The Steady Plane Couette flow and heat transfer of a rarefied electron gas

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Abstract:

Moments method has been used to replace a model kinetic equation for a rarefied electron gas by non-linear moments equations. These equa ions are solved by small parameter method to get insight into the behaviour of the flow for any degree of rarefaction.

Introduction and basic equations:

The electron gas is considered filling the space between two parallel impermeable non-heat conducting planes distance d apart. The upper plane is moving with a constant velocity $(+U)$ in the x-direction, and is kept at a constant temperature T_o.

The lower plane is moving with velocity $(-U)$ and is kept at a constant temperature T_o. There is an external magentic induction B_o in the z-direction which is imposed on the system. The moments equations of the model kinetic equation [1] are:

$$\int \varphi_i \left(\vec{c}\,\frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m}\,\frac{\partial f}{\partial \vec{c}}\right) dc = \int \frac{n\,R\,T}{\mu}(f_o - f)\,\varphi_i \, d\,\vec{c} \qquad (1)$$

where f is the distribution function of the charges, f_o

the local Maxwell distribution, n the density, R is the gas constant, μ the coefficient of viscosity, T the temperature, φ_i is a function of velocity, $\vec{c}$ the velocity of the charges and $\vec{r}$ its position vector. The moments equations from equations (1) are non-linear differential equations and they are solved by using of the small parameter method assuming a small relative temperature difference ($\chi = 1 + \chi'$, $\chi'^2 \ll 1$) between the two plates. We shall assume also that the flow occurs with a small Mach number and that:

$$\left(\gamma^2 = \left(\frac{e\ B_o\ U}{K\ T_o}\right)^2 \ll 1 \right)$$

From dimensional analysis it is clear that the induced magenetic field can be neglected relative to the induced electric field E. The latter is assumed to act in the y-direction. It can be considered composing of two fields E_1 emerging from electrons for which $c_y < 0$ and E_2 emerging from electrons for which $c_y > 0$. The flow velocity v is assumed also in the x-direction. As for the dirtribution function f we shall use two-stream local Maxwellian distribution function(2):

$$f = n_1 \left(\frac{m}{2\pi K T_1}\right)^{\frac{3}{2}} \exp\left\{- m \frac{(c_x - v_1)^2 + c_y^2 + c_z^2}{K T_1}\right\}, \ c_y$$

$$= n_2 \left(\frac{m}{2\pi K T_2}\right)^{\frac{3}{2}} \exp\left\{- m \frac{(c_x - v_2)^2 + c_y^2 + c_z^2}{K T_2}\right\}, \ c_y >$$

where n_1, n_2, v_1, v_2, T_1 and T_2 are six unknowns to be determined from equations of type (1).

Boundary conditions:

Taking $\varphi_1 = 1$, $\varphi_2 = c_x$, $\varphi_3 = c_y$, $\varphi_4 = c_x^2 + c_y^2 + c_z^2$, $\varphi_5 = c_x\, c_y$, $\varphi_6 = c_x\, c_y^2$, and taking f in the form (2) substituting in equation (1) we get:

$$n_1 \sqrt{T_1} = n_2 \sqrt{T_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$n_1 \sqrt{T_1}\,(v_2 - v\ ) = \alpha_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{d}{dy}(n_1\, T_1 + n_2 T_2) + \frac{eE}{k}(n_1 + n_2) - \frac{eB_o}{k}(n_1\, v_1 + n_2 v_2) = 0 \;\ldots (5)$$

$$n_2 T_2^{\frac{3}{2}} - n_1 T_1^{\frac{3}{2}} + \frac{m n_1 \sqrt{T_1}}{4\ K}(v_2^2 - v_1^2) = \alpha_3 \quad \ldots\ldots\ldots\ldots (6)$$

$$\frac{d}{dy}(n_1\, T_1\, v_1 + n_2 T_2 V_2) + \frac{eE}{k}(n_1 v_1 + n_2 v_2) - \frac{eB_o}{k}(\,n_1 v_1^2 + n_2 v_2^2\,) =$$

$$= \frac{-\ n\ k\ T}{2m\ (1 + \chi^2)^{3/4}\mu_o}\sqrt{\frac{k}{2\ \pi\ m}}\ \alpha_1 \ldots\ldots (7)$$

$$\frac{d}{dy}\left[(n_1 T_1^2 + n_2 T_2^2) + \frac{k}{5m}(n_1 T_1 v_1^2 + n_2 T_2 v_2^2)\right] \pm \frac{eE}{m}\left[\frac{k}{m}(n_1 T_1 + n_2 T_2) +\right.$$

$$\left. + \frac{1}{5}(m_1 v_1^2 + m_2 v_2^2)\right] - \frac{eB_o}{m}\left[\frac{k}{m}(n_1 T_1 v_1 + n_2 T_2 v_2) + \frac{1}{5}(v_1^3 + v_2^3)\right] =$$

$$= \frac{-8n\ T_o\ T}{5(1 + \chi)^{3/4}\mu_o} \cdot \frac{k^2}{m^2}\sqrt{\frac{k}{2\pi m}}\ \alpha_3 \ldots\ldots\ldots (8)$$

