

CHAPTER 4

The Power of Correlation

4.1 Correlation: Making Sense out of Nonsense:

In the analysis of experimental data involving two random variables X, Y , a plot of the two variables against each other may or may not indicate some interdependence. Although the two random variables are seemingly unrelated yet they may show that somehow there is a discreet relation between them. It may well be true that for such random variables, there is no closed form formula that relates each value of Y to the corresponding value of X , yet we may deduce from the analysis of all values of Y and all values of X some tendency of behavior. This tendency is called correlation. It is this correlation that we are after, because it tells us whatever information we can deduce from the experiment, and it bears sense out of apparent complete chaos and randomness. Such correlation or joint moment is defined as

$$\text{correlation} = E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy \quad (4-1)$$

We also define the correlation of centered random variables $X - E[X]$ and $Y - E[Y]$, i.e., joint moment or covariance $\text{cov}[XY]$ as

$$\text{cov}[XY] = E\{(X - E[X])(Y - E[Y])\} \quad (4-2)$$

or
$$\text{cov}[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f_{X,Y}(x,y) dx dy \quad (4-3)$$

where $\mu_X = E[X] = \bar{X}$ and $\mu_Y = E[Y] = \bar{Y}$

Then

$$\text{cov}[XY] = E[XY] - \mu_X \mu_Y \quad (4-4)$$

We then define the correlation coefficient ρ as

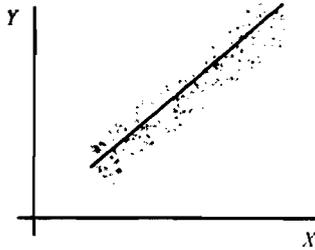
$$\rho = \frac{\text{cov}[XY]}{\sigma_X \sigma_Y} \quad (4-5)$$

We say that the two random variables X, Y are uncorrelated if and only if their covariance is zero

$$\text{cov}[XY] = 0 \quad (XY \text{ Uncorrelated}) \quad (4-6)$$

For statistically independent variables, we note $f_{X,Y}(xy) = f_X(x) f_Y(y)$

$$\text{cov}[XY] = \int_{-\infty}^{\infty} (x - \mu_X) f_X(x) dx \int_{-\infty}^{\infty} (y - \mu_Y) f_Y(y) dy \quad (4-7)$$



(Fig. 4.1) Scatter diagram for wealth (Y) vs age (X)

or
$$\text{cov}[XY] = E[(X - \mu_X)(Y - \mu_Y)] \quad (4 - 8)$$

$$= E[X - \mu_X]E[Y - \mu_Y] = \{E(X) - \mu_X\}\{E(Y) - \mu_Y\} \\ = 0 \text{ (X, Y statistically independent)} \quad (4 - 9)$$

Hence, statistically independent variables are uncorrelated. But the converse is not necessarily true. We say that two random variables are orthogonal if and only if their correlation is zero

$$E[XY] = 0 \text{ (X, Y orthogonal)} \quad (4 - 10)$$

From eqn. (4 - 4), we observe that if one of the variables X, Y or both have zero mean, (μ_X or μ_Y) and if they are orthogonal, i.e., $E[XY] = 0$, then X, Y are uncorrelated and vice versa.

As an example for correlation, let us assume that X represents age and Y represents wealth, and we plot values of X and Y for a large group of people (Fig. 4.1) such a plot is called scatter diagram. We observe - despite sporadic dispersion of points - that there is a tendency that wealth increases with age. A measure of the closeness of the values of X and Y from the straight line $y = mx + c$ is monitored by the correlation coefficient ρ

$$\rho = \frac{\overline{(X - \mu_X)(Y - \mu_Y)}}{\sigma_X \sigma_Y} \quad (4 - 11)$$

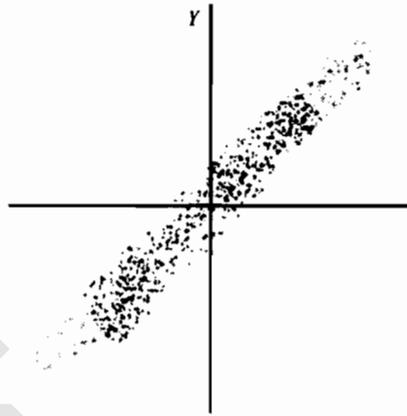
In the special case when $\mu_X = 0$, $\mu_Y = 0$.

$$\rho = \frac{\overline{XY}}{\sigma_X \sigma_Y} \quad (\mu_X = \mu_Y = 0) \quad (4 - 12)$$

We note that

$$-1 \leq \rho \leq +1 \quad (4 - 13)$$

If $\rho = +1$ or -1 , there is complete relation and the points fall on the straight line (Fig. 4.2).



(Fig. 4.2) $Y = X$ for $\mu_X = \mu_Y = 0$

Ex. 4.1 Prove $-1 \leq \rho \leq +1$

Solution

To show this assume that X, Y are completely dependent so that $X = Y$.

For $\mu_X = \mu_Y = 0$, $\text{cov}[XY] = \sigma_X^2 = \sigma_Y^2 = \sigma_X \sigma_Y$

Thus, $\rho = 1$ from eqn. (4-3) and (4-12).

For $Y = -X$, $\text{cov}[XY] = -\sigma_X^2 = -\sigma_Y^2 = -\sigma_X \sigma_Y$. Thus, $\rho = -1$, from eqn. (4-12). When $\rho = 0$, X, Y are uncorrelated. Note that when two variables are statistically independent then they are uncorrelated although the converse is not necessarily true.

Ex. 4.2

Two variables X_1 and X_2 have same average \bar{x} and variances σ_1^2 and σ_2^2 . Determine the variance of y where $y = x_1 + kx_2$

Solution

$$\sigma_y^2 = \overline{(y - \bar{y})^2} = \overline{[(x_1 + kx_2) - (\bar{x}_1 + k\bar{x}_2)]^2}$$

Since

$$\bar{x}_1 + k\bar{x}_2 = \bar{x}_1 + k\bar{x}_2$$

$$\sigma_y^2 = \overline{[(x_1 - \bar{x}_1) + k(x_2 - \bar{x}_2)]^2}$$

Since

$$\bar{x}_1 = \bar{x}_2 = \bar{x}$$

$$\sigma_y^2 = \overline{[(x_1 - \bar{x}) + k(x_2 - \bar{x})]^2}$$

$$= \overline{(x_1 - \bar{x})^2} + 2k\overline{(x_1 - \bar{x})(x_2 - \bar{x})} + k^2\overline{(x_2 - \bar{x})^2}$$

(4-14)

$$\rho = \frac{\sigma_1^2 + 2k(x_1 - \bar{x})(x_2 - \bar{x}) + k^2 \sigma_2^2}{(x_1 - \bar{x})(x_2 - \bar{x}) \sigma_1 \sigma_2}$$

Now

Thus,

$$\sigma_y^2 = \sigma_1^2 + 2k \rho \sigma_1 \sigma_2 + k^2 \sigma_2^2 \quad (4 - 14)$$

4.2 Autocorrelation Function:

We may classify signals into three classes. The first class is periodic signals of period T , whether the signal is continuous or has discontinuities. The second class is transient aperiodic signals which occur only once and do not repeat themselves. The third class is random signals which are non periodic, i.e., neither periodic nor aperiodic yet are extended in time. The first class is called power signal in which the average power is given by

$$P_{av} = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt \quad (4 - 15)$$

where x is voltage or current assuming 1Ω resistance .

