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المرحوم الأستاذ الدكتور عبد الفتاح اسماعيل

استاذ الكيمياء ، وكيل اول وزارة التعليم العالى

تنفيذا لقرار مجلس كلية البنات فى جلسته بتاريخ ٨ من ابريل سنة ١٩٧٨ يشرف حولية الكلية ان تنشر فيما يلى تعريفا بشخصية علمية كان لها دورها الريادى فى شتى المجالات العلمية والجامعية ، فى مصر ، وفى العالم العربى . شخصية المرحوم الأستاذ الدكتور عبد الفتاح اسماعيل .

هيئة التحرير

CURRICULUM VITAE

(February 1974)

Name : Prof. Dr. Abdel Fattah Ismail.

Position : Senior Under-Secretary of State,
Ministry of Higher Education.
Cairo Egypt.

Adress : 4, Ibrahim Naguib Street, Garden City, Cairo

Nationality : Egyptian

Marital Status : Married to :

— Samiha, Ph.D. Professor of Organic Chemistry, Ain Shams University, Cairo.

— Having a son and a Daughter :
Mohamed, Ph.D. Civil Eng.,
Riee University, U.S.A.

— Nabila, Ph.D. Assistant Professor
Biochemistry Cairo University.

Qualifications :

— B.Sc. (First Class Honours-Chemistry) Cairo University (1937)

— M.Sc. Organic Chemistry (1940)

— Ph.D. Organic Chemistry (1945)

Professional Career :

1. Occupied various academic posts at the Universities of Cairo (1937-44), Alexandria (1944-50), Baghdad (1949—delegated), Ain-Shams 1950-1951) Alexandria (1951-1957) and Kuwait (1966-72).

2. Secretary General of the Supreme Science Council, (1957-61) now Academy of scientific Research and Technology, Egypt.

3. Permanent Under-Secretary of State, Ministry of Higher Education, Egypt (1961-1966).

4. During the last two posts, he was offered the post of non-resident Professor of Physico-organic Chemistry at the University of Alexandria, Egypt.

5. In 1965-1966 delegated to Kuwait Government to establish Kuwait University, then appointed as Professor of Physico-organic Chemistry and Rector Kuwait University, Kuwait (1966-1972).

6. Senior Under-Secretary of State, Ministry of Higher Education Egypt, (1972 — ......)

Activities : (1972—1977)

1. Secretary-General, Egyptian National Commission for UNESCO.

2. Member of the Board of Directors of co-ordination Centre among the Arab National Commissions for UNESCO-RABAT.

3. President of the Egyptian National Commission for ALECSO.

4. Member of the Executive Board of ALECSO.

5. Member of the Board of the National Research Centre of Pedagogy, Egypt.

6. Member of the University Council of Cairo University.

7. Member of Board of Research in Basic Sciences, Academy of Scientific Research and Technology, Egypt.

8. Member of the supreme Council of Art, Literature and Social Sciences, Egypt.

9. Member of the Board of the Faculty of Science, Ain-Shams University, Cairo.

10. Member of the Supreme International Cultural Relation Committee, Egypt.

11. Member of the National Technical Assistance Commission, Egypt.

12. Member of the National Commission for International Conferences, Egypt.

Some Prdvious Activities :

1. Established the System of Admission and co-ordination of Students in Egyptian Universities (1953 — ......)

2. Supervised the first plan for scientific Research in Egypt (1960-1965). It included the establishment of a number of scientific Research Centers, as well as 5000 graduates whom he sponsored sending on scholarships abroad to qualify for the Ph.D. in various fields of pure and applied sciences. The Plan also comprised problems facing the National Economy in matters related to industry mining, agriculturr, medicine ...... etc. to be solved through financed scientific research programmes. This was accompanied by issuring the first guide for scientific personnel in Egypt (1958).

3. Participated on behalf of UAR in the «Auger Project» sponsored by UNESCO (1960) dealing with current trends in scientific Policies. Subsequently he was elected as Vice-President of the Commission held at UNESCO to put the outline of World Scientific co-operation during the period (1960-1970), then asked by UNESCO to write an article on «Current Trends of Scientific Policy in U.A.R.» published in «Impact of Science on Society» (1962).

4. Rector of Kuwait University (1966-1972).

5. Planning and Executing a new programme for sending 2000 graduates (1973-1976) to be sent on scholarships abroad to qualify for the Ph.D. in various fields. The Porgramme is dedicated for staffing Egyptian Universities and Higher Institutes.

6. Represented Egypt at several regional and international Conferences and presided or membered several delegations for drawing cultural and exective agreements with many countries.

7. Chairman of the Pullbright Commission for Exchange of Professors and students between Egypt and USA (1960-1966).

8. Member of the Supreme Council of Youth Wellfare in Egypt (1954-58) where he presided the Social Commission.

9. Member of the Egyptian National Commission for NUESCO (1959-1966).

10. President of the Executive Board of the Egyptian National Commission for UNESCO (1961-1966).

11. Member of the Board of the Union of Arab Universities (1966-1972).

12. Member of the Council of Cairo University (1961-1966).

13. Member of the Council of Alexandria University (1961-66).

14. Chairman of the Beards of Higher Institutes, Ministry of Higher Education (1961-1966), Egypt.

Research Work :

Published several research papers in different scientific journals in England, in the Field of polynuclear aromatic chemistry, some of which were mentioned as references in scientific text books. He has also written some books and articles in organic chemistry, scientific planning and scientific research policies.

