CARTWRIGHT'S THEOREM FOR VECTOR VALUED
ENTIRE FUNCTIONS OF n-COMPLEX VARIABLES

M.A. KANDIL

Department of Mathematics, Assiut University, Egypt

In this paper we deduce an interpolation formula for vector valued entire functions of n-complex variables of exponential types. This formula is used to derive a sufficient condition for these functions to have compact trajectories. The result leads to a generalization of Cartwright's theorem[3] for this class of functions.

§1. Introduction:

The vector valued functions are defined on the field of complex variables $C_z^n - z = (z_1, z_2, \ldots, z_n)$, $z_k = x_k + iy_k$, $k = 1,2,3,\ldots,n$ - to Banach space X. Let $E_{\mu_1, \mu_2, \ldots, \mu_n}(X)$ be the space of such vector valued functions satisfying the following two conditions :

(1) They are entire functions of exponential types $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ relative to $z_1, z_2, \ldots, z_n$ respectively, where $\mu_k < \tau, k = 1,2, \ldots, n$.

(2) They are bounded functions on the n-dimensional Euclidean Space R^n. So if $F(z_1, z_2, \ldots, z_n) \in E_{\mu_1, \mu_2, \ldots, \mu_n}(X)$ then

(1) $F(z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} a_{i_1, i_2, \ldots, i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}$

(ii) $\| F(z_1, z_2, \ldots, z_n) \|_x < A \exp \sum_{k=1}^{n} (\mu_k + \varepsilon) | z_k |$

where ε is an arbitrary positive number , A is a constant, $\mu_k < \pi$ for all $k = 1,2,\ldots,n$.

(iii) $\text{Sup } \left| \left| F(x_1, x_2, \ldots, x_n) \right| \right|_X < \infty$

§2. Generalization of Cartwright's Theorem.

Theorem (1). If $F(z_1, z_2, \ldots, z_n) \; E_{\mu_1, \mu_2, \ldots, \mu_n}(X)$, then

$$F(z_1, z_2, \ldots, z_n) = \sum_{m_1 = -\infty}^{+\infty} \sum_{m_2 = -\infty}^{+\infty} \cdots \sum_{m_n = -\infty}^{+\infty} L_{m_1, m_2, \ldots, m_n}(z_1, z_2, \ldots, z_n) \; F(m_1, m_2, \ldots, m_n)$$

where

$$L_{m_1, m_2, \ldots, m_n}(z_1, z_2, \ldots, z_n) = (-1)^{m_1 + m_2 + \ldots + m_n} \prod_{i=1}^{n} \frac{\sin \pi z_i}{\pi} \frac{\sin \omega(m_1 - z_i)}{(m_i - z_i)^2}$$

$$\omega < \pi - \max_k \mu_k \; , \; k = 1, 2, \ldots, n$$

Proof. This is just a generalization of the one variable expansion $\left[4\right]$.

Theorem (2) . Let $F(z_1, z_2, \ldots, z_n)$ be a vector valued function defined on the n-dimensional Euclidean space R^n to Banach space X with expansion,

$$F(x_1, x_2, \ldots, x_n) = \sum_{m_1 = -\infty}^{+\infty} \sum_{m_2 = -\infty}^{+\infty} \cdots \sum_{m_n = -\infty}^{+\infty} L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) \; F(\alpha_{m_1}, \alpha_{m_2}, \ldots, \alpha_{m_n})$$

such that :

(i) The set of the numerical function $L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n)$ satisfies the condition

$$\sum_{m_1 = -\infty}^{+\infty} \sum_{m_2 = -\infty}^{+\infty} \cdots \sum_{m_n = -\infty}^{+\infty} \left| L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) \right| < M$$

(M is a constant)

and ,

(ii) If the set of points

$$\bar{z} = (\alpha_{m_1}, \alpha_{m_2}, \ldots, \alpha_{m_n}) \subset R^n$$

is defined for all combinations $(m_1, m_2, \ldots, m_n)$ of positive and negative integers, and if the set $F\left(\begin{matrix} x \\ m_1 \end{matrix}, \begin{matrix} x \\ m_2 \end{matrix}, \ldots, \begin{matrix} x \\ m_n \end{matrix}\right)$ of values of the function $F(x_1, x_2, \ldots, x_n)$ at the set is compact, then the function $F(x_1, x_2, \ldots, x_n)$ is of compact trajectory.