The second class is an energy signal in which the energy is defined by

$$E = \int_{-\infty}^{\infty} x^2(t) dt \quad (4 - 16)$$

The third cases is power signal in which the average power is defined as

$$P_{av} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt \quad (4 - 17)$$

$$= E[X^2(t)] = \overline{x^2(t)} \quad (4 - 18)$$

The autocorrelation function $R(\tau)$ is defined as the average of the signal multiplied by itself after shifting in time by an amount τ .

$$R_x(\tau) = E[X(t)X(t-\tau)] = E[X(t)X(t+\tau)] \quad \text{for all } \tau \quad (4 - 19)$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t-\tau) dt \quad (4 - 20)$$

This definition holds, for both random and deterministic power signals whether periodic or non periodic power. From eqns. (4 - 18) and (4 - 17), we note by putting $\tau = 0$

$$R_x(0) = E[X^2(t)] \quad (4 - 21)$$

We also note from eqn. (4 - 20)

$$R_x(\tau) = R_x(-\tau) \quad (4 - 22)$$

This means that $R_x(\tau)$ is an even function of τ . The autocorrelation function $R_x(\tau)$ has its maximum magnitude at $\tau = 0$

$$|R_x(\tau)| \leq R_x(0) \quad (4 - 23)$$

To prove this, we consider

$$E\{[X(t+\tau) \pm X(t)]^2\} \geq 0 \quad (4 - 23)$$

$$E[X^2(t+\tau) \pm 2E\{X(t+\tau)X(t)\} + E\{X^2(t)\}] \geq 0 \quad (4 - 24)$$

We note that

$$E[X^2(t+\tau)] = E[X^2(t)] = R_x(0) \quad (4 - 25)$$

Thus, eqn. (4 - 24) becomes

$$2R_x(0) \pm 2R_x(\tau) \geq 0 \quad (4 - 26)$$

Hence

$$-R_x(0) \leq R_x(\tau) \leq R_x(0) \quad (4 - 27)$$

The physical significance of the autocorrelation function is that it provides a means for describing the interdependence (correlation) of a random signal with itself after time τ . It shows whether or not there is coherence or continuity of phase or behavior between the sample and what becomes of it, τ seconds later.

The more rapidly the random signal changes with time, the more rapidly will the autocorrelation function $R_x(\tau)$ decrease from its maximum value $R_x(0)$ as τ increases. This decrease may be described by decorrelation time τ_0 , such that for $\tau > \tau_0$, $R_x(\tau)$ is below $\frac{1}{100}R_x(0)$. After τ_0 , the signal loses correlation with itself.

Fig. (4.3) shows a slowly varying random signal for which τ_0 is large and a fast varying random signal for which τ_0 is small

We therefore see that *pdf* is insufficient to fully describe a random signal, because it contains no information about the signal's rate of change. The autocorrelation gives this temporal (time) information.

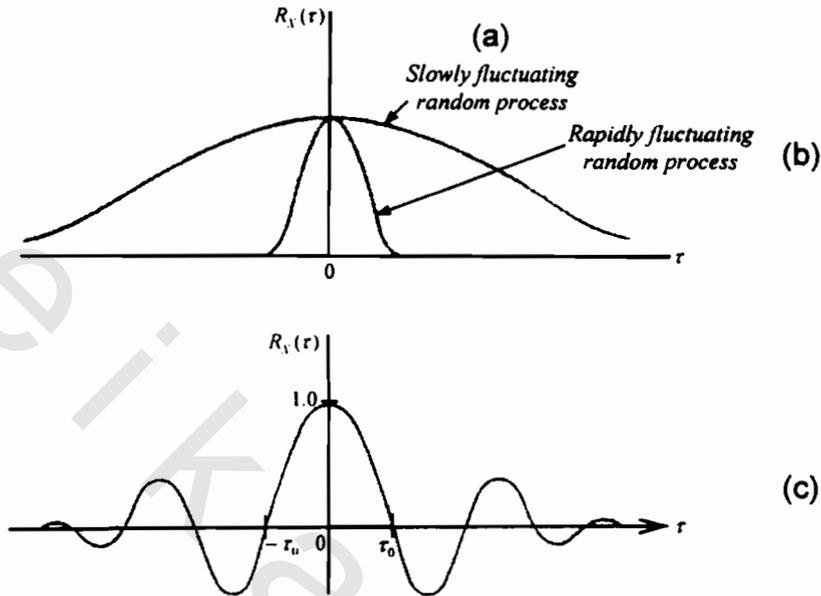
Ex. 4.3

For a sinusoidal wave with random phase, find the autocorrelation function.

Solution

$$X(t) = A \cos(2\pi f_c t + \theta) \quad (4 - 28)$$

where θ is a uniformly distributed random variable in the interval $-\pi, \pi$ so that



(Fig. 4.3) Decorrelation time and time dependence of autocorrelation function
 a) a slowly varying signal b) a fast varying signal
 c) general behavior of $R_X(\tau)$. First null gives the decorrelation time.

$$f_g(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases} \quad (4-29)$$

Then

$$R_X(\tau) = E[X(t+\tau)X(t)] \\ = E[A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cos(2\pi f_c t + \theta)] \quad (4-30)$$

$$= \frac{A^2}{2} E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)] + \frac{A^2}{2} E[\cos 2\pi f_c \tau] \quad (4-31)$$

Substituting eqn. (4-29) in the expectation operation of eqn. (4-31)

$$= \frac{A^2}{2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) d\theta + \frac{A^2}{2} \cos(2\pi f_c \tau)$$

The first integral vanishes. Thus,

$$R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau) \quad (4-32)$$

Thus the autocorrelation function of a sinusoidal wave with random phase is another sinusoid at the same frequency in τ domain

Ex. 4.4

For a square wave signal obtain $R_X(\tau)$

Solution

When we overlap $x(t)$ and $x(t - \tau)$, we have

$$\begin{aligned}
 R_X(\tau) &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t - \tau) dt \\
 &= \frac{1}{T} \left[\int_{-T/2}^{-T/2 + \tau} (-V_0)V_0 dt + \int_{-T/2 + \tau}^0 (-V_0)(-V_0) dt \right. \\
 &\quad \left. + \int_0^{\tau} V_0(-V_0) dt + \int_{\tau}^{T/2} V_0(V_0) dt \right] \\
 &= \frac{1}{T} \left[-V_0^2 \left(-\frac{T}{2} + \tau + \frac{T}{2} \right) + V_0^2 \left(\frac{T}{2} - \tau \right) - V_0^2 \tau + V_0^2 \left(\frac{T}{2} - \tau \right) \right] \\
 &= V_0^2 \left[1 - 4 \frac{\tau}{T} \right] \tag{4 - 33}
 \end{aligned}$$

This result is shown (Fig. 4.4c)

Ex. 4.5

A random binary wave consists of a random sequence of binary symbols 1 and 0. The symbol 1 is represented by a pulse of amplitude $+A$ and the symbol 0 is represented by a pulse of amplitude $-A$. The duration of the pulse is T second. The pulses are not synchronized. The starting time of the first complete pulse t_d is equally likely to lie anywhere between 0 and T . The 0's and 1's are equally likely. Obtain $R(\tau)$.

