ANGLES BETWEEN SUBSPACES IN N-DIMENSIONAL VECTOR SPACE

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ABSTRACT : In this paper, the concept of angles between two subspaces V and W in n-dimensional vector space R_n is introduced, and some properties of these angles are given to be used in the proof of one theoren which determines in R_n the number of subspaces w of different dimensions, inclining to a fixed subspace V at the same fixed angle ψ . It is of interist to note that this theorem remains valid in 3-dimensional vector space even for ordinary angles between vectors.

INTRODUCTION : It is clear that in 3-dimensional vector space there exist only 2 subspaces of different dimensions, which are inclined to a fixed subspace V at the same angle α , $0 \leqslant \alpha \leqslant 90°$. Obvieusly, one of the two integers (3-dim V) and (4-dim V) equals 2, for dim V = 1 if V is a straight line, and dim V = 2 if V is a plane. Thus in 3-dimensional vector space the number of subspaces W of different dimensions, which are inclined to a fixed subspace V at the same angle α is equal to one of the two integers (3-dim V) and (4-dim V). This fact is generalized in the present paper, where the concept of angles between subspaces in unitary n-dimensional vector space R_n is introduced and a theorem is proved to assure that if in R_n a subspace V of dimension m is fixed, and an angle ψ is given in V, then the number of subspaces W of different dimensions, which are inclined to subspace V at the same angle ψ equals one of the two integers (n - m) and (n + 1 - m).

BASIC LEMMAS : Let V and W be two subspaces in R_n, and let P_V and P_W be projection operators of R_n onte subspaces V and W respectively. If P_V^W is the restriction of P_V on W, and P_W^V is the restriction of P_W on V, then

$$\psi = P_V^W P_W^V , \qquad \phi = P_W^V P_V^W \tag{1}$$

are two linear operators in V and W respectively.

DEFINITION : Linear operators ψ and ϕ determined by equalities (1) are called the angles between subspaces V and W. For angles ψ and ϕ we can use the notations:

$$\psi = \measuredangle (V,W) , \quad \phi = \measuredangle (W,V).$$

(ii) $\alpha_0 = 0$. Here we have

$$m = \alpha_1 + \alpha_2 + \dots + \alpha_s + \alpha ,$$

$$\dim W = m' = \alpha'_0 + m ,$$

$$\alpha'_0 = \dim (W \cap V^{\perp}).$$

Since W is a proper subspace in R_n , α'_0 must satisfy the condition

$$0 \leqslant \alpha'_0 \leqslant n-m-1$$

and so in this case there are (n-m) subspaces of dimensions m, m+1, ... , n-1.

(iii) $m > \alpha_0 > 0$. In this case

$$m = \alpha_0 + \alpha_1 + \alpha_2 \dots + \alpha_s + \alpha ,$$

$$\dim W = m' = \alpha'_0 + m - \alpha_0 ,$$

where α'_0 must satisfy the condition

$$0 \leqslant \alpha'_0 \leqslant n-m$$

and so in this case there are (n+1-m) subspaces W of dimensions $m-\alpha_0$, $m-\alpha_0+1$, ... , $n-\alpha_0$, and the Theorem is proved.

REMARK 1 : In cases (ii) and (iii) it is easy to show how to construct any subspace W of given dimension m' to be inclined to V at the angle $\psi = \measuredangle\ (V,W)$, where

$$m \leqslant m' \leqslant n-1 \qquad \text{in case (ii)}$$

$$m-\alpha_0 \leqslant m' \leqslant n-\alpha_0 \qquad \text{in case (iii)}$$

REMARK 2 : Examples show that two different subspaces W_1 and W_2 of the same dimension may satisfy the condition $\measuredangle(V,W_1) = \measuredangle\ (V,W_2)$. It is not difficult to prove that two subspaces W_1 and W_2 of the same dimension m' are inclined to subspace V at the same angle $\psi = \measuredangle\ (V,W_1) = \measuredangle\ (V,W_2)$ if and only if there exists in R_n a unitary operator τ, which transforms W_1 into W_2 and induces in V a unit operator, i.e. τ is such that

$$\tau W_1 = W_2\ , \quad \tau v = v \quad \forall v \in V.$$

REMARK 3 : The concept of angles between subspaces in R_n is a rich concept which has important applications (see [2] and [3]).

