MHD PLANE AND CYLINDRICAL FLOW IN THE SLIP REGIME

By

A.G. EL-SAKKA, M.A. KHIDR and M. HAMDY

ABSTRACT

The flow of a slightly rarefied electrically conducting gas between two flat infinite plates and between two concentric cylinders in relative motion are considered. The slip velocity and temperature jump at the boundaries are taken into account to investigate the effect of rarefaction on the nature of the flow.

NOTATIONS

H — the magnetic field, u — the flow velocity , σ — the electric conductivity , — the mass density , μ — coefficient of viscosity , μ_e — the magnetic permeability , p — the pressure , T — the temperature , R — the gas constant , K — the thermal conductivity , h — the enthalpy , R_e Reynolds number $R_e = uL/\varrho\,\mu$, $\gamma = c_p/c_v$ the ratio of the two specific heats , p_r — Prandtl number : $P_r = \frac{c_p}{K}$, R_v — Magnetic Reynold number , R_H — magnetic pressure number

$$R_H = \frac{\mu_e H_o^2}{\varrho u^2} \text{ , Hartmann number } \alpha = \sqrt{R_H R_e R_v}$$

INTRODUCTION

The flow between two infinite plates and between concentric cylinders in relative tangential motion is called a Couette flow problem , actually , one of the motives for studying the Couette flow is its usefulness in studying boundary layer flows. Although it is not exactly analogous to a true boundary layer , it is sufficiently similar and censiderably easier to solve.

In classical hydrodynamics a complete integral to incompressible Couette flow was obtained in (1,2). Illingworth (3) obtained a numerical solution for compressible plane and cylindrical Couette flows of compressible gases. Shidlevsky (4) used Illingwerth results to obtain approximate solutions for the problems in the slip regime.

The MHD treatment of the problem started by Hartmann and Lazarus (5). They obtained the solution of plane incompressible MHD Couette flow problems. Bleviss (6) treated the plane Couette flow for the hypersonic motion of high temperature ionized air. Ramamoorthy (7) obtained an approximate solution for the incompressible cylindrical Couette flow problem. Arora and Gupta (8) found the complete exact solution of that problem.

In this work we shall study the MHD flow of an incompressible slightly rarefied gas. A small relative temperature between the two surfaces is assumed, hence, the electrical conductivity and coefficient of viscosity can be considered as constants. We shall use slip boundary conditions as obtained in (4).

THE PLANE FLOW

Consider the flow between two infinite plates.
The upper plate moves in the x-direction with constant velocity U , while the lower plate is fixed, the upper and lower plates are kept at constant temperatures T_o and T_o respectively. The applied magnetic field is (O , H_o , O) while the induced magnetic field will be assumed to be (H , O , O) and considered as a function of y only. The flow velocity is taken in the x-direction and is considered as a function of y . The equations govering the flow may be written in the form (9) :

$$H_o \frac{du}{dy} = - \frac{1}{\sigma \mu_e} \frac{d^2H}{dy^2} \qquad \ldots(1)$$

$$\mu_e H_o \frac{dH}{dy} = \frac{d}{dy} \left(\mu \frac{du}{dy} \right) \qquad \ldots(2)$$

$$\frac{d}{dy} (P + \mu_e H^2) = 0 \qquad \ldots(3)$$

$$\mu \left(\frac{du}{dy} \right)^2 + \frac{d}{dy} \left(K \frac{dT}{dy} \right) + \frac{1}{\sigma} \left(\frac{dH}{dy} \right)^2 = 0 \qquad \ldots(4)$$

$$P = \rho R T \qquad \ldots(5)$$

$$h = c_p T$$

If we consider that the gas is slightly rarefied and that there are complete energy and momentum accomodation between the surfaces of the plates and the gas, then we can use the boundary conditions decduced in (4) for the slip regime. These conditions may be written in the form :

at the lower plate $y = -d$ we have :