Solution

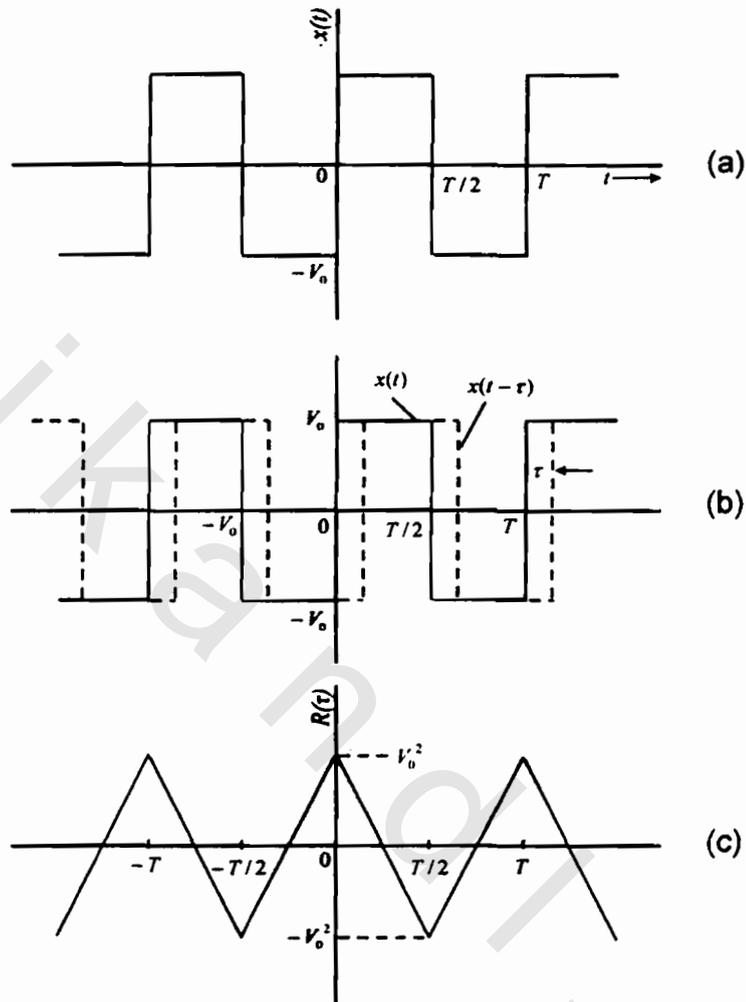
Since t_d is a uniformly distributed random variable

$$f(t_d) = \begin{cases} 1/T & 0 \leq t_d \leq T \\ 0 & \text{otherwise} \end{cases}$$

Since the amplitudes levels $-A, A$ occur with equal probability

$$E[X(t)] = 0 \text{ For all } t$$

Thus, the mean is zero.



(Fig. 4.4). $R(\tau)$ of a square wave
 a) square wave $x(t)$ b) shifting by τ c) $R(\tau)$

To evaluate $E[X(t)X(t-\tau)]$, we consider first the case when $\tau > T$, then $X(t)$ and $X(t-\tau)$ occur in different pulse intervals and are thus independent

$$E[X(t)X(t-\tau)] = E[X(t)] E[X(t-\tau)] = 0$$

$X(t-\tau)$ occurs in the same pulse interval if $t_d < T - \tau$

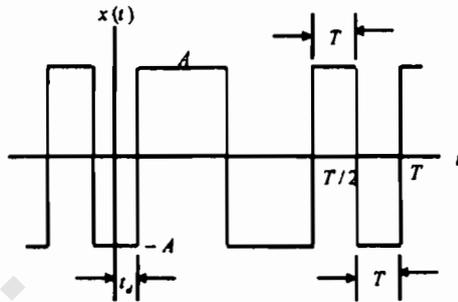


Fig. (4.5) Random binary wave

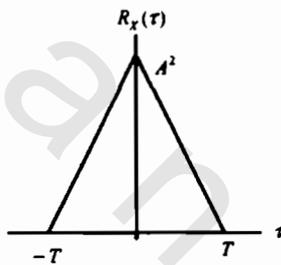


Fig. (4.6) $R(\tau)$ for random binary wave

Thus,

$$\begin{aligned}
 E[X(t)X(t-\tau)] &= \int_0^{T-\tau} A^2 f(t_d) dt_d \\
 &= A^2 \left(1 - \frac{|\tau|}{T}\right) \quad |\tau| < T \\
 &= \begin{cases} A^2 \left(1 - \frac{|\tau|}{T}\right) & |\tau| < T \\ 0 & |\tau| \geq T \end{cases} \quad (4-34)
 \end{aligned}$$

This result is shown (Fig. 4.5)

4.3 Cross correlation:

We define the cross correlation of two random signals $X(t)$ and $Y(t)$ as

$$R_{XY}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) y(t - \tau) dt \quad (4 - 35)$$

It signifies intersymbol interference or cross talk. We note that

$$R_{XY}(\tau) = R_{YX}(\tau) \quad (4 - 36)$$

$$R_{XY}(\tau) \neq R_{XY}(-\tau) \quad (4 - 37)$$

$$R_{XY}(\tau) \leq \sqrt{R_X(0) R_Y(0)} \quad (4 - 38)$$

For statistically independent random processes,

$$R_{XY}(\tau) = R_{YX}(\tau)$$

and if either process has zero mean, then

$$R_{xy}(\tau) = R_{yx}(\tau) = 0 \text{ for all } \tau \quad (4 - 39)$$

Ex 4.6 Two random signals

$$m(t) = A \sin(\omega_1 t + \theta)$$

$$n(t) = B \sin(\omega_2 t + \varphi)$$

where θ, φ vary uniformly from 0 to 2π . Show that the cross correlation

$$R_{mn}(\tau) = 0$$

Solution

$$\begin{aligned} R_{mn}(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} m(t) n(t - \tau) dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} A \sin(\omega_1 t + \theta) B \sin\{\omega_2 t - (\omega_2 \tau - \varphi)\} dt \\ &= \frac{AB}{T} \int_{-T/2}^{T/2} (\sin \omega_1 t \cos \theta + \cos \omega_1 t \sin \theta) \times \\ &\quad \{\sin \omega_2 t \cos(\omega_2 \tau - \varphi) - \cos \omega_2 t \sin(\omega_2 \tau - \varphi)\} dt \\ &= AB \left[\cos \theta \cos(\omega_2 \tau - \varphi) \frac{1}{T} \int_{-T/2}^{T/2} \sin \omega_1 t \sin \omega_2 t dt \right. \\ &\quad \left. + \sin \theta \cos(\omega_2 \tau - \varphi) \frac{1}{T} \int_{-T/2}^{T/2} \cos \omega_1 t \sin \omega_2 t dt \right] \end{aligned}$$

$$\left. \begin{aligned} & -\cos\theta \sin(\omega_2\tau - \varphi) \frac{1}{T} \int_{-T/2}^{T/2} \sin \omega_1 t \cos \omega_2 t \, dt \\ & -\sin\theta \sin(\omega_2\tau - \varphi) \frac{1}{T} \int_{-T/2}^{T/2} \cos \omega_1 t \cos \omega_2 t \, dt \end{aligned} \right] = 0$$

where $\omega_1 \neq \omega_2$, Thus,

$$R_{mn}(\tau) = 0$$

Hence the two signals are statistically independent.

