# ELECTROHYDRODYNAMIC STABILITY OF A STAGNATION POINT

By

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*Abstract* .

The stability of a stagation point on the parablolidal surface of seperation between two dielecric fluids under the influence of on electric field is studied. A localised perturbation technique is used in the neighborhood of the stagnation point. It is found that the electric fied looses its destablising effect on the stability of the stagnation point.

## 1. *Introduction :*

The published literature in electrohydrodynamics deals mainly with surface instabilites. It is quite wellknown that (4) the electric fields which are normal to surfaces seperating two fluids tend to destabilise the interaface, while tangential fields has stablising effect. An exhaustive collection of works on hydrodynamic stabilities of surfaces may be found in ref. (2). Unfortunately, a little interest is paid to point instability. Recently, Berghmans (1) studied the hydrodynamic stability of a stagation point on a parabolic interface under gravity force and surface tension. He found that the instability can occur for certain displacements of the interface. He also noted that the surface tension has its maximum effect at the stagnation point.

The aim of this note is to study the stability of the stagnation point under the influence of an electric field. It is found that the electric has no effect on the stability of the stagnation point.

2. *Formulation of the problem :*

The problem considered here deals with a parabolic interface formed by a jet impinging vertically a horzontal surface of a fluid at rest. The surface equation can then be written as (3)

$$F(r,z) = z - b r^2 = 0, \quad (1)$$

where (r, z) are cylindrical coordinates and the z-axis is vertically upwards. The steady state solutions for the velocity of the upper fluid are (1)

$$v_{1r} = a r \quad (2)$$

$$v_{1\theta} = 0, \quad (3)$$

$$v_{1z} = 2 a (2 b r^2 - z) \quad (4)$$

The lower fluid is assumed to be at rest, i.e.

$$v_{2r} = v_{2\theta} = v_{2z} = 0, \quad (5)$$

where the subscripts 1,2 refer to upper and lower fluids respectively. It is evident that the above expressions for the velocity satisfy the equations governing motion which are

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla \pi + \rho \underline{g}$$

$$\text{where } \pi = p - \tfrac{1}{2}\epsilon \underline{E}^2 ,$$

along with

$$\nabla \cdot (\epsilon \underline{E}) = 0 ,$$

$$\nabla \wedge \underline{E} = 0 ,$$

where p is the pressure, is the fluid denisty, g is the gravity acceleration acting in the negative z direction and $\underline{E}$ is the electric field.

We assume that the electric field consistant with eps. (7) and (8) to be of the form

$$E_r = E^* r, \quad (9)$$

$$E_z = -2 E^* z \quad (10)$$

The surface charge denisty q on the parabloic surface is zero for the steady stae provided that the above assumption for the field is taken near the origin. We notice that

$$q = \underline{n}.( \quad \underline{E}_1 — \quad \underline{E}_2)$$

and n is the unit normal to the surface.

We note that eqs, (2) — (4) show that the origin is stagnant.

## 3. *Perturbation Equations*

We assume that the interface is slightly deformed and the resulting surface is given by

$$F(r,z,t) = z - b\,r^2 - \bar{\eta}(r,t) = 0 \qquad (11)$$

Consequently ,the dependant variables $\underline{v}$ , $\underline{E}$, $\pi$ receive increments to be of the form

$$v^o = \underline{v} + \bar{\underline{v}} \quad , \underline{E}^o = \underline{E} + \bar{\underline{E}} \quad , \quad \pi^o = \pi + \bar{\pi} \quad , \qquad (12)$$

whre the superscript o refer to the perturbed quantities and the bar denotes the perturabtion. We assume that the perturabation flow is irrotaional and hence there exists a velocity potential $\bar{\ }$ such that

$$\bar{\underline{v}} = - \nabla \bar{\phi} \qquad (13)$$

The z component of eq.(6)

$$\rho \frac{\partial v_z^o}{\partial t} + \rho\, v_r^o \frac{\partial v_z^o}{\partial r} + \rho\, v_z^o \frac{\partial v_z^o}{\partial z} = - \frac{\partial \pi^o}{\partial z} - \rho g \qquad (14)$$

For simplicity, we shall only consider symmetric perturbations, i.e.

