# ON THE SMOOTHNESS OF THE SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF QUASILINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

By

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§ 1. Def :

1. A function u (x) defined on the set of points A is said to satisfy Hölder's condition with exponent $\alpha$ $(0 < \alpha < 1)$ if the ratio

$$\frac{| u(P) - u(Q) |}{\overline{PQ}^{\alpha}}$$

is uniformly bounded above for any two points P and Q from A.

2. The space $C_{m+\alpha}$ (A) ( m nonnegative integer, and $0 < \alpha < 1$) is the space of all m-times differentiable functions in A, the $m^{th}$ derevative of which satisfy Holder's condition with exponent $\alpha$.

§ 2. Consider the convex domain G, with boundary $\Gamma$ consisting of finite number of curves $\Gamma_k$ . $k = 1, 2, \ldots, n$. The two curves $\Gamma_k$ and $\Gamma_{k+1}$ intersect at $( a_k , b_k )$ with angle $\Theta_k$ ,$(0 < \Theta_k < \pi)$. The curve $\Gamma_k$ may be represented parametrically in the form $x = x(s)$ and $y = y(s)$ where s is the arc length measured from an end of $\Gamma_k$ and $x(s) \varepsilon C_{m+2+\alpha}$ and $y(s) \varepsilon C_{m+2+\alpha}$ $(m \geq 0 , 0 < \alpha < 1)$.

On $\Gamma_k$, a function $\Phi_k(s) \varepsilon C_{m+2+\alpha}$ is defined, such that $\Phi_k = \Phi_{k+1}$ at the common point of $\Gamma_k$ and $\Gamma_{k+1}$.

Consider Drichlet's problem

$$a_{ij} ( x , y , u , u_x , u_y ) u_{ij} = b ( x , y , u , u_x , u_y ) , ( x , y ) \varepsilon G$$

$$u|_{\Gamma} = \Phi(s)$$

where :

i) $\Phi(s) = \Phi_k(s)$ on $\Gamma_k$ $\qquad k = 1, 2, \ldots , n$

and $u_{ij} = \frac{\partial^2 u}{\partial x_i \, \partial x_j}$ $(x_1 = x, \; x_2 = y)$ and repeated indices mean summation.

ii) The functions $a_{ij}$ (x, y, u, p, q) and b (x,y,u,p,q) are defined for (x,y) $\varepsilon$ G and all finite values of u, p and q, and belong to $C_{m+\alpha}$

iii) For (x, y) $\varepsilon$ G and any positive number K, such that $|u| + p + q \leq K$, $a_{ij}$ satisfy the inequality $m(K)(\eta_1^2 + \eta_2^2) \leq a_{ij}\,\eta_i\,\eta_j \leq M(K)(\eta_1^2 + \eta_2^2)$ for any real numbers $\eta_1$ and $\eta_2$, where m(K) and M(K) depend only upon K.

It is known [1] that, the solution u(x,y) of this problem, belongs to $C_{m+2+\beta}$ $(0 < \beta < \alpha)$every where in G except at the angular points

We now define the following numbers

1. $\zeta_{ij}^{(k)} = a_{ij} \{ a_k, b_k, \Phi_{l_1^k}(a_k, b_k) \cos \alpha_k + \Phi_{l_2^k}(a_k, b_k) \cos(\theta_k + \alpha_k), \Phi_{l_1^k}(a_k, b_k) \sin \alpha_k + \Phi_{l_2^k}(a_k, b_k) \sin(\theta_k + \alpha_k) \}$

where $\Phi_{l_1^k}(a_k, b_k)$ and $\Phi_{l_2^k}(a_k, b_k)$ are the derevatives of $\Phi$ in the directions of the tangents to the two curves $\Gamma_k$ and $\Gamma_{k+1}$ respectively.

i.e. In the directions making with the x-axis the angles $\theta = \alpha_k$ and $\theta = \alpha_k + \theta_k$

2. $$\omega_k = \tan^{-1} \frac{[\zeta_{11}^{(k)} \zeta_{22}^{(k)} - \zeta_{12}^{(k)2}]^{1/2}}{\zeta_{22}^{(k)} \cot \theta_k - \zeta_{12}^{(k)}}$$

**Theorem :**

In the neighbourhood of the angular point $(a_k, b_k)$, there exists a number $\beta \leq \alpha$ such that

i) If $\frac{\pi}{\omega_k} > m + 2 + \beta$, then $u \; \varepsilon \; C_{m+2+\beta}$

ii) If $\frac{\pi}{\omega_k} \leq m + 2 + \beta$, then $u \; \varepsilon \; C_{\frac{\pi}{\omega_k} - \varepsilon}$ $(\varepsilon > 0)$

iii) If $\frac{\pi}{\omega_k} < m + 2$, then there exist two numbers $\tau$ and $\tau_0$, $(0 < \tau, \tau_0 < 1)$ such that

$$r^{\tau} u^{\left( \left[ \frac{\pi}{\omega_k} - \varepsilon \right] + 1 \right)} \varepsilon\ C_{\tau_0}$$

where r is the distance from (x, y) to ( $a_k$ , $b_k$ ).

