YOUNG OPERATORS FOR THE REPRESENTATIONS $[n-1,1]$ and $[2\ 1^{n-1}]$ IN STANDARD ORTHOGONAL FORM

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Abstract :

Explicit expressions in term of ket and bra tableau operators formed from symmetrisers and antisymmetrisers and tableau permutations are given for the Young operators for the representations $[n-1,1]$ and $[2\ 1^{n-1}]$ of the symmetric group S_n in standard orthogonal form.

1. *Introduction :*

Jahn (1960) gave the young operators of S_n as linear combinations of two-sided products of Young operators of S_{n-1} with the particular transposition $P_{n,n-1}$. To reach an explicit expression a long chain of calculations is required. It is the aim of the present paper to simplify the young operatorexpansions for the two representations $[n-1,1]$ and $[2\ 1^{n-1}]$ of S_n in standard orthogonal from. A complete set of young operators for the two particular representations is explicity constructed and tabulated.

2. Young operators: notation and properties

The $(n-1)^2$ Young operators O^n_{ab} $(a,b = 2,3,\ldots,n)$ for the representation $[n-1,1]$ of S_n in standard orthogonal form are required to satisfy

$$O^n_{ab}\, O^n_{cd} = \delta_{bc}\, O^n_{ad} \;, \qquad \ldots (2.1)$$

$$P_{a,a+1}\, O^n_{ab} = (1/a)\, O^n_{ab} + \{(a^2-1)^{\frac{1}{2}}/a\} O^n_{a+1,b} \;, \qquad \ldots (2.2)$$

$$O^n_{ab}\, P_{b,b+1} = (1/b)\, O^n_{ab} + \{(b^2-1)^{\frac{1}{2}}/b\} O^n_{a,b+1} \;. \qquad \ldots (2.3)$$

Here the numerical labels a, b are abbreviations for the standard tableau labels

$$A_n = \begin{matrix} 1 \ldots \dot{a} \ldots n \\ a \end{matrix} = \begin{matrix} 1\;2 \ldots a-1\;\; a+1 \ldots n-1\;\; n \\ a \end{matrix} \;, \qquad \ldots (2.4)$$

$$B_n = \begin{matrix} 1 \ldots \dot{b} \ldots n \\ b \end{matrix} = \begin{matrix} 1\;2 \ldots b-1\;\; b+1 \ldots n-1\;\; n \\ b \end{matrix} \;, \qquad \ldots (2.5)$$

where, following the author's notation (El-Sharkaway 1975), $\dot{a}$ or $\dot{b}$ is used to denote the omission of a or b from 2...n. The coefficient 1/a occurs in (2.2) and 1/b in (2.3) because the Young axial distance from a+1 to a in A_n is +a and from b+1 to b in B_n is +b.

The $(n-1)^2$ Young operators O^n_{a*b*} $(a,b = 2,3,\ldots,n)$ for the representation $[21^{n-2}]$ of S_n taken in standard orthogonal form are required to satisfy

$$P_{a,a+1}\, O^n_{a*b*} = -(1/a) O^n_{a*b*} + \{(a^2-1)^{\frac{1}{2}}/a\} O^n_{a+1*b*} \;, \qquad \ldots (2.6)$$

$$O^n_{a*b*}\, P_{b,b+1} = -(1/b) O^n_{a*b*} + \{(b^2-1)^{\frac{1}{2}}/b\} O^n_{a*b+1*} \;. \qquad \ldots (2.7)$$

Here the starred numerical labels a*, b* are abbreviations for the standard tableau labels

$$A^*_n = 1a\times\ldots\dot{a}\ldots n = 1a\times 2\ldots a-1\ a+1\ldots n-1\ n\ , \qquad \ldots (2.8)$$

$$B^*_n = 1b\times\ldots\dot{b}\ldots n = 1b\times 2\ldots b-1\ b+1\ldots n-1\ n\ , \qquad \ldots (2.9)$$

where, using again the author's notation (El-Sharkaway 1975) the abbreviations

$$1a\times\ldots\dot{a}\ldots n = \begin{matrix}1a\\ \vdots\\ a-1\\ a+1\\ \vdots\\ n\end{matrix}\ , \quad 1b\times\ldots\dot{b}\ldots n = \begin{matrix}1b\\ \vdots\\ b-1\\ b+1\\ \vdots\\ n\end{matrix}\ , \qquad \ldots (2.10)$$

are employed to simplify the printing. The coefficients $-(1/a)$ and $-(1/b)$ occur in (2.6) and (2.7) because the Young axial distance from a+1 to a in (2.10) is $-a$ and the distance from b+1 to b is $-b$.

3. Symmetrisers and antisymmetrisers: properties

$\mathcal{S}$ and $\mathcal{A}$ are used to denote symmetrisers and antisymmetrisers defined by

$$\mathcal{S}_{1\ldots n} = (1/n!) \sum_{\text{all } n!\ P \text{ in } S_n} P \qquad \ldots (3.1)$$

$$\mathcal{A}_{1\ldots n} = (1/n!) \sum_{\text{all } n!\ P \text{ in } S_n} (-1)^p P\ , \quad (-1)^p \text{ parity of } P. \qquad \ldots (3.2)$$

$\mathcal{S}_{1\ldots n}$ is the Young operator for the totally symmetric representation $[n]$ of S_n and is totally symmetric i.e.

$$P\,\mathcal{S}_{1\ldots n} = \mathcal{S}_{1\ldots n}\,P = \mathcal{S}_{1\ldots n},\quad P \text{ in } S_n\ , \qquad \ldots (3.3)$$

$\mathcal{A}_{1\ldots n}$ is the Young operator for $[1^n]$ and satisfies

$$(-1)^p P\mathcal{A}_{1\ldots n} = \mathcal{A}_{1\ldots n}(-1)^p P = \mathcal{A}_{1\ldots n},\ (-1)^p \text{ parity of } P,\ P \text{ in } S_n.$$

. . . (3.4)

It follows from (3.3) and (3.4) that

$$\mathcal{S}_{1\ldots a}\mathcal{S}_{1\ldots n} = \mathcal{S}_{1\ldots n}\mathcal{S}_{1\ldots a} = \mathcal{S}_{1\ldots n},\ a \leq n, \quad \ldots (3.5)$$

$$\mathcal{A}_{1\ldots a}\mathcal{A}_{1\ldots n} = \mathcal{A}_{1\ldots n}\mathcal{A}_{1\ldots a} = \mathcal{A}_{1\ldots n},\ a \leq n, \quad \ldots (3.6)$$

these being special cases of a general relation (Jahn 1960 (3.14)). From

$$\mathcal{A}_{1a}\mathcal{S}_{1a} = \tfrac{1}{2}(I-P_{1a})\ \tfrac{1}{2}(I+P_{1a}) = 0 = \mathcal{S}_{1a}\mathcal{A}_{1a} \quad \ldots (3.7)$$

follows

$$\mathcal{A}_{1a}\mathcal{S}_{1\ldots n} = \underline{\mathcal{A}_{1a}\mathcal{S}_{1a}}\,\mathcal{S}_{1\ldots n} = \mathcal{S}_{1\ldots n}\,\underline{\mathcal{S}_{1a}\mathcal{A}_{1a}} = \mathcal{S}_{1\ldots n}\mathcal{A}_{1a} = 0$$

. . . (3.8)

and more generally

$$\mathcal{A}_{1\ldots a}\mathcal{S}_{1\ldots n} = \mathcal{S}_{1\ldots n}\mathcal{A}_{1\ldots a} = \mathcal{A}_{1\ldots n}\mathcal{S}_{1\ldots a} = \mathcal{S}_{1\ldots a}\mathcal{A}_{1\ldots n} = 0,$$