Decorations :

Awarded several medals and decorations from Egypt, Lebanon and Italy.

CONTENTS

SYMMETRISED SPIN STATES : A SIMPLE EXPLICIT EXPANSION

By

NAHID G. I. EL SHARKAWAY * and F. AYOUB **

ABSTRACT

A simple explicit expansion (requiring neither Young operators vector-coupling coefficients) is given for the states of n spin ½ ticles characterized by the Young tableaux

$$\begin{vmatrix} 1 & & & \ldots n-1 \\ p_1 p_2 & \ldots & p_{m-1} n & \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & & & \ldots n \\ p_1 p_2 & \ldots & p_{m-1} n-1 & \end{vmatrix}$$

he irreducible representation [n-m, m] of S_n, taken in the standard ng ortyogonal form. The correctness of the expansion is established verifying that the states satisfy Young's theorem for the matrix $P_{n,n-1}$

Young's theorem :

$$\psi_n = \begin{vmatrix} 1 & & & \ldots n-1 \\ p_1 p_2 & \ldots & p_{m-1} n & \end{vmatrix} SM_s \rangle \qquad \text{— (1)}$$

d

$$\psi_{n-1} = \begin{vmatrix} 1 & & & \ldots n \\ p_1 p_2 & \ldots & p_{m-1} n-1 & \end{vmatrix} SM_s \rangle, \qquad \text{— (2)}$$

th

$$S = \frac{1}{2}(n-2m), \quad M_s = S, S-1, \ldots, -S \qquad \text{— (3)}$$

d

$$p_1 < p_2 < \ldots < p_{m-1} \qquad \text{— (4)}$$

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integers taken from the set (2, 3, ..., n-2), denote two vector-coupled spin states of n spin ½ particles, the transformation proparties of these states under permutation of the particle numbers being determined by the given Young-Yamanonchi's labels of the irreducible representation [n-m, m] of S_n the standard Young orthogonal form for this representation being understood.

Young's theorem states that

$$P_{n,n-1}\,\psi_n = -\frac{1}{h}\,\psi_n + \frac{\sqrt{(h^2-1)}}{h}\,\psi_{n-1}, \quad \text{— (5)}$$

$$P_{n,n-1}\,\psi_{n-1} = +\frac{1}{h}\,\psi_{n-1} + \frac{\sqrt{(h^2-1)}}{h}\,\psi_n, \quad \text{— (6)}$$

where

$$h = n - 2m + 1 \quad \text{— (7)}$$

is the axial distance from n to n-1 for these state labels (see Figure 1).

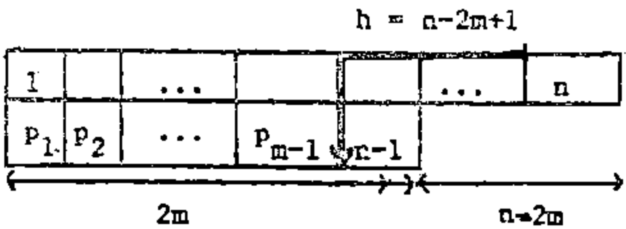

Figure 1. *Axial distance from square labelled n to spuare labelled n-1 in a Young tableau of [n-m, m].*

§ 2. *Notation and lemma*

Using the usual notation,

$$\alpha, \beta \quad \text{——— (8)}$$

denote single particle spin states having

$$m_s = +\tfrac{1}{2}, -\tfrac{1}{2} \quad \text{——— (9)}$$

respectively. We use the abbreviation

$$\begin{vmatrix}1\\2\end{vmatrix}^{\,3} = (\alpha_1\beta_2 - \beta_1\alpha_2)\alpha_3, \quad \text{——— (10)}$$

$$\alpha_i, \quad \beta_i \quad \text{——— (11)}$$

being the spin states for particle i.

We make repeated use of the lemma

$$\begin{vmatrix}1\\2\end{vmatrix}^{3} + \begin{vmatrix}2\\3\end{vmatrix}^{1} + \begin{vmatrix}3\\1\end{vmatrix}^{2} = 0. \qquad \text{——— (12)}$$

which follows immediately from

$$\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \alpha_1 & \alpha_2 & \alpha_3\end{vmatrix} = 0. \qquad \text{——— (13)}$$

§ 3. *The explicit expansions in the case* $M_s = S$:

Taking first, for simplicity, the special case $M_s = S$, we postulate that the following two sums (equations (14) and (20)) of m-fold products of two-particle determinants into (n-2m)-fold products of single particle α states, with the normalisation factors indicated, are the correctly normalised and symmetrised n-particle states.