Ex 4.7

Show that the quadrature modulated signals

$$X_1(t) = X(t) \cos(2\pi f_c t + \theta)$$

$$X_2(t) = X(t) \sin(2\pi f_c t + \theta)$$

where θ is a uniformly distributed random variable from 0 to π . are orthogonal.

Solution

$$\begin{aligned} R_{12}(\tau) &= E[X_1(t) X_2(t - \tau)] \\ &= E[X(t) X(t - \tau) \cos(2\pi f_c t + \theta) \sin(2\pi f_c t - 2\pi f_c \tau + \theta)] \\ &= E[X(t) X(t - \tau)] E[\cos(2\pi f_c t + \theta) \sin(2\pi f_c t - 2\pi f_c \tau + \theta)] \\ &= \frac{1}{2} R_X(\tau) E[\sin(4\pi f_c t - 2\pi f_c \tau + 2\theta) - \sin 2\pi f_c \tau] \\ &= -\frac{1}{2} R_X(\tau) \sin(2\pi f_c \tau) \end{aligned}$$

$$\text{At } \tau = 0 \quad \sin 2\pi f_c \tau = 0$$

Thus,

$$R_{12}(0) = E[X_1(t) X_2(t)] = 0$$

Therefore the two signals $X_1(t)$ and $X_2(t)$ are orthogonal to each other from eqn. (4 - 10).

4.4 Characterization of Randomness:

In randomly varying phenomena such as noise, wind, earthquakes, epidemics etc, we may take different sets of measurements or waveforms either for different parameters or same parameter at different, locations or conditions or for the same parameter and same location but at different times. Each sample

measurement at any instant of time is called random variable X whose value varies from one instant of time to another because it is random (stochastic).

Now if we observe the sample measurement of the same measurable quantity at the same location as time proceeds, we have a random process. In terms of photography, a random variable is a still shot in time while a random process is a video or motion picture shooting. When we take a set of sample functions of different measurable quantities or the same measurable quantity but at different conditions at a fixed instant of time, then we have a group of random variables, representing the measurement values at an instant of time, we have again a case representing the measured values at an instant of time i.e., similar to still photography. In this case, the group of random variables $X_1(t_1), X_2(t_1), X_1(t_3) \dots$ are called an ensemble (Fig. 4.7). We may define the ensemble average at instant t_1 as

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx \quad (4-40)$$

where $f(x)$ is the pdf of the random variable constituting the ensemble.

However, we may consider the time average of one measurable quantity at a fixed location and fixed conditions or parameters averaged over time as $\langle x(t) \rangle$

$$\langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt \quad (4-41)$$

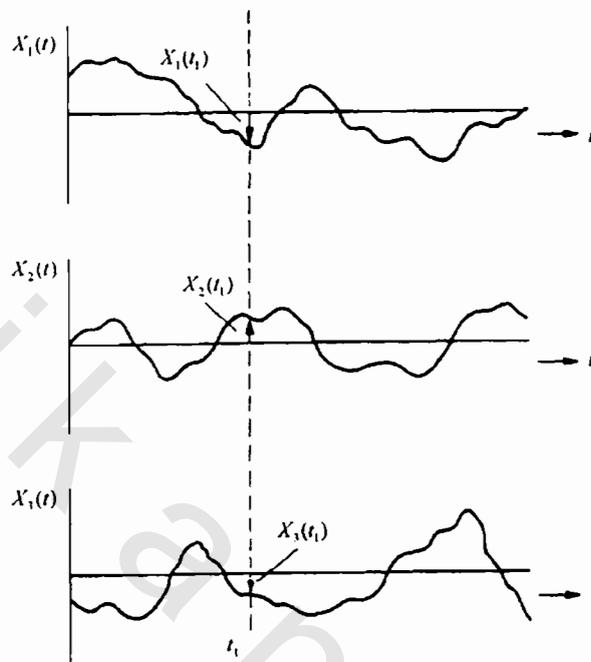
In other words, the output of a random experiment - if considered a random variable - is an unpredictable number, and if considered a random process is, an unpredictable waveform in time.

We may also define the autocorrelation function of the random process $X(t)$ as the expectation value of the product $X(t_1) X(t_2)$

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1) X(t_2)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x(t_2) f\{x(t_1), x(t_2)\} dx(t_1) dx(t_2) \end{aligned} \quad (4-42)$$

where $f\{x(t_1), x(t_2)\}$ is the second order pdf of the process .

Consider now a random process in which the average values of the random variable X is the same using ensemble averaging at an instant t_1 or another instant t_2 , i.e., $E[X]$ is independent of time. We call this case a stationary random process. There are two levels of stationarity.



(Fig. 4.7) Random variables, ensemble, and random process

- i) each of $X_1(t_1), X_2(t_1), X_3(t_1)$ at instant t_1 is a random variable*
- ii) the group of $X_1(t_1), X_2(t_1), X_3(t_1)$ form an ensemble*
- iii) each of $X_1(t)$ or $X_2(t)$ or $X_3(t)$ as time proceeds gives a random process*

One is called strict sense stationarity, which means that the ensemble average is independent of time, and also the autocorrelation function is independent of time. The second case is called wide sense stationarity. It means that the ensemble average is independent of time, but the autocorrelation function depends only on the time interval between the instants of measurement, i.e., eqn. (4 – 42) reduces to

$$R_Y(\tau) = E[X_1(t_1) X_1(t_1 - \tau)] \quad (4 - 43)$$

A further classification of stochastic processes is ergodicity. We define a random process in which the ensemble averaging leads to the same result as time averaging i.e., the process is both stationary and ergodic. Stationarity is essential to be ergodic, but not all stationary processes are ergodic. In ergodic processes, the statistical behavior may be obtained by focusing on one single sample function. Its time characterization gives all statistical properties of the ensemble, i.e.,

$$E[X] = \langle x \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt \quad (4-44)$$

$$E[X^2] = \langle x^2 \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt \quad (4-45)$$

$$R_X(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau) dt \quad (4-46)$$

We may also define the autocovariance function of a stationary random process $X(t)$ as

$$\begin{aligned} C_X(t_1, t_2) &= E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)] \\ &= R_X(t_2 - t_1) - \mu_X^2 \end{aligned} \quad (4-47)$$

Like the autocorrelation function, the autocovariance function of a stationary random process $X(t)$ depends only on the time difference $t_2 - t_1$. If we know the mean and the autocorrelation function of the process, we can readily determine the autocovariance function. Thus the mean and the autocorrelation function are sufficient to describe the first two moments of the process.