$a \leq n$. . . . (3.9)

A basic property of the symmetrisers and antisymmetrisers, following directly from the definitions (3.1), (3.2) is given by

$$\mathcal{S}_{1\ldots n} = \{(I+P_{1n}+P_{2n}+\ldots+P_{n-1,n})/n\}\,\mathcal{S}_{1\ldots n}$$

$$= S_{1\ldots\dot{n}}\,\{(I+P_{1n}+P_{2n}+\ldots+P_{n-1,n})/n\}\,, \qquad \ldots (3.10)$$

$$\mathcal{A}_{1\ldots n} = \{(I-P_{1n}-P_{2n}-\ldots-P_{n-1,n})/n\}\mathcal{A}_{1\ldots\dot{n}}$$

$$= \mathcal{A}_{1\ldots\dot{n}}\,\{(I-P_{1n}-P_{2n}-\ldots-P_{n-1,n})/n\}\,. \qquad \ldots (3.11)$$

4. Reduction of multiple symmetriser antisymmetriser products

We show that the following four relations hold:

(i) $(\mathcal{A}_{1a}S_{1\ldots\dot{a}})(\mathcal{A}_{1a}S_{1\ldots\dot{a}}) = a/\{2(1-a)\}(\mathcal{A}_{1a}S_{1\ldots\dot{a}}) \quad (a \le n)$

$\ldots (4.1)$

(ii) $(\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots b})(\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots b}) = b/\{2(b-1)\}(\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots b}) \quad (a \le b \le n),$

$\ldots (4.2)$

(iii) $(S_{1\ldots\dot{a}}\mathcal{A}_{1a})(S_{1\ldots\dot{a}}\mathcal{A}_{1a}) = a/\{2(a-1)\}(S_{1\ldots\dot{a}}\mathcal{A}_{1a}) \quad (a \le n)$

$\ldots (4.3)$

(iv) $(S_{1\ldots\dot{a}\ldots b}\mathcal{A}_{1a})(S_{1\ldots\dot{a}\ldots b}\mathcal{A}_{1a}) = b/\{2(b-1)\}(S_{1\ldots\dot{a}\ldots b}\mathcal{A}_{1a}) \quad (a \le b \le n).$

$\ldots (4.4)$

A further set of four relations (used in connection with $[21^{n-2}]$ may be obtained from the above, interchanging $\mathcal{A}$ by S and S by $\mathcal{A}$ throughout.

Proof of (ii)

It will be sufficient to establish the second relation as (i) follows from it (b = a) and (iii) and (iv) are a direct consequence of (i) and (ii). Relation (ii) may be rewritten as

$$[\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots b} - bI/\{2(b-1)\}]\,\mathcal{A}_{1a}\,S_{1\ldots\dot{a}\ldots b} = 0\,. \qquad \ldots (4.5)$$

Putting (from (3.10))

$$S_{1\ldots\dot{a}\ldots b} = \{(I+P_{12}+\ldots+P_{1\dot{a}}+\ldots+P_{1b})/(b-1)\}\, S_{2\ldots\dot{a}\ldots b} \,, \qquad \ldots (4.6)$$

commuting $S_{2\ldots\dot{a}\ldots b}$ with A_{1a}, using (from (3.5))

$$S_{2\ldots\dot{a}\ldots b}\, S_{1\ldots\dot{a}\ldots b} = S_{1\ldots\dot{a}\ldots b} \,, \qquad \ldots (4.7)$$

writing $A_{1a} = (I-P_{1a})/2$, removing the common factor $2/(b-1)$ there remains

$$\left[(I-P_{1a})(I+P_{12}+\ldots+P_{1\dot{a}}+\ldots+P_{1b}) - bI\right] A_{1a} S_{1\ldots\dot{a}\ldots b} = 0. \qquad \ldots (4.8)$$

Now

$$P_{1a}\, P_{1j} = P_{a1j} = P_{aj}\, P_{1a} \quad (j = 2,\ldots,\dot{a},\ldots,b) \qquad \ldots (4.9)$$

and (from (3.4))

$$-P_{1a}\, A_{1a} = +A_{1a} \,. \qquad \ldots (4.10)$$

There remains

$$(I+P_{12}+\ldots+P_{1\dot{a}}+\ldots+P_{1b}$$

$$+I+P_{a2}+\ldots+P_{a\dot{a}}+\ldots+P_{ab}-bI)\, A_{1a} S_{1\ldots\dot{a}\ldots b} = 0 \qquad \ldots (4.11)$$

or, changing the overall sign,

$$\sum_{j=2,\ldots,\dot{a},\ldots b} (I-P_{1j}-P_{aj}) A_{1a} S_{1\ldots\dot{a}\ldots b} = 0 \,. \qquad \ldots (4.12)$$

Now (from 3.11))

$$3\mathcal{A}_{1aj} = (I - P_{1j} - P_{aj})\mathcal{A}_{1a} \quad (j = 2,\ldots,\dot{a},\ldots,b) \quad \ldots (4.13)$$

and (from (3.9))

$$\mathcal{A}_{1aj}\mathcal{S}_{1\ldots\dot{a}\ldots b} = \mathcal{A}_{1aj}\underline{\mathcal{A}_{1j}\mathcal{S}_{1j}}\mathcal{S}_{1\ldots\dot{a}\ldots b} = 0 \quad (j = 2,\ldots,\dot{a},\ldots b) \quad \ldots (4.14)$$

Hence relation (ii) has been established.

Clearly the further four relations used in connection with $[21^{n-2}]$ obtained from (4.1) to (4.4) by interchanging $\mathcal{A}$ and $\mathcal{S}$ may be established in a very similar manner.

5. Bra and ket tableau operators

Bra and ket tableau operators $\langle A_n|$ and $|B_n\rangle$ are defined for $[n-1,1]$ by

$$\left\langle A_n\right| = \left\langle {1\ldots\dot{a}\ldots n \atop a}\right| = 2\{(a-1)(n-1)/(an)\}^{1/2}\,\mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n} \quad \ldots (5.1)$$

$$\left|B_n\right\rangle = \left|{1\ldots\dot{b}\ldots n \atop b}\right\rangle = 2\{(b-1)(n-1)/(bn)\}^{1/2}\,\mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}} \quad \ldots (5.2)$$

and for $[21^{n-2}]$ by

$$\left\langle A_n^*\right| = \left\langle\left({1\ldots\dot{a}\ldots n \atop a}\right)^*\right| = \langle 1a_{\wedge}\ldots\dot{a}\ldots n| = 2\{(a-1)(n-1)/(an)\}^{1/2}\,\mathcal{A}_{1\ldots\dot{a}}\mathcal{S}_{1a}\mathcal{A}_{1\ldots\dot{a}\ldots n} \quad \ldots (5.3)$$

$$\left|B_n^*\right\rangle = \left|\left({1\ldots\dot{b}\ldots n \atop b}\right)^*\right\rangle = |1b_{\wedge}\ldots\dot{b}\ldots n\rangle = 2\{(b-1)(n-1)/(bn)\}^{1/2}\,\mathcal{A}_{1\ldots\dot{b}\ldots n}\mathcal{S}_{1b}\mathcal{A}_{1\ldots\dot{b}} \quad \ldots (5.4)$$