$$\psi_n = \begin{vmatrix}1 & & & & \ldots n-1\\ p_1 p_2 & \cdots & p_{m-1} & n & \end{vmatrix} SS\Big\rangle$$

$$= \frac{1}{\sqrt{(N_n)}} \sum_{q_1=1}^{p_1-1} \begin{vmatrix}q_1\\p_1\end{vmatrix} \sum_{\substack{q_2=1\\ \neq q_1\\ \neq p_1}}^{p_2-1} \begin{vmatrix}q_2\\p_2\end{vmatrix} \cdots \sum_{\substack{q_{m-1}=1\\ \neq q_1,q_2,\ldots,q_{m-2}\\ \neq p_1,p_2,\ldots,p_{m-2}}}^{p_{m-1}-1} \begin{vmatrix}q_{m-1}\\p_{m-1}\end{vmatrix} \sum_{\substack{q_m=1\\ \neq q_1,q_2,\ldots,q_{m-1}\\ \neq p_1,p_2,\ldots,p_{m-1}}}^{n-1} \begin{vmatrix}q_m\\n\end{vmatrix} a_1 a_2 \ldots a_{h-1}. \qquad \text{——— (14)}$$

where

$$a_1 < a_2 < \ldots \leq a_{h-1} \qquad \text{——— (15)}$$

are the

$$h - 1 = n - 2m \qquad \text{——— (16)}$$

numbers remaining when the 2m-1 distinct numbers

$$q_1, \ldots, q_m \;, p_1, P_2, \ldots, P_{m-1} \qquad \text{——— (17)}$$

are removed from the set (1,2, ..., n-1) and the normalising factor N_n is given by

$$N_n = p_1(p_1-1)(p_2-2)(p_2-3)\ldots\{p_{m-1}-(2m-4)\}\{p_{m-1}-(2m-3)\}\{n-(2m-2)\}\{n-(2m-1)\} \qquad \text{——— (18)}$$

Iote, for later use, that the last two terms in N_n may be written as

$$\{n - (2m-2)\}\{n-(2m-1)\} = (h+1)h.$$

Similarly

$$\psi_{n-1} = \begin{vmatrix} 1 & & & & \dots n \\ p_1 p_2 & \dots & p_{m-1} & n-1 & \end{vmatrix} SS\rangle \qquad \text{——— (19)}$$

$$= \frac{1}{\sqrt{(N_{n-1})}} \sum_{q_1=1}^{p_1-1} \begin{vmatrix} q_1 \\ p_1 \end{vmatrix} \sum_{\substack{q_2=1 \\ \neq q_1 \\ \neq p_1}}^{p_2-1} \begin{vmatrix} q_2 \\ p_2 \end{vmatrix} \cdots \sum_{\substack{q_{m-1}=1 \\ \neq q_1,q_2,\dots,q_{m-2} \\ \neq p_1,p_2,\dots,p_{m-2}}}^{p_{m-1}-1} \begin{vmatrix} q_{m-1} \\ p_{m-1} \end{vmatrix} \sum_{\substack{q_m=1 \\ \neq q_1,q_2,\dots,q_{m-1} \\ \neq p_1,p_2,\dots,p_{m-1}}}^{n-2} \begin{vmatrix} q_m \\ n-1 \end{vmatrix} a_1 a_2 \dots a_{h-2} n,$$

——— (20)

th now

$$a_1 < a_2 < \dots \leq a_{h-2} \qquad \text{——— (21)}$$

ing the

$$h - 2 = n - 2m - 1 \qquad \text{——— (22)}$$

mbers remaining when the 2m-1 distinct numbers

$$q_1, q_2, \dots, q_m \quad p_1, p_2, \dots, p_{m-1} \qquad \text{——— (23)}$$

removed from the set (1,2, ..., n-2) and

$$\cdot\; p_1(p_1-1)(p_2-2)(p_2-3)\dots\{p_{m-1}-(2m-4)\}\{p_{m-1}-(2m-3)\}\{(n-1)-(2m-2)\}\{(n-1)-(2m-1)\}$$

——— (24)

ote that the last two terms in N_{n-1} may be written as

$$n\text{-}1) - (2m-2)\ ]\ [\ (n-1) - (2m-1)\ ] = h\ (h-1). \qquad \text{——— (25)}$$

Proof of Young's theorem for ψ_n *and* Ψ_{n-1}

before we use throughout

$$h = n - 2m + 1 \qquad \text{——— (26)}$$

l use

$$a_2 \dots a_i \dots a_{h-1} \qquad (i = 1,2, \dots h-1) \qquad \text{——— (27)}$$

denote the set $a_1 a_2 \dots a_{h-1}$ with a omitted.