$$u = \sqrt{\frac{\gamma \pi}{2h(\gamma - 1)}} \frac{\mu}{\rho} \frac{du}{dy}$$

$$\frac{h}{h_o} = \varphi + \frac{15}{8} \sqrt{\frac{\gamma \pi}{2h(\gamma - 1)}} \frac{\mu}{\rho h} \frac{dh}{dy}$$

and at the upper plate $y = d$ we have :

$$u = U - \sqrt{\frac{\gamma \pi}{2h(\gamma - 1)}} \frac{\mu}{\rho} \frac{du}{dy}$$

$$\frac{h}{h_o} = 1 - \frac{15}{8} \sqrt{\frac{\gamma \pi}{2h(\gamma - 1)}} \frac{\mu}{\rho h} \frac{dh}{dy}$$

If we take $u = uU$, $H = H_o H'$, $P = \rho_o U^2 p'$, $h = h_o h$ then the equations (1 – 6) will take the form

$$\frac{d^2 u}{d y'^2} = - R_H R_e \frac{dH'}{dy'} \qquad \ldots(7)$$

$$\frac{d u'}{d y'} = - \frac{1}{R_\nu} \frac{d^2 H'}{dy^2} \qquad \ldots(8)$$

$$\frac{d}{d y} (p' + \frac{R_H}{2} H'^2) = 0 \qquad \ldots(9)$$

$$(\frac{d u'}{d y'})^2 + \frac{1}{(\gamma-1) p_r M_o^2} \frac{d^2 h'}{dy^2} + \frac{R_H R_e}{R_\nu} (\frac{dH'}{dy'})^2 = 0 \qquad \ldots(10)$$

$$p' = \frac{1}{\gamma M_o^2} h' \qquad \ldots(11)$$

$$h' = T' \qquad \ldots(12)$$

The boundary conditions may be written in the form :

$$u' = K_n \frac{du}{dy} , h' = \varphi + K_n \frac{dh'}{dy'} , H' = 0$$

at $y' = 1$

$$u' = 1 - K_n \frac{du'}{dy} , h' = 1 - K_h \frac{dh'}{d y} , H' = 0$$

Considering that K_n and K_h are of the order of Knudsen number , which is considered small in the slip regime and solving the system (7 — 12) and applying the boundary conditions we get :

$$u' = \sinh \alpha y' / 2 \, (\sinh \alpha + \alpha K_n \cosh \alpha) + \frac{1}{2}$$

$$h' = \left\{ (\gamma - 1) P_r M_0^2 / 2 (\sinh \alpha + \alpha K_n \cosh \alpha) \right\} \left\{ \cosh 2\alpha - \right.$$

$$\left. - \cosh 2\alpha y' + 2 \alpha K_n \sinh 2\alpha + \frac{\chi'}{2} (1 - y') + 1 \right\} \quad \ldots (14)$$

$$H' = - \frac{R_\nu}{2\alpha} \left[\frac{\cosh \alpha - \cosh \alpha y'}{\sinh \alpha + \alpha K_n \cosh \alpha} \right] \quad \ldots (15)$$

$$p' = \frac{1}{\gamma M_0^2} h' = \frac{1}{\gamma M_0^2} T' \quad \ldots (16)$$

From (13) and Fig. (1) we notice that :

1) At any point in the lower half of the flow ($y' < 1$) the velocity increases with α , while the converse occurs in the upper half.

2) For $y' > 1$ the velocity value at any point in the rarefied case is larger than that of the continuous medium, while the converse occurs for $y' < 1$.

From (14) and Fig. (2) we can see that :

1) At any point the temperature increases with the degree of rarefaction and with α to a limiting value and then decreases. The value of α corresponding to maximum temperature can be found analytically.

2) For constant α , K_n and K_h the points of maximum temperature can be found.

Also from (15) and Fig. (3) we can see that the magnitude of the induced magnetic field decreases with α and with the degrees of rarefaction (K_n and K_h). For large values of α the induced magnetic field is nearly constant except at narrow layers near the boundaries.

THE CYLINDRICAL CASE

Here we shall consider the gas to be bounded by two coaxial cylinders of radi a and b ; $b < a$.