$\langle A_n|$ and $|B_n\rangle$ become equal when a = b = n :

$$\langle N_n| = \left\langle {}^{1\ldots\dot{n}}_{n}\right| = |N_n\rangle = \left|{}^{1\ldots\dot{n}}_{n}\right\rangle = 2\{(n-1)/n\}S_{1\ldots\dot{n}}A_{1n}S_{1\ldots\dot{n}} \quad \ldots (5.5)$$

and likewise with $\langle A_n^*|$ and $|B_n^*\rangle$:

$$\langle N_n^*| = \langle 1n*\ldots\dot{n}| = |N_n^*\rangle = |1n*\ldots\dot{n}\rangle = 2\{(n-1)/n\}A_{1\ldots\dot{n}}S_{1n}A_{1\ldots\dot{n}} \quad \ldots (5.6)$$

We may put n = a in these expressions and obtain

$$\langle A_a| = \left\langle {}^{1\ldots\dot{a}}_{a}\right| = |A_a\rangle = \left|{}^{1\ldots\dot{a}}_{a}\right\rangle = 2\{(a-1)/a\}S_{1\ldots\dot{a}}A_{1a}S_{1\ldots\dot{a}}\,, \quad \ldots (5.7)$$

$$\langle A_a^*| = \langle 1a*\ldots\dot{a}| = |A_a^*\rangle = |1a*\ldots\dot{a}\rangle = 2\{(a-1)/a\}A_{1\ldots\dot{a}}S_{1a}A_{1\ldots\dot{a}}\,, \quad \ldots (5.8)$$

these being tableau operators for the representations $[a-1,1]$ and $[21^{a-2}]$ respectively of the symmetric group S_a.

6. "Diagonal" bracket and ket-bra tableau operators

Compound bracket and ket-bra tableau operators are defined as simple products of the corresponding bra and ket operators. Included in our main theorem is the statement that the "diagonal" Young operator O^n_{aa} for $[n-1,1]$ is equal to the bracket tableau operator $\langle A_n|A_n\rangle$. and similarly for $[21^{n-2}]$ O^n_{a*a*} is equal to $\langle A_n^*|A_n^*\rangle$. Thus from (5.1) – (5.4)

$$O^n_{aa} = \langle A_n|A_n\rangle = \left\langle {}^{1\ldots\dot{a}\ldots n}_{a}\right.\left|{}^{1\ldots\dot{a}\ldots n}_{a}\right\rangle$$

$$= 4\{(a-1)(n-1)/(an)\}\, S_{1\ldots\dot{a}}\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}S_{1\ldots\dot{a}} \quad \ldots \quad (6.1)$$

$$O^{n}_{a^*a^*} = \langle A^*_n | A^*_n \rangle = \langle 1a^*\ldots a\ldots n | 1a^*\ldots\dot{a}\ldots n \rangle$$

$$= 4\{(a-1)(n-1)/(an)\}\mathcal{A}_{1\ldots\dot{a}}S_{1a}\mathcal{A}_{1\ldots\dot{a}\ldots n}S_{1a}\mathcal{A}_{1\ldots\dot{a}} \quad \ldots \quad (6.2)$$

Putting n=a in (6.1) and (6.2) and using the reduction formula (4.3) and the analogous one with S and A interchanged, we find

$$O^{a}_{aa} = \langle A_a | A_a \rangle = \left\langle {1\ldots\dot{a} \atop a} \middle| {1\ldots\dot{a} \atop a} \right\rangle = 2\{(a-1)/a\}S_{1\ldots\dot{a}}\mathcal{A}_{1a}S_{1\ldots\dot{a}}$$

$$= \langle A_a | = | A_a \rangle = | A_a \rangle\langle A_a | , \quad \ldots \quad (6.3)$$

$$O^{a}_{a^*a^*} = \langle A^*_a | A^*_a \rangle = \langle 1a^*\ldots\dot{a} \,|\, 1a\ldots\dot{a} \rangle = 2\{(a-1)/a\}\mathcal{A}_{1\ldots\dot{a}}S_{1a}\mathcal{A}_{1\ldots\dot{a}}$$

$$= \langle A^*_a | = | A^*_a \rangle = | A^*_a \rangle \langle A^*_a | . \quad \ldots \quad (6.4)$$

Here the equality of the ket-bra tableau operator $|A_a\rangle\langle A_a|$ with the bracket tableau operator $\langle A_a|A_a\rangle$ is a consequence of $\langle A_a|$ and $|A_a\rangle$ being equal and therefore commuting. The same applies to the operators obtained by putting a = n (and A = N):

$$O^{n}_{nn} = \langle N_n | N_n \rangle = \left\langle {1\ldots\dot{n} \atop n} \middle| {1\ldots\dot{n} \atop n} \right\rangle = 2\{(n-1)/n\}S_{1\ldots\dot{n}}\mathcal{A}_{1n}S_{1\ldots\dot{n}}$$

$$= \langle N_n | = | N_n \rangle = | N_n \rangle\langle N_n | , \quad \ldots \quad (6.5)$$

$$O^{n}_{n^*n^*} = \langle N^*_n | N^*_n \rangle = \langle 1n^*\ldots\dot{n} | 1n^*\ldots\dot{n} \rangle = 2\{(n-1)/n\}\mathcal{A}_{1\ldots\dot{n}}S_{1n}\mathcal{A}_{1\ldots\dot{n}}$$

$$= \langle N^*_n | = | N^*_n \rangle = | N^*_n \rangle\langle N^*_n | . \quad \ldots \quad (6.6)$$

In the general case $\langle A_n|$ and $|A_n\rangle$ are neither equal nor commute , and we must distinguish the ket-bra tableau operator $|A_n\rangle\langle A_n|$ from the bracket tableau operator $\langle A_n | A_n \rangle = O^n_{aa}$. We use a tilde $\sim$ to make the distinction and find

$$\tilde{O}^n_{aa} = |A_n\rangle\langle A_n| = \left|\begin{smallmatrix}1\ldots\dot{a}\ldots n\\ a\end{smallmatrix}\right\rangle \left\langle\begin{smallmatrix}1\ldots\dot{a}\ldots n\\ a\end{smallmatrix}\right|$$

$$= 4\{(a-1)(n-1)/(an)\}\mathcal{S}_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots a\ldots n}$$

$$= 4\{(a-1)(n-1)/(an)\}\mathcal{S}_{1\ldots\dot{a}\ldots n}\underline{\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}}}\mathcal{S}_{1\ldots\dot{a}\ldots n}$$

$$= 2\{(n-1)/n\}\mathcal{S}_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n}, \qquad \ldots (6.7)$$

using (4.1). Similarly, for $[21^{n-2}]$,

$$\tilde{O}^n_{a*a*} = |A^*_n\rangle\langle A^*_n| = |1a_*\ldots\dot{a}\ldots n\rangle\langle 1a_*\ldots\dot{a}\ldots n|$$

$$= 2\{(n-1)/n\}\mathcal{A}_{1\ldots\dot{a}\ldots n}\mathcal{S}_{1a}\mathcal{A}_{1\ldots\dot{a}\ldots n}. \qquad \ldots (6.8)$$

We tabulate some properties of the "diagonal" operators O^n_{aa} and $\tilde{O}^n_{aa}$ for $[n-1,1]$. (Similarly results hold for the operators of $[21^{n-2}]$ with appropriare stars introduced).