Supplemented with Poisson's equation

$$\frac{dE_1}{dy} = -\frac{e}{2\varepsilon}n_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\frac{dE_2}{dy} = \frac{e}{2\varepsilon}n_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where e is the charge of the electron and ε the dieletric constant Assuming complete momentum and energy accomodation at the two plates the boundary conditions will be:

$$\left.\begin{aligned} & v_1 = U\ ,\ T_1 = T_o\ ,\quad E_1 = 0 \text{ at } y = \frac{d}{2} \\ & v_2 = -U\ ,\ T_2 = \chi T_o,\ E_2 = 0,\ n_2 = n_o \text{ at } y = -\frac{d}{2}\ , \end{aligned}\right\} \quad (11)$$

assuning that n_2 is given at the lower plate.

<u>Method of solution</u>:

Let $y = y' d$, $n_i = n'_i\, n_o$, $T_i = T'_i\, T_o$, $v_i = v'_i\, U$, $E_i = E'_i B_o U$; $i = 1,2$ and neglecting terms of order M_o^2 $\left(M_o = \dfrac{U}{\sqrt{\gamma_o m k T_o}}\right)$ we get:

$$n'_1 \sqrt{T'_1} = n'_2 \sqrt{T'_2} \qquad \ldots\ldots\ldots\ldots \quad (12)$$

$$n'_1 \sqrt{T'_1}\ (v'_2 - v'_1) = \alpha'_1 \qquad \ldots\ldots\ldots\ldots \quad (13)$$

$$\frac{d}{dy}(n_1 T_1 + n_2 T_2) + \gamma E(n_1 + n_2) - \gamma(n_1 v_1 + n_2 v_2) = 0 \ldots. \quad (14)$$

$$n'_1 \sqrt{T'_1}\ (T'_2 - T'_1) = \alpha'_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

$$\frac{d}{dy'}(n'_1 T'_1 v'_1 + n'_2 T'_2 v'_2) + \gamma E'(n'_1 v'_1 + n'_2 v'_2) - \gamma(n'_1 v'^2_1 + n'_2 v'^2_2) = $$
$$= -\frac{1}{2}\delta\,\alpha'_1 (n'_1 + n'_2) \qquad \ldots\ldots\ldots\ldots \quad (16)$$

$$\frac{d}{dy'}(n'_1 T'^2_1 + n'_2 T'^2_2) + \gamma E'(n'_1 T'_1 + n'_2 T'_2) - \gamma(n'_1 T'_1 v'_1 + n'_2 T'_2 v'_2)$$
$$= -\frac{8}{5}\delta\,\alpha'_3\ (n'_1 + n'_2) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

$$\frac{dE'}{dy'} = -\gamma'\, n'_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

$$\frac{dE_2'}{dy'} = -\gamma' \; n_2' \qquad \ldots\ldots\ldots\ldots\ldots (19)$$

where $\delta = \dfrac{d}{\mu_0 (1+\chi')^{3/4}} \sqrt{\dfrac{mkT_0}{\pi}}$ is a measure of rarefaction

$\gamma = \dfrac{eB_0 Ud}{k\ T_0}$ is a measure of magnetic field

$\gamma' = \dfrac{n_0\ k\ T_0}{2\ B_0\ U^2}$ is a measure of the induced electric field, and $_0 = c_p/c_v$

Together with the non-dimenional boundary conditions:

$$n_2' = T_2' = 1,\ v_2' = -1\ ,\ E_2' = 0 \text{ for } y' = -\tfrac{1}{2} \qquad \ldots\ldots\ldots(20)$$

$$T_1' = 1+\chi',\ v_I' = 1,\quad E_1' = 0 \text{ for } y' = \tfrac{1}{2} \qquad \ldots\ldots\ldots(21)$$