Using cylindrical coordinate system with the z-axis taken to be the axis of the cylinders and assuming :

1) The outer cylinder is rotating with a constant angular velocity Ω and is kept at a constant temperature T_0.

   The inner cylinder is stationary with temperature T_0.

2) The applied field $H_o = \frac{K}{r}$ is radial, while the induced magnetic field H is in the θ-direction.

3) The unknowns v , H and h depend on r only.

4) The flow velocity v is in the θ -direction.
Then the basic MHD Navier-Stokes equations will be :

$$H_o\left(\frac{v}{r} - \frac{dv}{dr}\right) = \frac{1}{\sigma\mu_e} \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\,H)\right] \qquad \ldots(1)$$

$$\frac{v^2}{r} + \frac{\mu_e}{\rho}\left(H_o\frac{dH_o}{dr} - \frac{H^2}{r}\right) = \frac{1}{\rho}\frac{d}{dz}\left[p + \frac{\mu_e}{2}(H_o^2 + H^2)\right] \qquad \ldots(2)$$

$$\frac{1}{r^2}\frac{d}{dr}\left[\mu r^2\left(\frac{dv}{dr} - \frac{v}{r}\right)\right] = -\mu_e\left(\frac{H_o\,H}{r} + H_o\frac{dH}{dr}\right) \qquad \ldots(3)$$

$$\frac{1}{r}\frac{d}{dr}\left[\mu r v\left(\frac{dv}{dr} - \frac{v}{r}\right)\right] + \frac{1}{rP_r}\left(\mu r\frac{dh}{dr}\right) + \frac{1}{\sigma}\left[\frac{1}{r}\frac{d}{dr}(rH)\right]^2 +$$

$$+ \mu_e H_o \frac{v}{r}\frac{d}{dr}(rH) = 0 \qquad \ldots(4)$$

$$p = \rho RT \qquad \ldots(5)$$

$$h = c_p T \qquad \ldots(6)$$

Solution :

Let $r = r'a$, $v = r'\omega'\Omega a$, $h = h_o h'$, $p = \rho\Omega^2 a^2 p'$,

$H = \frac{k}{a}H'$, then we have the non-dimensional system (dropping the primes) :

$$\frac{d\omega}{dr} = -\frac{1}{R_v}\frac{d}{dr}\left[\frac{1}{r}(rH)\right] \qquad \ldots(7)$$

$$\frac{d}{dr}\left(r^3\frac{d\omega}{dr}\right) = -R_H R_e \frac{d}{dr}(rH) \qquad \ldots(8)$$

$$r^2\left(\frac{d\omega}{dr}\right)^2 + \frac{1}{(\gamma-1)p_r M_o^2}\frac{1}{r}\frac{d}{dr}\left(r\frac{dh}{dr}\right) + \frac{R_H R_e}{R_v}\left[\frac{1}{r}\frac{d}{dr}(rH)\right]^2 = 0 \quad \ldots(9)$$

$$p = \frac{1}{\gamma M_o^2}h$$

where ω' is the non-dimensional angular velocity .
Subject to non-dimensional modified boundary conditions in the slip velocity regime

$$\omega = m_\omega \frac{d\omega}{dr}\ ,\ h = \gamma + m_\omega \frac{dh}{dr}\ ;\ H = 0 \text{ for } r = q\ ;\ q = \frac{b}{a}$$

$$\omega = 1 - m_\omega \frac{d\omega}{dr}\ ,\ h = 1 - m_h \frac{dh}{dr}\ ,\ H = 0 \text{ for } r = 1$$

System (7 - 10) has the solution .