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PROOF : Here also the proof is given only for ψ . Let $\psi(x) = \lambda\, x$. Since the length of any vector cannot be increased by projection, therefore

$$\|x\| \geqslant \|P_W(x)\| \geqslant \|P_V P_W(x)\| = \|\psi(x)\| = \lambda \|x\|$$

Corollary 1 and relation $\|x\| \geq \lambda \|x\|$ imply that $0 \leqslant \lambda \leqslant 1$. To find the multiplicity of eigenvalue 1 for operator ψ let, on one hand, $x \in (V \cap W)$; therefore

$$\psi(x) = P_V\, P_W\, (x) = P_V\, (x) = x\,.$$

On the other hand, if $\psi(x) = x$, then

$$\|x\| \geqslant \|P_W(x)\| \geqslant \|\psi(x)\| = \|x\|$$

i.e. $\|x\| = \|P_W(x)\|$, which can be valid only when $P_W(x) = x$, and consequently $x \in (V \cap W)$. Actually we obtained that $\psi(x) = x$ if and only if $x \in (V \cap W)$, and the Lemma is proved.

LEMMA 3 : Operators ψ and ϕ have the same eigenvalues, different from zero. Moreover, each of these eigenvalues has the same multiplicity for ψ and ϕ.

PROOF : Let $\psi(x) = \lambda x$, where $x \in V$, $\lambda \neq 0$. Then we obtain

$$\phi(P_W(x)) = P_W P_V (P_W(x))$$
$$= P_W (P_V P_W(x))$$
$$= P_W (\psi(x))$$
$$= P_W (\lambda x)$$
$$= \lambda (P_W(x))$$

i.e. λ is also an eigenvalue for operator ϕ, where corresponding eigenvector is $P_W(x)$. Analogically, if $\phi(y) = \mu y$, $y \in W$, $\mu \neq 0$, then we obtain $\psi(P_V(y)) = \mu (P_V(y))$. Thus we proved that ψ and ϕ have the same eigenvalues, different from zero. Now, suppose that $\lambda \neq 0$ is an eigenvalue for ψ and ϕ with multiplicities β and γ rrespectively. Let $x_1, x_2, \ldots, x_\beta$ be complete orthonormal system of eigenvectors for operator ψ, which corresponds to eigenvalue λ. From equality (2) we obtain

$$(P_W(x_i).P_W(x_j)) = (\psi(x_i).x_j)$$

$$= \lambda (x_i . x_j)$$

$$= \lambda \delta_{ij}$$

where δ_{ij} is the Kronecker sympol, i.e. $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$; $i,j = 1,2, \dots, \beta$. Consequently, $P_W(x_1), \dots, P_W(x_\beta)$ is an orthogenal system of eigenvectors for operator $\varnothing$, corresponding to eigenvalue λ. Actually we proved that $\beta \leqslant \gamma$. Analogically we can prove that $\gamma \leqslant \beta$ and consequently $\beta = \gamma$. Proof is accomplished

From lemmas 1-3 we deduce that if $0, \lambda_1, \lambda_2, \dots, \dots, \lambda_s, 1$, $(0 < \lambda_1 < \lambda_2 \dots < \lambda_s < 1)$, are the eigenvalues of the operator ψ with multiplicities $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_s, \alpha$ respectively, then $0, \lambda_1, \lambda_2, \dots, \lambda_s, 1$ are also the eigenvalues of the operator $\varnothing$ with multiplicities $\alpha_0', \alpha_1, \alpha_2, \dots, \alpha_s, \alpha$ where

$$\alpha_0 = \dim (V \cap W^{\perp}), \quad \alpha_0' = \dim (W \cap V^{\perp}), \quad \alpha = \dim (V \cap W).$$

It is also clear that

$$\dim V = m = \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_s + \alpha,$$

$$\dim W = m' = \alpha_0' + \alpha_1 + \alpha_2 + \dots + \alpha_s + \alpha.$$

and, consequently, $m = m'$ if and only if $\alpha_0 = \alpha'_0$.

