

الباب الرابع

المتابعات والمتسلسلات

Sequences and infinite series

أولاً: المتابعة: Sequence

$$Z_1, Z_2, \dots, Z_N \quad (1)$$

Convergence $\{Z_n\}$:

$$|Z_n - C| \leq \varepsilon, \forall n \geq N \quad (1)$$

$$\lim_{n \rightarrow \infty} Z_n = C$$

at $Z_n \rightarrow C$

Divergent

$(n \rightarrow \infty)$

: 1

$$1) \{Z_n\} = 1 + \frac{2}{n}$$

:

:

1 ∞ n

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right) = 1$$

:

∞ n

(1)

1

$$|Z_n - c| = \left|1 + \frac{2}{n} - 1\right| = \left|\frac{2}{n}\right| \leq \varepsilon, \Rightarrow n \geq \frac{2}{\varepsilon}$$

$$\frac{2}{n} < 0.01$$

$$\varepsilon = 0.01$$

$$n < 200$$

. 1

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:2

$$Z_n = \left\{ \frac{n}{n+1} \right\}$$

:

$$(1) \quad "1" \quad \left\{ \frac{n}{n+1} \right\}$$

$$Z_n - C = \frac{n}{n+1} - 1 = \frac{-1}{n+1}$$

$$|Z_n - c| = \left| \frac{-1}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| \langle \varepsilon,$$

$$\frac{n+1}{1} \rangle \frac{1}{\varepsilon} \Rightarrow n+1 \rangle \frac{1}{\varepsilon} \Rightarrow n \rangle \frac{1}{\varepsilon} - 1$$

$$\cdot \frac{1}{\varepsilon} - 1$$

N

$$\varepsilon = 0.01$$

. 1

N=99

$$\frac{1}{\varepsilon} - 1 = 99$$

:

$$1) \{Z_n\} = \frac{2n^2}{n+1}$$

$$2) \{Z_n\} = \frac{1}{n-3}$$

$$3) \{Z_n\} = \frac{-1}{n-3}$$

Series

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-1

$\{w_1, w_2, \dots, w_n\}$

$$\sum_{n=1}^{\infty} w_n = w_1 + w_2 + w_3 \dots (1)$$

infinite series

S_n

n

$$S_n = w_1 + \dots + w_n$$

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$\{S_1, S_2, \dots, S_n\}$

:

-2

(1

$$\lim_{n \rightarrow \infty} (S_n) = S$$

Converge (1) :

S

$$S = \sum_{n=1}^{\infty} w_n = w_1 + w_2 + \dots$$

: (2)

$$\lim_{n \rightarrow \infty} (S_n) = \infty$$

Divergent

(Oscillating) S_n (3)

$\infty, -\infty$ S_n (4)

:3

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots (1)$$

(1) S_n

$$S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{2} \quad 1$$

n

$$S_n = \left(1 - \left(\frac{1}{2}\right)^n \right) / \left(1 - \frac{1}{2} \right) = 2 - \frac{1}{2^{n-1}}$$

(1)

$$S = \lim_{n \rightarrow \infty} (S_n) = \lim_{n \rightarrow \infty} \left(2 - \frac{1}{2^{n-1}} \right) = 2$$

(1)

:4

$$\sum_{n=1}^{\infty} n = 1 + 2 + \dots + n + \dots (1)$$

:

$$S_n \qquad (1)$$

$$S_n = 1 + 2 + \dots + \frac{n(n+1)}{2}$$

: (1)

$$S = \lim_{n \rightarrow \infty} (S_n) = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = \infty$$

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$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - \dots + (-1)^{n-1} + \dots (1)$$

$$\Rightarrow n=1$$

$$\Rightarrow n=0$$

$$S_n = \sum_{r=1}^{\infty} (-1)^{r-1} = \begin{cases} 1 \\ 0 \end{cases}$$

(1)

S

-3

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

:

$$S_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \frac{1}{n}$$
$$1 + \frac{(n+1)}{2}$$

:

$$\lim_{n \rightarrow \infty} (S_n) = \lim_{n \rightarrow \infty} 1 + \frac{(n+1)}{2} = \infty$$

-4

$$w_1 + w_2 + w_3 + \dots$$

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$$\sum_{n=1}^{\infty} |w_n| = |w_1| + |w_2| + \dots \quad (3)$$

Absolutely

convergence

$$w_1 + w_2 + w_3 + \dots$$

(3)

(Conditionally Convergent)

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$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$$