$$\omega = c_1 + c_2 r^{B-1} + c_3 r^{-B-1} \qquad \ldots(11)$$

$$h = T = \frac{-(\gamma-1)p_r M_o^2}{2B}\left\{(B-1)c_2^2 r^{2B} + (B+1)c_3^2 r^{-2B} + c_5 \ln r + c_6\right\} \quad .(12)$$

$$\frac{H}{R_\nu} = \frac{-1}{B^2-1}\left\{(B-1)c_2 r^B - (B+1)c_3 r^{-B} + c_4 r^{-1}\right\} \qquad \ldots(13)$$

where

$$c_1 = 1 - c_2\left[1 + m_\omega(B-1)\right] - c_3\left[1 - m_\omega(B+1)\right] \qquad \ldots(14)$$

$$c_2 = \frac{(B+1)(q^{-1} - q^{-B})}{(B+1)(q^{-1}-q^{-B})(1-q^{B-1}) + (B-1)(q^{-1}-q^{B}) + m_\omega(B^2-1)\left[(q^{-1}-q^{-B})(1+q^{B-2}) - (q^{-1}-q^{B})(1+q^{-B-2})\right]} \qquad \ldots(15)$$

$$c_3 = \frac{B-1}{B+1}\ \frac{q^{-1}-q^{B}}{q^{-1}-q^{-B}}\ c_2 \qquad \ldots(16)$$

$$c_4 = (B+1)c_3 - (B-1)c_2 \qquad \ldots(17)$$

$$c_5 = \frac{1}{\ln q - 2Bm_h(1+q^{-1})}\left\{(B-1)\left[1-q^{2B}+2Bm_h(1+q^{2B-1})\right]c_2^2 + (B+1)\cdot\left[1-q^{-2B}-2Bm_h(1+q^{-2B-1})\right]c_3^2 - \frac{2B\chi}{(\gamma-1)p_r M_o^2}\right\} \qquad \ldots(18)$$

$$c_6 = -\left[\frac{2B}{(\gamma-1)p_r H_o^2} + (B-1)(1+2Bm_h)c_2^2 + (B+1)(1-2Bm_h)c_3^2 + 2Bm_h\ c_5\right] \quad ..(19)$$

and . $B = \sqrt{\alpha^2 + 1}$

From Maxwells equations the dimensionless electric field and current density are given by :

$$j_\theta = -r^{\cdot}\left[c_2 r^{B} + c_3 r^{-B} + \frac{c_4}{B^2-1} r^{-2}\right] \qquad \ldots(20)$$

$$E_z = j^{\cdot} - v^{\cdot} r^{-1} \qquad \ldots(21)$$

The analytical solution of the problem is represented by the equations (11-21). We restrict ourselves to the values : $m_\omega = \frac{m}{V} = \frac{5}{100}$

$q = \frac{1}{2}$, $M_o = \frac{1}{2}$ $\frac{100}{5}$ of the paramenters. We observe the following :

1) As shown in table (1), we may see that :

a- At any point the magnitude of velocity is larger in the rarefied case than that of the continuum case, except at points in the region near to the outer plate.

b- Near the inner cylinder, the magnitude of velocity increases with B. The converse occurs near the outer cylinder.

c- The slip velocity at each plante increases with B.

2) From table (2) we may see that the temperature at any point near the inner plante is larger in the continuum case. The converse happens near the outer cylinder. For large values of B, rarefaction tends to increase the temperature at any point.

3) From table (3) we may see that .

a- The induced magnetic field increases with r to a maximum value and then decreases.

b- Rarefaction tends to decrease the magnitude of the induced magnetic field at any point.

c- At any point the induced magnetic field decreases as B increases.

4) From table (4) we may see that :

a- The magnitude of the induced electric field, in general, increases with r.

b- The magnitude of the induced electric field decreases with rarefaction.

c- The magnitude of the induced electric field decreases with B.

5) From table (5) we may see that the magnitude of the current density at any point decreases with the increase of rarefaction and with the increase of B. We may also see that its magnitude decreases with r to a minimum value and then increases.

References

1. Pai. S.I. Viscous flow theory. 1. Laminar flow. Van Norstand, Princeton N.J. 1956.

2. Sommerfield A. «Mechanics of deformable bodies Academic Press N.Y. 1950.

3. Illingworth C.R., Some solutions of the equations of flow of a viscous compressible fluids. Proc. Cambridge Phil. Soc. 46, (1950).

