0]{} 0$ such that if

$$v(x) \geq y^2 P^2(y)$$

then for sufficiently small $|y|$ and $|x|$

$$\dot{v} = \frac{dv}{dt} = \frac{\partial v}{\partial x} [Ax + X(y, x)] < 0,$$

where $v(x)$ is a quadratic form satisfying the equation

$$\frac{\partial v}{\partial x} Ax = - \| x \|^2, \quad [2].$$

Lemma (2) :

There exists constants $E > 0$, $\lambda > 0$, $c_0 > 0$, $a_0 > 0$ and a scalar function f (c, a, to) , where c, to are scalars and a is an n-dimensional vector, such that :

1. the function f is defined for any $c \in C$, $|c| \leqslant c_0$, $\|a\| \leqslant a_0$ and for all $t_0 \in (0, \infty)$,
2. f is continuous with respect to a and t_0 for any fixed $c \in C$,
3. $|f(c, a, t_0)| \leqslant \|a\|$,
4. for any solution of the system (6) with initial conditions :

$$t = t_0, \ \Psi = a, \ \alpha = f(c, a, t_0), \ \|a\| \leqslant a_0$$

the following inequalities

$$\|x(t)\| \leqslant \|a\| e^{-\lambda(t-t_0)},$$

$$\|\Psi(t)\| \leqslant E \|a\| e^{-\lambda(t-t_0)},$$

hold for all $t \geq t_0$.

We turn now to the proof of the Theorem.

i) For the proof of the first part, we take arbitrary

$$\varepsilon > 0 \quad , \quad \varepsilon < a_0 \quad .$$

for this ε we can find two numbers

$$\left.\begin{array}{l} c_1 > 2c > 0 \quad , \quad c_1 \varepsilon C \quad , \quad c_1 \leqq c_0 \ , \\ c_2 < 0 \qquad\quad , \quad c_2 \varepsilon C \quad , \quad |c_2| \leqq c_0 \ , \end{array}\right\} \tag{7}$$

such that the following inequalities hold

$$\bar{c} = \max \{ c_1 , |c_2| \} < \varepsilon/_2 , \tag{8}$$

$$\bar{P}(2\bar{c}) < {}^1/_8 \tag{9}$$

where $\bar{P}(y)$ is an increasing function such that if $v(x) \leqq y^2 P^2(y)$ we have $\|x\| < y\bar{P}(y)$.

For the chosen numbers c_1, and c_2 we take $\delta > 0$ such that the following inequalities hold

$$\delta < {}^1/_8 \min \{ c_1 , |c_2| \} , \tag{10}$$

and

$$v(x(t) - u(c)) < 4\bar{c}^2 P^2(2\bar{c}) \text{ for } \|x(t) - u(c)\| < \delta . \tag{11}$$

We shall try to prove now that for every solution of system (1), for which the initial data satisfy conditions :

$$|y(t_0) - c| < \delta \ , \ \|x(t_0) - u(c)\| < \delta \tag{12}$$

the following inequalities

$$|y(t) - c| < 2\bar{c}, \ \ldots \tag{13}$$

$$\|x(t) - u(c)\| < 2\bar{c} , \ \ldots \tag{14}$$

hold for all $t \geq t_0$

These two inequalities combined with (8) show that the nonzero solution (5) is stable.

On the contrary, suppose that, there exists a solution $y(t)$, $x(t)$ of system (1) for which the initial data satisfy conditions (12), and a number $T > t_0$, such that for any $t \in [t_0, T)$ inequalities (13) and (14) satisfied, save for at $t = T$ where at least one of them turn into an equality.

We prove first that for this solution the inequality

$$v(x(t) - u(c)) < 4\bar{c}^2 P^2(2\bar{c}), \cdots \qquad (15)$$

holds for all $t \in [t_0, T]$.

According to (11) inequality (15) takes place for $t = t_0$.

Suppose, that inequality (15) is broken first at

$$t = t^* \varepsilon (t_0, T].$$

From inequality (13) and in virtue of the monotonic behaviour of the function $P(y)$, it follows that

$$v(x(t^*) - u(c)) \geq (y - c)^2 P^2(y - c),$$

and then from Lemma (1), it follows that

$$\dot{v}(x(t^*) - u(c)) < 0.$$

This contradicts the definition of t^*.