(i) $O^n_{aa}\langle A_n| = \langle A_n|A_n\rangle\langle A_n| = \langle A_n|\tilde{O}^n_{aa} = \langle A_n|$ $\qquad \ldots (6.9)$

(ii) $|A_n\rangle\, O^n_{aa} = |A_n\rangle\langle A_n|A_n\rangle = \tilde{O}^n_{aa}|A_n\rangle = |A_n\rangle$ $\qquad \ldots (6.10)$

(iii) $|A_n\rangle\, O^n_{aa}\,\langle A_n| = |A_n\rangle\langle A_n|A_n\rangle\langle A_n| = \tilde{O}^n_{aa}\,\tilde{O}^n_{aa} = \tilde{O}^n_{aa}$ $\qquad \ldots (6.11)$

(iv) $\langle A_n|\tilde{O}^n_{aa}|A_n\rangle = \langle A_n|A_n\rangle\langle A_n|A_n\rangle = O^n_{aa}O^n_{aa} = O^n_{aa}$ $\qquad \ldots (6.12)$

Since $\tilde{O}^n_{aa}$ has a simpler explicit expression than O^n_{aa}, we verify the last relations in (6.9) and (6.10) from which the rest follows. We find

$$\langle A_n|\tilde{O}^n_{aa} = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\ldots\dot{a}}\underline{\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots n}}$$
$$\times\ 2\{(n-1)/n\}$$
$$= 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\ldots\dot{a}}\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots n} = \langle A_n| \,, \qquad \ldots (6.13)$$

making use of (4.2) with b = n. Similarly

$$\tilde{O}^n_{aa}|A_n\rangle = 2\{(n-1)/n\} \times 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} \underline{S_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}S_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}}S_{1\ldots\dot{a}}$$
$$= 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}S_{1\ldots\dot{a}} = |A_n\rangle \,, \qquad \ldots (6.14)$$

using (4.4) with b = n.

These relations may be summarised in the statements that the ket $|A_n$

is a left-hand eigenstate of O^n_{aa}

and a right-hand eigenstate of $\tilde{O}^n_{aa}$

whilst the

bra $\langle A_n|$ is a right-hand eigenstate of O^n_{aa}

and a left-hand eigenstate of $\tilde{O}^n_{aa}$,

the eigenvalue being +1 in all cases. Further

O^n_{aa} and $\tilde{O}^n_{aa}$ are idempotents.

It is clear that similar relations hold for the starred operators of $[21^{n-2}]$.

7. $O^n_{nn} = S_{1\ldots\dot{n}} - S_{1\ldots n}$, $O^n_{n*n*} = \mathcal{A}_{1\ldots\dot{n}} - \mathcal{A}_{1\ldots n}$

Using (3.10)

$$S_{1\ldots\dot{n}} - S_{1\ldots n} = S_{1\ldots\dot{n}}\{I - (I+P_{1n}+P_{2n}+\ldots+P_{n-1,n})/n\}S_{1\ldots\dot{n}}$$

$$= (1/n)S_{1\ldots\dot{n}}\{(n-1)I - (P_{1n}+P_{2n}+\ldots+P_{n-1,n})\}S_{1\ldots\dot{n}}$$

$$= (2/n) S_{1\ldots\dot{n}} (A_{1n}+A_{2n}+\ldots+A_{n-1,n}) S_{1\ldots\dot{n}} \,. \qquad \ldots (7.1)$$

Now, for $2 \leq a \leq n-1$,

$$S_{1\ldots a\ldots\dot{n}} \mathcal{A}_{an} S_{1\ldots a\ldots\dot{n}}$$

$$= S_{1\ldots a\ldots\dot{n}} P_{1a} \mathcal{A}_{an} P_{1a}^{-1} S_{1\ldots a\ldots\dot{n}}$$

$$= S_{1\ldots\dot{n}} \mathcal{A}_{1n} S_{1\ldots\dot{n}}$$

Hence

$$S_{1\ldots\dot{n}} - S_{1\ldots n} = 2(n-1)/n \, S_{1\ldots\dot{n}} \mathcal{A}_{1n} S_{1\ldots\dot{n}}$$

$$= 0^n_{nn} \,, \qquad \ldots (7.3)$$

by, (6.5).

Similarly

$$\mathcal{A}_{1\ldots\dot{n}} - \mathcal{A}_{1\ldots n} = \mathcal{A}_{1\ldots\dot{n}}\{I - (I-P_{1n}-P_{2n}-\ldots-P_{n-1,n})/n\}\mathcal{A}_{1\ldots\dot{n}}$$

$$= (1/n)\mathcal{A}_{1\ldots\dot{n}}\{(n-1)I + (P_{1n}+P_{2n}+\ldots+P_{n-1,n})\}\mathcal{A}_{1\ldots\dot{n}}$$

$$= (2/n)\mathcal{A}_{1\ldots\dot{n}}(S_{1n}+S_{2n}+\ldots+S_{n-1,n})\mathcal{A}_{1\ldots\dot{n}}$$

$$= \{2(n-1)/n\}\mathcal{A}_{1\ldots\dot{n}} S_{1n} \mathcal{A}_{1\ldots\dot{n}} = 0^n_{n^*n^*} \,, \qquad \ldots (7.4)$$

by (6.6), since

$$\mathcal{A}_{1\ldots a\ldots\dot{n}} S_{an} \mathcal{A}_{1\ldots a\ldots\dot{n}}$$

$$= \mathcal{A}_{1\ldots\dot{n}} P_{1a} S_{an} P_{1a}^{-1} \mathcal{A}_{1\ldots\dot{n}} = \mathcal{A}_{1\ldots\dot{n}} S_{1n} \mathcal{A}_{1\ldots\dot{n}} \quad (a = 2,3,\ldots,n-1).$$

$$\ldots (7.5)$$

8. $|A_n\rangle\langle B_n| = \delta_{ab}\, \tilde{0}^n_{aa}$, $|A^*_n\rangle\langle B^*_n| = \delta_{ab}\, \tilde{0}^n_{a^*a^*}$

With

$$|A_n\rangle = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} S_{1\ldots\dot{a}\ldots n} \mathcal{A}_{1a} S_{1\ldots\dot{a}} \qquad \ldots (8.1)$$

$$\langle B_n| = 2\{(b-1)(n-1)/(bn)\}^{\frac{1}{2}} S_{1\ldots\dot{b}} \mathcal{A}_{1b} S_{1\ldots\dot{b}\ldots n} \qquad \ldots (8.2)$$

we see that $|A_n\rangle\langle B_n|$ involves the product

$$Q_{ab} = \mathcal{A}_{1a} S_{1\ldots\dot{a}} S_{1\ldots\dot{b}} \mathcal{A}_{1b} \,. \qquad \ldots (8.3)$$

Now if a < b we have by (3.5)

$$S_{1\ldots\dot{a}} S_{1\ldots\dot{b}} = S_{1\ldots\dot{b}} \,, \qquad \ldots (8.4)$$

so that

$$Q_{ab}\ (a < b) = \mathcal{A}_{1a} S_{1\ldots\dot{b}} \mathcal{A}_{1b} = 0 \,, \qquad \ldots (8.5)$$

by (3.8). On the other hand, if a > b, we have

$$S_{1\ldots\dot{a}} S_{1\ldots\dot{b}} = S_{1\ldots\dot{a}} \,, \qquad \ldots (8.6)$$

so that

$$Q_{ab}\ (a > b) = \mathcal{A}_{1a} S_{1\ldots\dot{a}} \mathcal{A}_{1b} = 0 \,, \qquad \ldots (8.7)$$

again by (3.8). Thus

$$|A_n\rangle\langle B_n| = \delta_{ab}|A_n\rangle\langle A_n| = \delta_{ab}\,\tilde{O}^n_{aa}$$

$$= \delta_{ab}\, 2\{(n-a)/n\}\mathcal{A}_{1\ldots\dot{a}\ldots n} S_{1a} \mathcal{A}_{1\ldots\dot{a}\ldots n} \,. \qquad \ldots (8.8)$$

The proof that

$$|A^*_n\rangle\langle B^*_n| = \delta_{ab}\, |A^*_n\rangle\langle A^*_n| = \delta_{ab}\,\tilde{O}^n_{a^*a^*}$$

$$= \delta_{ab}\, 2\{(n-1)/n\}\, S_{1\ldots\dot{a}\ldots n} \mathcal{A}_{1a} S_{1\ldots\dot{a}\ldots n} \qquad \ldots (8.9)$$

follows in exactly the same way, with $\mathcal{A}$ and S interchanged throughout.