4. Shidlouskey V.P. Rarefied Gas Dynamics. N.Y. 1969.

5. Hartmann J. Lazarus F., Kgl. Donske. Videnskab, Selskabs, Mat. Fys, Medd, 15, (6,7) (1937).

6. Bleviss Z.O., Magnetohydrodynamics of hypersonic Couette flow J. Aerospace. Soci. 25, 601 (1958).

7. Ramamoorthy P. Phys. Fluids, 4, 1444 (1961).

8. Arora K.L., Gupta P.R. Phys. Flüids, 6 1146 (1972).

9. Pai. S.I. Magnetodyrodynamics and plazma physics.

TABLE 1

$$V\ (r),\ \Psi' = \frac{5}{100}\ ,\quad m_w = 0,\ \frac{5}{100}$$

r / m_w	B = 1		B = 2		B = 5		B = 10		B = 20	
	0	5/100	0	5/100	0	5/100	0	5/100	0	5/100
. 5	0	.1154	0	.1107	0	.1518	0	.2032	0	.3145
.55	.1275	.2248	.1365	.2242	.1878	.2901	.2658	.3708	.3697	.4470
. 6	.2445	.3266	.2558	.3246	.3160	.3886	.3952	.4572	.4592	.5016
.65	.3538	.4222	.3635	.4066	.4018	.4676	.4715	.5171	.5071	.5457
. 7	.4571	.5130	.4636	.5027	.4921	.5287	.5291	.5676	.5480	.5884
.75	.5556	.6004	.5584	.5849	.5714	.6039	.5814	.6157	.5875	.6307
. 8	.6500	.6846	.6395	.6566	.6455	.6679	.6346	.6388	.6293	.6733
.85	.4708	.7662	.7385	.4722	.7197	.7320	.6958	.7176	.6746	.7177
. 7	.8297	.8459	.8263	.8189	.8028	.7982	.7662	.7760	.7292	.7656
.95	.9158	.9237	.9131	.8954	.9034	.8832	.8472	.8544	.8294	.8306
1	1	1	.9714	1	.9620	1	1	.9427	1	.9223

TABLE 2

$h(r)$, $\Psi' = \frac{5}{100}$. $m_h = 0.\ \frac{5}{100}$

	B = 1		B = 2		B = 5		B = 10	
r \ m_w	0	5/100	0	5/100	0	5/100	0	5/100
. 5	1.0500	1.0415	1.0500	1.0310	1.0500	1.0271	1.0500	1.0503
.55	1.0474	1.0405	1.0473	1.0306	1.0481	1.0270	1.0426	1.0500
. 6	1.0446	1.0368	1.0443	1.0293	1.0433	1.0270	1.0426	1.0500
.65	1.0381	1.0326	1.0379	1.0266	1.0365	1.0244	1.339	1.0400
. 7	1.0328	1.0292	1.0361	1.0243	1.0305	1.0225	1.0300	1.0440
.75	1.0287	1.0264	1.0222	1.0254	1.0207	1.0207	1.0268	1.0425
. 8	1.0220	1.0222	1.0227	1.0197	1.0188	1.0161	1.0233	1.0408
.85	1.0165	1.0186	1.0165	1.0168	1.0053	1.0119	1.0187	1.0374
. 9	1.0111	1.0151	1.0110	1.0142	1.0010	1.0063	1.0149	1.0359
.95	1.0059	1.0117	1.0059	1.0176	1.0122	1.0153	1.0194	1.0389
1	1.0000	1.0084	1.0000	1.0093	1.0000	1.9998	1.0000	1.0345