the

By comparing ψ_n and ψ_{n-1}, taking note of the form of the last two terms of N_n and N_{n-1}, we see, since $P_{n,n-1}$ affects only the following respective parts of ψ_n and ψ_{n-1} denoted by R_n and R_{n-1}:

$$R_n = \frac{1}{\sqrt{\{(h+1)h\}}}\left[\sum_{i=1}^{h-1}\left\{\left|\begin{matrix}a_i\\ n\end{matrix}\right| a_1 a_2 \ldots \dot{a}_i \ldots a_{h-1}\right\}(n-1) + \left|\begin{matrix}n-1\\ n\end{matrix}\right| a_1 a_2 \ldots a_{h-1}\right]. \qquad (28)$$

$$R_{n-1} = \frac{1}{\sqrt{\{(h-1)h\}}}\left[\sum_{i=1}^{h-1}\left\{\left|\begin{matrix}a_i\\ n-j\end{matrix}\right| a_1 a_2 \ldots \dot{a}_i \ldots a_{h-1}\right\} n\right]. \qquad (29)$$

Young's theorem for Ψ_n, Ψ_{n-1} is then established if we can prove that

$$P_{n,n-1} R_n = -\frac{1}{h} R_n + \frac{\sqrt{(h^2-1)}}{h} R_{n-1}, \qquad (30)$$

$$P_{n,n-1} R_{n-1} = +\frac{1}{h} R_{n-1} + \frac{(h^2-1)}{h} R_n. \qquad (31)$$

For the L.H.S. of (30) we have

$$P_{n,n-1} R_n = \frac{1}{\sqrt{\{(h+1)h\}}}\left[\sum_{i=1}^{h-1}\left\{\left|\begin{matrix}a_i\\ n-1\end{matrix}\right| a_1 a_2 \ldots \dot{a}_i \ldots a_{h-1}\right\} n - \left|\begin{matrix}n-1\\ n\end{matrix}\right| a_1 a_2 \ldots a_{h-1}\right]. \qquad (32)$$

whilst for the R.H.S. of (30) we have

$$-\frac{1}{h} R_n + \frac{\sqrt{(h^2-1)}}{h} R_{n-1}$$

$$+ \frac{1}{h\sqrt{\{(h+1)h\}}} \left[\begin{matrix} -\sum_{i=1}^{h-1} \left\{ \left| \begin{matrix} a_i \\ n \end{matrix} \right| a_1 a_2 \dots \dot{a}_i \dots a_{h-1} \right\} (n-1) \\ - \left| \begin{matrix} n-1 \\ n \end{matrix} \right| a_1 a_2 \dots a_{h-1} \\ + (h+1) \sum_{i=1}^{h-1} \left\{ \left| \begin{matrix} a_i \\ n-1 \end{matrix} \right| a_1 a_2 \dots \dot{a}_i \dots a_{h-1} \right\} n \end{matrix} \right] \qquad (33)$$

Using then the lemma (12) in the form

$$\left| \begin{matrix} a_i \\ n-j \end{matrix} \right| n + \left| \begin{matrix} n-1 \\ n \end{matrix} \right| a_i + \left| \begin{matrix} n \\ a_i \end{matrix} \right| (n-1) = 0, \qquad (34)$$

we have

$$\sum_{i=1}^{n-1} \left\{ \left| \begin{matrix} a_i \\ n-1 \end{matrix} \right| a_1 a_2 \dots \dot{a}_i \dots a_{h-1} \right\} n$$

$$= -(h-1) \left| \begin{matrix} n-1 \\ n \end{matrix} \right| a_1 a_2 \dots a_i \dots a_{h-1}$$

$$+ \sum_{i=1}^{h-1} \left\{ \left| \begin{matrix} a_i \\ n \end{matrix} \right| a_1 a_2 \dots \dot{a}_i \dots a_{h-1} \right\} (n-1) \qquad (35)$$

and applying this to the + 1 part of the last term in (33) we find R.H.S. of (30)

$$= \frac{1}{h\sqrt{\{(h+1)h\}}} \left[\begin{matrix} + h \sum_{i=1}^{h-1} \left\{ \left| \begin{matrix} a_i \\ n-1 \end{matrix} \right| a_1 a_2 \dots \dot{a}_i \dots a_{h-1} \right\} n \\ - h \left| \begin{matrix} n-1 \\ n \end{matrix} \right| a_1 a_2 \dots a_{h-1} \end{matrix} \right] \qquad (36)$$

= L.H.S. of (30).

Similarly

L.H.S. of (31) $= P_{n,n-1} \; R_{n-1}$

$$= \frac{1}{\sqrt{\{(h-1)h\}}} \left[\sum_{i=1}^{h-1} \left\{ \begin{vmatrix} a_i \\ n \end{vmatrix} a_1 a_2 \ldots \dot{a}_i \ldots a_{h-1} \right\} (n-1) \right] \qquad (37)$$

and R.H.S. of (31) $= +\frac{1}{h} R_{n-1} + \frac{\sqrt{(h^2-1)}}{h} R_n$

$$= \frac{1}{h\sqrt{\{h(h-1)\}}} \left[\begin{array}{l} \sum_{i=1}^{h-1} \left\{ \begin{vmatrix} a_i \\ n-1 \end{vmatrix} a_1 a_2 \ldots \dot{a}_i \ldots a_{h-1} \right\} n \\ + (h-1) \sum_{i=1}^{h-1} \left\{ \begin{vmatrix} a_i \\ n \end{vmatrix} a_1 a_2 \ldots a_i \ldots a_{h-1} \right\} (n-1) \\ + (h-1) \begin{vmatrix} n-1 \\ n \end{vmatrix} a_1 a_2 \ldots a_{h-1} \end{array} \right] \qquad (38)$$

Application to this of (35) shows directly that R.H.S. of (31) = L.H.S. of (31) and Young's theorem is established.