To solve the system (12 - 19) we put in it:

$$n_i' = n_i^{(o)'} + \gamma\, n_i'^{(1)} \qquad \ldots\ldots\ldots(22)$$

$$T_i' = T_i'^{(o)} + \gamma\, T_i'^{(1)} \qquad \ldots\ldots\ldots(23)$$

$$v_i' = v_i'^{(o)} + \gamma\, v_i'^{(1)} \qquad \ldots\ldots\ldots(24)$$

$$E_i' = E_i'^{(o)} + \gamma\, E_i'^{(1)} \qquad \ldots\ldots\ldots(25)$$

$i = 1\ ,\ 2$

Then, equating in both the sides of the equations of the system (12 - 19) termes free of γ gives:

$$n_1'^{(o)} \sqrt{T_1'^{(o)}} = n_2'^{(o)} \sqrt{T_2'^{(o)}} \qquad \ldots\ldots\ (26)$$

$$n_1'^{(o)} \sqrt{T_1'^{(o)}}\ (v_2'^{(o)} - v_1'^{(o)}) = \alpha_1'^{(o)} \qquad \ldots\ldots\ (27)$$

$$n_1'^{(o)}\ T_1'^{(o)} + n_2'^{(o)}\ T_2'^{(o)} = \alpha_2'^{(o)} \qquad \ldots\ldots,\ (28)$$

$$n_1^{\backslash(o)}\sqrt{T_1^{\backslash(o)}}\,(T_2^{\backslash(o)}-T_1^{\backslash(o)})=\alpha_2^{\backslash}(o)\alpha_3^{\backslash}(o) \quad\ldots\ldots \quad (29)$$

$$\frac{d}{dy}(n_1^{\backslash(o)}T_1^{\backslash(o)}v_1^{\backslash(o)}+n_2^{\backslash(o)}T_2^{\backslash(o)}v_2^{\backslash(o)})=-1/2\,\delta\alpha_1^{(o)}(n_1^{\backslash(o)}+n_2^{\backslash(o)}) \quad (30)$$

$$\frac{d}{dy}(n_1^{\backslash(o)}T_1^{\backslash(o)2}+n_2^{\backslash(o)}T_2^{\backslash(o)2})=-\frac{8}{5}\delta\alpha_2^{\backslash}(o)\alpha_3^{\backslash}(o)(n_1^{\backslash(o)}+n_2^{\backslash(o)}) (31)$$

$$\frac{dE_1^{\backslash(o)}}{dy}=-\gamma^{\backslash}n_1^{\backslash(o)} \quad\ldots\ldots\ldots\ldots \quad (32)$$

$$\frac{dE_2^{\backslash(o)}}{dy}=-\gamma^{\backslash}n_2^{\backslash(o)} \quad\ldots\ldots\ldots\ldots \quad (33)$$

which has a solution of the form:

$$n_1^{\backslash(o)}=1-\frac{\chi^{\backslash}}{2}+\frac{16}{5}\delta\alpha_3^{\backslash}(o)_y \quad\ldots\ldots\ldots \quad (34)$$

$$n_2^{\backslash(o)}=1-\frac{\chi^{\backslash}}{2}+\alpha_3^{\backslash}(o)+\frac{16}{5}\delta\alpha_3^{\backslash}(o)_y \quad\ldots\ldots\ldots;. \quad (35)$$

$$v^{\backslash}(o)=\frac{\delta\alpha_1^{\backslash}(o)}{2}\left[(1-\frac{\chi^{\backslash}}{2}-\frac{8}{5}\delta\alpha_3^{\backslash}(o))_y+\frac{8}{5}\delta\alpha_3^{\backslash}(o)_y{}^2\right]-$$
$$-\frac{\alpha_1^{\backslash}(o)}{2}(1+\frac{\chi^{\backslash}}{4})+\alpha_5^{\backslash}(o) \quad\ldots\ldots\ldots \quad (36)$$

$$v_2^{\backslash(o)}=-\frac{\delta\alpha_1^{\backslash}(o)}{2}\left[(1-\frac{\chi^{\backslash}}{2}+\frac{8}{5}\alpha_3^{\backslash}(o))_y+\frac{8}{5}\delta\alpha_3^{\backslash}(o)_y{}^2\right]+$$
$$+\frac{\alpha_1^{\backslash}(o)}{2}(1+\frac{\chi^{\backslash}}{4}+\alpha_3^{\backslash}(o))+\alpha_5^{\backslash}(o) \quad\ldots\ldots \quad (37$$

$$T^{\backslash}(o)=\frac{n_1^{\backslash(o)}T_1^{\backslash(o)}+n_2^{\backslash(o)}T_2^{\backslash(o)}}{n_1^{\backslash(o)}+n_2^{\backslash(o)}}=1+\frac{\chi^{\backslash}}{2}-\frac{16}{5}\delta\alpha_3^{\backslash}(o)y \quad (38)$$