To conclude, for strict sense stationarity, μ_X is independent of t while R_X , and C_X are independent of τ . For wide sense stationarity μ_X is independent of t , but R_X and C_X depend on τ . For ergodicity, time averaging leads to the same results as ensemble averaging for all statistical properties $E[X]$, $E[X^2]$, R_X , C_X .

4.5 Wiener Khintchine Theorem:

Assume a random process $X(t)$ applied as an input to a linear time invariant filter of impulse response $h(t)$ producing a new random output $Y(t)$ (Fig. 4.8).

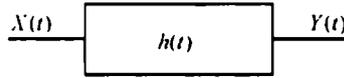
Assuming $X(t)$ to be a wide sense stationary random process.

$$\mu_Y(t) = E[Y(t)] = E \int_{-\infty}^{\infty} h(\tau_1) X(t - \tau_1) d\tau_1 \quad (4-48)$$

For stable (time invariant) system and finite $E X(t)$, we may interchange the order of the expectation and the integration with respect to τ_1 .

$$\begin{aligned} \mu_Y(t) &= \int_{-\infty}^{\infty} h(\tau_1) E[X(t - \tau_1)] d\tau_1 \\ &= \int_{-\infty}^{\infty} h(\tau_1) \mu_X(t - \tau_1) d\tau_1 \end{aligned}$$

But $\mu_X(t)$ is constant for a wide sense stationary random process



(Fig. 4.8) Random process inputted to a time invariant filter

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau_1) d\tau_1 = \mu_X H(0) \quad (4 - 49)$$

where $H(0)$ is the zero frequency transfer function given by

$$H(0) = \int_{-\infty}^{\infty} h(\tau) d\tau \quad (4 - 50)$$

Thus, the mean of a random process produced at the output of a linear time invariant system in response to $X(t)$ is equal to the mean of $X(t)$ multiplied by the dc response of the system. Consider now the autocorrelation function of the output random process $Y(t)$

$$\begin{aligned} R_Y(t_1, t_2) &= E[Y(t_1) Y(t_2)] \\ &= E\left[\int_{-\infty}^{\infty} h(\tau_1) X(t_1 - \tau_1) d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) X(t_2 - \tau_2) d\tau_2 \right] \end{aligned} \quad (4 - 51)$$

If $E[X^2(t)]$ is finite for all t and for a stable system, we may interchange the order of the expectation and integration,

$$\begin{aligned} R_Y(t_1, t_2) &= \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2) E[X(t_1 - \tau_1) X(t_2 - \tau_2)] \\ &= \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2) R_X(t_1 - \tau_1, t_2 - \tau_2) \end{aligned} \quad (4 - 52)$$

When the input $X(t)$ is a wide sense stationary random process, the autocorrelation function of $X(t)$ is only a function of the difference between the observation times $t_1 - \tau_1$ and $t_2 - \tau_2$. Putting $\tau = t_1 - t_2$

$$\begin{aligned} R_X(t_1, t_2) &= R_X(t_2 - t_1) \\ R_X(t_1 - \tau_1, t_2 - \tau_2) &= R_X(t_1 - t_2 - \tau_1 + \tau_2) \\ &= R_X(\tau - \tau_1 + \tau_2) \end{aligned}$$

Thus,

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \quad (4 - 53)$$

We see from eqns. (4 - 53) and (4 - 49) that if the input to a stable linear time invariant filter is a wide sense stationary random process, then the output of the filter is also a wide sense stationary random process. From eqn. (4 - 21)

$$R_Y(0) = E[Y^2(t)] \quad (4 - 54)$$

Thus, the mean square value of the output of the random process $Y(t)$ is obtainable by putting $\tau = 0$ in eqn. (4 - 53)

$$E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \quad (4 - 55)$$

Thus far we have considered the response of a time invariant linear filter to a wide sense stationary random process input in the time domain. We now turn to the frequency domain. We note

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f) e^{j2\pi f\tau_1} df \quad (4 - 56)$$

Thus,

$$\begin{aligned} E[Y^2(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(f) e^{j2\pi f\tau_1} df \right] h(\tau_2) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} H(f) df \int_{-\infty}^{\infty} h(\tau_2) d\tau_2 \int_{-\infty}^{\infty} R_X(\tau_2 - \tau_1) e^{j2\pi f\tau_1} d\tau_1 \end{aligned} \quad (4 - 57)$$

Defining $\tau' = \tau_2 - \tau_1$, $\tau_1 = \tau_2 - \tau'$, $|d\tau_1| = |d\tau'|$ since we are integrating for all possible values of τ'

$$E[Y^2(t)] = \int_{-\infty}^{\infty} H(f) df \int_{-\infty}^{\infty} h(\tau_2) e^{j2\pi f\tau_2} d\tau_2 \int_{-\infty}^{\infty} R_X(\tau') e^{-j2\pi f\tau'} d\tau' \quad (4 - 58)$$

Noting

$$H^*(f) = \int_{-\infty}^{\infty} h(\tau_2) e^{j2\pi f\tau_2} d\tau_2 \quad (4 - 59)$$

where $H^*(f)$ is the conjugate of $H(f)$

Thus,

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} R_X(\tau') e^{-j2\pi f\tau'} d\tau' \quad (4 - 60)$$

Noting that the second integral is the Fourier transform of $R_X(\tau')$. Defining it as

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau') e^{-j2\pi f\tau'} d\tau' \quad (4 - 61)$$

We call $S_X(f)$ the power spectral density (PSD) or power spectrum of the wide sense stationary random process $X(t)$

Thus,

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df \quad (4 - 62)$$

Let us define the power spectral density of the output $S_Y(f)$ as

$$S_Y(f) = |H(f)|^2 S_X(f) \quad (4 - 63)$$

Thus,

$$E[Y^2(f)] = \int_{-\infty}^{\infty} S_Y(f) df \quad (4 - 64)$$

Thus, the mean square value of the output of a stable linear time invariant filter in response to a wide sense stationary process is equal to the area under the output power spectral density. The output PSD is the product of the input PSD multiplied by the square of the transfer function. We see that the PSD and the autocorrelation function form a Fourier transform pair

$$S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau \quad (4 - 65)$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} S_Y(f) e^{j2\pi f\tau} df \quad (4 - 66)$$

This is called Wiener Khintchine Theorem. Alternatively,

$$S_Y(\omega) = \int_{-\infty}^{\infty} R_Y(\tau) e^{-j\omega\tau} d\tau \quad (4 - 67)$$

$$R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) e^{j\omega\tau} d\omega \quad (4 - 68)$$

Ex. 4.8

A random process $X(t)$ is passed through an ideal narrow band filter with a transfer characteristic $H(f) = 1$ for $|f \pm f_c| < \Delta f / 2$ and 0 for $|f \pm f_c| > \Delta f / 2$, where f_c is the center frequency and Δf is the filter bandwidth (Fig. 4.9). Find the output power P .