5. Note on the normalisation : non-orthogonality of terms in the expansion.

It follows immediately, from the restrictions placed on the summation variables $q_1, q_2, \ldots q_{m-1}, q_m$, that the number of terms σ_n in the expansion for ψ_n is given by

$$\sigma_n = (p_1-1)(p_2-1-2) \ldots \{p_{m-1}-1-2(m-2)\}\{n-1-2(m-1)\}$$

$$= (p_1-1)(p_2-3) \ldots \{p_{m-1} - (2m-3)\}\{n-(2m-1)\} . \qquad (39)$$

Each term in this expansion is normalised by the factor.

$$1/\sqrt{(2^m)} \qquad \text{——— (40)}$$

(due to the occurrence in each term of m two-rowed determinants).
However N_n is not simply the product $2^m\sigma_n$: there is a correction due to non-orthogonality of the individual terms in the sum. In fact we have

$$N_n = 2^m \sigma_n \tau_n, \qquad \text{——— (41)}$$

where

$$\tau_n = p_1(p_2-2) \ldots \{p_{m-1} - (2m-4)\}\{n - (2m-2)\}/2^m$$

——— (42)

is the correction due to non-orthogonality. Similarly for Ψ_{n-1}

6. *Removal of the restriction $M_s = S$*

The expansion (14) for Ψ_n is valid only when $M_s = S$. This restriction is however easily removed. In each term of the expansion the m two-particle determinantal functions are singlet spin states and contribute nothing to M_s. Thus the maximum value $M_s = S$ is contributed by the totally symmetric spin state

$$\phi_{SS} = a_1 a_2 \ldots a_{h-1} \qquad \text{——— (43)}$$

for the remaining

$$h-1 = n - 2m \qquad \text{——— (44)}$$

particles. These thus have their spins coupled to the maximum resultant S = ½ (n—2m) and the corresponding spin states are totally symmetric.
The general spin state with $M_s = S-r$ is given by

$$\phi_{SM_S} = \frac{1}{\sqrt{\binom{h-1}{r}}} \sum_{i_1<i_2\ldots<i_r}^{h-1} a_1 a_2 \ldots \dot{a}_{i_1} \bar{a}_{i_1} \ldots \dot{a}_{i_2} \bar{a}_{i_2} \ldots \dot{a}_{i_r} \bar{a}_{i_r} \ldots a_{h-1},$$

——— (45)

where, in keeping with the previous notation,

a_i represents the α spin state α_{a_i} for particle numbered a_i and now, using a new notation,

$\bar{a}_i$ represents the β spin state β_{a_i}. for particle numbered a_i and $\dot{a}_i\bar{a}_i$ is used to represent the replacement of α_{a_i} by β_{a_i}. (45) is thus the totally symmetric normalised combination obtained from (43) by replacing in all possible ways r of the α states by β states leading to $M_S = S-r$. It is well-known also that this is the correct vector-coupled state when the coupling, as here, is to a maximum resultant.

Thus the expansion for the general case

$$\Psi_n = \left| \begin{matrix} 1 & & \\ p_1 p_2 \ldots p_{m-1} n & & \ldots n-1 \end{matrix} \right| SM_S \Big\rangle \qquad (46)$$

with $M_S = S-r, \quad r = 0,1,2, \ldots, 2S = h-1 \qquad (47)$

is obtained from the expansion (14) valid for $M_s = S$ by replacing in each term

$$a_1 a_2 \ldots\ldots a_{h-1}$$

by the normalised totally symmetric linear combination (45).

The similar result holds also for Ψ_{n-1} (equation 20) where now, of course, we may have n replaced by $\bar{n}$ (i.e. α_n replaced by β_n).

7. *The special case of states symmetric or antisymmetric in the particle numbers n—1 and n*

For the representation [n—m.m] or S_n there are standard Young tableaux in which n—1 and n occur at the end of a row and Young's theory demands in this case for the corresponding base vectors symmetry in the particle numbers n—1 and n. In addition for n even (= 2m) there are standard tableaux in which n—1 and n occur in the last column corresponding to antisymmetry in n—1 and n.

Expansions for these special cases can be written down in the same way as for Ψ_n and Ψ_{-} and the above properties verified.

We consider three cases in turn, taking again for simplicity $M_s = S$, the extension to general M_s being made as in §6.