TABLE 3

$$\frac{H(r)}{R_v} \cdot m = O \cdot \frac{5}{100}$$

r	B = 2		B = 5		B = 10		B = 20	
	0	5/100	0	5/100	0	5/100	0	5/100
. 5	0	0	0	0	0	0	0	0
.55	.0281	.0211	.0268	.0175	.0213	.0119	.0186	.0069
. 6	.5439	.0330	.0389	.0254	.0280	.0156	.0171	.0063
.65	.0513	.0386	.0436	.0285	.0292	.0160	.0157	.0058
. 7	.0528	.0397	.0437	.0286	.0283	.0154	.0144	.0053
.75	.0501	.0376	.0412	.0270	.0246	.0144	.0131	.0050
. 8	.0338	.0254	.0365	.0239	.0239	.0130	.0117	.0045
.85	.0357	.0299	.0293	.0189	.0206	.0112	.0110	.0041
. 9	.0313	.0190	.0224	.0147	.0165	.0089	.0093	.0035
.95	.0133	.0111	.0113	.0073	.0091	.0049	.0074	.0044
1	0	0	0	0	0	0	0	0

TABLE 4

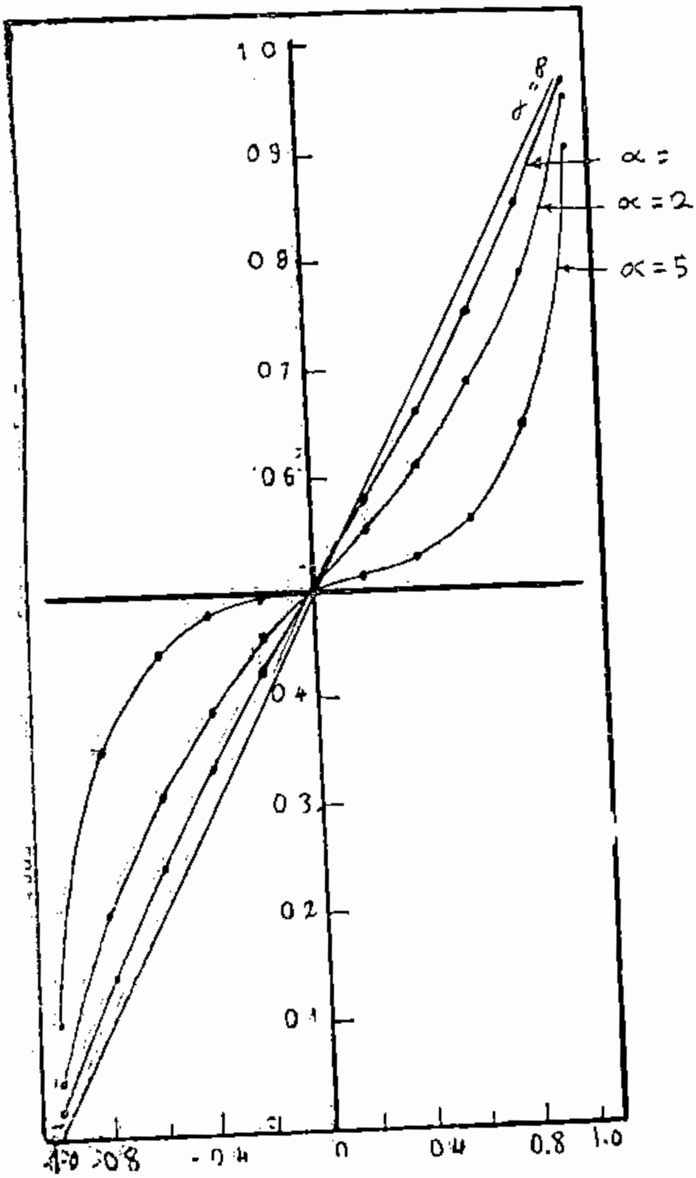
$j(r),\ m_w = O,\ \frac{5}{100}$

r / m_w	B = 2		B = 5		B = 10		B = 20	
	0	5/100	0	5/100	0	5/100	0	5/100
. 5	1.2408	—.9454	.3540	.2352	.6484	.3166	.7564	.1786
.55	1.0007	—.7618	.1104	.0742	.1555	.0720	.0492	.0044
. 6	.8402	.6388	.0048	.0023	.0093	0003	.0245	.0135
.65	.7309	.5539	.0662	.0429	. 365	.0220	.0340	.0134
. 7	.6559	.4974	.1050	.0687	.0501	.0261	.0296	.0113
.75	.6044	.4579	.1349	.0884	.0591	.0325	.0251	.0046
. 8	.5843	.4424	.1653	.0885	.0708	.0387	.0243	.0091
.85	.5515	.4134	.1840	.1153	.0868	.0473	.0269	.0102
. 9	.5331	.4027	.2169	.1426	.1144	.0622	.0407	.0154
.95	.5259	.3969	.2660	.1705	.1785	.0970	.1016	.0383
1	.5239	.3951	.3071	.2020	.2548	.1384	.2268	.0857