Now we are ready to prove the following theorem which is the main aim of the present paper.

<u>THEOREM</u> : Let in R_n a proper subspace V of dimension m is fixed, and let φ be a given angle in V. Then the number of proper subspaces W of different dimensions, which are inclined to V at the same angle $\varphi = \measuredangle (V,W)$ is equal to $(n+1-m)$ or $(n-m)$ according as α_0 satisfies or does not satisfy the condition $m > \alpha_0 > 0$ respectively.

<u>PROOF</u> : Let the eigenvalues of operator φ be $0, \lambda_1, \lambda_2, \ldots, \lambda_s, 1$, $(0 < \lambda_1 < \lambda_2 \ldots < \lambda_s < 1)$, with multiplicities $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_s, \alpha$ respectively. Here α_0 plays the leading part. There are three possible cases.

(i) $\alpha_0 = m$. In this case V and W are multally orthogonal, and so W coincides with any subspace in $V^{\perp}$. Since $\dim V^{\perp} = n-m$, we can construct in $V^{\perp}$ subspaces W of dimensions 1,2, ..., n-m, where subspace W with dimension (n-m) coincides with $V^{\perp}$.

<u>LEMMA 1</u> : ψ and ϕ are positive-semidefinite operators in V and W respectively. Moreover, ψ and ϕ will be positive-definite if and only if $(V \cap W^{\perp}) = 0$ and $(W \cap V^{\perp}) = 0$ respectively.

<u>PROOF</u> : We prove the lemma only for the operator ψ . For any two vectors x and y from V we obtain

$$(\psi(x).y) = (P_V^W \, P_W^V \, (x).y)$$

$$= (P_V \, P_W \, (x).y)$$

$$= ((I-P_{V^{\perp}}) \, P_W \, (x).y)$$

$$= (P_W \, (x).y)$$

$$= (P_W(x).(P_W + P_{W^{\perp}}) \, (y))$$

$$\therefore \quad (\psi(x).y) = (P_W(x).P_W(y)) \qquad (2)$$

$$= ((I-P_{W^{\perp}})(x).P_W(y))$$

$$= (x.P_W(y))$$

$$= (x.(P_V + P_{V^{\perp}}) \, P_W(y))$$

$$= (x.P_V P_W \, (y))$$

$$= (x. \, P_V^W \, P_W^V \, (y))$$

$$\therefore \quad (\Psi(x).y) = (x.\Psi(y)) \qquad (3)$$

Equality (3) proves that Ψ is hermitian operator in V.

From (2) we obtain for every vector $x \in V$

$$(\Psi(x).x) = (P_W(x).P_W(x)) = \| P_W(x) \|^2 \geqslant 0$$

i.e. Ψ is positive-semidefinite operator in V. It is clear that Ψ will be positive-definite if and only if $P_{W^\perp}(x) \neq 0 \ \forall x \in V$, i.e. if and only if $(V \cap W^\perp)=0$, and the proof is finished.

<u>COROLLARY 1</u> : All eigenvalues of operators Ψ and $\emptyset$ are non-negative (see [1] , P. 274).

<u>COROLLARY 2</u>: Zere is an eigenvalue for Ψ and $\emptyset$ with multiplicities α_0 and α_0' respectively, where

$$\alpha_0 = \dim (V \cap W^\perp), \quad \alpha_0' = \dim (W \cap V^\perp).$$

<u>LEMMA 2</u> : All eigenvalues of operators Ψ and $\emptyset$ are included between 0 and 1, moreover, eigenvalue 1 has the same multiplicity α for Ψ and $\emptyset$, where $\alpha = \dim (V \cap W)$.

REFERENCES

1- F.R. Gantmacher, The theory of matrices, volume 1, 1960.

2- L.A. Kaluznin, H.M. Hawidi, Geometric theory of the unitary equivalence of matrices. (Russian). Dok l & Akad. Nauk SSSR, 169 (1966) 1009-1012.

3- H.M. Hawidi, Relative position of subspaces in a finite dimensional unitary space. (Russian). Ukrain. Math. Z. 18 (1966), No. 6, 130-134.

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