Case 1 : n—1 and n both in the second row

with

$$N_{n-1,n} = p_1(p_1-1)(p_2-2)(p_2-3) \dots \{p_{m-2}-(2m-6)\}\{p_{m-2}-(2m-5)\}$$

$$\times \{n-1-(2m-4)\}\{n-1-(3m-3)\}\{n-(2m-2)\}\{n-(2m-1)\},$$

We have $\qquad$ ——— (48)

$$\psi_{n-1,n} = \begin{vmatrix} 1 & & & & & \dots \\ p_1 & p_2 & \dots & p_{m-2} & n-1 \;\; n & \end{vmatrix} SS\Big\rangle$$

$$= \frac{1}{\sqrt{(N_{n-1,n})}} \left[\sum_{q_1=1}^{p_1-1} \begin{vmatrix} q_1 \\ p_1 \end{vmatrix} \sum_{\substack{q_2=1 \\ \neq q_1 \\ \neq p_1}}^{p_2-1} \begin{vmatrix} q_2 \\ p_2 \end{vmatrix} \dots \sum_{\substack{q_{m-2}=1 \\ \neq q_1, q_2, \dots, q_{m-1} \\ \neq p_1, p_2, \dots, p_{m-1}}}^{p_{m-2}-1} \begin{vmatrix} q_{m-2} \\ p_{m-2} \end{vmatrix} \right.$$

$$\left. \times \sum_{\substack{i<j \\ 1}}^{n-2m+2} \left\{ \begin{vmatrix} a_i \\ n-1 \end{vmatrix} \begin{vmatrix} a_j \\ n \end{vmatrix} + \begin{vmatrix} a_j \\ n-1 \end{vmatrix} \begin{vmatrix} a_i \\ n \end{vmatrix} \right\} a_1 a_2 \dots \dot{a}_i \dots \dot{a}_j \dots a_{n-2m+2} \right]$$

(49)

where

$$(a_1 a_2 \dots a_i \dots a_j \dots a_{n-2m+2}), \quad i < j, \qquad \text{——— (50)}$$

is the set $(1,2 \dots\dots n-2)$ with $q_1 q_2 \dots\dots q_{m-2}\, p_1 p_2 \dots\dots p_{m-2}$ removed. Remembering that $P_{n,n-1}$ acts only on particle numbers and not on coefficients, it is immediately obvious from the form of the last term, which is the only place where the particle numbers n—1 and n occur, that

$$P_{n,n-1}\, \psi_{n-1,n} = +\, \psi_{n-1,n} \qquad \text{——— (51)}$$

as required.

Case 2 : n — 1 and n both in the first row

with

$$N^{n-1,n} = p_1(p_1-1)(p_2-2)(p_2-3) \ldots \{p_m-(2m-2)\}\{p_m-(2m-1)\}, \qquad (52)$$

we have

$$\psi^{n-1,n} = \left| \begin{matrix} 1 \\ p_1 p_2 \ldots p_m \end{matrix} \ldots n-1 \; n \right| SS\rangle$$

$$\frac{1}{\sqrt{(N^{n-1,n})}} \sum_{q_1=1}^{p_1-1} \left|\begin{matrix} q_1 \\ p_1 \end{matrix}\right| \sum_{\substack{q_2=1 \\ \neq q_1}}^{p_2-1} \left|\begin{matrix} q_2 \\ p_2 \end{matrix}\right| \ldots \sum_{\substack{q_m=1 \\ \neq q_1, q_2, \ldots, q_{m-1} \\ \neq p_1, p_2, \ldots, p_{m-1}}}^{p_m-1} \left|\begin{matrix} q_m \\ p_m \end{matrix}\right| \overline{a_1 a_2 \ldots a_{n-2m-2}(n-1)n} \; . \qquad (53)$$

with

$$(a_1 \; a_2 \ldots a_{n-2m-2}) = (1,2, \ldots n-2)_{\dot{q}_1 \dot{q}_2 \ldots \dot{q}_m \dot{p}_1 \dot{p}_2 \ldots \dot{p}_m} \qquad (54)$$

the particle numbers omitted from the set (1, 2 ...... n—2) being indicated by the particle numbers with dots, we see that each term in the expansion for $\psi^{n-1,n}$ contains the particle numbers n—1 and n only at the end in the form

$$(n-1)n = \alpha_{n-1} \; \alpha_n \qquad (55)$$

Case 3 : n even (= 2 m) with n—1 and n in the last column

which is symmetric in n—1 and n as required.

with

$$N^n_{n-1} = p_1(p_1-1)(p_2-2)(p_2-3) \ldots \left\{p_{\frac{n}{2}-1}-(n-4)\right\}\left\{p_{\frac{n}{2}-1}-(n-3)\right\}.2.1, \qquad (56)$$