Solution

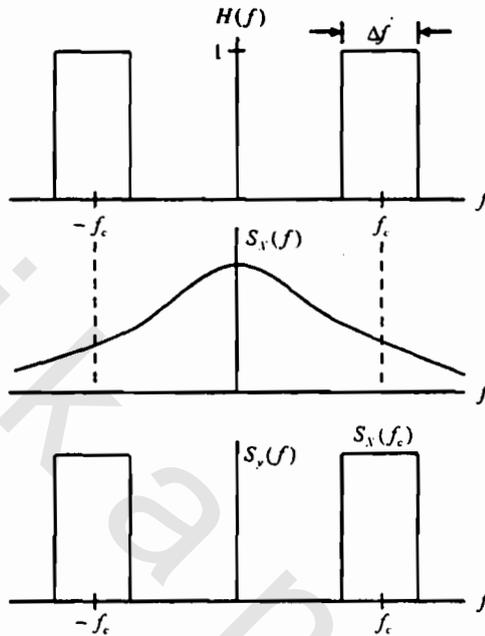
If Δf is sufficiently small compared to f_c , the filter picks up the area under $S_X(f)$ within its passband. Thus, $S_Y(f_c)$ represents the output PSD within the passband of the filter $S_Y(f)$ and the total output power is the area under $S_Y(f)$.

$$\begin{aligned} E[Y^2(t)] &= \int_{-\infty}^{\infty} S_Y(f) df \\ &= 2\Delta f S_Y(f_c) \end{aligned} \quad (4 - 69)$$

We note

$$P = \int_{-\infty}^{\infty} S(f) df \quad (4 - 70)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \quad (4 - 71)$$



(Fig. 4.9). PSD of the output from narrow BPF

4.6 Properties of the Power Spectral Density:

There are important properties for the PSD function:

1.
$$S_x(0) = \int_{-\infty}^{\infty} R_x(\tau) d\tau \quad (4 - 72)$$

This follows directly from eqn. (4 - 74) by putting $t = 0$. Thus the zero frequency value of the PSD of a wide sense stationary random process is equal to the total area under the graph of the autocorrelation function.

2.
$$R_x(0) = E[X^2(t)] = \int_{-\infty}^{\infty} S_x(f) df \quad (4 - 73)$$

This follows directly from eqn. (4 - 65) and eqn. (4 - 21) by putting $\tau = 0$. Thus, the mean square value of a wide sense stationary random process is equal to the total area under the graph of the PSD

3.
$$S_x(f) \geq 0 \text{ for all } f \quad (4 - 74)$$

The PSD of a wide sense stationary random process is always non negative. This follows from eqn. (4 - 62) where $E[Y^2(t)]$ is always non negative.

$$4. \quad S_X(-f) = S_X(f) \quad (4 - 75)$$

The PSD of a real valued random process is an even function of f . This follows from eqn. (4 - 65) by substituting $-f$ for f and from eqn. (4 - 22)

$$S_X(-f) = \int_{-\infty}^{\infty} R(\tau) e^{j2\pi f\tau} d\tau$$

Now substitute $-\tau$ for τ , noting $R_X(-\tau) = R_X(\tau)$

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau = S_X(f)$$

5. The PSD appropriately normalized has the properties of *pdf*. We define

$$f_X(f) = \frac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f) df} \geq 0 \text{ for all } f \quad (4 - 76)$$

$$\int_{-\infty}^{\infty} f_X(f) df = 1 \quad (4 - 77)$$

This is called the normalized form of the PSD.

6. We define the *rms* bandwidth of a wide sense stationary process $X(t)$, B_{rms} as

$$B_{rms} = \sqrt{\int_{-\infty}^{\infty} f^2 f_X(f) df} \quad (4 - 78)$$

$$= \sqrt{\frac{\int_{-\infty}^{\infty} f^2 f_X(f) df}{\int_{-\infty}^{\infty} S_X(f) df}} \quad (4 - 79)$$

7. We may relate the PSD to Fourier transform as follows. For a signal of finite energy, Parseval's theorem dictates.

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad (4 - 80)$$

where E is the energy across 1Ω resistor for $f(t)$ being voltage or current.

Thus, $|F(\omega)|^2$ is the normalized energy per unit frequency in 1Ω resistance, hence, called energy spectral density or energy density. The total area under $|F(\omega)|^2$ is the energy of the signal.

For power signals,

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt \quad (4 - 81)$$

Assume a truncated portion of a power signal in the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$. The truncation or gating function is $g_T(t/T)$. Thus,

$$F_T(\omega) = \mathcal{F} [f(t) g_T(t/T)] \quad (4-82)$$

From eqn. (4-81)

$$P = \langle f^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} |f(t)|^2 dt \quad (4-83)$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega \quad (4-84)$$

$$\begin{aligned} P &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega \end{aligned} \quad (4-85)$$

Thus

$$S(\omega) = \lim_{T \rightarrow \infty} \frac{|F_T(\omega)|^2}{T} \quad (4-86)$$

Ex 4.9

A sinusoidal wave with random phase $X(t) = A \cos(2\pi f_c t + \theta)$, where θ is a uniformly distributed random variable over the interval $-\pi, \pi$. Find the PSD and the total power.

Solution

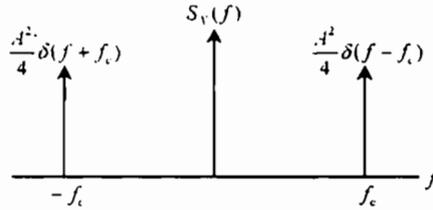
We have seen in Ex 4.1, eqn. (4-32)

$$R_X(\tau) = \frac{A^2}{2} \cos 2\pi f_c \tau \quad (4-87)$$

Using eqn. (4-67), i.e., taking Fourier transform of $R_X(\tau)$

$$S_X(f) = \frac{A^2}{4} [\delta(f - f_c) + \delta(f + f_c)] \quad (4-88)$$

which consists of a pair of delta functions weighted by the factor $\frac{A^2}{4}$ and located at $\pm f_c$ (Fig. 4.10). Since the total area under the delta function is 1, the total area under $S_X(f)$ becomes $\frac{A^2}{2}$, which is $R_X(0)$ in eqn. (4-87) according to eqn. (4-73)



(Fig. 4.10) PSD of a sinusoidal wave with random phase

Ex 4.10

A random binary wave consists of a random sequence of binary symbols 1 and 0. The symbol 1 is represented by a pulse of amplitude $+A$ and the symbol 0 is represented by a pulse of amplitude $-A$. The duration of the pulse is T second. The starting time of the first complete pulse is equally likely to lie anywhere between 0 and T . Find the PSD and total power.