9. Tableau permutations

We denote the permutation which converts, for $[n-1,1]$, the standard tableau

$$N_n = \begin{matrix} 1 & 2 & \ldots & a-1 & a & a+1 & \ldots & n-2 & n-1 \\ n \end{matrix} \qquad \ldots (9.1)$$

into the standard tableau

$$A_n = \begin{matrix} 1 & 2 & \dots & a-1 & a+1 & a+2 & \dots & n-1 & n \\ a \end{matrix} \quad \dots (9.2)$$

by

$$(A_n|P|N_n) = P_{(a,a+1,\dots,n)} \quad \dots (9.3)$$

and the inverse permutation, converting A_n into N_n by

$$(N_n|P|A_n) = P_{(n,n-1,\dots,a)} \quad \dots (9.4)$$

The permutation which converts

$$B_n = \begin{matrix} 1 \dots \hat{b} \dots n \\ b \end{matrix} \quad \text{into } A_n = \begin{matrix} 1 \dots \hat{a} \dots n \\ a \end{matrix}$$

may then be written as

$$(A_n|P|B_n) = (A_n|P|N_n)(N_n|P|B_n)$$

$$= P_{(a,a+1,\dots,n)} P_{(n,n-1,\dots,b)} \cdot \quad \dots (9.5)$$

This may be evaluated for the two cases $a < b$ and $a > b$ as follows

If $a < b$ $\quad P_{(a,a+1,\dots,n)} = P_{(a,a+1,\dots,b)} P_{(b,b+1,\dots,n)}$

$$= P_{(a,a+1,\dots,b)} P^{-1}_{(n,n-1,\dots,b)}$$

$$\dots (9.6)$$

so that

$$(A_n|P|B_n)_{a<b} = P_{(a,a+1,\dots,b)} \cdot \quad \dots (9.7)$$

If $a > b$ $\quad P_{(n,n-1,\dots,b)} = P_{(n,n-1,\dots,a)} P_{(a,a-1,\dots,b)}$

$$= P^{-1}_{(a,a+1,\dots,n)} P_{(a,a-1,\dots,b)}$$

$$\dots (9.8)$$

so that

$$(A_n|P|B_n)_{a>b} = P_{(a,a-1,\dots,b)} \cdot \quad \dots (9.9)$$

It is easy to verify this directly from the form the tableaux A_n, B_n have in the two cases:

<u>a < b</u>

$$B_n = \begin{matrix} 1 & 2 & \ldots a-1 & a & a+1 \ldots & b-1 & b+1 \ldots & n \\ b \end{matrix} \qquad \ldots (9.10)$$

$$A_n = \begin{matrix} 1 & 2 & \ldots a-1 & a+1 & a+2 & \ldots & b & b+1 \ldots & n \\ a \end{matrix} \qquad \ldots (9.11)$$

$$(A_n|P|B_n) = P_{(a,a+1,\ldots,b)} \qquad \ldots (9.12)$$

<u>a b</u>

$$B_n = \begin{matrix} 1 & 2 & \ldots & b-1 & b+1 & \ldots & a-1 & a & a+1 & \ldots & n \\ b \end{matrix} \qquad \ldots (9.13)$$

$$A_n = \begin{matrix} 1 & 2 & \ldots & b-1 & b & \ldots & a-2 & a-1 & a+1 & \ldots & n \\ a \end{matrix} \qquad \ldots (9.14)$$

$$(A_n|P|B_n) = P_{(a,a-1,\ldots,b)} \qquad \ldots (9.15)$$

10. <u>$O^n_{ab} = \langle A_n|(A_n|P|B_n)|B_n\rangle$ implies $O^n_{an} = \langle A_n|P_{(a,a+1,\ldots,n)}$,</u>

<u>$O^n_{nb} = P_{(n,n-1,\ldots,b)}|B_n\rangle$.</u>

<u>$O^n_{a^*b^*} = \langle A^*_n|(A_n|P|B_n)|B^*_n\rangle$ implies $O^n_{a^*n^*} = \langle A^*_n|P_{(a,a+1,\ldots,n)}$,</u>

<u>$O^n_{n^*b^*} = P_{(n,n-1,\ldots,b)}|B^*_n\rangle$.</u>

Since from (6.5), (6.7)

$$|N_n\rangle\langle N_n| = 2\{(n-1)/n\}\mathcal{S}_{1\ldots\dot{n}}\mathcal{A}_{1n}\mathcal{S}_{1\ldots\dot{n}} \qquad \ldots (10.1)$$

$$|A_n\rangle\langle A_n| = 2\{(n-1)/n\}\mathcal{S}_{1\ldots\dot{a}\ldots n}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n} \qquad \ldots (10.2)$$

$$|B_n\rangle\langle B_n| = 2\{(n-1)/n\}\mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}\ldots n} \qquad \ldots (10.3)$$

with the same coefficient $2\{(n-1)/n\}$ it follows

$$P_{(a,a+1,\ldots,n)}|N_n\rangle\langle N_n|P^{-1}_{(a,a+1,\ldots,n)} = |A_n\rangle\langle A_n| \qquad \ldots (10.4)$$

or, equivalently,

$$P^{-1}_{(n,n-1,\ldots,b)}|N_n\rangle\langle N_n|P_{(n,n-1,\ldots,b)} = |B_n\rangle\langle B_n| \qquad \ldots (10.5)$$

[Note that the equality of the coefficient 2 (n-1)/n in (10.1), (10.2), (10.3) is needed for (10.4), (10.5) to hold, since transformation by P or by P^{-1} affects only the symmetriser and antisymmetrisers: it is a pitfall to expect the transformation to change coefficients!].

It follows

$$(A_n|P|N_n)|N_n\rangle\langle N_n| = P_{(a,a+1,\ldots,n)}|N_n\rangle\langle N_n|$$

$$= P_{(a,a+1,\ldots,n)}|N_n\rangle\langle N_n|\, P^{-1}_{(a,a+1,\ldots,n)} P_{(a,a+1,\ldots,n)}$$

$$= |A_n\rangle\langle A_n|P_{(a,a+1,\ldots,n)} \qquad \ldots (10.6)$$

$$|N_n\rangle\langle N_n|(N_n|P|B_n) = |N_n\rangle\langle N_n|\, P_{(n,n-1,\ldots,b)}$$

$$= P_{(n,n-1,\ldots,b)}|P^{-1}_{(n,n-1,\ldots,b)}|N_n\rangle\langle N_n|P_{(n,n-1,\ldots,b)}$$

$$= P_{(n,n-1,\ldots,n)}|B_n\rangle\langle B_n| \,. \qquad \ldots (10.7)$$

Using then the relation (from (6.5))

$$|N_n\rangle = |N_n\rangle\langle N_n| = \langle N_n| \qquad \ldots (10.8)$$

it follows (using the statement of the theorem to follow)

$$O^n_{an} = \langle A_n|(A_n|P|N_n)|N_n\rangle = \langle A_n|A_n\rangle\langle A_n|P_{(a,a+1,\ldots,n)} = \langle A_n|P_{(a,a+1,\ldots,n)}$$

$$\ldots (10.9)$$

$$O^n_{nb} = \langle N_n|(N_n|P|B_n)|B_n\rangle = P_{(n,n-1,\ldots,b)}|B_n\rangle\langle B_n|B_n\rangle = P_{(n,n-1,\ldots,b)}|B_n\rangle$$

$$\ldots (10.10)$$

The starred relations are established in an identical manner.