: "1"

$$S_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

$$\frac{a(1-r^n)}{1-r}$$

$$\lim_{n \rightarrow \infty} S_n = 1$$

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$w_1 + w_2 + \dots$$

: 1

$$w_1 + w_2 + \dots$$

: 2

$$w_1 + w_2 + \dots$$

:

$$\lim_{n \rightarrow \infty} w_n = 0$$

: -1

$$\lim_{n \rightarrow \infty} (S_n) = \begin{cases} 0 & \text{test fails} \\ \text{otherwise} & \text{div} \end{cases}$$

Cauchy test : -2

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L$$

< 1 conv

$$\lim_{n \rightarrow \infty} S_n = \begin{cases} < 1 & \text{div} \\ = 1 & \text{test fails} \end{cases}$$

$= 1$ test fails

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$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

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$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^n}} =$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$$

Comparison test :

-3

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$$\sum_{n=1}^{\infty} a_n,$$

...

(1)

$$\sum_{n=1}^{\infty} b_n \quad \dots \quad (2)$$

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If $a_n \leq b_n$ and b_n convergent then a_n convergent,

If $a_n \geq b_n$ and b_n divergent then a_n divergent

Division Test : **-4**

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$$\sum_{n=1}^{\infty} a_n < 1 \dots \dots \dots \text{conv}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \gamma > 1 \dots \dots \dots \text{div}$$

$$= 1 \dots \dots \dots \text{test fails}$$

Integral Test: **-5**

:

$$\sum_{n=1}^{\infty} a_n$$

$$\int_0^{\infty} a_n dn = \begin{cases} a \dots \dots \dots \text{conv} \\ \infty, \dots \text{or}, \dots -\infty \dots \dots \dots \text{div} \end{cases}$$

:1

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

:

$$a_n = \frac{1}{2^{n-1}}, \quad a_{n+1} = \frac{1}{2^n}$$

:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{2^n} \div \frac{1}{2^{n-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{n-1}}{2^n} = \frac{1}{2} < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \quad (\text{convergent})$$

:2

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$$\sum_{n=1}^{\infty} n$$

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$$a_n = n, \Rightarrow \lim_{n \rightarrow \infty} (n) = \infty \neq 0$$

divergent ()

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$$\sum_{n=1}^{\infty} \frac{n+2}{n^2}$$

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$$a_n = \frac{n+2}{n^2}, \quad b_n = \frac{1}{n},$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent}$$

$$\therefore \frac{n+2}{n^2} = \frac{1}{n} + \frac{2}{n^2} \geq \frac{1}{n},$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n+2}{n^2} \geq \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{n+2}{n^2}$$

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$$\sum_{n=1}^{\infty} \frac{3n^2 + n + 1}{n^5 + 1}$$

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$$\therefore a_n = \frac{3n^2 + n + 1}{n^5 + 1}, \quad b_n = \frac{1}{n^3}, \quad \sum_{n=1}^{\infty} \frac{1}{n^3}$$

convergent Theorem

$$\therefore \frac{3n^2 + n + 1}{n^5 + 1} \leq \frac{1}{n^3},$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{3n^2 + n + 1}{n^5 + 1} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\sum_{n=1}^{\infty} \frac{3n^2 + n + 1}{n^5 + 1}$$

: 5

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

$$\int_1^{\infty} \frac{dn}{1+n^2} = \tan^{-1} n \Big|_1^{\infty} = \tan^{-1}[\infty - \tan^{-1} 1]$$

$$= 90 - 45 = 45$$

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

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$$\therefore a_n = \frac{n}{3^n}, \quad a_{n+1} = \frac{n+1}{3^{n+1}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+1}{3^{n+1}} \div \frac{n}{3^n}$$

$$= \lim_{n \rightarrow \infty} \frac{3^n}{3^{n+1}} \times \frac{n+1}{n} = \frac{1}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ (convergent)}$$

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$$\sum_{n=1}^{\infty} \frac{3^n}{n^2}$$

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$$\therefore a_n = \frac{3^n}{n^2}, \quad a_{n+1} = \frac{3^{n+1}}{(n+1)^2} \Rightarrow$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{3^{n+1}}{(n+1)^2} \div \frac{3^n}{n^2} =$$

$$= \lim_{n \rightarrow \infty} \frac{3^{n+1}}{3^n} \times \frac{n^2}{(n+1)^2} = 3 > 1$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2}$$

divergent

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$$\sum_{n=1}^{\infty} \frac{1}{[Ln(n)]^n}$$