Solution

We have seen in Ex 4.5, eqn. (4 – 34)

$$R_X(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{T}\right) & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

From eqn. (4 – 65)

$$S_X(f) = \int_{-T}^T A^2 \left(1 - \frac{|\tau|}{T}\right) e^{-j2\pi f\tau} d\tau$$

This is Fourier transform of a triangular wave which is

$$S_X(f) = A^2 T \text{sinc}^2(f T) \tag{4 – 89}$$

Noting that $R_X(0) = A^2$, and using eqn. (4 – 73)

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df = R_X(0) = A^2 \tag{4 – 90}$$

We may also note that the energy spectral density of a rectangular pulse $g_T(t/T)$ of amplitude A and duration T is given by

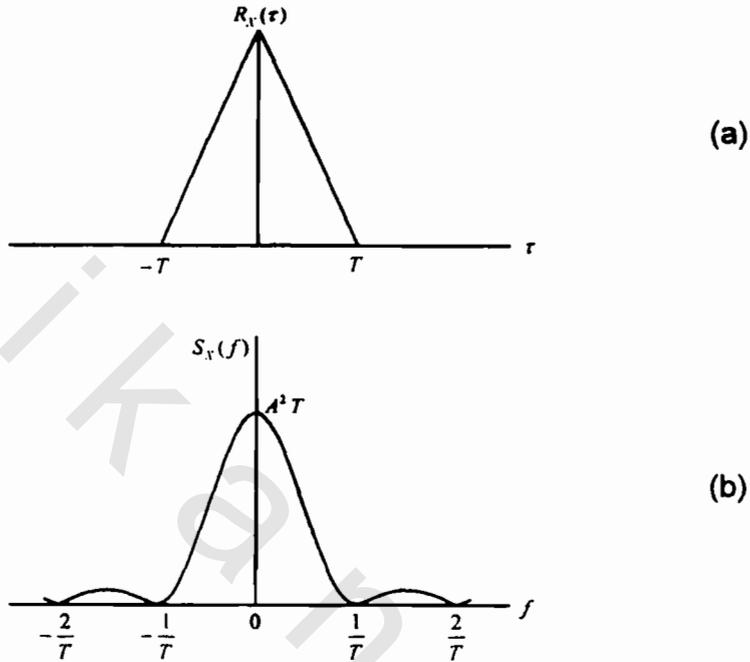
$$E(f) = A^2 T^2 \text{sinc}^2(f T) \tag{4 – 91}$$

But

$$S(f) = A^2 T \text{sinc}^2(f T)$$

Therefore,

$$S(f) = \frac{E(f)}{T} \tag{4 – 92}$$



(Fig. 4.11) Random binary wave

a) $R_x(\tau)$ b) $S_x(f)$

Thus, for a random binary wave, the PSD is the bit energy divided by the bit duration

Ex 4.11

Consider a wide sense random process $X(t)$ and a sinusoidal waveform $\cos(2\pi f_c t + \theta)$, where θ is a random variable which is uniformly distributed over the interval $0, 2\pi$. Find the PSD of the composite function $Y(t) = X(t) \cos(2\pi f_c t + \theta)$

Solution

The random phase θ signifies that the time origin of the sinusoidal wave is arbitrarily chosen, so that we have two physically independent random processes $X(t)$ and $\theta(t)$

From eqn. (4 - 51)

$$R_y(\tau) = E[Y(t+\tau) Y(t)]$$

$$\begin{aligned}
&= E[X(t+\tau) \cos(2\pi f_c t + 2\pi f_c \tau + \theta) X(t) \cos(2\pi f_c t + \theta)] \\
&= E[X(t+\tau) X(t)] E[\cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cos(2\pi f_c t + \theta)] \\
&= \frac{1}{2} R_X(\tau) E[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)] \\
&= \frac{1}{2} R_X(\tau) \cos(2\pi f_c \tau) \tag{4-93}
\end{aligned}$$

Thus, the autocorrelation of the output is the same as the autocorrelation of the sinusoidal wave modulated by the autocorrelation of the modulating signal. From eqn. (4-65)

$$S_Y(f) = \frac{1}{4} [S_X(f - f_c) + S_X(f + f_c)] \tag{4-94}$$

Thus, the PSD of the output is the same as the PSD of the modulating random signal but shifted $\pm f_c$ in the frequency spectrum and divided by 4.

Ex 4.12

Consider the comb filter (Fig. 4.12) consisting of a delay line and a summing device obtain the PSD of the output.

Solution

The transfer function of the filter is

$$\begin{aligned}
H(f) &= 1 - e^{-j2\pi f T} \\
&= 1 - \cos 2\pi f T + j \sin 2\pi f T \tag{4-95}
\end{aligned}$$

$$\begin{aligned}
|H(f)|^2 &= [(1 - \cos(2\pi f T))]^2 + \sin^2(2\pi f T) \\
&= 2[1 - \cos(2\pi f T)] \\
&= 4\sin^2(\pi f T) \tag{4-96}
\end{aligned}$$

From eqn. (4-72)

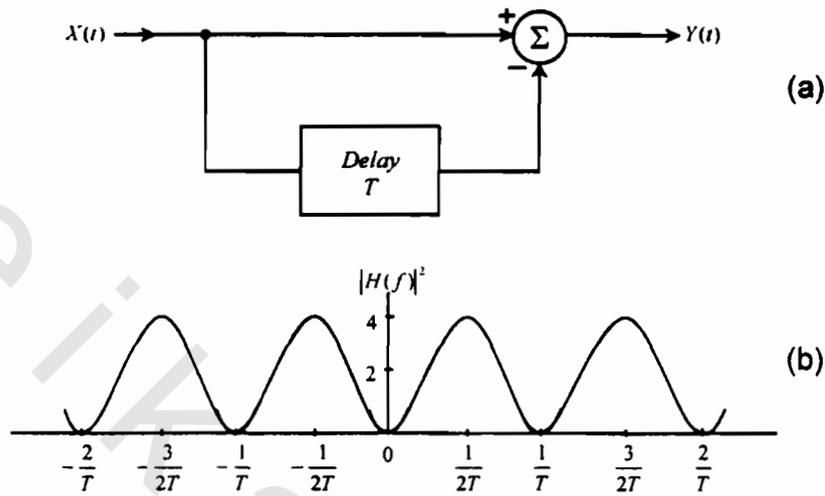
$$S_Y(f) = 4\sin^2(\pi f T) S_X(f) \tag{4-97}$$

For $f < \frac{1}{T}$, $\sin \pi f T \approx \pi f T$

Thus

$$S_Y(f) \cong 4\pi^2 f^2 T^2 S_X(f) \tag{4-98}$$

Thus, the comb filter acts as a differentiator for low frequencies.



(Fig. 4.12) PSD of the comb filter
 a) block diagram b) frequency response

4.7 Cross Spectral Density:

The PSD provides a measure of the frequency distribution of a single random process. Similarly, we may define the cross spectral density to provide the frequency dependence between two random processes. Let $X(t)$ and $Y(t)$ be two wide sense stationary random processes with their cross correlation functions denoted by $R_{XY}(\tau)$ and $R_{YX}(\tau)$. We define the cross spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ as

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f\tau} d\tau \quad (4-99)$$

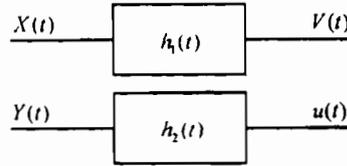
$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j2\pi f\tau} d\tau \quad (4-100)$$

Thus, the cross correlation functions and the cross spectral densities form Fourier transform pairs. Thus, using the inverse Fourier transformation

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(f) e^{j2\pi f\tau} df \quad (4-101)$$

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} S_{YX}(f) e^{j2\pi f\tau} df \quad (4-102)$$

We note that the cross spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ are real functions of $R_{XY}(\tau)$ is real and