11. Statement of theorem

For the representation $[n-1,1]$ of S_n in standard orthogonal form the Young operators are given by

$$O^n_{an} = \langle A_n | P_{(a,a+1,\ldots,n)}, \quad O^n_{ab} = \langle A_n | (A_n | P | B_n) | B_n \rangle ,$$

$$O^n_{nb} = P_{(n,n-1,\ldots,b)} | B_n \rangle ,$$

$$(a,b=2,3,\ldots,n), \qquad \ldots (11.1)$$

$$\langle A_n | = \left\langle {}^{1\ldots\dot{a}\ldots n}_{a} \right| = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} \mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a} \mathcal{S}_{1\ldots\dot{a}\ldots n}, \qquad \ldots (11.2)$$

$$| B_n \rangle = \left| {}^{1\ldots\dot{b}\ldots n}_{b} \right\rangle = 2\{(b-1)(n-1)/(bn)\}^{\frac{1}{2}} \mathcal{S}_{1\ldots\dot{b}\ldots n} \mathcal{A}_{1b} \mathcal{S}_{1\ldots\dot{b}}, \qquad \ldots (11.3)$$

$$(A_n | P | B_n) = P_{(a,a+1,\ldots,n)} P_{(n,n-1,\ldots,b)} = P_{(a,a+1,\ldots,b)}, (a<b)$$

$$= P_{(a,a-1,\ldots,b)}, \ (a>b) . \qquad \ldots (11.4)$$

For the representation $[21^{n-2}]$ of S_n in standard orthogonal form the Young operators are given by

$$O^n_{a^*n^*} = \langle A^*_n | P_{(a,a+1,\ldots,n)}, \quad O^n_{a^*b^*} = \langle A^*_n | (A_n | P | B_n) | B^*_n \rangle ,$$

$$O^n_{n^*b^*} = P_{(n,n-1,\ldots,b)} | B^*_n \rangle ,$$

$$(a,b=2,3,\ldots,n), \qquad \ldots (11.5)$$

$$\langle A^*_n | = \left\langle \left({}^{1\ldots\dot{a}\ldots n}_{a} \right)^* \right| = \langle 1a^*\ldots\dot{a}\ldots n | = 2\{(a-1)(n-1)/(an)\}$$

$$\mathcal{A}_{1\ldots\dot{a}} \mathcal{S}_{1a} \mathcal{A}_{1\ldots\dot{a}\ldots n} , \qquad \ldots (11.6)$$

$$|B^*_n\rangle = \left|\begin{pmatrix}1\ldots\dot{b}\ldots n\\ b\end{pmatrix}^*\right\rangle = |1b^*\ldots\dot{b}\ldots n\rangle = 2\{(b-1)(n-1)/(bn)\}$$

$$\mathcal{A}_{1\ldots\dot{b}\ldots n}\mathcal{S}_{1b}\mathcal{A}_{1\ldots\dot{b}} \cdot \qquad \ldots (11.7)$$

These expressions satisfy the relations

$$O^n_{ab}\, O^n_{cd} = \delta_{bc}\, O^n_{ad} \;, \qquad \ldots (11.8)$$

$$O^n_{a*b*}\, O^n_{c*d*} = \delta_{bc}\, O^n_{a*d*} \;, \qquad \ldots (11.9)$$

$$P_{a,a+1}\, O^n_{ab} = (1/a)\, O^n_{ab} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{a+1,b} \;, \qquad \ldots (11.10)$$

$$O^n_{ab}\, P_{b,b+1} = (1/b)\, O^n_{ab} + \{(b^2-1)^{\frac{1}{2}}/b\}O^n_{a,b+1} \;, \qquad \ldots (11.11)$$

$$P_{a,a+1}\, O^n_{a*b*} = -(1/a)\, O^n_{a*b*} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{a+1*b*} \;, \qquad \ldots (11.12)$$

$$O^n_{a*b*}\, P_{b,b+1} = -(1/b)\, O^n_{a*b*} + \{(b^2-1)^{\frac{1}{2}}/b\}O^n_{a*b+1*} \;. \qquad \ldots (11.13)$$

12. Proof that $O^n_{ab}\, O^n_{cd} = \delta_{bc}\, O^n_{ad}$ (and $O^n_{a*b*}\, O^n_{c*d*} = \delta_{bc}\, O^n_{a*d*}$)

We have already shown (8.8) that

$$|B_n\rangle\langle C_n| = \delta_{bc}|B_n\rangle\langle B_n| \qquad \ldots (12.1)$$

It remains to show

$$O^n_{ab}\, O^n_{bc} = O^n_{bc} \;. \qquad \ldots (12.2)$$

$$\text{L.H.S.} = \langle A_n|(A_n|P|B_n)|B_n\rangle\langle B_n|(B_n|P|C_n)|C_n\rangle \qquad \ldots (12.3)$$

$$\text{R.H.S.} = \langle A_n|(A_n|P|C_n)|C_n\rangle \qquad \ldots (12.4)$$

We have shown (6.7) that

$$|B_n\rangle\langle B_n| = 2\{(n-1)/n\}\, \mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}\ldots n} \quad \ldots \quad (12.5)$$

Hence with

$$\langle A_n| = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}}\, \mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n} \quad \ldots \quad (12.6)$$

$$|C_n\rangle = 2\{(c-1)(n-1)/(cn)\}^{\frac{1}{2}}\, \mathcal{S}_{1\ldots\dot{c}\ldots n}\mathcal{A}_{1c}\mathcal{S}_{1\ldots\dot{c}} \quad \ldots \quad (12.7)$$

and taking, for simplicity, $a < b < c$ we have

$$\text{L.H.S.} = 8\{(n-1)/n\}^2\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}}\, \mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n}$$

$$\times P_{(a,a+1,\ldots,b)}\, \mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}\ldots n}P_{(b,b+1,\ldots,c)}$$

$$\mathcal{S}_{1\ldots\dot{c}\ldots n}\mathcal{A}_{1c}\mathcal{S}_{1\ldots\dot{c}} \quad \ldots \quad (12.8)$$

$$= 8\{(n-1)/n\}^2\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}}\mathcal{S}_{1\ldots\dot{a}}\, P_{(a,a+1,\ldots,b)}$$

$$\times \underline{\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}\ldots n}}\,\mathcal{A}_{1b}P_{(b,b-1,\ldots,c)}\mathcal{S}_{1\ldots\dot{c}} \quad \ldots \quad (12.9)$$

$$= 4\{(n-1)/n\}\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}}\, \mathcal{S}_{1\ldots\dot{a}}\, P_{(a,a+1,\ldots,b)}$$

$$\times\, \mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\, P_{(b,b+1,\ldots,c)}\, \mathcal{S}_{1\ldots\dot{c}} \quad \ldots \quad (12.10)$$

Using (4.2) with a replaced by b and b by n,

$$\text{R.H.S.} = 4\{(n-1)/n\}\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}}\, \mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n}$$

$$\times P_{(a,a+1,\ldots,b)}\, P_{(b,b+1,\ldots,c)}\, \mathcal{S}_{1\ldots\dot{c}\ldots n}\mathcal{A}_{1c}\mathcal{S}_{1\ldots\dot{c}} \quad \ldots \quad (12.11)$$

$$= 4\{(n-1)/n\}\{(a-1)(c-1)/(ac)\}^{\frac{1}{2}}\, \mathcal{S}_{1\ldots\dot{a}}\, P_{(a,a+1,\ldots,b)}$$

$$\mathcal{A}_{1b}\mathcal{S}_{1\ldots b\ldots n}\mathcal{A}_{1b}\, P_{(b,b+1,\ldots,c)}\mathcal{S}_{1\ldots\dot{c}} = \text{L.H.S.} \quad \ldots \quad (12.12)$$

The proof is similar with the restriction $a < b < c$ removed, thus (11.8) is proved and (11.9) by interchanging $\mathcal{S}$ and $\mathcal{A}$ throughout.