$$\bar{v}_r = o \qquad \text{at} \qquad r = o , \qquad (15)$$

and from eqs. (2), (5) we also se that $v_r$ vanishes at the stagnation point. Thus eq. (14) takes the form

$$\rho \frac{\partial \bar{v}_z}{\partial t} + \tfrac{1}{2} \rho \frac{\partial ( v_z + \bar{v}_z )^2}{\partial z} = - \frac{\partial ( \pi + \bar{\pi} )}{\partial z} - \rho g, \qquad \text{at } r=0 \quad (16)$$

Eq.(16) can then be linearised to the form

$$-\rho \frac{\partial}{\partial t}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) + \rho \frac{\partial}{\partial z}(v_z \bar{v}_z) = -\rho g - \frac{\partial \bar{\pi}}{\partial z} \quad . \qquad (17)$$

Integrating eq.(17) w.r.t. z from 0 to $\bar{\eta}$ we get

$$\bar{\pi} = +\rho \frac{\partial \bar{\phi}}{\partial t} - \rho g \bar{\eta} - \rho v_z \bar{v}_z \qquad \text{at } r=0, \; z = \bar{\eta}(0,t) \qquad (18)$$

Eq. (18) is valied for both regions of the fluids. At the stagnation point $v_z = 0$ and in the upper region $v_z = -2a\bar{\eta}_1$ (0,t) which is a first order quantity. It follows that in both cases the last term of eq. (18) is a second order quantity which can be neglected.

Thus

$$\bar{\pi} = +\rho \frac{\partial \bar{\phi}}{\partial t} - \rho g \bar{\eta} \qquad (19)$$

at $r = 0$ and $z = \bar{\eta}(0,t)$.

It is also assumed that the perturbation of the velocity satisfies the equation of continuity

$$\nabla \cdot \underline{\bar{v}} = 0 \qquad (20)$$

and from eq.(13)

$$\nabla^2 \bar{\phi} = 0 \qquad (21)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \; .$$

$\bar{\eta}$ is related to $\bar{\phi}$ by the surface equation

$$\frac{D\, F^0(r,z,t)}{Dt} = 0.$$

It follows that

$$\frac{\partial \bar{\eta}}{\partial t} + a r \frac{\partial \bar{\eta}}{\partial r} + 2 b r \frac{\partial \bar{\phi}_1}{\partial r} - \frac{\partial \bar{\phi}_1}{\partial z} = 0 \qquad (22)$$

The perturbation in the electric field $\underline{E}$ satisfies eqs. (7) and (8) i.e.

(8) i.e.

$$\nabla \cdot \epsilon \underline{E} = 0 \tag{23}$$

$$\nabla \wedge \underline{E} = 0 \tag{24}$$

It follows that there exists a potential function $\bar{\Psi}$ such that

$$\underline{E} = \quad \bar{\Psi} \tag{25}$$

and

$$\nabla^2 \bar{\Psi} = 0 \quad . \tag{26}$$

Since we are interested in the stability of the stagnation point, we may expand $\bar{\Phi}$ , $\bar{\eta}$ , $\bar{\Psi}$ in powers of r,z of the form[1]

$$\bar{\Phi}_1 = A \quad e^{i\alpha t}( 1 - k^2r^2+ \ldots)( 1 + fz + 2 k^2z^2 +\ldots) \tag{27}$$

$$\bar{\Phi}_2 = A \quad e^{i\alpha t}( 1 - q^2r^2+ \ldots)( 1 + fz + 2 q^2z^2+\ldots) \tag{28}$$

$$\bar{\eta} = Q \quad e^{i\alpha t}( \xi - s\, r^2 +\ldots) \tag{29}$$

$$\bar{\Psi}_1 = A_1 \, e^{i\alpha t}( 1 - k^2r^2 +\ldots)( 1 + fz + 2 k^2z^2 +\ldots) \tag{30}$$

$$\bar{\Psi}_2 = A_2 \, e^{i\alpha t}( 1 -q^2r^2 +\ldots)( 1 + fz + 2 q^2z^2 +\ldots) \tag{31}$$

where A, $A_1$, $A_2$, Q, f, s, k, q, are constants. $\alpha$ being real or complex, determines the nature of the perturbation where stable or unstable.