Therefore

$$v(x(t) - u(c)) < 4\bar{c}^2 P^2(2\bar{c})$$

for all $t \varepsilon [t_0 \cdot T]$, and consequently

$$\| x(t) - u(c) \| < 2\bar{c}\bar{P}(2\bar{c}) < 2\bar{c} \text{ for all } t \varepsilon [t_0, T]. \qquad (16)$$

Thus from the definition of T, it follows that

$$| y(t) - c | = 2\bar{c},$$

which means that

either $y(T) - c = 2\bar{c}$,

or $\quad y(T) - c = -2\bar{c}$.

We shall prove that, these two possibilities are not true.

Consider the following function

$$\gamma(t) = c_1 - c + f(c_1, x(t) - u(c_1), t) - (y(t) - c) \qquad (17)$$

where y (t) , x (t) is the chosen solution and the function f is defined in Lemma (2).

Since from Lemma (1).

$$\| u(c_j) \| < | c_j | \overline{P}(c_j), \ (j = 1, 2), \tag{18}$$

then we have from inequalities (7), (8), (9) and (16)

$$\| x(t) - u(c_1) \| \leq \| x(t) - u(c) \| + \| u(c) \| + \| u(c_1) \| <$$

$$< 2\bar{c}\,\overline{P}(2\bar{c}) + c\,\overline{P}(c) + c_1\overline{P}(c_1) < \frac{7}{16}\bar{c} < a_0,$$

for all $t \in [t_0, T]$.

Thus, the function $\gamma(t)$ is defined and continuous for all $t \in [t_0, T]$.

From inequalities (7) and (10) and from lemma (2) it follows that,

$$c_1 - c + f(c_1, x(t_0) - u(c_1), t_0) > c_1 - c - |f| > \frac{3}{16} c_1. \tag{19}$$

From the inequality $| y(t_0) - c | < \delta < {}^1/_8\, c_1$

it then follows that

$$\gamma(t_0) > 0 \tag{20}$$

By the same way, using (7), (9), (16), (18) and lemma (2) we can prove, that

$$c_1 - c + f(c_1, x(T) - u(c_1), T) < \frac{29}{16}\bar{c},$$

and since $y(T) - c = 2\bar{c}$ then , it follows, that

$$\gamma(T) < 0 \tag{21}$$

From (20) and (21) it follows that, there exists an instance $t' \in (t_0, T)$, where $\gamma(t') = 0$,

i.e. $$y(t') - c = c_1 - c + f(c_1, x(t') - u(c_1), t') \tag{22}$$

Considering now the variables $\alpha(t)$ and $\Psi(t)$, we have

$$\alpha(t) = y'(t) - c_1, \tag{23}$$

$$\psi(t) = x(t) - u(c_1) \tag{24}$$

From (22), it follows that, for $t = t'$

$$\psi = x(t') - u(c_1) = \bar{a} \qquad \text{say,} \qquad (25)$$

and

$$\alpha = f(c_1, \bar{a}, t'). \qquad (26)$$

As we have proved above, the following inequality

$$\| x(t) - u(c_1) \| < \frac{7}{16} \bar{c} < a_0$$

holds for all $t \varepsilon [t_0, T]$.

Then from (25) it follows that $\| \bar{a} \| < a_0$, and from lemma (2), we have

$$| \alpha(t) | \leq \| x(t) - u(c_1) \| < \frac{7}{16} \bar{c},$$

for all $t \geq t'$.

Now from (7), (8) and (23) we have

$$| y(t) - c | \leq | y(t) - c_1 | + c_1 + c < \frac{7}{16} \bar{c} + c_1 + \frac{1}{2} c_1 < 2 \bar{c}$$

for all $t \geq t'$.

This inequality contradicts our assumption that

$$y(t) - c = 2 \bar{c} \text{ for } t = T > t'.$$

This proves that

$$y(T) - c < 2 \bar{c}. \qquad (27)$$

Similarly we can prove, that

$$y(T) - c > -2 \bar{c} \qquad (28)$$

From (27) and (28) it follows that

$$| y(T) - c | < 2 \bar{c},$$

which proves the first part of the theorem.

ii) The second part can be proved on similar lines as the proofs of Krasovesky's theorem [3] and the first part of this theorem.

REFERENCES

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[3] Krasovesky N. N. « The stability of motion in the case of one Zero root » (Mat. Sbornik), vol. 37, 1955 (Russian).