TABLE 5

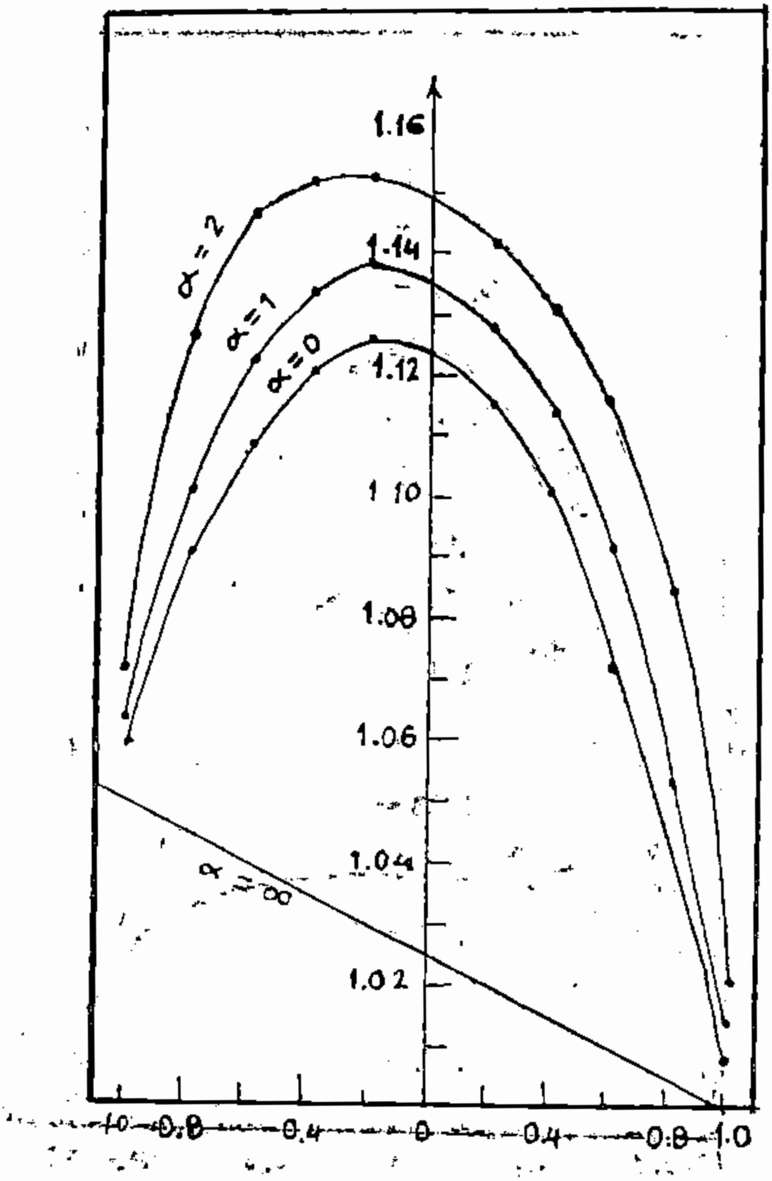
$E(r)\,;\, m_w = 0\,,\ \frac{5}{100}$

r \ m_w	B = 2: 0	B = 2: 5/100	B = 5: 0	B = 5: 5/100	B = 10: 0	B = 10: 5/100	B = 20: 0	B = 20: 5/100
. 5	—1.2408	—1.1762	.3540	.1772	.6484	— .0898	— .7564	— .4504
.55	—1.2322	—1.1795	— .2311	— .4533	— .3287	— .6021	— .6230	— .8083
. 6	—1.2477	—1.1831	— .5315	— .6500	— .6494	— .7617	— .7358	— .8405
.65	—1.2752	—1.2035	— .6844	— .7623	— .7419	— .8176	— .8141	— .8530
. 7	—1.2888	—1.2302	— .8080	— .8240	— .8060	— .8370	— .8124	— .8519
.75	—1.3452	—1.2584	— .8969	— .8936	— .8443	— .8534	— .8084	— .8505