**Proof :** Without loss of generality, we can suppose that there exists only a single angle on the boundary Γ of the region, and that the x axis ($x \geq 0$) touches one branch of Γ

$$( \text{i. e. } n = 1 \ , \ \alpha_1 = 0 \ , \ 0_1 = 0 ).$$

Substituting the value of u(x,y) $\varepsilon$ $C_{1+\alpha}$ in the functions $a_{ij}$ and b, we obtain

$$a_{ij} ( x,y , u ( x , y ) , u_x ( x , y ) , u_y ( x , y ) ) = A_{ij} ( x , y )$$
$$b ( x,y , u ( x , y ) , u_x ( x , y ) , u_y ( x , y ) ) = B ( x , y ) .$$

The functions $a_{ij}$ and b belong to $C_{m+\alpha}$ thus the functions $A_{ij}$ and B belong to $C_\beta$ , where $\beta = \alpha$ if m > 0 and $\beta = \alpha^2$ if m = 0.

Consider now Drichlet's problem for the linear elliptic equation

$$A_{ij} ( x , y ) u_{ij} = B ( x , y ) \qquad ( x , y ) \varepsilon G$$
$$u_{|\Gamma} = \Phi \varepsilon C_{m+2+\alpha}$$

For this problem, it was proved [1] that if

$$A_{ij} ( x , y ) \varepsilon C_{m+\beta} \ , B ( x , y ) \varepsilon C_{m+\beta}$$

$$\text{and } \omega = \tan^{-1} \frac{[A_{11}(0,0) A_{22}(0,0) - A_{12}^2(0,0)]^{1/2}}{A_{22}(0,0) \cot \theta - A_{12}(0,0)}$$

then the following results are valid

i) If $\frac{\pi}{\omega} > m + 2 + \beta$ , then $u(x,y) \varepsilon C_{m+2+\beta} (\overline{G})$

ii) If $\frac{\pi}{\omega} \leqslant m + 2 + \beta$ , then $u(x,y) \varepsilon C_{\frac{\pi}{\omega} - \varepsilon} (\overline{G})$

iii) If $\frac{\pi}{\omega} < m + 2$, then there exist two constants $\tau$ and $\tau_0$ ; $0 < \tau , \tau_0 < 1$, such that

$$r^{\tau} u^{\left( \left[ \frac{\pi}{\omega} - \varepsilon \right] + 1 \right)} \varepsilon\ C_{\tau_0} (\overline{G})$$

where r is the distance from (x,y) to the origin.

Thus u(x,y) $\varepsilon \, C_{2+\beta}(\bar{G})$ if $\frac{\pi}{\omega} > 2 + \beta$ otherwise u $\varepsilon \, C_{\frac{\pi}{\omega}-\varepsilon}(\bar{G})$.

If $\frac{\pi}{\omega} > 2 + \beta$, we substitute the value of u(x,y) $\varepsilon \, C_{2+\beta}(\bar{G})$ in $a_{ij}$ and b to obtain $A_{ij}$ and B which belong to $C_{1+\beta}(\bar{G})$ and using the same result of the linear equations we conclude that u $\varepsilon \, C_{3+\beta}(\bar{G})$ if $\frac{\pi}{\omega} > 3 + \beta$ otherwise u $\varepsilon \, C_{\frac{\pi}{\omega}-\varepsilon}(\bar{G})$.

Repeating this process, we obtain that if $\frac{\pi}{\omega} > m + 2 + \beta$, then u $\varepsilon \, C_{m+2+\beta}(\bar{G})$ otherwise u $\varepsilon \, C_{\frac{\pi}{\omega}-\varepsilon}(\bar{G})$,

Similarly if $\frac{\pi}{\omega} < m + 2$, then we can find $\tau$ and $\tau_0$, $(0 < \tau, \tau_0 < 1)$, such that

$$r^{\tau} \; u^{\left(\left[\frac{\pi}{\omega} - \varepsilon\right] + 1\right)} \varepsilon \, C_{\tau_0}(\bar{G})$$

[1] A. H. Azzam ; Ph.D. thesis. Moscow State University 1967, Unpublished.