4. Dispersion Relation

The solutions for $\bar{\Phi}$ , $\bar{\eta}$ , $\bar{\Psi}$ should satisfy eqs.(21),(22), (25)and the following boundary conditions as well. Eq.(22)relates A to Q.On subistitution for $\bar{\Phi}$ . $\bar{\eta}$ into eq.(22) we get

$$A = i\alpha\, Q \tag{32}$$

The radius of curvature of the boundary surface is given by

$$\frac{1}{R_a^0} = \frac{1}{r}\,\frac{dz}{dr}\,\left[ 1 + (\frac{dz}{dr})^2 \right]^{-\frac{1}{2}} + \frac{d^2z}{dr^2}\left[1 + (\frac{dz}{dr})^2 \right]^{-3/2} \tag{33}$$

Subistituting for z(r) from eq.(11) we find that

$$\frac{1}{R_a^0} = ( 2b + \frac{1}{r}\frac{\partial \bar{\eta}}{\partial r} )\left\{ 1 - \tfrac{1}{2}( 2br + \frac{\partial \bar{\eta}}{\partial r})^2 \right\}$$
$$+ ( 2b + \frac{\partial^2 \bar{\eta}}{\partial r^2})\left\{ 1 - \tfrac{3}{2}( 2br - \frac{\partial \bar{\eta}}{\partial r})^2 \right\} \tag{34}$$

$1/R_a^0$ should be finite in order to have finite pressure difference across the interface.Therefore

$$\frac{\partial \bar{\eta}}{\partial r} = 0 \qquad \text{at } r=0 \tag{35}$$

and hence

$$\frac{1}{R_a^0} = 4\,b + 2\,\frac{\partial^2 \bar{\eta}}{\partial r^2} \,. \tag{36}$$

Condition (35) is satisfied by the solution given by eq.(29).It is also observed from eqs.(27) and (28) that $\bar{v}_r = \frac{-\partial \Phi}{\partial r} = 0$ at $r = 0$.

The electric potential should be continuous at the interface, therefore

$$A_1 = A_2 \tag{37}$$

The normal electric displacement $\epsilon\, E_n$ should be continuous at the interface

$$\underline{n}.(\epsilon_1\, \underline{E}_1^0 - \epsilon_2\, \underline{E}^0{}_2) = 0. \tag{38}$$

Subistituting from eqs.(30)and (31) into eq.(38) we get

$$A_1 = A_2 = -iQ\, E^{*}. \tag{39}$$

The normal component of the stress tensor $T_{ij}$ where[4]

$$T_{ij} = -(\pi + \tfrac{1}{2}\epsilon E_k E_k)\,\delta_{ij} + \epsilon\, E_i\, E_j \tag{40}$$

is discontinuous at the interface by the surface tension,thus

$$-(\pi_1^0 + \tfrac{1}{2}\epsilon_1 {E^0_1}^2) + (\pi_2^0 - \tfrac{1}{2}\epsilon_2 {E_2^0}^2) = \frac{T}{R_a^0} \tag{41}$$

where T is the surface tension.

On subistitution into eq.(41) for $\pi^0$, $E^0$, $R_a^0$ and linearising we get in the limit as $r \to 0$ the following dispersion equation

$$\alpha^2 = \frac{4\,T\,e}{\rho_2 - \rho_1} + fg \tag{42}$$

We observe that the radial electric field $E_r$ vanishes as $r \to 0$ while the vertical component $E_z$ is of order $\gamma_l$ Therefore the electric field is mainly perpndicular to the surface at the stagnation point.

Eq. (42) is the same as obtained by Berghans for his study of the stability of the stagnation point in absence of electric field. Thus it is clear from eq. (42) that the electric field has no effect on the stability of the stagnation point. Therefore the normal field looses its destablising effect on the stability of the stagnation point in contrast with the surface tension effect (1). The reason is that the terms containing electric field in eq. (41) become of the second order due to the surface deformation. However, in the neighborhood of the stagnation point the electric field may affect the surface stability. The investigation of the latter case may not be carried out easily by the localised perturbation technique.

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