$$E_1^{\backslash(o)}=-\gamma^{\backslash}\left[(1-\frac{\chi^{\backslash}}{2})y+\frac{8}{5}\delta\alpha_3^{\backslash}(o)\;y^2+\alpha_{61}^{\backslash}(o))\right]\ldots\ldots \quad (3$$

$$E_2^{\backslash(o)}=-\gamma^{\backslash}\left[(1-\frac{\chi^{\backslash}}{2}-\alpha_3^{\backslash}(o))y+\frac{8}{5}\delta\alpha_3^{\backslash}(o)\;y^2+\alpha_{62}^{\backslash}(o)\right] \quad (4$$

where

$$\alpha'_1(o) = \frac{-8}{\delta(2-\chi')+4+\chi'+2\alpha'_3(o)} \quad \ldots\ldots\ldots\ldots \quad (41)$$

$$\alpha'_2(o) = 2-\alpha'_3(o) \quad \ldots\ldots\ldots\ldots \quad (42)$$

$$\alpha'_3(o) = \frac{-5\chi'}{2(5+8\delta)} \quad \ldots\ldots\ldots\ldots \quad (43)$$

$$\alpha'_5(o) = \frac{\alpha'_1(o)\,\alpha'_3(o)}{20}\,(4\delta^2-8\delta-5) \quad \ldots\ldots\ldots \quad (44)$$

$$\alpha'_{61}(o) = -1/2\,\left(1-\frac{\chi'}{2}\right)-\frac{2}{5}\delta\,\alpha'_3(o) \quad \ldots\ldots\ldots \quad (45)$$

$$\alpha'_{62}(o) = 1/2\left(1-\frac{\chi'}{2}-\alpha'_3(o)\right)-\frac{2}{5}\delta\,\alpha'_3(o) \quad \ldots\ldots \quad (46)$$

The coefficients of γ in system (12 - 19) give the system of equations for frist appoximation:

$$2n'_1(1)-2n'_2(1)+T'_1(1)-T'_2(1) = 0 \quad \ldots\ldots\ldots \quad (47)$$

$$v'_2(1)-v'_1(1) = \alpha'_1(1)-\alpha'_1(o)\,(n'_1(1)+1/2\;T'_1(1)) \ldots \quad (48)$$

$$n'_1(1)+n'_2(1)+T'_1(1)+T'_2(1) = \alpha'_2(1)+y^2\left(2\gamma'-\frac{\delta\,\alpha'_1(o)}{2}\right) \ldots (49)$$

$$T'_2(1)-T'_1(1) = \alpha'_3(1) \quad \ldots\ldots\ldots\ldots\ldots (50)$$

$$\frac{d}{dy}\left[v'_1(1)+v'_2(1)+v'_1(o)(n'_1(1)+T'_1(1))+v'_2(o)(n'_2(1)+T'_2(1))\right]+$$
$$+\,\delta\,\alpha'_1(o)\left(2\gamma'-\frac{\delta\,\alpha'_1(o)}{2}\right)y^2 =$$
$$= \frac{\alpha'_1(o)^2}{2}-1/2\,\delta\left[\alpha'_1(o)(n'_1(1)+n'_2(1))+2\,\alpha'_1(1)\right] \quad (51)$$

$$n'_1(1)+n'_2(1)+2T'_1(1)+2T'_2(1) = \alpha'_4(1)-\frac{18}{5}\delta\alpha'_3(o)\,y+$$
$$+\left(2\gamma'-\frac{\delta\,\alpha'_1(o)}{2}\right)y^2 \quad \ldots\ldots\ldots\ldots\ldots (52)$$

$$\frac{dE_1'^{(1)}}{dy} = - \gamma' n_1'^{(1)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (53)$$

$$\frac{dE_2'^{(1)}}{dy} = - \gamma' n_2'^{(1)} \qquad \ldots\ldots\ldots\ldots\ldots (54)$$

with boundary conditions

$$n_2'^{(1)} = T_2'^{(1)} = v_2'^{(1)} = E_2'^{(1)} = 0 \text{ for } y = -1/2 \ldots \quad (55)$$

$$T_1'^{(1)} = v_1'^{(1)} = E_1'^{(1)} = 0 \qquad \text{for } y = 1/2 \ldots \quad (56)$$