(with the last two terms coming from $\{n-(2m-2)\}\{n-(2m-1)\}$ since $2m = n$),

$$\psi^n_{n-1} = \left| \begin{matrix} 1 & & n-1 \\ p_1 p_2 \ldots p_{\frac{n}{2}-1} & & n \end{matrix} \right| 00\rangle$$

$$= \frac{1}{\sqrt{(N^n_{n-1})}} \sum_{q_1=1}^{p_1-1} \begin{vmatrix} q_1 \\ p_1 \end{vmatrix} \sum_{\substack{q_2=1 \\ \neq q_1 \\ \neq p_1}}^{p_2-1} \begin{vmatrix} q_2 \\ p_2 \end{vmatrix} \cdots \sum_{\substack{q_{\frac{n}{2}-1}=1 \\ \neq q_1, q_2, \ldots, q_{\frac{n}{2}-2} \\ \neq p_1, p_2, \ldots, p_{\frac{n}{2}-2}}}^{p_{\frac{n}{2}-1}-1} \begin{vmatrix} q_{\frac{n}{2}-1} \\ p_{\frac{n}{2}-1} \end{vmatrix} \times \begin{vmatrix} n-1 \\ n \end{vmatrix} . \qquad (57)$$

Here the particle numbers n—1,n occur only in the last determinant

$$\begin{vmatrix} n-1 \\ n \end{vmatrix} \equiv \alpha_{n-1}\beta_n - \beta_{n-1}\alpha_n , \qquad (58)$$

which is clearly antisymmetric in n—1 and n.

Thus the constructed states have been shown to satisfy Young's requirements for any position of n—1 and n in the state labels.

§ 8. *«Internal» permutation symmetry : effect of $P_{n'-1}$ with $n' < n$.*

For the expansion in this paper to be proved correct not only must they agree with Young's theorem for $P_{n,n-1}$ («external» symmetry), they must also agree for all $P_{n'-1}$ with $n' < n$ («internal» symmetry).

We look in turn at the various cases.

Internal case 1 : $n'-1$ and n' on the second row

In this case we may write
so that the function label (for ψ_n, for example) has the from

$$p_{i-1} = n'-1, \quad p_i = n', \qquad (59)$$

$$\psi_{(n'-1,n')n} = \begin{vmatrix} 1 & & & & \cdots & n-1 \\ p_1 p_2 & \cdots & p_{i-2}(n'-1)n' & p_{i+1} & \cdots & n \end{vmatrix} SS\rangle . \qquad (60)$$

In the expansion for this state the particle numbers n'—1 and n' will occur only in the sum

$$\sum_{\substack{i<j \\ 1 \\ \neq q_1, q_2, \ldots, q_{i-2} \\ \neq p_1, p_2, \ldots, p_{i-2}}}^{n'-2} \left\{ \left|{}^{i}_{n'-1}\right| \left|{}^{j}_{n'}\right| + \left|{}^{j}_{n'-1}\right| \left|{}^{i}_{n'}\right| \right\} ,$$

——— (61)

the sums before this term all variables are less than n'—1 and ums after this term they are by definition $(p_{i-1}, p_i) - (n'-1, n')$ follows that the expansion has the required property

$$'-1 \quad (n'-1,n')n \;=\; +\psi_{(n'-1,n')n} \; .$$ ——— (62)

this argument holds for all the various possible external ry types

$$\psi_n, \; \psi_{n-1}, \; \psi_{n-1,n}, \; \psi^{n-1,n}, \; \psi_n^{n-1}$$ —— (63)

case 2 : n'—1 and n' in the first row

$$\psi_n = \left|{}^{1}_{p_1 p_2 \ldots p_{i-1} p_i \ldots p_{m-1} n} {}^{\ldots n-1}\right| SS\rangle$$ —— (64)

els in the first row are

$$(1 \ldots \dot{p}_1 \ldots \dot{p}_2 \ldots \ldots \dot{p}_{i-1} \ldots \dot{p}_i \ldots \ldots \dot{p}_{m-1} \ldots n-1)$$

ps at ——— (65)

$$p_1, p_2, \ldots, p_{i-1}, p_i, \ldots, p_{m-1},$$ ——— (66)

cutive pair of particle numbers n'—1, n' must avoid these gaps.

ollows that if, on the one hand, (n'—1,n') is in the interval

$$1 \ldots \dot{p}_i), \quad i = 1,2, \ldots, m-1 \quad \text{with} \quad \dot{p}_0 \equiv \dot{0},$$

——— (67)

sion for

$$\psi^{(n'-1,n')} = \left|\begin{matrix} 1\ldots\dot{p}_1\ldots\dot{p}_2\ldots\;\ldots\ddot{p}_{i-1}\ldots(n'-1)n'\ldots\dot{p}_i\ldots\;\ldots\dot{p}_{m-1}\ldots n-1 \\ p_1 p_2 \;\cdots\; p_{i-1} p_i \;\cdots\; p_{m-1} \end{matrix}\right| SS\rangle \qquad (68)$$

with contain the particle numbers n'—1,n' always in combinations of one of the following three forms :

$$\left|\begin{matrix} n'-1 \\ p_i \end{matrix}\right| \left|\begin{matrix} n' \\ p_k \end{matrix}\right| + \left|\begin{matrix} n' \\ p_i \end{matrix}\right| \left|\begin{matrix} n'-1 \\ p_k \end{matrix}\right| , \quad k = i+1, i+2, \ldots, m \quad \text{with} \quad p_m = n, \qquad (69)$$