(Fig. 4.13) Cross spectral density

$$R_{XY}(\tau) = R_{YX}(-\tau) \quad (4-103)$$

$$S_{XY}(f) = S_{YX}(-f) = S_{YX}^*(f) \quad (4-104)$$

Ex 4.13

Consider two jointly wide sense stationary random processes passing through two separate time invariant linear filters. $X(t)$ is the input to the filter whose impulse response is $h_1(t)$, and $Y(t)$ is the input of the filter whose impulse response is $h_2(t)$. $V(t)$ is the output the first filter and $u(t)$ is the output of the second filter. Obtain the cross spectral density $S_{VU}(f)$

Solution

We find the cross correlation of the two outputs

$$\begin{aligned} R_{VU}(t_1, t_2) &= E[V(t_1) U(t_2)] \\ &= E\left[\int_{-\infty}^{\infty} h_1(\tau_1) X(t_1 - \tau_1) d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2) Y(t_2 - \tau_2) d\tau_2\right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) E[X(t_1 - \tau_1) Y(t_2 - \tau_2)] d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) R_{XY}(t_1 - \tau_1, t_2 - \tau_2) d\tau_1 d\tau_2 \quad (4-105) \end{aligned}$$

where $R_{XY}(t_1, t_2)$ is the cross correlation of the two inputs $X(t)$, $Y(t)$.

Because the input random processes are jointly wide sense stationary we may set $\tau = t_1 - t_2$

$$R_{VU}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) R_{XY}(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \quad (4-106)$$

Taking Fourier transform of both sides, as we did in eqn. (4-57)

$$S_{VU}(f) = H_1(f) H_2^*(f) S_{XY}(f) \quad (4-107)$$

This is the relation between the cross spectral density of the two outputs and the cross spectral density of the two inputs.

Ex 4.14

Suppose that the random processes $X(t)$ and $Y(t)$ have zero mean and are individually wide sense stationary. Consider the sum random process

$$Z(t) = X(t) + Y(t)$$

Determine the PSD of $Z(t)$

Solution

$$\begin{aligned} R_Z(t_1, t_2) &= E[Z(t_1) Z(t_2)] \\ &= E[\{X(t_1) + Y(t_1)\}\{X(t_2) + Y(t_2)\}] \\ &= E[X(t_1) X(t_2)] + E[X(t_1) Y(t_2)] \\ &\quad + E[Y(t_1) X(t_2)] + E[Y(t_1) Y(t_2)] \\ &= R_X(t_1, t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2) + R_Y(t_1, t_2) \end{aligned}$$

Defining $\tau = t_1 - t_2$

$$R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$$

Taking Fourier transform of both sides

$$S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f) \tag{4 -108}$$

When the wide sense stationary random processes $X(t)$ and $Y(t)$ are uncorrelated, the cross spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ are zero.

$$S_Z(f) = S_X(f) + S_Y(f) \tag{4 -109}$$

Thus, for a sum of zero mean wide sense stationary random processes that are uncorrelated with each other, the PSD of the sum is equal to the sum of their individual power spectral densities. This result will have an important bearing in the discussion on noise in the next chapter.

Ex 4.14

Show how $R_X(\tau)$ relates to power

Solution

For a random variable $X(t)$, we have

$$R_X(0) = \langle x^2 \rangle = \text{total power} \tag{4 -110}$$

The dc power $\langle x(t) \rangle^2$ is given by

$$R_X(\infty) = \langle x(t) \rangle^2 \tag{4 -111}$$

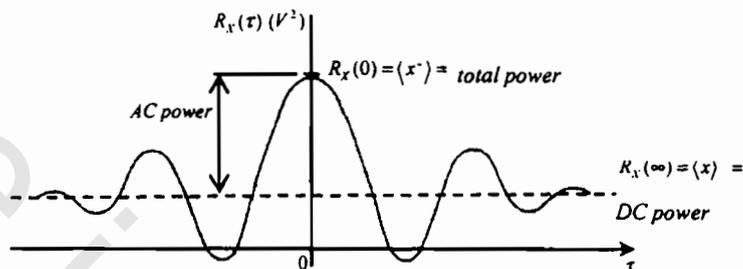


Fig. (4.14) Power relations

From (Fig. 4.14), $R_x(0) - R_x(\infty)$ is the variance σ^2 of $x(t)$ given by

$$R_x(0) - R_x(\infty) = \langle x^2(t) \rangle - \langle x(t) \rangle^2 = \sigma^2 \quad (4-112)$$

Since the total power is $R_x(0) = \langle x^2 \rangle$.

$$R_x(0) = \langle x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 + \langle x \rangle^2 \quad (4-113)$$

$$\text{Total power} = \text{AC power} + \text{DC power} \quad (4-114)$$

This is to be compared with the following

$$\sigma_x^2 = E[(x - \mu_x)^2] \quad (4-115)$$

$$= E[x^2] - 2\mu_x[E] + \mu_x^2$$

$$= E[x^2] - 2\mu_x^2 + \mu_x^2$$

$$= E[x^2] - \mu_x^2 \quad (4-116)$$

$$E[x^2] = \sigma_x^2 + \mu_x^2 \quad (4-117)$$

The total power $E[x^2]$ is equal to the sum of the AC power σ_x^2 and the DC power μ_x^2

Problems

1. Compare the result of Ex 4.9 with that of a sinusoidal wave with constant phase. What do you conclude?
2. Verify eqn. (4 – 91) for a rectangular pulse showing that for a binary wave the PSD is the bit energy divided by the bit duration.
3. Repeat Ex 4.10 where the symbol 1 is $+AV$ and the symbol 0 is $0V$.
4. A carrier of frequency f_c is amplitude modulated by a random signal in the form of a random binary wave. Obtain the PSD of the output and the total power. What do you conclude?
5. Two random signals $X(t)$ and $Y(t)$ of the same amplitude and duration each of which is a random binary wave. Obtain the PSD of their sum. Calculate the cross power densities. What do you conclude?
6. Repeat the problem above if the two waves differ in amplitude and duration.
7. If the two waves above are each inputted to a band pass filter centered at $\pm f_c$ and having one sided width Δf find the cross spectral density of the two inputs and the two outputs. What do you conclude?
8. Find B_{rms} for a sinusoidal wave with random phase.
9. Find the autocorrelation function of a square pulse. Sketch the result.
10. Find the cross correlation function of two square pulse waves of the same frequency and same amplitude separated by a constant time shift τ_0 . Sketch the result.
11. Find the cross correlation function of a sinusoid and a square wave of the same period.
12. Find the autocorrelation function, and the decorrelation time of a rectangular pulse, and hence find the energy spectral density
Hint: for a transient signal the energy spectral density and the autocorrelation function form a Fourier transform pair.

13. Find and sketch the autocorrelation function of a stationary random signal whose power spectral density is triangular. Then find the decorrelation time. Take $f_c = 5 \text{ kHz}$, $\Delta f = \pm 1 \text{ kHz}$.

14. If $X(t)$ and $Y(t)$ are random variables, show that $\frac{1}{2}[R_X(0) + R_Y(0)] > R_{XY}(\tau)$ for all τ .

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