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$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{[Ln(n)]^n}} =$$

$$\lim_{n \rightarrow \infty} \frac{1}{Ln(n)} = 0 < 1$$

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$$\sum_{n=1}^{\infty} \frac{1}{n[Ln(n)]}$$

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$$\int_1^{\infty} \frac{dn}{n[Ln(n)]} = [Ln.Ln(n)]_1^{\infty} = \infty$$

:10

$$\sum_{n=1}^{\infty} \frac{(3-4i)^n}{n!}$$

:

$$a_{n+1} = \frac{(3-4i)^{n+1}}{(n+1)!}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(3-4i)^{n+1}}{(n+1)!} \div \frac{(3-4i)^n}{n!} =$$

$$= \lim_{n \rightarrow \infty} \frac{1}{(n+1)} \times (3-4i) = 0 < 1$$

$$\sum_{n=1}^{\infty} \frac{(3-4i)^n}{n!} \text{ convergent}$$

: 11

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$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$$

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$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+2}}$$

$$\int_1^{\infty} \frac{dn}{\sqrt{n+2}} = \frac{2}{1} [\sqrt{n+2}]_0^{\infty} = \infty$$

:

$$1) \frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots$$

$$2) \sum_{n=1}^{\infty} \frac{n-1}{n+1}$$

$$3) \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

$$4) \sum_{n=1}^{\infty} \frac{n+1}{n2^n}$$

$$5) \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

$$6) \sum_{n=1}^{\infty} \left(\frac{n}{3n-1} \right)^n$$

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Alternating Series

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$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - \dots$$

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:

$$\sum_{n=0}^{\infty} a_n$$

a_n

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:

1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$

:

$$\sum_{n=0}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

$$2) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n-1}$$

:

$$\sum_{n=0}^{\infty} \frac{1}{2n-1}$$

$$\sum_{n=0}^{\infty} \frac{1}{2n-1} \geq \sum_{n=0}^{\infty} \frac{1}{n} \text{ (divergent)}$$

$$\sum_{n=0}^{\infty} \frac{1}{2n-1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0,$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n-1}$$

$$3) \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n}$$

:

$$\sum_{n=0}^{\infty} \frac{3^n}{n}$$

$$\lim_{n \rightarrow \infty} \frac{3^n}{n} = \infty$$

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Power series

:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

x

$$a_0, a_1, a_2, \dots, a_n, \dots$$

:

x

(1)

.R

0

=

.∞

(2)

. R

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(3)

. [a, b]

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$$1) \sum_{n=0}^{\infty} \frac{x^n}{n}$$

x

$$a_n = \frac{x^n}{n}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)} \div \frac{x^n}{n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{(n+1)} \times x \right|$$

$$= |x| < 1$$

$$\Rightarrow -1 < x < 1$$

$$x = -1 \quad x = 1$$

$$x = 1 : \quad :$$

$$\left(\quad \right) \quad \sum_{n=0}^{\infty} \frac{1}{n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

$$x = -1$$

$$-1 \leq x < 1$$

$$\left(\quad \right)$$

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$$2) \sum_{n=0}^{\infty} \frac{n! x^{2n}}{n+1}$$

:

∞
x

R

$$\therefore a_n = \frac{n! x^{2n}}{n+1},$$

$$a_{n+1} = \frac{x^{2n+2} (n+1)!}{(n+2)}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2} (n+1)!}{(n+2)} \div \frac{n! x^{2n}}{n+1} \right| =$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n+1)}{(n+2)} \times x^2 \right| = \infty > 1$$

R

$$3) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

:

x

.∞

R

$$\because a_n = \frac{x^{2n+1}}{(2n+1)!}, \quad a_{n+1} = \frac{x^{2n+3}}{(2n+3)!}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \div \frac{x^{2n+1}}{(2n+1)!} \right| =$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{(2n+2)(2n+3)} x^2 \right| = 0 < 1$$

R

.∞

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$$1) \sum_{n=0}^{\infty} (x+i)^n$$

$$2) \sum_{n=0}^{\infty} \frac{x^{2n+1} n!}{(2n+1)!}$$

$$3) \sum_{n=0}^{\infty} n^n x^n$$

$$4) \sum_{n=0}^{\infty} \frac{(x-a)^n}{(2n+1)!}$$

$$5) \frac{x}{e^x} + \frac{2x}{e^{2x}} + \dots,$$

$$6) \frac{1}{1+x} + \frac{1}{1+x^2} + \dots,$$

$$7) \text{Ln}(x) + \text{Ln}^2 x + \dots$$

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