. 8	—1.3968	—1.2982	— .9722	— .4234	— .8641	— .8672	— .8109	— .8507
.85	—1.4231	—1.3148	—1.0307	— .9765	— .8054	— .8916	— .8146	— .8546
. 9	—1.4550	—1.3426	—1.1089	—1.0295	— .9657	— .9244	— .8509	— .8661
.95	—1.4888	—1.3692	—1.2169	—1.1002	—1.0987	— .9963	— .9746	— .9126
1	—1.5239	—1.3951	—1.3071	—1.1640	—1.2458	—1.0811	—1.2268	—1.0080



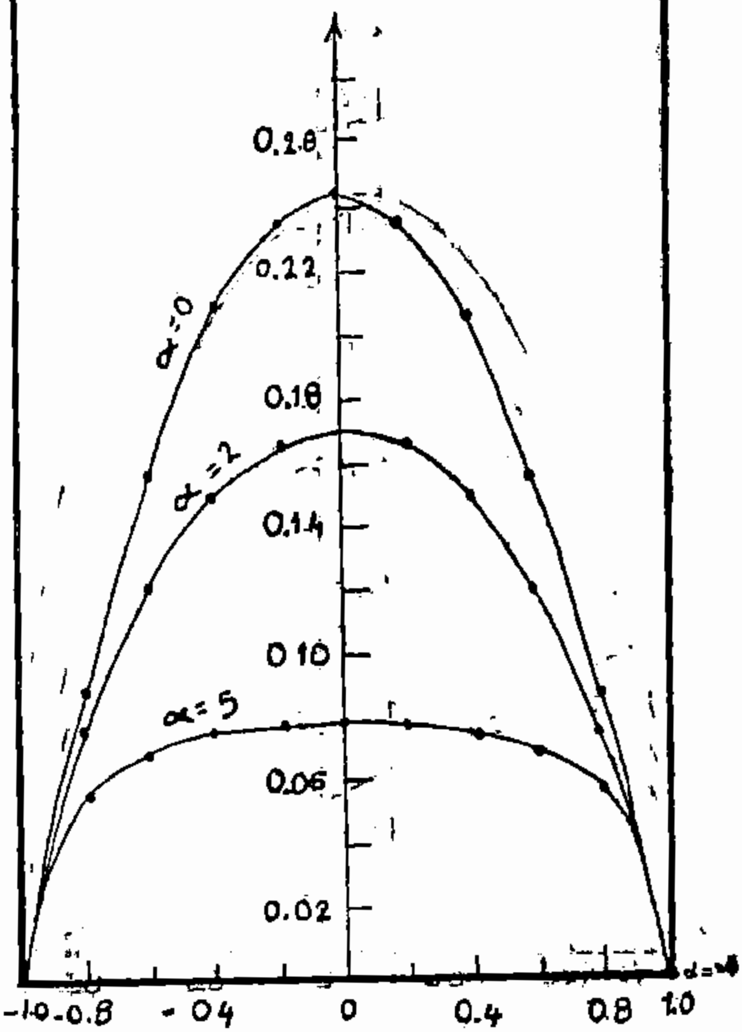
$v(y')$, $\chi' = k_n = \frac{5}{100}$

Fig. (1)



$T(y')\ ,\quad \chi' = k_4 = \frac{5}{100}$

Fig. (2)



$-\frac{H_x}{R_v}, x' = k_n = \frac{5}{100}$

Fig. (3)