13. $\underline{P_{a,a+1}\ O^n_{an} = (1/a)O^n_{an} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{a+1,n},\ P_{a,a+1},O^n_{a*n*} = -(1/a)O^n_{a*b*} +}$
$\underline{\{(a^2-1)^{\frac{1}{2}}/a\}O^n_{a+1*n*}}$

$\underline{O^n_{na}\ P_{a,a+1} = (1/a)O^n_{na} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{n,a+1},\ O^n_{n*a*}\ P_{a,a+1} = -(1/a)O^n_{n*a*} +}$
$\underline{\{(a^2-1)^{\frac{1}{2}}/a\}O^n_{n*a+1*}}$

From (10.9) we have

$$O^n_{an} = \langle A_n|P_{(a,a+1,a+2,\ldots,n)} = \langle A_n|P_{a,a+1}\ P_{(a+1,a+2,\ldots,n)} \qquad \ldots (13.1)$$

$$O^n_{a+1,n} = \langle (A+1)_n|P_{(a+1,a+2,\ldots,n)}\ . \qquad \ldots (13.2)$$

Hence

$$P_{a,a+1}\ O^n_{an} = (1/a)\ O^n_{an} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{a+1,n} \qquad \ldots (13.3)$$

requires (cancelling the common term $P_{(a+1,a+2,\ldots,n)}$ on the right)

$$P_{a,a+1}\langle A_n|P_{a,a+1} = (1/a)\langle A_n|P_{a,a+1} + \{(a^2-1)^{\frac{1}{2}}/a\}\ \langle (A+1)_n| \qquad \ldots (13.4)$$

With, from (5.1),

$$\langle A_n| = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}}\ \mathcal{S}_{1\ldots a}\mathcal{A}_{1a}\mathcal{S}_{1\ldots a\ldots n} \qquad \ldots (13.5)$$

we deduce, by some cancellation,

$$\{(a^2-1)^{\frac{1}{2}}/a\}\ \langle (A+1)_n| = 2\{(a-1)(n-1)/(an)\}\ \mathcal{S}_{1\ldots a+1}\mathcal{A}_{1a+1}\mathcal{S}_{1\ldots a+1\ldots n} \qquad \ldots (13.6)$$

and also, noting that $\mathcal{S}_{1\ldots\dot{a}} = \mathcal{S}_{1\ldots a-1}$ is independent of a and a+1

$$P_{a,a+1}\langle A_n | P_{a,a+1} = 2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}} \mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a+1} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} \quad \ldots (13.7)$$

Hence, removing the common factor $2\{(a-1)(n-1)/(an)\}^{\frac{1}{2}}$ equation (13.4) requires

$$\mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a+1} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} = (1/a)\mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a} \mathcal{S}_{1\ldots\dot{a}\ldots n} P_{a,a+1} + \mathcal{S}_{1\ldots\dot{a+1}} \mathcal{A}_{1a+1} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} \quad \ldots (13.8)$$

Now from (7.3) with n = a we have

$$\mathcal{S}_{1\ldots\dot{a}} - \mathcal{S}_{1\ldots\dot{a+1}} = 2\{(a-1)/a\} \mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a} \mathcal{S}_{1\ldots\dot{a}} \quad \ldots (13.9)$$

Hence it remains to show (multiplying through by a)

$$2(a-1)\mathcal{S}_{1\ldots a} \mathcal{A}_{1a} \mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a+1} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} = \mathcal{S}_{1\ldots\dot{a}} \mathcal{A}_{1a} \mathcal{S}_{1\ldots\dot{a}\ldots n} P_{a,a+1}. \quad (13.10)$$

Cancelling the common factor $\mathcal{S}_{1\ldots\dot{a}}$ on the left, putting

$$2\mathcal{A}_{1a} = I - P_{1a}, \quad (a-1)\mathcal{S}_{1\ldots\dot{a}} = (I+P_{12}+P_{13}+\ldots+P_{1,a-1})\mathcal{S}_{2\ldots\dot{a}} \quad \ldots (13.10)$$

and writing

$$\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n} P_{a,a+1} = P_{a,a+1}\mathcal{A}_{1a+1}\mathcal{S}_{1\ldots\dot{a+1}\ldots n} \quad \ldots (13.12)$$

it remains to show

$$(I-P_{1a})(I+P_{12}+P_{13}+\ldots+P_{1,a-1})\mathcal{S}_{2\ldots\dot{a}} \mathcal{A}_{1a+1} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} = P_{a,a+1}\mathcal{A}_{1a+1}\mathcal{S}_{1\ldots\dot{a+1}\ldots n} \quad \ldots (13.13)$$

Now, by (3.1),

$$\mathcal{S}_{2\ldots\dot{a}} \mathcal{A}_{1a+1} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} = \mathcal{A}_{1a+1}\mathcal{S}_{2\ldots\dot{a}} \mathcal{S}_{1\ldots\dot{a+1}\ldots n} = \mathcal{A}_{1a+1}\mathcal{S}_{1\ldots\dot{a+1}\ldots n} \quad \ldots (13.14)$$

Further

$$(I-P_{1a})P_{1r}\mathcal{A}_{1a+1}S_{1\ldots\dot{a+1}\ldots n} = (I-P_{1a})P_{1r}\mathcal{A}_{1a+1}P_{1r}S_{1\ldots\dot{a+1}\ldots n}$$

$$= (I-P_{1a})\mathcal{A}_{ra+1}S_{1\ldots\dot{a+1}\ldots n} \quad \text{for } r = 2,3,\ldots,a-1 \;. \qquad \ldots (13.15)$$

The since

$$P_{1a}\mathcal{A}_{ra+1}S_{1\ldots\dot{a+1}\ldots n} = \mathcal{A}_{ra+1}P_{1a}S_{1\ldots\dot{a+1}\ldots n}$$

$$= \mathcal{A}_{ra+1}S_{1\ldots\dot{a+1}\ldots n} \, , \qquad \ldots (13.16)$$

it follows

$$(I-P_{1a})(P_{12}+P_{13}+\ldots+P_{1,a-1})\mathcal{A}_{1a+1}S_{1\ldots\dot{a+1}\ldots n} = 0$$

and it remains to show that

$$(I-P_{1a}-P_{a,a+1})\mathcal{A}_{1a+1}S_{1\ldots\dot{a+1}\ldots n} = 0 \qquad \ldots (13.17)$$

or $\quad \mathcal{A}_{1,a,a+1}S_{1\ldots\dot{a+1}\ldots n} = 0$

or $\quad \mathcal{A}_{1,a,a+1}\underline{\mathcal{A}_{1a}S_{1a}}\;S_{1\ldots\dot{a+1}\ldots n} = 0 \, , \qquad \ldots (13.18)$

which is true and establishes the required result.

The relation

$$O^n_{na}\,P_{a,a+1} = (1/a)O^n_{na} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{n,a+1} \qquad \ldots (13.19)$$

is established in the same way with multiplication from the right.