The system of equations (47 – 54) has the solution:

$$n_1'^{(1)} = n_2'^{(1)} = 1/2(2\gamma' - \frac{\delta\alpha_1'^{(o)}}{2})(y'^2 - 1/4) \quad \ldots\ldots \quad (57)$$

$$T_2'^{(1)} = T_1'^{(1)} = \alpha_3'^{(1)} = \alpha_5'^{(1)} = 0 \qquad \ldots\ldots\ldots\ldots (58)$$

$$\alpha_2'^{(1)} = \alpha_4'^{(1)} = 1/4(2\gamma' - \delta\alpha_1'^{(o)} \qquad \ldots\ldots\ldots\ldots (59)$$

$$v_1'^{(1)} = 1/2\left\{\alpha_1'^{(1)}(1 - \delta y') + \alpha_1'^{(o)2} y' - \alpha_1'^{(o)}(2\gamma' - \delta\alpha_1'^{(o)}).\right.$$

$$\left. \cdot \left[\frac{\delta}{3} y'^3 + (y'^2 - 1/4)\right]\right\} \qquad \ldots\ldots\ldots \quad (60)$$

$$\alpha_1'^{(1)} = \frac{12\,\alpha_1'^{(o)} - \delta\,\alpha_1'^{(o)}(4\gamma' - \delta\alpha_1'^{(o)})}{12\,(2 + \delta)} \ldots\ldots \quad (61)$$

$$E_1'^{(1)} = -\gamma'\left[(2\gamma' - \frac{\delta\alpha_1'^{(o)}}{2})(\frac{y'^3}{3} - \frac{y'}{4}) + \alpha_{61}'^{(1)} = E_2'^{(1)} .(6\right.$$

$$\alpha_{61}'^{(1)} = -\frac{1}{12}(2\gamma' - \frac{\delta\alpha_1'^{(o)}}{2}) \qquad \ldots\ldots\ldots\ldots\ldots (63$$

Discussion and numerical results:

We have obtained the following analytical expressio for the density, temperature, velocity , and electric, field

$$n' = 1 - \frac{\chi'}{2} - \frac{\alpha_3'(o)}{2} + \frac{16}{5}\delta\alpha_3'(o)y' + \gamma(2\gamma' - \frac{\delta\alpha_1'(o)}{2})(y'^2 - 1/4)$$

$$T' = 1 + \frac{\chi'}{2} - \frac{16}{5}\delta\alpha_3'(o)y' + \gamma(2\gamma' - \frac{\delta\alpha_1'(o)}{2})(y'^2 - 1/4)$$

$$v' = -\frac{\delta\alpha_1'(o)}{2}\left[(1 - \frac{\chi'}{2})\, y' + \frac{8}{5}\delta\alpha_3'(o)y'^2\right] + \frac{\alpha_1'(o)}{5} + \frac{\gamma}{2}\left[(\frac{\alpha_1'(o)}{2} - \delta\alpha_1'(1))y' - \frac{\delta\alpha_1'(o)}{2}(2\gamma' - \frac{\delta\alpha_1'(o)}{2})\, y'^3\right]$$

$$E' = -\gamma'\left[(2 - \alpha_3'(o) - \chi')y' + \frac{16}{5}\delta\alpha_3'(o)y'^2 - \frac{4}{5}\delta\alpha_3'(o) + \gamma(4\gamma' - \delta\alpha_1'(o))(\frac{y'^3}{3} - \frac{y'}{4})\right.$$

We may note the following:

(I) For the number density $n'(y)$:

i) There is a density drop at the lower plate which equals to $\frac{-\chi'}{2(5 + 8\delta)}$.

It is independent of the fields and increases with δ

ii) For $\gamma = o$, the density varies linearly with y, agreeing with the carresponding neutral case.

(II) For the temperature distribution we may note the following:

i) There is a temperature jump at the two plates

$$T_s(\pm 1/2) = \pm \frac{5\,\chi'}{2(5 + 8\delta)}$$

It is independent of the fields and decreases with δ

In the continum case ($\delta \longrightarrow \infty$) it vanishes.

ii) For the neutral case T varies linearly with y. For charged case T decreases with y to a minimum value then increases. It is larger at any point for the neutral case than that for the charged case.

(III) For the mean velocity we may note the following:

i) There is a slip velocity at the two plates

$v_s(\pm 1/2) = \pm 1 - v(\pm 1/2)$, the magritude of the slip velocity at the lower plate is larger in the neutral case than that for the charged case. For the charged case the slip velocity decreases with γ', and for both cases it decreases with δ and vanishes as $\delta \longrightarrow \infty$.

ii) At any point the magnitude of velocity is larger for the neutral case.

(IV) The non-dimensional shear stress $p'_{xy} = \alpha'_1(o) + \gamma\,\alpha'_1(1)$ is constant and we may note that it decreases with δ and γ'.

(V) The heat flux vector $q'_y = \frac{-5\gamma'}{(5 + 8\delta)}$ is constant and is independent of the fields. It decreases wi and vanishes in the continuum limit.

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