$$\left|\begin{matrix} n'-1 \\ p_i \end{matrix}\right| n' + \left|\begin{matrix} n' \\ p_i \end{matrix}\right| (n'-1), \qquad (70)$$

$$(n'-1)n'. \qquad (71)$$

If, on the other hand, (n'—1) occurs in the last interval :

$$(\dot{p}_{m-1} \;\ldots\ldots\; n-1) \qquad (72)$$

then it follows that in each term of the expansion the particle numbers n'—1 and n' will occur either in the combination

$$\left|\begin{matrix} n'-1 \\ n \end{matrix}\right| n' + \left|\begin{matrix} n' \\ n \end{matrix}\right| (n'-1) \qquad (73)$$

$$\text{or} \quad (n'-1)n'. \qquad (74)$$

Thus

$$P_{n',n'-1}\, \psi_n^{(n'-1,n')} = +\, \psi_n^{(n'-1,n')}. \qquad (75)$$

Similarly for others forms of external symmetry and general M .

Internal case 3 : n'—1 and n' (= 2m') in the same column

This case is easily treated because the particle numbers n'—1 and n' will occur in each term of the expansion in the form of a two-particle determinant (singlet state) and cannot therefore appear elsewhere in the expansion either before or after the determinant. In fact his singlet state for particles n'—1 and n' separates a singlet state for particles 1,2, ..., n'—2 from a spin state for particles n'+1, n'+2, ..., n.

An example will make this clear :

$$\begin{vmatrix} 1 & 2 & 5 & 7 & 8 \\ 3 & 4 & 6 & 9 & \end{vmatrix} \tfrac{1}{2}\tfrac{1}{2}\Big\rangle$$

$$= \frac{1}{\sqrt{(3.2.2.1.2.1.3.2)}} \left\{ \begin{vmatrix} 1 \\ 3 \end{vmatrix} \begin{vmatrix} 2 \\ 4 \end{vmatrix} + \begin{vmatrix} 2 \\ 3 \end{vmatrix} \begin{vmatrix} 1 \\ 4 \end{vmatrix} \right\} \begin{vmatrix} 5 \\ 6 \end{vmatrix} \left\{ \begin{vmatrix} 7 \\ 9 \end{vmatrix} 8 + \begin{vmatrix} 8 \\ 9 \end{vmatrix} 7 \right\}$$

$$= \frac{1}{2\sqrt{3}} \begin{Bmatrix} (1\bar{3}-\bar{1}3)(2\bar{4}-\bar{2}4) \\ +(2\bar{3}-\bar{2}3)(1\bar{4}-\bar{1}4) \end{Bmatrix} \sqrt{\frac{1}{2}}(5\bar{6}-\bar{5}6) \sqrt{\frac{1}{6}} \begin{Bmatrix} (7\bar{9}-\bar{7}9)8 \\ +(8\bar{9}-\bar{8}9)7 \end{Bmatrix}$$

$$= \frac{1}{\sqrt{12}} \{ 2\ (12\bar{3}\bar{4} + \bar{1}\bar{2}34) - (1\bar{2}\bar{3}4 + 1\bar{2}3\bar{4} + \bar{1}23\bar{4} + \bar{1}2\bar{3}4) \}$$

$$\times \frac{1}{\sqrt{2}}(56 + 56) \frac{1}{\sqrt{6}} \{2\ 78\bar{9} - (789 + 789)\}$$

$$= \phi_{oo}(1,2,3,4) \sqrt{\frac{1}{2}} \begin{vmatrix} 5 \\ 6 \end{vmatrix} \phi_{\frac{1}{2}\frac{1}{2}}(7,8,9).$$

In this example $n'-1 = 5$, $n' = 6$ and the singlet state $\sqrt{\frac{1}{2}}\begin{vmatrix} n'-1 \\ n' \end{vmatrix}$ separates a normalised singletstate $\phi_{oo}(1,2,\ldots,n'-2)$ from a normalised spin state $\phi_{ss}(n'+1, n'+2, \ldots, n)$. Note that n' must be even ($= 2m'$) and the state for particles $1,2, \ldots, n'-2$ is always singlet. Thus we may write, for $n' = p_i$,

$$\psi\begin{vmatrix} n'-1 & \\ & n' \end{vmatrix}_n = \begin{vmatrix} 1 & & & & n'-1 & & & \ldots n-1 \\ p_1 p_2 & \ldots & p_{i-2} & n'-2 & n' & p_{i+1} & \ldots\ p_{m-1} & n \end{vmatrix} SS\Big\rangle$$

$$= \phi_{oo}(1,2,\ \ldots,\ n'-2) \sqrt{\frac{1}{2}} \begin{vmatrix} n'-1 \\ n' \end{vmatrix} \phi_{SS}(n'+1, n'+2, \ldots, n) \qquad (77)$$

and we clearly always have

$$P_{n_1' n'-1}\, \psi\begin{pmatrix} n'-1 \\ n' \end{pmatrix} = -\psi\begin{pmatrix} n'-1 \\ n' \end{pmatrix}$$

for all kinds of external symmetry and all values of M_s .

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