The starred relations

$$P_{a,a+1}\,O^n_{a^*n^*} = -(1/a)O^n_{a^*n^*} + \{(a^2-1)^{\frac{1}{2}}/a\}\,O^n_{a+1^*n^*} \qquad \ldots (13.20)$$

$$O^n_{n^*a^*}\,P_{a,a+1} = -(1/a)O^n_{n^*a^*} + \{(a^2-1)^{\frac{1}{2}}/a\}O^n_{n^*a+1^*} \qquad \ldots (13.21)$$

are also established in a similar manner (with : and interchanged throughout, with the minus sign in front of 1/a being required because the final relation is

$$(I+P_{1a})\,\mathcal{S}_{1a+1}\,\mathcal{A}_{1\ldots\dot{a+1}\ldots n} = -\,P_{a,a+1}\,\mathcal{S}_{1a+1}\,\mathcal{A}_{1\ldots\dot{a+1}\ldots n}\,. \qquad \ldots (13.22)$$

reducing to

$$\mathcal{S}_{1,a,a+1}\,\underline{\mathcal{S}_{1a}\,\mathcal{A}_{1a}}\,\mathcal{A}_{1\ldots\dot{a+1}\ldots n} = 0\,, \qquad \ldots (13.23)$$

which is true as before.

Multiplication of (13.3) on the right by O^{n}_{nb} leads to the general relation (11.10). The other general relations (11.11) – (11.13) may be established in the same way from the corresponding special cases established above.

14. Results in tabular form

Table 1. Young operators $\left(\begin{matrix}1\ldots\dot{a}\ldots n\\ a\end{matrix}\right|0\left|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right)$ for $[n-1,1]$

$a < n$	$2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}\mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots a\ldots n}\left(\begin{matrix}1\ldots\dot{a}\ldots n\\ a\end{matrix}\right\|$	p	$\left\|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right)\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}}\,2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}$	$b < n$
			$\left\|\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right)$	$b = n$
$a = n$	$\sqrt{\left\{\frac{2(n-1)}{n}\right\}}\mathcal{S}_{1\ldots\dot{n}}\mathcal{A}_{1n}\left(\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right\|$		$\left\|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right)\mathcal{S}_{1\ldots\dot{b}}\sqrt{\left\{\frac{2(b-1)}{b}\right\}}$	$b < n$
			$\left\|\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right)\mathcal{S}_{1\ldots\dot{n}}\sqrt{\left\{\frac{2(n-1)}{n}\right\}}$	$b = n$

e.g.

$$O^n_{ab} = \left(\begin{matrix}1\ldots\dot{a}\ldots n\\ a\end{matrix}\right|0\left|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right) = 4\left(\frac{n-1}{n}\right)\sqrt{\left\{\frac{(a-1)(b-1)}{ab}\right\}}\mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n}\left(\begin{matrix}1\ldots\dot{a}\ldots n\\ a\end{matrix}\right|p\left|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right)\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}}$$

$$O^n_{cn} = \left(\begin{matrix}1\ldots\dot{a}\ldots n\\ a\end{matrix}\right|0\left|\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right) = 2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}\mathcal{S}_{1\ldots\dot{a}}\mathcal{A}_{1a}\mathcal{S}_{1\ldots\dot{a}\ldots n}\left(\begin{matrix}1\ldots\dot{a}\ldots n\\ a\end{matrix}\right|p\left|\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right)$$

$$O^n_{nb} = \left(\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right|0\left|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right) = 2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}\mathcal{S}_{1\ldots\dot{n}}\mathcal{A}_{1n}\left(\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right|p\left|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right)\mathcal{S}_{1\ldots\dot{b}} = 2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}\left(\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right|p\left|\begin{matrix}1\ldots\dot{b}\ldots n\\ b\end{matrix}\right)$$

$$\times\,\mathcal{S}_{1\ldots\dot{b}\ldots n}\mathcal{A}_{1b}\mathcal{S}_{1\ldots\dot{b}}$$

$$O^n_{nn} = \left(\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right|0\left|\begin{matrix}1\ldots\dot{n}\\ n\end{matrix}\right) = \frac{2(n-1)}{n}\mathcal{S}_{1\ldots\dot{n}}\mathcal{A}_{1n}\mathcal{S}_{1\ldots\dot{n}}$$

Table 2. Young operators $\left[\begin{array}{c|c|c} 1a & & 1b \\ \vdots & 0 & \vdots \\ \dot a & & \dot b \\ \vdots & & \vdots \\ n & & n \end{array}\right]$ **for $[21^{n-2}]$**

$a < n$	$2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}\mathcal{A}_{1\ldots\dot a}S_{1a}\mathcal{A}_{1\ldots\dot a\ldots n}$ $(1a \ldots \dot a \ldots n)$	p	$(1b \ldots \dot b \ldots n)$ $S_{1b}\mathcal{A}_{1\ldots\dot b}\;2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}$	$b < n$
			$(1n \ldots n)$	$b = n$
$a = n$	$\sqrt{\left\{\frac{2(n-1)}{n}\right\}}\mathcal{A}_{1\ldots\dot n}\dot S_{1n}$ $(1n \ldots n)$		$(1b \ldots \dot b \ldots n)$ $\mathcal{A}_{1\ldots\dot b}\sqrt{\left\{\frac{2(b-1)}{b}\right\}}$	$b < n$
			$(1n \ldots \dot n)$ $\mathcal{A}_{1\ldots\dot n}\sqrt{\left\{\frac{2(n-1)}{n}\right\}}$	$b = n$

e.g.

$$O^n_{a^*b^*} = (1a^* \ldots \dot a \ldots n|0|1b^* \ldots \dot b \ldots n) = 4\left(\frac{n-1}{n}\right)\sqrt{\left\{\frac{(a-1)(b-1)}{ab}\right\}}\mathcal{A}_{1\ldots\dot a}S_{1a}\mathcal{A}_{1\ldots\dot a\ldots n}\left(1\ldots\dot a\ldots n\,\middle|\,p\,\middle|\,1\ldots\dot b\ldots n\right)S_{1b}\mathcal{A}_{1\ldots\dot b}$$

$$O^n_{a^*n^*} = (1a^* \ldots \dot a \ldots n|0|1n^* \ldots \dot n) = 2\sqrt{\left\{\frac{(a-1)(n-1)}{an}\right\}}\mathcal{A}_{1\ldots\dot a}S_{1a}\mathcal{A}_{1\ldots\dot a\ldots n}\left(\begin{matrix}1\ldots\dot a\ldots n\\ a\end{matrix}\,\middle|\,p\,\middle|\,\begin{matrix}1\ldots\dot n\\ n\end{matrix}\right)$$

$$O^n_{n^*b^*} = (1n^* \ldots \dot n|0|1b^* \ldots \dot b \ldots n) = 2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}\mathcal{A}_{1\ldots\dot n}S_{1n}\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\,\middle|\,p\,\middle|\,\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)\mathcal{A}_{1\ldots\dot b} = 2\sqrt{\left\{\frac{(b-1)(n-1)}{bn}\right\}}\left(\begin{matrix}1\ldots\dot n\\ n\end{matrix}\,\middle|\,p\,\middle|\,\begin{matrix}1\ldots\dot b\ldots n\\ b\end{matrix}\right)$$

$$\times\,\mathcal{A}_{1\ldots\dot b\ldots n}S_{1b}\mathcal{A}_{1\ldots\dot b}$$

$$O^n_{n^*n^*} = (1n^* \ldots \dot n|0|1n^* \ldots \dot n) = \frac{2(n-1)}{n}\mathcal{A}_{1\ldots\dot n}S_{1n}\mathcal{A}_{1\ldots\dot n}$$