Proof. For any arbitrary positive number ε, there exists a finite set of points in R^n,

$$\tau = \left[(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \right]; \quad r = 1, 2, \ldots, s;$$

The set generates a finite number of divisions of the set of points

$$\varepsilon = \sum_{r=1}^{s} \beta_r$$

such that for any point $(\alpha_1, \alpha_2, \ldots, \alpha_n) \varepsilon \beta_r$ we get ,

$$\left| F (\alpha_1, \alpha_2, \ldots, \alpha_n) - F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \right|\Big|_x < \varepsilon.$$

Consider the following equality :

$$F(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{s} \sum_{\beta_r} \sum_{m_1, m_2, \ldots, m_n} L(x_1, x_2, \ldots, x_n) \left\{ F(\alpha_{m_1}, \ldots, \alpha_{m_n}) - F(\gamma_1^r, \ldots, \gamma_n^r) \right\} +$$

$$+ \sum_{r=1}^{s} \sum_{\beta_r} \sum_{m_1, m_2, \ldots, m_n} L(x_1, x_2, \ldots, x_n) \; F(\gamma_1^r, \ldots, \gamma_n^r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

For the last term in (1) we get

$$\sum_{r=1}^{s} \sum_{\beta_r} \sum_{m_1, m_2, \ldots, m_n} L(x_1, x_2, \ldots, x_n) \; F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) =$$

$$= \sum_{r=1}^{s} F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \sum_{\beta_r} \sum_{m_1, m_2, \ldots, m_n} L(x_1, x_2, \ldots, x_n)$$

Let
$$\sum_{\beta_r \ m_1, m_2, \ldots, m_n} L(x_1, x_2, \ldots, x_n) = G_r(x_1, x_2, \ldots, x_n)$$

From condition (1) of the theorem we get that the numerical function $G_r(x_1, x_2, \ldots, x_n)$ is bounded, and so it has a compact trajectory.

Let
$$F_s(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{s} G_r(x_1, x_2, \ldots, x_n) \ F(\gamma_1^r, \gamma_2^r, \ldots, \gamma_n^r) \quad (2)$$

It is clear that $F_s(x_1, x_2, \ldots, x_n)$ is a polynomial in the bounded numerical functions $G_r(x_1, x_2, \ldots, x_n)$, $r = 1, 2, \ldots, s$, and so the vector valued function $F_s(x_1, x_2, \ldots, x_n)$ is of compact trajectory .

From (1) and (2) we have

$$F(x_1, x_2, \ldots, x_n) - F_s(x_1, x_2, \ldots, x_n) =$$
$$= \sum_{r=1}^{s} \sum_{\beta_r \ m_1, m_2, \ldots, m_n} L(x_1, x_2, \ldots, x_n) \left\{ F(\alpha_{m_1}, \ldots, \alpha_{m_n}) - F(\gamma_1^r, \ldots, \gamma_n^r) \right\}$$

$$\therefore \ ||F(x_1, x_2, \ldots, x_n) - F_s(x_1, x_2, \ldots, x_n)|| < \epsilon M.$$

Thus the trajectory of the vector valued function $F_s(x_1, x_2, \ldots, x_n)$ is ϵM net for the trajectory of the function $F(x_1, x_2, \ldots, x_n)$, and so the trajectory of the vector valued function $F(x_1, x_2, \ldots, x_n)$ is compact, Q.E.D.

Theorem (3). If $F(z_1, z_2, \ldots, z_n) \epsilon \ E_{\mu_1, \mu_2, \ldots, \nu_n}(\overline{X})$, and if the set $\left[F(m_1, m_2, \ldots, m_n) \right]$ of values of the function $F(x_1, x_2, \ldots, x_n)$ at the lattice point * is compact, then the function $F(x_1, x_2, \ldots, x_n)$ is of a compact trajectory.
Proof. From theorem (1), we have the expansion

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* All combinations $(m_1, \ldots, m_n)$ of positive and negative integers

$$F(x_1, x_2, \ldots, x_n) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_n=-\infty}^{+\infty} L(x_1, x_2, \ldots, x_n) \, F(m_1, m_2, \ldots, m_n)$$

where

$$L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) = (-1)^{m_1+m_2+\ldots+m_n} \prod_{i=1}^{n} \frac{\sin \pi z_i \, \sin\omega(m_i - z_i)}{\pi^n \, \omega^n \, (m_i - z_i)^2}$$

So we have

$$\sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_n=-\infty}^{+\infty} L_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) =$$

$$= \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_n=-\infty}^{+\infty} \prod_{i=1}^{n} \left| \frac{\sin z_i \, \sin \omega(m_i - z_i)}{\pi^n \, \omega^n \, (m_i - z_i)^2} \right|$$

$$= \sum_{m_1=-\infty}^{+\infty} \left| \frac{\sin \pi z_1 \, \sin \omega(m_1 - z_1)}{\pi\omega \, (m_1 - z_1)} \right| \cdots \sum_{m_n=-\infty}^{+\infty} \left| \frac{\sin z_n \, \sin\omega (m_n - z_n)}{\pi \, \omega(m_n - z_n)} \right| \leq \left(\frac{\pi}{\omega}\right)^n$$

From theorem (2) we get the required result .

From the above we can generalize the Cartwright's theorem to be :

If $F(z_1, z_2, \ldots, z_n)$ is a vector valued entire function of n-variables $z_1, z_2, \ldots, z_n$ of exponential types $\mu_1, \mu_2, \ldots, \mu_n$ repectively , $\mu_i < k$, $i = 1, 2, \ldots, n$, and if the set of its values at the lattice points is compact, then the function $F(x_1, x_2, \ldots, x_n)$ is of compact trajectory.

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